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Integrable Solutions and Continuous Dependence of a Nonlinear Singular Integral Inclusion of Fractional Orders and Applications

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Abstract. Let $\mathcal{E}$ be a reflexive Banach space. In this article we study the existence of integrable solutions in the space of all lesbesgue integrable functions on $\mathcal{E}$, $L^1([0,\mathcal{T}],\mathcal{E})$, of the nonlinear singular integral inclusion of fractional orders beneath the assumption that the multi-valued function G has Lipschitz selection in $\mathcal{E}$. The main tool applied in this work is the Banach contraction fixed point theorem. Moreover, the paper explores a qualitative property associated with these solutions for the given problem such as the continuous dependence of the solutions on the set of selections $S^1_{G(\tau,x(\tau))}$. As an application, the existence of integrable solutions of the two nonlocal and weighted problems of the fractional differential inclusion is investigated. We additionally provide an example given as a numerical application to demonstrate the effectiveness and value of our results.

1. Introduction

Let $\mathcal{I} = [0,\mathcal{T}]$ and let $\mathcal{E}$ be a reflexive Banach space with the norm $\|.\|_{\mathcal{E}}$. Indicate by $L^1(\mathcal{I},\mathcal{E})$ the Banach space of all Lebesgue integrable functions $x : \mathcal{I} \to \mathcal{E}$ defined on the interval $\mathcal{I}$ and taking values in $\mathcal{E}$ with the norm

$$\|x\|_{L^1} = \int_0^{\mathcal{T}} \|x(\tau)\|_{\mathcal{E}} d\tau.$$

Functional inclusions and functional differential inclusions have been broadly examined by several creators and there are numerous curiously comes about concerning these issues(see [1]- [8]). The nonlinear integral equations recently studied by many authors for example (see [9]- [11]), where authors examine the solvability of non-linear 2D Volterra integral equations through Petryshyn fixed point theorem in Banach space, two systems of nonlinear Volterra integral equation and Volterra integro-differential equation through Banach's contraction principle and a nonlinear integral equation with multiple variable time delays and a nonlinear integro-differential equation

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without delay by the fixed point method using progressive contractions. Also, consider some properties of this solution. Also, a functional integral inclusion was discussed by B.C. Dhage and D. O'Regan (see [12]- [13]), they proved the existence of extremal solutions using Caratheodory's conditions on the multi-valued function. However, in this article, we establish our results using Lipschitz condition on the multi-valued function. Theorems which guarantee the existence of the solutions for the inclusions problems are generally obtained under the assumptions that the multi-valued function is either lower or upper semi-continuous (see [13]- [14]) and for the discontinuity of the multi-valued function (see [15]). In addition the fractional differential equations was investigated by a number of authors (see [16]- [19]) in which the authors in [16] establish sufficient conditions for the existence of solutions of such problems and the authors in [17] and [19] investigated the boundedness of solutions and investigate qualitative properties of solutions of these equations. The integrable solution for some functional equations and functional integral equations was discussed by Banas (see [20]- [21]) based on the technique associated with the notion of a measure of weak noncompactness.

In ([22]- [27]) the Lipschitz selections of the multi-valued functions was investigated.

Assume the nonlinear singular integral inclusion of fractional orders, $\alpha, \beta \in (0,1)$,

$$x(\tau) \in \frac{\tau^{\alpha-1}}{\Gamma(\alpha)} A + I^{\beta} G(\tau, x(m(\tau))), \ \tau \in \mathcal{I} \tag{1.1}$$

where $G : \mathcal{I} \times \mathcal{E} \to \chi(\mathcal{E})$ is a nonlinear multi-valued mapping and $\chi(\mathcal{E})$ is the power set of nonempty subsets of $\mathcal{E}$.

Here, in our article we study the existence of integrable solutions $x \in L^1(\mathcal{I}, \mathcal{E})$ of the nonlinear singular integral inclusion of fractional orders (1.1) in $\mathcal{E}$. We proves the existence theorem of that inclusion in the space $L^1(\mathcal{I}, \mathcal{E})$ using Banach contraction fixed point theorem and with the assumption that the multi-valued function G satisfy Lipschitz condition.

We study a qualitative property associated with these solutions for the given problem such as the continuous dependence of the solutions on the set of selections $S^1_{G(\tau, x(\tau))}$. We provide an example given as numerical application to demonstrate the effectiveness and value of our results. Finally, As an application, we study the existence of integrable solutions of the two nonlocal and weighted problems of the fractional differential inclusion

$${}^{R}D^{\alpha} x(\tau) \in G(\tau, x(m(\tau))), \ \tau \in \mathcal{I} \tag{1.2}$$

with each one of the nonlocal condition

$$I^{1-\alpha} x(\tau)|_{\tau=0} = A, \ A \in \mathcal{E} \tag{1.3}$$

or the weighted condition

$$\tau^{1-\alpha} x(\tau)|_{\tau=0} = \frac{A}{\Gamma(\alpha)}, \ A \in \mathcal{E} \tag{1.4}$$

where ${}^{R}D^{\alpha}$ is Riemann-Liouville derivative.

