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A Study on the Two Forms of (2+1)-Fractional Order Burgers Equation

Fathea M. O. Birkea[1], Ali Satty[1], Abaker A. Hassaballa[1,*], Ahmed M. A. Adam[1], Elzain A. E. Gumma[1], Emad A-B. Abdel-Salam[3], Eltayeb A. Yousif[2] , Mohamed I. Nouh[4]

[1]*Department of Mathematics, College of Science, Northern Border University, Arar 91431, Saudi Arabia*

[2]*Department of Applied Mathematics, Faculty of Mathematical Sciences and informatics, University of Khartoum, Khartoum, Sudan*

[3]*Department of Mathematics, Faculty of Science, New Valley University, Egypt*

[4]*Astronomy Department, National Research Institute of Astronomy and Geophysics (NRIAG), Egypt*

**Corresponding author:* abakerh@gmail.com

ABSTRACT. This study examines the dynamics of two forms of (2+1)-dimensional fractional order Burgers equation, a paramount structure within the realm of nonlinear fractional calculus. The main objective is to acquire solutions for this equation through the application of the singular manifold (SM) method. The study successfully derives diverse solutions corresponding to varied fractional orders. Additionally, multiple-kink solutions are systematically derived and illustrated with graphical representations to highlight their intrinsic physical properties. Overall, the results demonstrate the effectiveness and reliability of the SM method in yielding precise solutions for the two forms of (2+1)-dimensional fractional order Burgers equation.

1. Introduction

Nonlinear partial differential equations (NPDEs) are fundamental to many physical theories and applications. These equations provide a mathematical foundation for understanding the complex behaviors observed in nature across various scientific disciplines. The nonlinearity inherent in these PDEs introduces a level of sophistication that mirrors the intricate nature of

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physical phenomena. This makes them a valuable tool for studies in fields such as engineering, mechanics, and chemistry; as indicated by Dodd et al. [1]. Attaining an in-depth understanding of these nonlinear phenomena can be accomplished through the finding of exact solutions to NPDEs. As a result, various methodologies have been developed for the precise resolution of NPDEs using analytical and numerical methods, such as symmetry reduction [2], Bäcklund and Darboux transformations ([3], [4]) and the singular manifold (SM) method [5].

The fractional order Burgers' equation [6] is a fascinating topic in the realm of NPDEs. This equation is a generalization of the traditional Burgers' equation, which is a crucial construct within the field of nonlinear fractional calculus used in various areas such as fluid dynamics, nonlinear acoustics, and traffic flow. The inclusion of fractional derivatives in the Burgers' equation introduces memory and hereditary properties to the model, allowing it to more accurately depict anomalous diffusion and non-local effects, which are observed in complex systems. Fractional calculus is an extension of classical calculus that allows for differentiation and integration to an arbitrary order, not necessarily an integer ([7], [8]). Several definitions for fractional derivatives exist, including the conformable fractional derivative (CFD) technique. Khalil et al. [9] introduced the concept of CFD, marking a notable progress in the domain of fractional calculus. This derivative is characterized by its unique attributes that render it particularly beneficial for a wide range of applications in diverse disciplines such as mathematics, engineering, and physics. Employing CFD in the study of soliton theory offers significant advantages for analyzing soliton wave dynamics and facilitates a profound comprehension of the associated physical processes. In light of these benefits, the present study utilizes the singular manifold (SM) method to derive solutions relevant to the two forms of (2+1)-dimensional fractional order Burgers equation.

The SM method [5] has proven to be an invaluable tool in the realm of nonlinear PDEs, allowing for the simplification of complex systems into more tractable substructures and revealing previously undiscovered classes of solutions. The investigation of the applicability of the SM method to the fractional-order Burger equation requires further investigation. This study introduces novelty by applying the SM method to the fractional-order Burger equation, a technique infrequently used for this specific equation. The aims to expand the array of exact solutions for the fractional order Burger's equation and to enhance comprehension regarding the complex dynamics inherent in fractional-order nonlinear systems. Several studies have utilized the SM method to examine diverse NPDEs, with a focus on obtaining traveling wave solutions. See ([10], [11], [12], [13], [14]) for examples.

This study examines the dynamics of the two forms of (2+1)-dimensional fractional order Burgers equation. The main objective of this study is to acquire solutions for this equation through the application of the SM method. To illustrate the influence of the fractional operator on the outcomes, the solutions obtained are displayed for various fractional orders. In addition, multiple-kink solutions are derived and elucidated with graphical depictions to shed light on their intrinsic physical characteristics. The structure of the manuscript is organized in the following manner: In Section 2, we delineate the attributes of CFD. Comprehensive exact wave solutions for the two forms of (2+1)-dimensional fractional order Burgers' equations are elaborated in Sections 3 and 4. In continuation, Sections 5 and 6 are dedicated to explicating the kink-solutions for the aforementioned equations. The article concludes with a final discourse in Section 7.

