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Classification of Random Walks and Green's Theorem on Infinite Homogeneous Trees

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Abstract. In the context of random walks whose states are the vertices of an infinite tree, a classification of random walks is given as transient or recurrent. On the infinite homogeneous trees with the assumption that the transition probability between any two neighboring states are the same, a form of the classical Green's formula is derived. As a consequence, two versions of the mean-value property for median functions are obtained.

1. Introduction

The formula known as Green's theorem in the continuous (Euclidean) case, for instance in $\mathbb{R}^2$, is expressed as follows: Suppose Ω represents a bounded domain with a smooth boundary, and assume u and v are C^2 functions defined in a neighborhood of the closure of Ω. Then, we have:

$$\int_{\Omega} u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n} ds,$$

where n denotes the outward unit normal. This relationship stems from the application of the Gauss Divergence Theorem.

Among many corollaries of this formula in the study of potential theory, we single out the equation

$$\int_{\Omega} \Delta v dA = \int_{\partial\Omega} \frac{\partial v}{\partial n} ds$$

and the two versions of mean-value property for median functions given in the classical case for example in Brelot [6] and Axler, Bourdon, Ramey [3]. For a detailed study of discrete potential theory on infinite networks, see Saordi [19]. For some other discrete versions of the Green's

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formula we can refer to Duffin [7] (on the set of lattice point in $\mathbb{R}^3$), Bendito et al. [5] and Kayano and Yamasaki [16] (on networks with symmetric conductance).

In this discussion, we address a related issue within the framework of a random walk $\{T, p(x, y)\}$, where the state space T constitutes an infinite tree and $P = \{p(x, y)\}$ represents the transition probability matrix, assumed to be irreducible. As we delve into the subsequent analysis leading to a generalized version of the discrete Green's theorem in $\{T, P\}$, discrepancies arise in the proofs contingent upon whether the random walk is transient or recurrent. (Conventionally, a random walk is deemed recurrent if, after initiating at state e, it repeatedly returns to e an infinite number of times; otherwise, it is transient.) We obtain a very useful sufficient condition for the walk $\{T, P\}$ to be recurrent, which helps us to identify easily in many cases whether the walk is transient or recurrent. Finally, in the special case when T is a homogeneous tree, we deduce two mean-value properties of median functions on $\{T, P\}$.

2. Preliminaries

Consider a random walk $\{T, p(x, y)\}$, where T represents an infinite tree as the state space, and $P = \{p(x, y)\}$ denotes the matrix of transition probabilities. Each $p(x, y)$ satisfies $\sum_y p(x, y) = 1$ for every x in T. We assume the irreducibility of the transition probability matrix P. Let e be a fixed state in T. If the random walk, beginning at e, revisits e infinitely many times, it is termed recurrent; otherwise, it is transient.

Write $x \sim y$ if and only if the transition probability $p(x, y) > 0$. Let F be a subset of T, a state x is said to be an interior state of T if x and all the states $y \sim x$ are in F. Write $\overset{\circ}{F}$ as the set consisting of all the interior states of F. Write also $\partial F = F \setminus \overset{\circ}{F}$. A real-valued function $f(x)$ defined on F is termed supermedian on F if, for any x in the interior of F, it satisfies $f(x) \geq Pf(x) = \sum_{y \sim x} p(x, y) f(y)$, and it is considered a median function if $f(x) = Pf(x)$. Suppose $s(x)$ is a non-negative supermedian function on F having the property that if $h(x)$ is any median function on F such that $h(x) \leq s(x)$, then $h(x) \leq 0$; such a function $s(x)$ is termed as an excessive function on T. It is established that the random walk $\{T, p(x, y)\}$ is transient if there exists an excessive function $s(x) > 0$ defined on T, and the walk is recurrent if every non-negative supermedian function on T is constant [2]. If $p^n(x, y)$ is the entry in the x^{th} row and y^{th} column in the matrix p^n, then in the case of transient walks the real-valued function $G_y(x) = G(x, y) = \sum_{n=0}^{\infty} p^n(x, y)$ represents the expected number of visits the walker makes to reach the state y starting from the state x; moreover for a fixed y, $G_y(x)$ is a bounded function such that $G_y(x) \leq G_y(y)$ and $PG_y(x) - G_y(x) = -\delta_y(x)$ for all x. In [13], Lyons has chosen an old statement of Royden and Suji and given a necessary and sufficient condition that a countable Markov chain which is reversible random walk then the network is transient. Some more results regarding recurrent and transient of a random walk in networks are given in [14] and [15].