2. Preliminaries

Here, we display a few documentations and assistant comes about that will be required in this work.

Definition 2.1. *[2] A multi-valued map G from $\mathcal{I} \times \mathcal{E}$ to the family of all nonempty closed subsets of $\mathcal{E}$ is called Lipschitzian if there exists $b > 0$ such that for all $\tau \in \mathcal{I}$ and all $x_1, x_2 \in \mathcal{E}$, we have*

$$\mathcal{H}(G(\tau, x_1(\tau)), G(\tau, x_2(\tau))) \leq b\|x_1(\tau) - x_2(\tau)\|_{\mathcal{E}}$$

where $\mathcal{H}(C, D)$ is the Hausdorff metric among the two subsets $C, \mathcal{D} \in \mathcal{I} \times \mathcal{E}$.

Denote $S^1_{G(\tau,x(\tau))} = Lip(\mathcal{I}, \mathcal{E})$ be the set of all Lipschitz selections of G.
Now, we state the Banach contraction fixed point theorem (see [28]).

Theorem 2.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a map such that $d(f(x), f(y)) \leq Cd(x, y)$ for some $0 \leq C < 1$ and all $x, y \in X$. Then f has a unique fixed point in X.*

3. Existence of solution

In this section, we introduce the main result by proving the existence of integrable solution $x \in L^1(\mathcal{I}, \mathcal{E})$ of the inclusion (1.1) in $\mathcal{E}$ with the assumption that the multi-valued function G satisfy Lipschitz condition.

Definition 3.1. *By integrable solution of the inclusion (1.1) in $\mathcal{E}$, we mean a single-valued function $x \in L^1(\mathcal{I}, \mathcal{E})$, which fulfills (1.1).*

Consider now the inclusion (1.1) with the assumptions:
(H1) The set $G(\tau, x)$ is compact and convex for all $(\tau, x) \in \mathcal{I} \times \mathcal{E}$.
(H2) The multi-valued map G is Lipschitzian with a Lipschitz constant $b > 0$ such that

$$\mathcal{H}(G(\tau, x_1(\tau)), G(\tau, x_2(\tau))) \leq b\|x_1(\tau) - x_2(\tau)\|_{\mathcal{E}}$$

for all $\tau \in \mathcal{I}$ and $x_1, x_2 \in \mathcal{E}$, where $\mathcal{H}(C, D)$ is the Hausdorff metric among the two subsets $C, \mathcal{D} \in \mathcal{I} \times \mathcal{E}$.
(H3) The set of selections $S^1_{G(\tau,x(\tau))}$ of Lipschitz type of the multi-valued function G is nonempty.
(H4) The function $m : \mathcal{I} \to \mathcal{I}$, $m(\tau) \leq \tau$ is continuous function.
(H5) $A \in \mathcal{E}$.
(H6) A constant $M > 0$ exist such that $m'(\tau) > M$, $\forall\, \tau \in \mathcal{I}$.

Remark 3.1. *According to the assumptions (H1)-(H3), there exists a Lipschitz selection $g \in S^1_{G(\tau,x(\tau))}$ such that*

$$\|g(\tau, x(m(\tau)))\|_{\mathcal{E}} \leq \|a_r(\tau)\|_{\mathcal{E}} + b\|x(m(\tau))\|_{\mathcal{E}}.$$

And this selection satisfy the nonlinear singular integral equation of fractional orders

$$x(\tau) = \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A + I^{\beta}g(\tau, x(m(\tau))),\ \tau \in \mathcal{I}. \tag{3.1}$$

Hence the solution of the nonlinear singular integral equation of fractional orders (3.1), if it exists, is a solution of the nonlinear singular integral inclusion of fractional orders (1.1).

Now, we seek about the existence of integrable solution of the nonlinear singular integral equation of fractional orders (3.1).