2. Fractional order Burger's equation

We engage in an analytical exploration of the order Burger's equation, as discussed in ([15], [16]). We articulate the definition of the conformable derivative of order α, wherein the range for α is confined to $\alpha \in (0,1]$, with respect to an independent variable denoted as t. We first discuss the core principles of CFD, as given by Khalil et al. [9]. We establish the definition of the conformable derivative of order α, with the condition that $0 < \alpha \leq 1$, in relation to an independent variable s.

$$\begin{aligned} D^{\alpha} M(s) &= \lim_{\varepsilon \to 0} \frac{M(s+\varepsilon s^{1-\alpha}) - M(s)}{\varepsilon}, \forall s > 0, \alpha \in (0,1], \\ M^{(\alpha)}(0) &= \lim_{s \to 0^+} M^{(\alpha)}(s). \end{aligned} \tag{1}$$

Upon setting α equal to 1 in the preceding equations, we observe that the fractional differential operator reverts to its classical integer-order counterpart. Consequently, the CFD fulfills the subsequent properties:

- $D^{\alpha} s^n = n s^{n-\alpha}, n \in R, D^{\alpha} a = 0,$
- $D^{\alpha}(aM + bN) = aD^{\alpha} M + bD^{\alpha} N, \forall a, b \in R,$
- $D^{\alpha}(MN) = MD^{\alpha} N + ND^{\alpha} M,$
- $D^{\alpha}\left(\frac{M}{N}\right) = \frac{MD^{\alpha} N - ND^{\alpha} M}{N^2},$
- $D^{\alpha} M(N) = \frac{dM}{dN} D^{\alpha} N, D^{\alpha} M(s) = s^{1-\alpha} \frac{dM}{ds}.$

In this context, M and N represent two functions that are α-differentiable with respect to a dependent variable denoted by s, and a denotes an arbitrary constant. According to [6], the (2+1)-dimensional Burgers equation is given by

$$M_t = M_{xx} + 2NM_x, M_x = N_y, \tag{2}$$

and the Burgers equation in a (2+1)-dimensional space, incorporating fractional derivatives with respect to both space and time, is formally expressed as

$$D_t^\alpha M = D_x^{\alpha\alpha} M + 2ND_x^\alpha M, D_x^\alpha M = D_y^\alpha N. \tag{3}$$

The (2+1)-dimensional higher-order Burgers equation [6] is

$$M_t = 4M_{xxx} + 12NM_{xx} + 12N_x M_x + 12N^2 M_x, M_x = N_y. \tag{4}$$

In equation (4), the CFDs concerning time (t) and space (x) are represented as D_t^α and D_x^α, correspondingly. The discussion is extended to higher-order processes, such as $D_x^{\alpha\alpha} u = D_x^\alpha (D_x^\alpha u)$, which characterize second-order fractional CFDs. In addition, the (2+1)-dimensional higher-order Burgers equation with fractional space and time derivative is

$$D_t^\alpha M = 4D_x^{\alpha\alpha\alpha} M + 12ND_x^{\alpha\alpha} M + 12D_x^\alpha N D_x^\alpha M + 12N^2 D_x^\alpha M, D_x^\alpha M = D_y^\alpha N. \tag{5}$$

Equations (3) and (5) can be written in another form by taking the transformation

$$M = D_y^\alpha N. \tag{6}$$

Substituting Equation (6) in Equation (3), we have

$$D_t^\alpha (D_y^\alpha N) = D_x^{\alpha\alpha} (D_y^\alpha N) + 2D_y^\alpha N D_x^\alpha (D_y^\alpha N). \tag{7}$$

Similarly, substituting Equation (6) into Equation (5), we get

$$D_t^\alpha (D_y^\alpha N) = 4D_x^{\alpha\alpha\alpha} (D_y^\alpha N) + 12D_x^\alpha N D_x^{\alpha\alpha} (D_y^\alpha N) + 12D_x^{\alpha\alpha} N\, D_x^\alpha (D_y^\alpha N) + 12(D_x^\alpha N)^2 D_x^\alpha (D_y^\alpha N). \tag{8}$$

In this study, we will utilize the CFD transformation alongside the SM method to rigorously address equations (3) and (5), aiming to derive exact solutions and multiple-kink solutions for each of these equations.