Dirichlet solution: Let F be a finite subset of T the state e in its interior. Let $z \in \partial F$. Write $B = \{e\} \cup \{\partial F \setminus z\}$. Let $\rho_z(x)$ denote the probability of the walker starting at a state x in F and reaches z before reaching any state in B. Then $\rho_z(x) = 0$ for any $x \in B$, and $\rho_z(z) = 1$ [1]; moreover if $a \in \overset{\circ}{F}$, then $\rho_z(a) = \sum_{y\sim a} p(a,y)\rho_z(y)$. Consequently $(I - P)\rho_z(x) = 0$ for any $x \in \overset{\circ}{F}$. Now let $\varphi(b)$ be any real-valued function on ∂F; then $h(x) = \sum_{b\in\partial F} \varphi(b)\rho_x(b)$ is a well-defined function such that $h(b) = \varphi(b)$ for any $b \in \partial F$ and $(I - P)h(x) = 0$ for any $x \in \overset{\circ}{F}$. The function $h(x)$ on F is referred to as the Dirichlet solution in F with boundary values $\varphi(b)$.

Suppose x is an arbitrary state in T in which the state e is fixed. Since T is a tree, there is a unique path $\{e = x_0, x_1, .., x_n = x\}$ connecting e to x. We say that the length of the path is n and write $|x| = dist(e, x) = n$. Note that if $|x| = n$, then there is only one state $x^{\sim}$ such that $x^{\sim} \sim x$ and $|x^{\sim}| = n - 1$.

Applications: Random walks are used in numerous fields. In social networking, it is used to detect fake user accounts and enhance privacy of the users. In [11], the authors used a ranking scheme for detecting fake accounts on Facebook profiles using random walk. In medical field, Zhang in [20] has used random walk to detect genetic diseases, issue associated to cancer genes, pathogenic genes and connection between micro organisms and diseases. In physics, Eisenriegler [8] structured long polymer chains and their properties using self avoiding walk, a walk on a lattice where the moves visit the same node only once. By ranking web pages using random walk in [12], the authors have magnified the quality of web search .
For example, let's examine a Lévy flight random walk. A Lévy flight constitutes a random walk characterized by step-lengths possessing a stable distribution, which is a probability distribution exhibiting heavy tails [4]. That is, length of the steps may vary significantly and sometimes there may occur a longer jumps. D. Reible and S. Mohanty in [17] constructed a Lévy flight model that indicates the square root dependents on bioturbation rates over extensive spread of biomass densities. The behaviour of bees and pollination of flowers are also analysed through Lévy random walk in [18] and [9] respectively.

3. Some Properties of Supermedian Function

- If $f_n(x)$ is a sequence of supermedian functions on F and if $f(x) = \lim_{n\to\infty} f_n(x)$ exists and is real valued, then $f(x)$ is supermedian on F, also $Pf(x) = \lim_{n\to\infty} Pf_n(x)$
- Let $f(x)$ be a supermedian function on F that dominates $\{v_i(x)\}$, which is the collection $\mathcal{F}$ of all submedian functions on F, then the collection $\mathcal{F}$ is an increasing function and $h(x) = \sup_{\mathcal{F}} v_i(x)$ is a median function $f(x)$ on F. Consequently, if a submedian function is dominated by a supermedian function $f(x)$ on F, then $f(x)$ is the total of greatest median function $(g.m.m)$ $h(x)$ and potential $p(x)$ on F. This breakdown as the total of $g.m.m.$ and potential is also unique and is called Riesz decomposition.