Theorem 3.1. *Consider the assumptions (H1)-(H6) be satisfied. Then ∃ an integrable solution $x \in L^1(\mathcal{I}, \mathcal{E})$ of (3.1).*

Proof. Define the operator $\mathcal{B}$ by

$$\mathcal{B}x(\tau) = \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A + I^{\beta}g(\tau, x(m(\tau))),\ \tau \in \mathcal{I}$$

consider the set Ω_r defined as
$\Omega_r = \{x \in L^1(\mathcal{I}, \mathcal{E}), \|x\|_{L^1} \leq r\}$; $r = \frac{\|A\|_{\mathcal{E}}K_1 + K_2\|a_r\|_{L^1}}{1-\frac{b}{M}K_2}$.
Hence, it is shown that Ω_r is nonempty, bounded, compact and convex set.
Let $x \in \Omega_r$ be arbitrary, then

$$\begin{aligned}
\| & \mathcal{B}x(\tau)\|_{\mathcal{E}} \\
= & \|\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A + I^{\beta}g(\tau, x(m(\tau)))\|_{\mathcal{E}} \\
\leq & \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\|A\|_{\mathcal{E}} + I^{\beta}\|g(\tau, x(m(\tau)))\|_{\mathcal{E}} \\
\leq & \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\|A\|_{\mathcal{E}} + I^{\beta}\{\|a_r(s)\|_{\mathcal{E}} + b\|x(m(\tau))\|_{\mathcal{E}}\} \\
\leq & \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\|A\|_{\mathcal{E}} + I^{\beta}\|a_r(s)\|_{\mathcal{E}} + bI^{\beta}\|x(m(\tau))\|_{\mathcal{E}} \\
\leq & \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\|A\|_{\mathcal{E}} + \int_0^{\tau}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|a_r(s)\|_{\mathcal{E}}ds + b\int_0^{\tau}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|x(m(s))\|_{\mathcal{E}}ds
\end{aligned}$$

taking $m(s) = u$ and $ds = \frac{du}{m'(s)}$, then

$$\begin{aligned}
\| & \mathcal{B}x(\tau)\|_{\mathcal{E}} \\
\leq & \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\|A\|_{\mathcal{E}} + \int_0^{\tau}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|a_r(s)\|_{\mathcal{E}}ds + b\int_{m(0)}^{m(\tau)}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|x(u)\|_{\mathcal{E}}\frac{du}{m'(s)} \\
\leq & \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}\|A\|_{\mathcal{E}} + \int_0^{\tau}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|a_r(s)\|_{\mathcal{E}}ds + \frac{b}{M}\int_0^{\tau}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|x(u)\|_{\mathcal{E}}du
\end{aligned}$$

Therefore

$$\begin{aligned}
\| & \mathcal{B}x\|_{L^1} \\
\leq & \int_0^{\mathcal{T}}\|A\|_{\mathcal{E}}\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}d\tau + \int_0^{\mathcal{T}}\int_0^{\tau}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|a_r(s)\|_{\mathcal{E}}dsd\tau + \frac{b}{M}\int_0^{\mathcal{T}}\int_0^{\tau}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|x(u)\|_{\mathcal{E}}dud\tau \\
\leq & \int_0^{\mathcal{T}}\|A\|_{\mathcal{E}}\frac{t^{\alpha-1}}{\Gamma(\alpha)}d\tau + \int_0^{\mathcal{T}}\|a_r(s)\|_{\mathcal{E}}(\int_s^{\mathcal{T}}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}d\tau)ds + \frac{b}{M}\int_0^{\mathcal{T}}\|x(u)\|_{\mathcal{E}}(\int_s^{\mathcal{T}}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}d\tau)du
\end{aligned}$$

$$\begin{aligned}
&\leq \|A\|_{\mathcal{E}}\frac{\mathcal{T}^{\alpha}}{\Gamma(\alpha+1)}+\int_0^{\mathcal{T}}\|a_r(s)\|_{\mathcal{E}}(\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)})ds+\frac{b}{M}\int_0^{\mathcal{T}}\|x(u)\|_{\mathcal{E}}(\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)})du\\
&\leq \|A\|_{\mathcal{E}}\frac{\mathcal{T}^{\alpha}}{\Gamma(\alpha+1)}+\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\|a_r\|_{L^1}+\frac{b}{M}\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\|x\|_{L^1}\\
&\leq \|A\|_{\mathcal{E}}K_1+K_2\|a_r\|_{L^1}+\frac{b}{M}K_2r=r,
\end{aligned}$$

where $r=\frac{\|A\|_{\mathcal{E}}K_1+K_2\|a_r\|_{L^1}}{1-\frac{b}{M}K_2}$, $K_1=\frac{\mathcal{T}^{\alpha}}{\Gamma(\alpha+1)}$, $K_2=\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}$.
Then

$$\|\mathcal{B}x\|_{L^1}\leq r$$

Hence, $\mathcal{B}x\in\Omega_r$, which proves that $\mathcal{B}\Omega_r\subset\Omega_r$ and $\mathcal{B}:\Omega_r\to\Omega_r$.
Now, we will show that $\mathcal{B}$ is continuous on Ω_r.
Choose a sequence $\{x_n\}$ from Ω_r converges to x $\forall\tau\in\mathcal{I}$ in Ω_r, i.e. $x_n\to x$, $\forall\tau\in\mathcal{I}$.
Now