3. General solutions of Equation (3)

Utilizing the SM method, the expansion of the Painlevé series ([5], [12]) for Equation (3) is truncated at the term representing a constant level

$$M = \varphi^{-1} M_0 + M_1, N = \varphi^{-1} N_0 + N_1, \tag{9}$$

where φ represents the singular manifold, and the pair $\{M_1, N_1\}$ corresponds to a particular seed solution of Equation (3). Upon inserting Equation (9) into Equation (3) and matching the coefficients corresponding to identical powers of φ, we obtain

$$M_0 = D_y^\alpha \varphi, N_0 = D_x^\alpha \varphi. \tag{10}$$

In this context, φ satisfies the following differential equation:

$$D_t^{\alpha}\varphi = D_x^{\alpha\alpha}\varphi + 2M_1 D_x^{\alpha}\varphi, \tag{11}$$

which is referred to as the singular manifold equation. Let us consider the following substitutions: $M_1 = \varphi$ and $D_y^{\alpha}N_1 = D_x^{\alpha}\varphi$. Consequently, we define M as:

$$M = \frac{1}{\varphi}D_x^{\alpha}\varphi + \varphi, \tag{12}$$

subject to φ fulfilling the fractional differential equation:

$$D_t^{\alpha}\varphi = D_x^{\alpha\alpha}\varphi + 2\varphi D_x^{\alpha}\varphi, D_y^{\alpha}\varphi = D_x^{\alpha}\varphi. \tag{13}$$

Equations (12) and (13) then represent an alternative representation of an auto-Bäcklund transformation of fractional order associated with Equation (3). Given the initial conditions $M_1 = 0$ and $N_1 = 0$, the ensuing transformation is analogously termed as the Cole-Hopf-type fractional transformation [15,16] or the hetero-Bäcklund fractional transformation, explicitly represented by:

$$M = \frac{1}{\varphi}D_x^{\alpha}\varphi, \tag{14}$$

where φ is subjected to satisfy the fractional partial differential equation:

$$D_t^{\alpha}\varphi = D_x^{\alpha\alpha}\varphi. \tag{15}$$

Now, consider the special exact solution of (2+1)-time space fractional Burger's equation within Equation (3) to be expressed as

$$M_1 = 0, N_1 = N_1\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\alpha}}{\alpha}\right), \tag{16}$$

wherein $N_1\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\alpha}}{\alpha}\right)$ represents a function subject to variation based on the variables indicated. Upon rigorous examination, one can confirm that Equation (11) admits a nonlinear separation solution

$$\varphi = F\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\alpha}}{\alpha}\right)G\left(\frac{y^{\alpha}}{\alpha}\right) + H\left(\frac{y^{\alpha}}{\alpha}\right), \tag{17}$$

in which $F\left(\frac{x^{\alpha}}{\alpha}, \frac{t^{\alpha}}{\alpha}\right)$, $G\left(\frac{y^{\alpha}}{\alpha}\right)$ and $H\left(\frac{y^{\alpha}}{\alpha}\right)$ are arbitrary functions corresponding to their respective variables. This holds true provided that

$$N_1 = \frac{D_t^{\alpha}F - D_x^{\alpha\alpha}F}{2D_x^{\alpha}F}. \tag{18}$$

Consequently, by direct computation from Equations (9), (10), (16), and (17), we derive a generalized functional separation solution for the (2+1)-dimensional Burger's equation characterized by fractional derivatives in both space and time as delineated in Equation (3). This solution can be expressed as:

$$M = \frac{F D_y^\alpha G + D_y^\alpha H}{FG + H}, \tag{19}$$

where $F = F\left(\frac{x^\alpha}{\alpha}, \frac{t^\alpha}{\alpha}\right), G = G\left(\frac{y^\alpha}{\alpha}\right)$ and $H = H\left(\frac{y^\alpha}{\alpha}\right)$ represent undefined functions of their respective indicated variables.

The resultant solution is encapsulated by three distinct functions that depend on time and space variables. Through judicious selection of these arbitrary functions in Equation (19), an extensive spectrum of solution structures for the fractional space-time (2+1) Burgers equation, as provided in Equation (3), can be thoroughly explored. It should be noted that when α equals 1, Equations (10) through (19) correspond to Equations (5) through (14) outlined in the work by Peng and Yamba [11].

4. General solutions of Equation (5)

Upon conducting a parallel analysis, we derive an auto-Bäcklund transformation corresponding to Equation (5):

$$M = \varphi^{-1} D_y^\alpha \varphi + M_1, N = \varphi^{-1} D_x^\alpha \varphi + N_1, \tag{20}$$

wherein φ represents the anomalous fractional manifold and constitutes an arbitrary function that satisfies Equation (5). Moreover, φ complies with the equation

$$D_t^\alpha \varphi = 4 D_x^{\alpha\alpha\alpha} \varphi + 12 N_1 D_x^{\alpha\alpha} \varphi + 12 D_x^\alpha N_1 D_x^\alpha \varphi + 12 N_1^2 D_x^\alpha \varphi. \tag{21}$$

Considering an alternative initial solution with $M_1 = 0$ and $N_1 = 0$, one can derive the Cole–Hopf-type fractional transformation denoted as Equation (13), in which φ is subject to $D_t^\alpha \varphi = 4 D_x^{\alpha\alpha\alpha} \varphi$ for Equation (5). Upon selecting a different exact solution, $M_1 = \varphi$ and $D_y^\alpha N_1 = D_x^\alpha \varphi$, another novel auto-Bäcklund fractional transformation materializes, represented by Equation (12), pertaining to Equation (5), with the stipulation that

$$D_t^\alpha \varphi = 4 D_x^{\alpha\alpha\alpha} \varphi + 12 \varphi D_x^{\alpha\alpha} \varphi + 12 D_x^\alpha \varphi D_x^\alpha \varphi + 12 \varphi^2 D_x^\alpha \varphi, D_x^\alpha \varphi = D_y^\alpha \varphi. \tag{22}$$