- **Dirichlet Problem**: Let E be a finite subset of $\{T, P\}$ and $F \subset \mathring{E}$. Let λE be finite set of states in $E \backslash F$ and $u(x)$ be a function in λE. Let $\mathcal{F}_i$ be the family of all real valued functions $\mu(x)$ on $F \cup \lambda E$ such that $\mu(x) = u(x)$ on λE and $\mu(x)$ is supermedian function at every state in F. Then there exists a median function $h(x)$ on F such that $h(x) = u(x)$ on λE and $h(x)$ is supermedian function at every state in F.
- **Reduced functions and Balayage**: Let $A \subset F$ and $s(x) \geq 0$ be supermedian function on a subset F. Let $\mathcal{F}$ be the family of all supermedian functions $f(x)$ on F that majorize $s(x)$ on A and are not negative. Then, the balance of $s(x)$ on A in the subset F is denoted as $\left(\Re_s^A(x)\right)_F = \inf_{f \in \mathcal{F}} f(x)$ (the reduced function in the case of potential theory on topological spaces). When it is the entire set T, we exclude F.
  Note that $v(x) = \left(\Re_s^A(x)\right)_F$ is a non negative supermedian function on F such that $v(x) \leq s(x)$ on F; $v(x) = s(x)$ on A and $\Delta v(x) = 0$ if $x \in F \backslash A$. If there exists a potential $p(x)$ on F such that $s(x) \leq p(x)$ on F, then $v(x)$ is a potential on F.

4. Classification of Random Walks $\{T, p(a, b)\}$

In classical case, that is, in $\mathbb{R}^n$, by considering the median measure at the point at infinity, the classification of Riemann surfaces is given. In the discrete case of an infinite network, V. Anandam has classified the infinite network as parabolic network and hyperbolic network. In this section, we construct a function on T which is supermedian outside a finite set. Based on this function we derive a necessary condition for a random walk to be recurrent or transient. We write $p^n(x, y)$ as $P_y^n(x)$.

Proposition 4.1. *The Markov chain is transient if and only if* $\sum_{n=0}^{\infty} P_e^n(e) < \infty$

Proof. Assume that $X_0 = e$ and fix a state e. According to Lawler ([10], section 2.2), analyzing the random variable R, which signifies the entire number of visits, including the primary visit to e. $R = \sum_{n=0}^{\infty} \chi\{T_n = e\}$ should be written, with χ symbolizing the characteristic function. In the context of recurrent chains, R is ∞. Stated differently, $R_m \to \infty$ when $m \to \infty$ if $\sum_{n=o}^{m} \chi\{T_n = e\}$. At this point, the forecast $E(R_m) = \sum_{n=0}^{m}$Probability of $T_n = e$. The expression $\sum_{n=0}^{m} p^n(e, e) = \sum_{n=0}^{m} P_e^n(e)$. For the recurrence, this means that $\sum_{n=0}^{\infty} P_e^n(e) = \infty$. Take note that if the chain is temporary, $R < \infty$ with probability 1. In this case the expectation of R is $E(R)$, where

$$\begin{aligned} E(R) =& E[\sum_{n=0}^{\infty} \chi\{T_n = e\}] \\ =& \sum_{n=0}^{\infty} Prob.\{T_n = e\} \\ =& \sum_{n=0}^{\infty} P_e^n(e) \end{aligned}$$

Accordingly, the chain cannot be recurrent if $\sum_{n=0}^{\infty} P^n < \infty$. Let us consider the opposite case, when the sequence T_n revisits e only a finite number of times due to the chain's transience. The probability of T_n returning to e for the first time is denoted by q. It is important to note that because the chain is temporary, $q \neq 1$. In contrast to the chain's temporary character, if $q = 1$, it would suggest that the chain regularly returns to e with probability 1, which would lead to the conclusion that the chain is recurrent.