$$\|g(\tau,x_n(m(\tau)))\|_{\mathcal{E}}\leq\|a_r(\tau)\|_{\mathcal{E}}+b\|x_n(m(\tau))\|_{\mathcal{E}},$$

and $x_n\to x$, then $g(\tau,x_n(m(\tau)))\to g(\tau,x(m(\tau)))$.
Since

$$\mathcal{B}x_n(\tau)=\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A+I^{\beta}g(\tau,x_n(m(\tau))),\ \tau\in\mathcal{I}.$$

Then

$$\begin{aligned}
\| & \mathcal{B}x_n(\tau)-\mathcal{B}x(\tau)\|_{\mathcal{E}}\\
&= \|(\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A+I^{\beta}g(\tau,x_n(m(\tau))))-(\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A+I^{\beta}g(\tau,x(m(\tau))))\|_{\mathcal{E}}\\
&= \|I^{\beta}g(\tau,x_n(m(\tau)))-I^{\beta}g(\tau,x(m(\tau)))\|_{\mathcal{E}}\\
&= I^{\beta}\|g(\tau,x_n(m(\tau)))-g(\tau,x(m(\tau)))\|_{\mathcal{E}}\\
&= \int_0^{\tau}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|g(s,x_n(m(s)))-g(s,x(m(s)))\|_{\mathcal{E}}ds.
\end{aligned}$$

taking $m(s)=u$ and $ds=\frac{du}{m'(s)}$, then

$$\begin{aligned}
\| & \mathcal{B}x_n(\tau)-\mathcal{B}x(\tau)\|_{\mathcal{E}}\\
&= \int_{m(0)}^{m(\tau)}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|g(s,x_n(u))-g(s,x(u))\|_{\mathcal{E}}\frac{du}{m'(s)}\\
&\leq \frac{1}{M}\int_0^{\tau}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|g(s,x_n(u))-g(s,x(u))\|_{\mathcal{E}}du.
\end{aligned}$$

Then

$$\begin{aligned}
\| & \mathcal{B}x_n-\mathcal{B}x\|_{L^1}\\
&\leq \frac{1}{M}\int_0^{\mathcal{T}}\int_0^{\tau}\frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|g(s,x_n(u))-g(s,x(u))\|_{\mathcal{E}}dud\tau
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{1}{M}\int_0^{\mathcal{T}} \|g(s,x_n(u)) - g(s,x(u))\|_{\mathcal{E}}\left(\int_s^{\mathcal{T}} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}d\tau\right)du \\
&\leq \frac{1}{M}\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\int_0^{\mathcal{T}} \|g(s,x_n(u)) - g(s,x(u))\|_{\mathcal{E}}du \\
&\leq \frac{1}{M}\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\|g(\tau,x_n) - g(\tau,x)\|_{L^1} \\
&\leq \frac{K_2}{M}\|g(\tau,x_n) - g(\tau,x)\|_{L^1} \\
&\leq \frac{K_2}{M}\frac{\varepsilon}{\frac{K_2}{M}} = \varepsilon.
\end{aligned}$$

Hence, $\mathcal{B}x_n \to \mathcal{B}x,\ \forall x_n \to x$, which proves the continuity of $\mathcal{B}$ on Ω_r.
Finally, we will prove that $\mathcal{B}$ is contraction. Let $x,\ y \in \Omega_r$ be arbitrary, then

$$\begin{aligned}
\| &\mathcal{B}x(\tau) - \mathcal{B}y(\tau)\|_{\mathcal{E}} \\
= &\|(\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A + I^{\beta}g(\tau,x(m(\tau)))) - (\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A + I^{\beta}g(\tau,y(m(\tau))))\|_{\mathcal{E}} \\
= &\|I^{\beta}g(\tau,x(m(\tau))) - I^{\beta}g(\tau,y(m(\tau)))\|_{\mathcal{E}} \\
= &I^{\beta}\|g(\tau,x(m(\tau))) - g(\tau,y(m(\tau)))\|_{\mathcal{E}} \\
= &bI^{\beta}\|x(m(\tau)) - y(m(\tau))\|_{\mathcal{E}} \\
= &b\int_0^{\tau} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|x(m(s)) - y(m(s))\|_{\mathcal{E}}ds
\end{aligned}$$

taking $m(s) = u$ and $ds = \frac{du}{m'(s)}$, then

$$\begin{aligned}
\| &\mathcal{B}x(\tau) - \mathcal{B}y(\tau)\|_{\mathcal{E}} \\
= &b\int_{m(0)}^{m(\tau)} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|x(u) - y(u)\|_{\mathcal{E}}\frac{du}{m'(s)} \\
\leq &\frac{b}{M}\int_0^{\tau} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|x(u) - y(u)\|_{\mathcal{E}}du.
\end{aligned}$$