Furthermore, selecting an initial solution of $M_1 = 0$ and $N_1\left(\frac{x^\alpha}{\alpha}, \frac{t^\alpha}{\alpha}\right)$ results in a solution expressed by Equation (19), which also satisfies Equation (5), provided that satisfies the subsequent condition as a surrogate for $N_1 = \frac{D_t^\alpha f - D_x^{\alpha\alpha} f}{2 D_x^\alpha f}$ in the Riccati equation:

$$D_x^\alpha N_1 = \frac{D_t^\alpha f - 4 D_x^{\alpha\alpha\alpha} f}{12 D_x^\alpha f} - \frac{D_x^{\alpha\alpha} f}{D_x^\alpha f} N_1 - N_1^2. \tag{23}$$

5. Kink-solutions of Equation (3)

To examine the multiple-kink wave solutions of Equation (3), let us consider

$$N = e^{(k_i x^\alpha + r_i y^\alpha - c_i t^\alpha)/\alpha}. \tag{24}$$

Upon substituting Equation (24) into the linear term of Equation (3), the dispersion relation can be derived as:

$$c_i = -k_i^2. \tag{25}$$

Consequently, we define

$$\theta_i = \frac{k_i x^\alpha + r_i y^\alpha + k_i^2 t^\alpha}{\alpha}. \tag{26}$$

The multiple-kink solution of Equation (3), using the Cole–Hopf transformation method, is posited as

$$N = R\, ln(f). \tag{27}$$

Consequently, it follows that

$$M = R\frac{D_y^\alpha f}{f}. \tag{28}$$

For the one-kink wave solution, let us consider the function:

$$f = 1 + e^{(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha}. \tag{29}$$

By substituting Equation (27) into Equation (7) and solving for R, we obtain

$$R = 1. \tag{30}$$

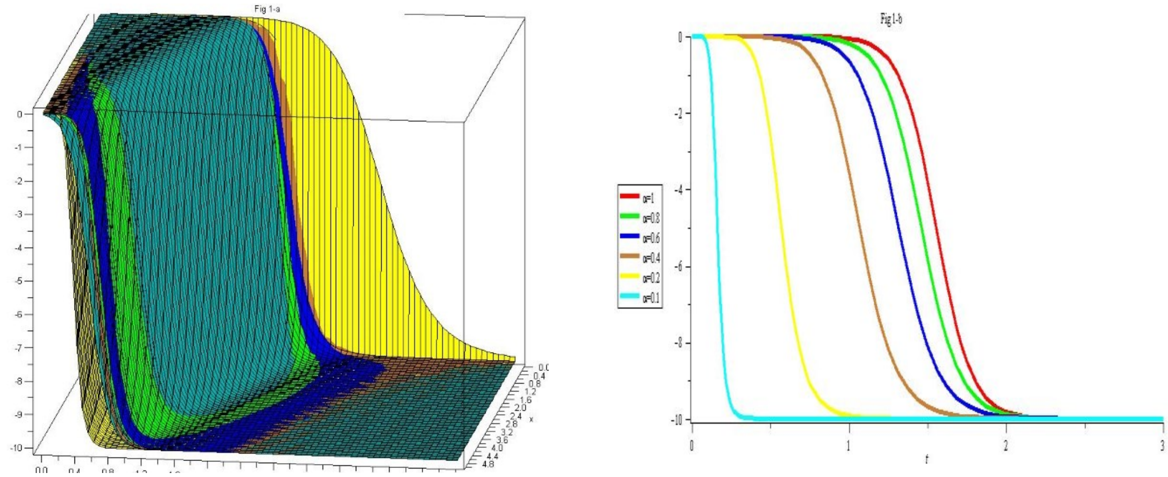
Consequently, we have:

$$N = ln(1 + e^{(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha}). \tag{31}$$

Therefore, the one-kink wave solution is given by:

$$M = \frac{r_1 e^{(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha}}{1 + e^{(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha}}. \tag{32}$$

Figure 1 depicts the evolutionary behavior of the one kink wave solution for $k_1 = 3$ and $r_1 = -5$ at various values of α. The red layer corresponds to $\alpha = 1$, the green layer to $\alpha = 0.9$, the blue layer to $\alpha = 0.8$, the gold layer to $\alpha = 0.7$, the yellow layer to $\alpha = 0.6$, and the cyan layer to $\alpha = 0.5$. Figure 1(a) demonstrates the cross-section at $y = 2$, while Figure 1(b) illustrates the cross-section at $y = 2$ and $x = 2$. From these figures, it is evident that the shape of the solution varies with the alteration of the fractional order. The modification of the solution's shape by adjusting the fractional order is pivotal in developing effective signal processing techniques capable of addressing the intricacies of real-world signals. Numerous real-world signals, including biomedical signals, seismic signals, and financial data, display non-stationary behavior, indicating that their statistical properties vary over time. Conventional approaches predicated on integer-order differential equations may fall short in adequately capturing the intricate dynamics of such signals. Incorporating fractional-order dynamics can significantly enhance signal processing techniques, enabling more precise modeling and analysis of non-stationary signals. Fractional calculus offers a robust framework to characterize the memory and long-range dependence attributes of signals, thereby facilitating the development of more accurate models and algorithms for applications such as signal denoising, feature extraction, and classification.