These are the case in the transience scenario, where $R = 1$ if and only if the chain never revisits e and $R = m$, if and only if the chain revisits $m-1$ times and fails to return for the mth time; otherwise, the probability is $q^{m-1}(1-q)$. Consequently,

$$E(R) = \sum_{m=1}^{\infty} mProb.\{R = M\}$$

$$= \sum_{m=1}^{\infty} m[q^{m-1}(1-q)]$$

$$E(R) = \frac{1}{1-q} < \infty$$

We conclude that the Markov chain is transient if and only if $\sum_{n=0}^{\infty} P_e^n(e) < \infty$. This may be done by comparing this to the previous expression for $E(R)$ in the transient case. □

Remark 4.1. *The walker can reach e from z in a finite number of steps for any state z. Therefore, the choice of the initially set state e has no bearing on the nature of transience.*

We will now examine $P_y^n(x)$, which is the chance that a walker starting at state x would reach state y in n steps, in place of circuit probabilities like $P_z^n(z)$. It is simpler to treat $\{T, P = [p(x,y)]\}$ as an infinite tree in this case, or as a random walk depending on the situation.

$P[P_y^n(x)] = \sum_z P_z(x)P_y^n(z) = \sum_z P_z(x)P^n(z,y) = P_y^{n+1}(x) = P_y^{n+1}(x)$.

Whereas $P_y^n(x)$ denotes the likelihood that a walker from state x would arrive at state y in n steps, $G_y(x) = \sum_{n=0}^{\infty} P_y^n(x)$ denotes the expected number of visits to state y from the beginning state x.

Proposition 4.2. *The infinite tree $\{T, P\}$ associated with a transient random walk $\{T, P = [p(x,y)]\}$ is hyperbolic.*

Proof. In fact, we will demonstrate that the Green's potential on the network $\{T, p(x,y)\}$ with median support at $\{y\}$ is denoted by $G_y(x)$.

From the infinite tree T, pick a state y. Given $G_y(x) = \sum_{n=0}^{\infty} P_y^n(x)$, $P[G_y(x)] = \sum_{n=1}^{\infty} P_y^n(x) \leq G_y(x)$. Hence, in the infinite tree $\{T, P\}$, $G_y(x)$ is a positive supermedian function.

Indeed, $G_y(x)$ is a potential function. For that, note that we can write $G_y(x) - P[G_y(x)] = \delta_y(x)$, which is the column vector with entry 1, when $y = x$ and 0 in other entries, when $G_y(x)$ is real valued. It follows that $-\Delta[G_y(x)] = (1-P)[G_y(x)] = \delta_y(x)$.

We have $h(x) = P^m h(x) \leq P^m[G_y(x)] = \sum_{n=m}^{\infty} P_y^n(x)$, which tends to 0 when $m \to \infty$, if $h \geq 0$ is a median function such that $h(x) \leq G_y(x)$. As a result, $h \equiv 0$. Thus, $G_y(x)$ is a potential, the Green's potential in this instance, with $\{y\}$ serving as its median support.

□

Proposition 4.3. *Let $\{T, p(x,y)\}$ be a random walk. Suppose there exists a function v defined outside a finite set A in T such that $Pf(x) \leq f(x)$ at every $x \in T \setminus A$ and $\lim_{x\to\infty} v(x) = \infty$. Then the random walk is recurrent.*

Proof. With the existence of a function $v(x)$ defined outside a finite set A in T, the random walk $\{T, p(x,y)\}$ has to be recurrent. For otherwise, for each vertex $y \in T$, we know that there exists a unique excessive function $G_y(x)$ which is bounded and $(I-P)G_y(x) = \delta_y(x)$. We shall choose a large finite set E, $\overset{\circ}{E} \supset A$. Let h be the Dirichlet solution on E with boundary values v on ∂E. Let v_1 be the function on T such that $v_1 = h$ on E and $v_1 = v$ on $T \setminus E$. Define for $x \in T$, $v_2(x) = v_1(x) + \sum_{y\in\partial E}(P-I)v_1(y)G_y(x)$.