Then

$$\begin{aligned}
\| &\mathcal{B}x - \mathcal{B}y\|_{L^1} \\
\leq &\frac{b}{M}\int_0^{\mathcal{T}}\int_0^{\tau} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|x(u) - y(u)\|_{\mathcal{E}}dud\tau \\
\leq &\frac{b}{M}\int_0^{\mathcal{T}} \|x(u) - y(u)\|_{\mathcal{E}}\left(\int_s^{\mathcal{T}} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}d\tau\right)du \\
\leq &\frac{b}{M}\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\|x-y\|_{L^1} \\
\leq &\frac{bK_2}{M}\|x-y\|_{L^1}.
\end{aligned}$$

If $\frac{bK_2}{M} < 1$, Then $\mathcal{B}$ is contraction mapping.
Therefore, according to Banach contraction mapping Theorem, then the operator $\mathcal{B}$ has a unique fixed point $x \in \Omega_r$, then $\exists$ integrable solution $x \in L^1(\mathcal{I}, \mathcal{E})$ of the equation (3.1).
Hence, $\exists$ integrable solutions $x \in L^1(\mathcal{I}, \mathcal{E})$ of the inclusion (1.1).

4. Continuous dependence on the set of selections $S^1_{G(\tau,x(\tau))}$

Here we study the continuous dependence of the solution on the set of selections $S^1_{G(\tau,x(\tau))}$ for the inclusion (1.1).

Definition 4.1. *The solution $x \in L^1(\mathcal{I}, \mathcal{E})$ of the inclusion (1.1) depends continuously on the set $S^1_{G(\tau,x(\tau))}$, if $\forall\ \varepsilon > 0$, and any two functions $g,\ h \in S^1_{G(\tau,x(\tau))}$, there exists $\delta > 0$ such that $\|g - h\|_{\mathcal{E}} < \delta$ implies $\|x_g - x_h\|_{L^1} < \varepsilon$, where $x_g,\ x_h$ are the two solutions of (1.1) and*

$$x(\tau) \in \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A + I^{\beta}h(\tau, x(m(\tau))),\ \tau \in \mathcal{I}$$

respectively.

Theorem 4.1. *Assume the assumptions (H1)-(H6) hold. Then the solution $x \in L^1(\mathcal{I}, \mathcal{E})$ of (1.1) depends continuously on $S^1_{G(\tau,x(\tau))}$.*

Proof. Let $g,\ h \in S^1_{G(\tau,x(\tau))}$ where

$$\|g(\tau, x_g(m(\tau))) - h(\tau, x_g(m(\tau)))\|_{\mathcal{E}} < \delta,\ \delta > 0,\ \tau \in \mathcal{I}$$

Then

$$\begin{aligned}
\| & x_g(\tau) - x_h(\tau)\|_{\mathcal{E}} \\
= & \|(\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A + I^{\beta}g(\tau, x_g(m(\tau)))) - (\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A + I^{\beta}h(\tau, x_g(m(\tau))))\|_{\mathcal{E}} \\
= & \|I^{\beta}g(\tau, x_g(m(\tau))) - I^{\beta}h(\tau, x_h(m(\tau)))\|_{\mathcal{E}} \\
= & I^{\beta}\|g(\tau, x_g(m(\tau))) - h(\tau, x_h(m(\tau)))\|_{\mathcal{E}} \\
= & I^{\beta}\|g(\tau, x_g(m(\tau))) - h(\tau, x_g(m(\tau))) + h(\tau, x_g(m(\tau))) - h(\tau, x_h(m(\tau)))\|_{\mathcal{E}} \\
\leq & I^{\beta}\{\|g(\tau, x_g(m(\tau))) - h(\tau, x_g(m(\tau)))\|_{\mathcal{E}} + \|h(\tau, x_g(m(\tau))) - h(\tau, x_h(m(\tau)))\|_{\mathcal{E}}\} \\
\leq & I^{\beta}\{\|g(\tau, x_g(m(\tau))) - h(\tau, x_g(m(\tau)))\|_{\mathcal{E}} + b\|x_g(m(\tau)) - x_h(m(\tau))\|_{\mathcal{E}}\} \\
\leq & I^{\beta}\|g(\tau, x_g(m(\tau))) - h(\tau, x_g(m(\tau)))\|_{\mathcal{E}} + bI^{\beta}\|x_g(m(\tau)) - x_h(m(\tau))\|_{\mathcal{E}} \\
\leq & I^{\beta}\delta + bI^{\beta}\|x_g(m(\tau)) - x_h(m(\tau))\|_{\mathcal{E}} \\
\leq & \int_0^{\tau} \frac{(\tau - s)^{\beta-1}}{\Gamma(\beta)}\delta ds + b\int_0^{\tau} \frac{(\tau - s)^{\beta-1}}{\Gamma(\beta)}\|x_g(m(s)) - x_h(m(s))\|_{\mathcal{E}} ds
\end{aligned}$$