(a) (b)

Figure 1. One kink wave solution structures with $k_1 = 3$*,* $r_1 = -5, \alpha = 1, 0.8, 0.6, 0.4, 0.2, 0.1$*: a) Cross-section at* $y = 2$ *and b) Cross-section at* $x = 2$.

The two-kink wave solution can be expressed as

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12}e^{\theta_1+\theta_2}. \tag{33}$$

By inserting Equation (32) into Equation (28) and subsequently substituting the outcome into Equation (7), we derive that $R = 1$ and

$$a_{12} = 0, \tag{34}$$

indicating no phase shifts. Consequently,

$$a_{ij} = 0, 1 \leq i < j \leq 3. \tag{35}$$

Therefore, we obtain

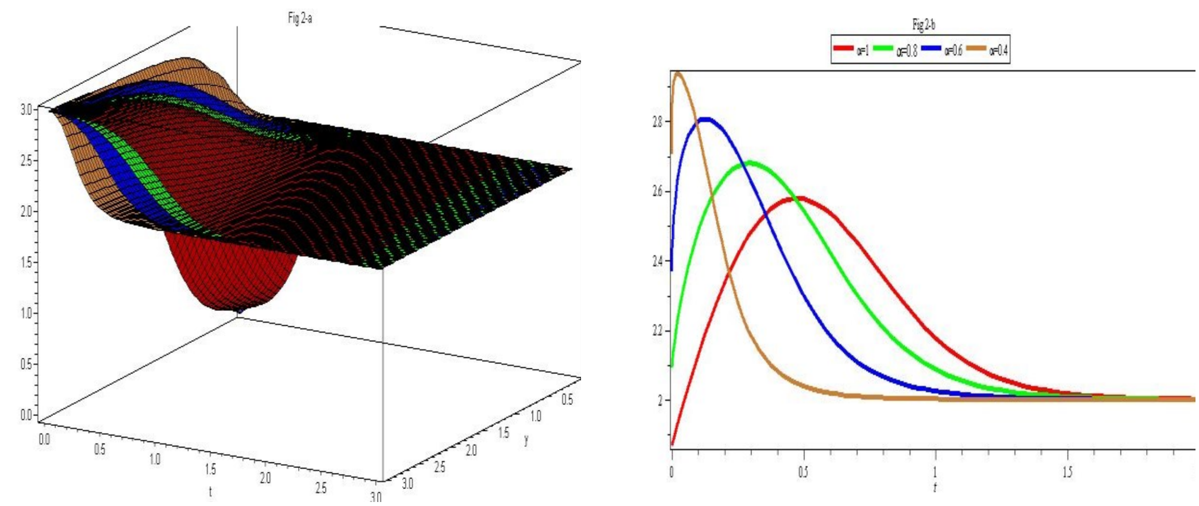
$$N = ln(1 + e^{(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha} + e^{(k_2x^\alpha+r_2y^\alpha+k_2^2t^\alpha)/\alpha}). \tag{36}$$

The two-kink wave solution for M is given by

$$M = \frac{r_1e^{(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}+r_2e^{(k_2x^\alpha+r_2y^\alpha+k_2^2t^\alpha)/\alpha}}{1+e^{(k_1x^\alpha+r_1y^\alpha+k_1^2t^\alpha)/\alpha}+e^{(k_2x^\alpha+r_2y^\alpha+k_2^2t^\alpha)/\alpha}}. \tag{37}$$

The evolutionary behavior of the two-kink wave solution is illustrated in Figure 2 with parameters $k_1 = -2, r_1 = 3, k_2 = -3, r_2$=2 at various values of α. The red layer corresponds to $\alpha = 1$, the green layer to $\alpha = 0.8$, the blue layer to $\alpha = 0.6$, and the gold layer to $\alpha = 0.4$. Figure 2(a) illustrates the cross-section at $x = 2$, while Figure 2(b) depicts the cross-section at $y = 1, x = 2$.

From these figures, it is evident that the solution's shape varies with changes in the fractional order. The system's behavior demonstrates non-integer order dynamics, suggesting that conventional integer-order models may not accurately represent its dynamics. In control theory, fractional-order controllers have garnered attention for their enhanced flexibility and performance over traditional integer-order controllers. Integrating fractional-order dynamics into controller design enables effective regulation of complex systems that exhibit non-linear and time-varying characteristics. This application underscores the significance of understanding and accurately modeling fractional-order dynamics to optimize the control of various engineering systems.