Then for $x \in \overset{\circ}{E}$, $(I-P)v_2(x) = 0$; for $x \in (T \setminus E)$, $(I-P)v_2(x) = (I-P)v_1(x) \geq 0$; for $x \in \partial E$, $(I-P)v_2(x) = (I-P)v_1(x) + \Delta v_1(x) = 0$. Thus $(I-P)v_2(x) \geq 0$ on T. Now $G_y(x)$ is bounded on T, so that $\lim_{x\to\infty} v_2(x) = \infty$. But this is not possible by the Minimum Principle for $v_2(x)$. Consequently, the assumption that $\{T, p(x,y)\}$ is not recurrent is invalid; that is $\{T, p(x,y)\}$ is recurrent. □

Consider the function $f(x) = (\frac{\alpha}{1-\alpha})^{|x|}$ where $0 < \alpha < 1$, defined on T. We can write $f(x)$ as $f(n) = (\frac{\alpha}{1-\alpha})^n$.

Lemma 4.1. *For the function f defined above, we have $Pf(n) - f(n) = [1 - \frac{p(x,x^{\sim})}{\alpha}]\frac{2\alpha-1}{1-\alpha}[\frac{\alpha}{1-\alpha}]^n$, $\forall n \in T$, where $x^{\sim}$ is the state with $d(e,x^{\sim}) = d(e,x) - 1 = n-1$, taking $d(e,e^{\sim}) = 0$.*

Proof. Let 'e' be a fixed state. Then

$$\begin{aligned} Pf(e) - f(e) &= \sum_{x^{\sim}\sim e} p(e,x^{\sim})f(x^{\sim}) - f(e) \\ &= \sum_{x^{\sim}\sim e} p(e,x^{\sim})\frac{\alpha}{1-\alpha} - 1 \\ &= \frac{2\alpha-1}{1-\alpha} \end{aligned}$$

Let 'x' be a state with $|x| = n$, then there is a unique state $x^{\sim}$, $x^{\sim} \sim x$, with $|x^{\sim}| = n-1$.

$$\begin{aligned} Pf(x) - f(x) &= \sum_{y\sim x} p(x,y)f(y) - f(x) \\ &= p(x,x^{\sim})f(x^{\sim}) \\ &+ \sum_{y\sim x, |y|=n+1} p(x,y)f(y) - f(x) \\ &= p(x,x^{\sim})(\frac{\alpha}{1-\alpha})^{n-1} \\ &+ (1-p(x,x^{\sim}))(\frac{\alpha}{1-\alpha})^{n+1} - (\frac{\alpha}{1-\alpha})^n \\ &= (\frac{\alpha}{1-\alpha})^n[p(x,x^{\sim})\frac{1-\alpha}{\alpha} \end{aligned}$$

$$+ \quad (1 - p(x, \tilde{x}))\frac{\alpha}{1-\alpha} - 1]$$
$$= \quad (\frac{\alpha}{1-\alpha})^n [\frac{2\alpha - 1}{1-\alpha}][1 - \frac{p(x, x^{\sim})}{\alpha}]$$

□

Theorem 4.1. (i) *If $p(x, x^{\sim}) \leq \alpha < \frac{1}{2}$ then the random walk $\{T, p(x, y)\}$ is transient;*

(ii) *If $p(x, x^{\sim}) \geq \alpha > \frac{1}{2}$ then the random walk is recurrent;*