taking $m(s) = u$ and $ds = \frac{du}{m'(s)}$, then

$$\| \quad x_g(\tau) - x_h(\tau)\|_{\mathcal{E}}$$

$$\leq \int_0^{\tau} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\delta ds + b\int_{m(0)}^{m(\tau)} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|x_g(u)-x_h(u)\|_{\mathcal{E}}\frac{du}{m'(s)}$$
$$\leq \int_0^{\tau} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\delta ds + \frac{b}{M}\int_0^{\tau} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|x_g(u)-x_h(u)\|_{\mathcal{E}}du.$$

Then

$$\begin{aligned}
\| & x_g - x_h\|_{L^1} \\
\leq & \int_0^{\mathcal{T}}\int_0^{\tau} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\delta ds d\tau + \frac{b}{M}\int_0^{\mathcal{T}}\int_0^{\tau} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}\|x_g(u)-x_h(u)\|_{\mathcal{E}}du d\tau \\
\leq & \int_0^{\mathcal{T}} \delta(\int_s^{\mathcal{T}} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}d\tau)ds + \frac{b}{M}\int_0^{\mathcal{T}} \|x_g(u)-x_h(u)\|_{\mathcal{E}}(\int_s^{\mathcal{T}} \frac{(\tau-s)^{\beta-1}}{\Gamma(\beta)}d\tau)du \\
\leq & \frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\delta\mathcal{T} + \frac{b}{M}\frac{\mathcal{T}^{\beta}}{\Gamma(\beta+1)}\|x_g - x_h\|_{L^1} \\
\leq & K_2\delta\mathcal{T} + \frac{bK_2}{M}\|x_g - x_h\|_{L^1}.
\end{aligned}$$

Therefore

$$\|x_g - x_h\|_{L^1} \leq \frac{K_2\delta\mathcal{T}}{1-\frac{bK_2}{M}} = \varepsilon.$$

Hence

$$\|x_g - x_h\|_{L^1} \leq \varepsilon.$$

Which complete our investigation.

5. An Example

Now we give an example given as numerical application to illustrate our main result contained in Theorem 3.1.

Let $\overline{\Omega} = \{x \in \mathcal{E}, \|x\|_{\mathcal{E}} \leq 1\}$ and $\mathcal{J} = [0,1]$. Assume the multi-valued function $G : \mathcal{J}\times \overline{\Omega} \to \chi(\mathcal{E})$ defined by

$$G(\tau, x(m(\tau))) = (a(\tau) + bx(m(\tau)))\overline{\Omega},\ \tau \in \mathcal{J}.$$

Then G is Lipschitz. Obviously we have

$$\begin{aligned}
\|G(\tau,x(m(\tau)))\|_{\mathcal{E}} &= sup\{\|g\| :\ g \in G(\tau,x(m(\tau)))\} \\
&= \|(a(\tau)+bx(m(\tau)))\overline{Q}\|_{\mathcal{E}} \\
&= \|a(\tau)+bx(m(\tau))\|_{\mathcal{E}} \\
&\leq \|a(\tau)\|_{\mathcal{E}} + b\|x(m(\tau))\|_{\mathcal{E}}.
\end{aligned}$$

Now let $g(\tau,x(m(\tau))) = a(\tau) + bx(m(\tau)) \in G(\tau,x(m(\tau)))$.

Hence, we can apply our results to the singular fractional order integral equation

$$x(\tau) = \frac{0.1}{\sqrt{\pi}}\tau^{-\frac{1}{2}} + I^{\frac{1}{2}}(\tau + 0.1x(m(\tau))),\ \tau \in \mathcal{J}. \tag{5.1}$$