(a) (b)

Figure 2. Two kink wave solution structures with $k_1 = -2,\ r_1 = 3, k_2 = -3, r_2 = 2, \alpha = 1, 0.8, 0.6, 0.4, 0.2, 0.1$*: a) Cross-section at* $x = 2$ *and b) Cross-section at* $y = 1, x = 2$.

To derive the three-kink wave solution, let us define

$$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3}. \tag{38}$$

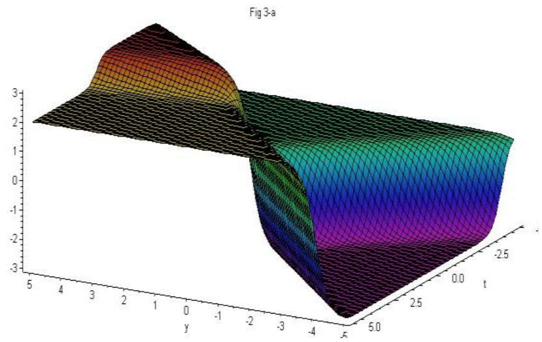
Following the same procedure as previously outlined, we obtain

$$N = ln(1 + e^{(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha} + e^{(k_2 x^\alpha + r_2 y^\alpha + k_2^2 t^\alpha)/\alpha} + e^{(k_3 x^\alpha + r_3 y^\alpha + k_3^2 t^\alpha)/\alpha}). \tag{39}$$

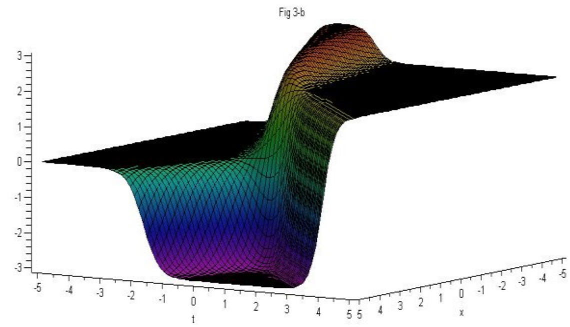
This results in the three-kink wave solution

$$M = \frac{r_1 e^{(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha} + r_2 e^{(k_2 x^\alpha + r_2 y^\alpha + k_2^2 t^\alpha)/\alpha} + r_3 e^{(k_3 x^\alpha + r_3 y^\alpha + k_3^2 t^\alpha)/\alpha}}{1 + e^{(k_1 x^\alpha + r_1 y^\alpha + k_1^2 t^\alpha)/\alpha} + e^{(k_2 x^\alpha + r_2 y^\alpha + k_2^2 t^\alpha)/\alpha} + e^{(k_3 x^\alpha + r_3 y^\alpha + k_3^2 t^\alpha)/\alpha}}. \tag{40}$$

The evolutionary behavior of the three-kink wave solution is depicted in Figure 3 with parameters $k_1 = -2, r_1 = 3, k_2 = -3, r_2 = 2, k_3 = 2, r_3 = -3$ for $\alpha = 1$. Figure 3(a) presents the cross section at $x = 1$, while Figure 3(b) illustrates the cross section at $y = 1$.



(a)



(b)

Figure 3. Three kink wave solution structures with $k_1 = -2, r_1 = 3, k_2 = -3, r_2 = 2, k_3 = 2, r_3 = -3$, $\alpha = 1$*: a) Cross-section at* $x = 1$ *and b) Cross-section at* $y = 1$.

This demonstrates that the (2 + 1)-dimensional Burger's equation, as represented by Equation (3), yields N-kink solutions for finite values of N, where $N \geq 1$. Based on the findings obtained, the general kink solutions can be represented in the following form:

$$M = \frac{\sum_{i=1}^{N} r_i e^{(k_i x^\alpha + r_i y^\alpha + k_i^2 t^\alpha)/\alpha}}{1 + \sum_{i=1}^{N} e^{(k_i x^\alpha + r_i y^\alpha + k_i^2 t^\alpha)/\alpha}}. \quad (41)$$

When $\alpha = 1$, the Equations (30), (31), (33), (34), (35), (36), and (37) correspond to Equations (14), (15), (19), (20), (22), (23), and (24) as presented in the study by Wazwaz [17].

6. Kink-solutions of Equation (5)

To examine the multiple-kink wave solutions of Equation (5), we consider

$$N = e^{(k_i x^\alpha + r_i y^\alpha - c_i t^\alpha)/\alpha}. \quad (42)$$

By substituting Equation (42) into the linear term of Equation (5), we derive the dispersion relation

$$c_i = -4k_i^3. \quad (43)$$

Consequently, we obtain

$$\theta_i = \frac{k_i x^\alpha + r_i y^\alpha + 4k_i^3 t^\alpha}{\alpha}. \quad (44)$$

Utilizing the Cole-Hopf fractional transformation, the multiple-kink wave solution of Equation (5) is given by

$$N = R\, ln(f). \quad (45)$$

and

$$M = R \frac{D_y^\alpha f}{f}. \quad (46)$$

In relation to the single-kink wave solution, consider the function

$$f = 1 + e^{(k_1 x^{\alpha} + r_1 y^{\alpha} + 4k_1^3 t^{\alpha})/\alpha}. \tag{47}$$

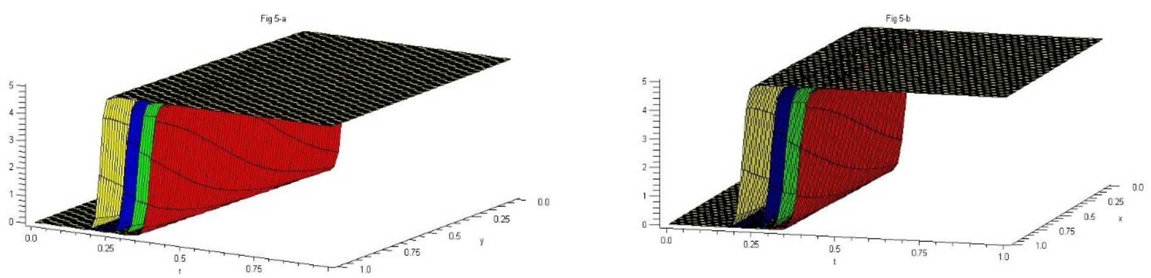
By substituting Equation (45) into Equation (8) and determining the solution for R, we obtain $R = 1$, and therefore

$$N = ln(1 + e^{(k_1 x^{\alpha} + r_1 y^{\alpha} + 4k_1^3 t^{\alpha})/\alpha + \rho_1}). \tag{48}$$

This yields the one kink wave solution:

$$M = \frac{r_1 e^{(k_1 x^{\alpha} + r_1 y^{\alpha} + 4k_1^3 t^{\alpha})/\alpha + \rho_1}}{1 + e^{(k_1 x^{\alpha} + r_1 y^{\alpha} + 4k_1^3 t^{\alpha})/\alpha + \rho_1}}. \tag{49}$$

The evolutionary behavior of the one-kink wave solution is illustrated in Figure 4 with parameters $k_1 = 3, r_1 = 5, \rho_1 = -50$, for varying values of α. The depiction includes the red layer for $\alpha = 1$, the green layer for $\alpha = 0.95$, the blue layer for $\alpha = 0.90$, and the gold layer for $\alpha = 0.80$. Figure 4(a) represents the cross-section at $y = 1$, while Figure 4(b) also represents the cross-section at $y = 1$.



(a) (b)

Figure 4. One kink wave solution structures with $k_1 = 3, r_1 = 5$, $\alpha = 1, 0.95, 0.90, 0.80$: a) Cross-section at $x = 1$ and b) Cross-section at $y = 1$.

In the context of the two-kink wave solution for Equation (5), we examine the function

$$f = 1 + e^{\theta_1} + e^{\theta_2} + a_{12} e^{\theta_1 + \theta_2}. \tag{50}$$

Incorporating Equation (50) into Equation (45) and subsequently substituting the outcome into Equation (7), we determine that ($R = 1$), with no phase shifts observed, i.e, $a_{12} = 0$. Therefore, we conclude that

$$a_{ij} = 0, 1 \leq i < j \leq 3. \tag{51}$$

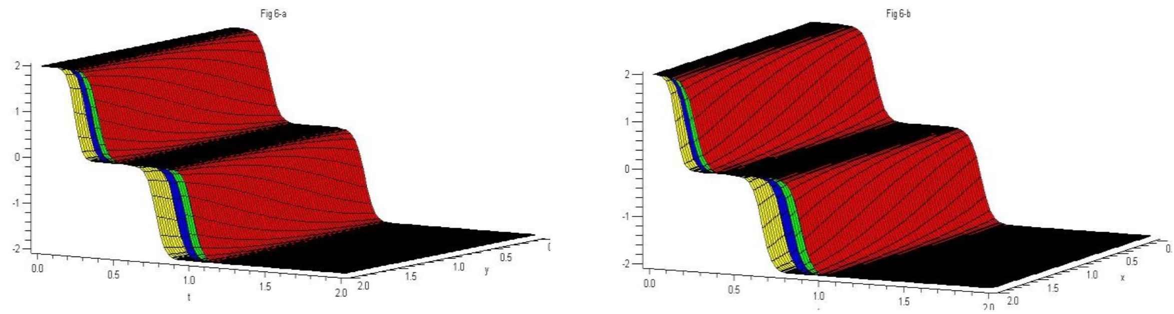
Consequently, we derive

$$N = ln(1 + e^{(k_1 x^{\alpha} + r_1 y^{\alpha} + 4k_1^3 t^{\alpha})/\alpha + \rho_1} + e^{(k_1 x^{\alpha} + r_1 y^{\alpha} + 4k_1^3 t^{\alpha})/\alpha + \rho_2}). \tag{52}$$