Here $g(\tau, x(m(\tau)) = (\tau + 0.1x(m(\tau))),\ m(\tau) \le \tau,\ \alpha = \beta = \frac{1}{2}$ and $A = 0.1$.
Now

$$\begin{aligned}
\|x(\tau)\|_{\mathcal{E}} &= \|\frac{0.1}{\sqrt{\pi}}\tau^{-\frac{1}{2}} + I^{\frac{1}{2}}(\tau + 0.1x(m(\tau)))\| \\
&\le \frac{0.1}{\sqrt{\pi}}|\tau^{-\frac{1}{2}}| + I^{\frac{1}{2}}(|\tau| + 0.1\|x(m(\tau))\|_{\mathcal{E}}) \\
&\le \frac{0.1}{\sqrt{\pi}}|\tau^{-\frac{1}{2}}| + I^{\frac{1}{2}}(|\tau| + 0.1\|x(m(\tau))\|_{\mathcal{E}}) \\
&\le \frac{0.1}{\sqrt{\pi}}|\tau^{-\frac{1}{2}}| + I^{\frac{1}{2}}|\tau| + 0.1I^{\frac{1}{2}}\|x(m(\tau))\|_{\mathcal{E}} \\
&\le \frac{0.1}{\sqrt{\pi}}|\tau^{-\frac{1}{2}}| + \int_0^\tau \frac{(\tau-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}|s|ds + 0.1\int_0^\tau \frac{(\tau-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\|x(m(s))\|_{\mathcal{E}}ds
\end{aligned}$$

taking $m(s) = u$ and $ds = \frac{du}{m'(s)}$, then

$$\begin{aligned}
\| & x(\tau)\|_{\mathcal{E}} \\
&\le \frac{0.1}{\sqrt{\pi}}\tau^{-\frac{1}{2}} + \int_0^\tau \frac{(\tau-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}sds + 0.1\int_{m(0)}^{m(\tau)} \frac{(\tau-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\|x(u)\|_{\mathcal{E}}\frac{du}{m'(s)} \\
&\le \frac{0.1}{\sqrt{\pi}}\tau^{-\frac{1}{2}} + \int_0^\tau \frac{(\tau-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}sds + \frac{0.1}{M}\int_0^\tau \frac{(\tau-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\|x(u)\|_{\mathcal{E}}du.
\end{aligned}$$

Then

$$\begin{aligned}
\|x\|_{L^1} &\le \frac{0.1}{\sqrt{\pi}}\int_0^1 \tau^{-\frac{1}{2}}d\tau + \int_0^1\int_0^\tau \frac{(\tau-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}sdsd\tau + \frac{0.1}{M}\int_0^1\int_0^\tau \frac{(\tau-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}\|x(u)\|_{\mathcal{E}}dud\tau \\
&\le \frac{0.1}{\sqrt{\pi}}\int_0^1 \tau^{-\frac{1}{2}}d\tau + \int_0^1 s\{\int_s^1 \frac{(\tau-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}d\tau\}ds + \frac{0.1}{M}\int_0^1 \|x(u)\|_{\mathcal{E}}\{\int_s^1 \frac{(\tau-s)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})}d\tau\}du \\
&\le \frac{0.1}{\sqrt{\pi}}\int_0^1 \tau^{-\frac{1}{2}}d\tau + \frac{1}{\frac{1}{2}\sqrt{\pi}}\int_0^1 sds + \frac{0.1}{M}\int_0^1 \|x(u)\|_{\mathcal{E}}\frac{1}{\frac{1}{2}\sqrt{\pi}}du \\
&\le \frac{0.2}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}} + \frac{2}{\sqrt{\pi}}r.
\end{aligned}$$

The assumptions (H1)-(H6) of Theorem 3.1 are satisfies with $a_r(\tau) = \tau,\ b = 0.1$ *and* $M = 0.1$. Therefore, by applying to Theorem 3.1, then the nonlinear singular fractional order integral equation (5.1) has a solution $x \in \mathcal{J}$.

6. Application

Consider the fractional differential inclusion (1.2) with each one of the nonlocal condition (1.3) or the weighted condition (1.4).

Remark 6.1. *According to assumptions (H1)-(H3), there exists a Lipschitz selection $g \in S^1_{G(\tau,x(\tau))}$ such that*

$$\|g(\tau, x_1(\tau)) - g(\tau, x_2(\tau))\|_{\mathcal{E}} \le b\|x_1(\tau) - x_2(\tau)\|_{\mathcal{E}}$$

for every $x_1, x_2 \in \mathcal{E}$ *and* $\tau \in \mathcal{I}$.
This selection satisfy the fractional differential equation

$$^{R}D^{\alpha}x(\tau) = g(\tau, x(m(\tau))),\ \tau \in \mathcal{I}. \tag{6.1}$$

Then any solution of the problems (6.1) and (1.3) or (6.1) and (1.4), if it exists, is a solution of the the problems (1.2) and (1.3) or (1.2) and (1.4).