This formulation yields the two-kink wave solution:

$$M = \frac{r_1 e^{(k_1 x^{\alpha} + r_1 y^{\alpha} + 4k_1^3 t^{\alpha})/\alpha + \rho_1} + r_2 e^{(k_1 x^{\alpha} + r_1 y^{\alpha} + 4k_1^3 t^{\alpha})/\alpha + \rho_2}}{1 + e^{(k_1 x^{\alpha} + r_1 y^{\alpha} + 4k_1^3 t^{\alpha})/\alpha + \rho_1} + e^{(k_1 x^{\alpha} + r_1 y^{\alpha} + 4k_1^3 t^{\alpha})/\alpha + \rho_2}}. \tag{53}$$

The evolutionary behavior of the two kink wave solution is illustrated in Figure 5 for the parameters $k_1 = -2, r_1 = 2, \rho_1 = 10, k_2 = -3, r_2 = 2, \rho_2 = -30$ under varying values of α. The red layer represents $\alpha = 1$, the green layer represents $\alpha = 0.95$, the blue layer represents $\alpha = 0.90$, and the gold layer represents $\alpha = 0.80$. Figure 5(a) shows the cross-section at $x = 1$, while Figure 5(b) depicts the cross-section at $y = 1$.



(a) (b)

Figure 5. Two kink wave solution structures with $k_1 = -2, r_1 = 2, \rho_1 = 10, k_2 = -3, r_2 = 2, \rho_2 = -30$, $\alpha = 1, 0.95, 0.90, 0.80$*: a) Cross-section at* $x = 1$ *and b) Cross-section at* $y = 1$.

Considering the three-kink wave solution of equation (5), we define

$$f = 1 + e^{\theta_1} + e^{\theta_2} + e^{\theta_3}. \tag{54}$$

Continuing as previously, we find

$$N = ln(1 + e^{(k_1 x^\alpha + r_1 y^\alpha + 4k_1^3 t^\alpha)/\alpha + \rho_1} + e^{(k_1 x^\alpha + r_1 y^\alpha + 4k_1^3 t^\alpha)/\alpha + \rho_2} + e^{(k_1 x^\alpha + r_1 y^\alpha + 4k_1^3 t^\alpha)/\alpha + \rho_3}). \tag{55}$$

The corresponding three-kink wave solution is given by

$$M = \frac{r_1 e^{(k_1 x^\alpha + r_1 y^\alpha + 4k_1^3 t^\alpha)/\alpha + \rho_1} + r_2 e^{(k_1 x^\alpha + r_1 y^\alpha + 4k_1^3 t^\alpha)/\alpha + \rho_2} + r_3 e^{(k_1 x^\alpha + r_1 y^\alpha + 4k_1^3 t^\alpha)/\alpha + \rho_3}}{1 + e^{(k_1 x^\alpha + r_1 y^\alpha + 4k_1^3 t^\alpha)/\alpha + \rho_1} + e^{(k_1 x^\alpha + r_1 y^\alpha + 4k_1^3 t^\alpha)/\alpha + \rho_2} + e^{(k_1 x^\alpha + r_1 y^\alpha + 4k_1^3 t^\alpha)/\alpha + \rho_3}}. \tag{56}$$

The (2+1)-dimensional space-time fractional higher-order Burgers equation, represented by Equation (5), yields N-kink solutions for finite N, where $N \geq 1$. From the results obtained, the general kink solutions can be expressed in the form:

$$M = \frac{\sum_{i=1}^{N} r_i e^{(k_i x^\alpha + r_i y^\alpha + 4k_i^3 t^\alpha)/\alpha + \rho_i}}{1 + \sum_{i=1}^{N} r_i e^{(k_i x^\alpha + r_i y^\alpha + 4k_i^3 t^\alpha)/\alpha + \rho_i}}. \tag{57}$$

Moreover, it should be noted that Equations (45), (46), (48), (49), (54), (55), and (56) correspond to Equations (33), (34), (36), (37), (39), (40), and (41) as presented in Wazwaz's study [17].

7. Conclusion

This paper has investigated two forms of the (2+1)-dimensional fractional Burger's equations, with a focus on deriving solutions through the singular manifold (SM) method. By integrating the conformable fractional derivative (CFD) into the singular manifold (SM) method, we have successfully obtained exact wave solutions. Furthermore, we identified multiple solutions, including kink wave solutions. Our research indicates that numerous nonlinear equations, even those involving higher derivatives, are capable of yielding comparable solutions. By selecting three functions arbitrarily, we elucidated the properties of these solutions and introduced new physical interpretations. Additionally, variations in the fractional order result in significantly more complex structures. When the fractional order is set to one, our findings align with those reported by Peng and Yamba [11] and Wazwaz [17].

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