Theorem 6.1. *Assume the assumptions (H1)-(H6) be satisfied. If the integrable solution* $x \in L^1(\mathcal{I}, \mathcal{E})$ *of the problems (6.1) and (1.3) or (6.1) and (1.4) exist, then it can be given by*

$$x(\tau) = \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A + I^{\alpha}g(\tau, x(m(\tau))) \tag{6.2}$$

Proof. Consider

$$^{R}D^{\alpha}x(\tau) = g(\tau, x(m(\tau))),\ \tau \in \mathcal{I}.$$

According to Riemann-Liouville fractional order derivative, we get

$$\frac{d}{d\tau}I^{1-\alpha}x(\tau) = g(\tau, x(m(\tau)))$$

Integrating both-sides, we get

$$I^{1-\alpha}x(\tau) - C = Ig(\tau, x(m(\tau))) \tag{6.3}$$

At $\tau = 0$, using the initial condition (1.3) we get $C = A$.
Hence from equation (6.3), we get

$$I^{1-\alpha}x(\tau) = A + Ig(\tau, x(m(\tau))).$$

Operating by I^{α} for both-sides and differentiation, we obtain

$$x(\tau) = \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A + I^{\alpha}g(\tau, x(m(\tau)))$$

This proves that the solution of (6.1) and (1.3) is given by equation (6.2).
Conversely, Operating equation (6.2) by $I^{1-\alpha}$, we have

$$\begin{aligned} I^{1-\alpha}x(\tau) &= I^{1-\alpha}\frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A + I^{1-\alpha}I^{\alpha}g(\tau, x(m(\tau))) \\ &= \int_0^{\tau}\frac{(\tau-s)^{-\alpha}}{\Gamma(1-\alpha)}\frac{s^{\alpha-1}}{\Gamma(\alpha)}A\,ds + Ig(\tau, x(m(\tau))) \\ &= \frac{A}{\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^{\tau}(\tau-s)^{-\alpha}s^{\alpha-1}ds + Ig(\tau, x(m(\tau))) \\ &= A + Ig(\tau, x(m(\tau))). \end{aligned}$$

Then

$$I^{1-\alpha}x(\tau) = A + Ig(\tau, x(m(\tau))). \tag{6.4}$$

Differentiate equation (6.4) with respect to τ we get equation (6.1).
At $\tau = 0$ in equation (6.4) we get condition (1.3).
Now operating equation (6.3) by I^α and differentiation, we conclude that

$$x(\tau) = \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}C + I^\alpha g(\tau, x(m(\tau))) \tag{6.5}$$

Multiplying this equation by $\tau^{1-\alpha}$, we obtain

$$\tau^{1-\alpha}x(\tau) = \frac{C}{\Gamma(\alpha)} + \tau^{1-\alpha}I^\alpha g(\tau, x(m(\tau)))$$

At $\tau = 0$ and using the initial condition (1.4) we deduce that $C = A$.
Then from equation (6.5), we get

$$x(\tau) = \frac{\tau^{\alpha-1}}{\Gamma(\alpha)}A + I^\alpha g(\tau, x(m(\tau)))$$

Then the solution of (6.1) and (1.4) is given by (6.2).
Conversely, Operating equation (6.2) by $I^{1-\alpha}$ we have equation (6.4).
Differentiate equation (6.4) with respect to τ we get equation (6.1).
Multiplying equation (6.2) by $\tau^{1-\alpha}$, we obtain

$$\tau^{1-\alpha}x(\tau) = \frac{A}{\Gamma(\alpha)} + \tau^{1-\alpha}I^\alpha g(\tau, x(m(\tau))).$$

At $\tau = 0$ we get condition (1.4).
When $\alpha = \beta$, in equation (3.1), we have equation (6.2).

7. Conclusions

In this paper we use a Lipschitz selection for a multi-valued function in the reflexive Banach space $\mathcal{E}$ to establish the solvability of a nonlinear singular functional integral inclusion (1.1). Our investigation is lying in the space of all integrable functions on a reflexive Banach space $\mathcal{E}$, $(L^1([0, \mathcal{T}], \mathcal{E}))$.

In the main result we introduced sufficient conditions and studied the existence of integrable solutions $x \in L^1([0, \mathcal{T}], \mathcal{E})$ of the nonlinear singular integral inclusion of fractional orders (1.1). We discussed the continuous dependence of the solutions on the set of selections $S^1_{G(\tau, x(\tau))}$ of that nonlinear singular integral inclusion of fractional order (1.1) and an numerical example is illustrated.

Finally, the existence of solutions $x \in L^1([0, \mathcal{T}], \mathcal{E})$ of the Riemann-Liouville fractional differential inclusion (1.2) with the nonlocal condition (1.3) and the weighted condition (1.4) is studied and investigated as an application.

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