(iii) *If $p(x, x^{\sim}) = \frac{1}{2}$ then also the walk is recurrent.*

Proof. If $p(x, x^{\sim}) \leq \alpha < \frac{1}{2}$ then by the Lemma 4.1, $Pf(e) - f(e) = \frac{2\alpha-1}{1-\alpha} < 0$ and for x, $|x| > 1$, $Pf(x) - f(x) = (\frac{\alpha}{1-\alpha})^n [\frac{2\alpha-1}{1-\alpha}][1 - \frac{p(x,x^{\sim})}{\alpha}] \leq 0$. Hence $f(n)$ is a supermedian non median function on T such that $\lim_{n\to\infty} f(n) = 0$. Therefore $f(x)$ is an excessive function on T, hence $\{T, p(x, y)\}$ is transient.

If $p(x, x^{\sim}) \geq \alpha > \frac{1}{2}$ then by Lemma 4.1, $Pf(e) - f(e) = \frac{2\alpha-1}{1-\alpha} > 0$ and for x, $|x| > 1$, $Pf(x) - f(x) = (\frac{\alpha}{1-\alpha})^n [\frac{2\alpha-1}{1-\alpha}][1 - \frac{p(x,x^{\sim})}{\alpha}] \leq 0$. Hence $f(n)$ is a supermedian function on $T \setminus e$ such that $\lim_{n\to\infty} f(n) = \infty$. Therefore by Proposition 4.3, T is recurrent.

Note that, when $\alpha = \frac{1}{2}$, $Pf(n) - f(n) = 0$, that is $f(n)$ is a median function on T. However, in this case also the random walk is recurrent. For, if $x \in T$, $|x| = n$ then for the function $\varphi(x) = n$, we have $(I - P)\varphi(x) = 0$ when $n \geq 1$. Hence $\varphi(x)$ is median in $T \setminus e$ and tends to ∞ at infinity. Hence the walk is recurrent. □

5. Mean-Value Property for Median Functions in a Homogeneous Tree

In this section, we derive a consequence of the classical Green's formula in the framework of a random walk on a homogeneous tree where each transition probability $p(x, x^{\sim}) = c$, a constant.

In the plane, suppose $f(x)$ is a continuous function in $\mathbb{R}^2$, let $B(x_0, \rho)$ be a disc with centre x_0 and radius ρ. Let $M_f(x_0, \rho)$ be the mean-value of $f(x)$ on the circumference $|x - x_0| = \rho$ and $A_f(x_0, \rho)$ be the areal mean. The function $f(x)$ is median if and only if the circumference mean $M_f(x_0, \rho)$ is equals to $f(x)$ and the function $f(x)$ is median if and only if the areal mean is equals to $f(x)$. The two theorems 5.2 and 5.3, that we prove in this section are the discrete versions of these relations.

Write $\Delta f(x) = (P - I)f(x)$, where $f(x)$ is a real valued function defined on T.

With the state e fixed in the random walk $\{T, p(x, y)\}$, for any state x there exists a unique path $\{e = x_0, x_1, x_2, ..., x_n = x\}$. Let $\varphi(x) = \frac{p(e,x_1)p(x_1,p_2)...p(x_{n-1},x)}{p(x,x_{n-1})p(x_{n-1},x_{n-2})...p(x_1,e)}$. Note that, if $y \sim x$, then $\varphi(x)p(x, y) = \varphi(y)p(y, x)$. Write $c(x, y) = \varphi(x)p(x, y)$. Then the conductance $\{c(x, y)\}$ is symmetric on T. Let Δ_c denote the Laplacian on the network $\{T, c(x, y)\}$. Then for any real-valued function

$f(x)$ on T, $\Delta_c f(x) = \sum_{y\sim x} c(x,y)[f(y) - f(x)] = \varphi(x)\sum_{y\sim x} p(x,y)[f(y) - f(x)] = \varphi(x)\Delta f(x)$. Note now that for any $f(x)$ if the operator Δ_c is restricted to B_n, then $\sum_{|x|\leq n} \Delta_c f(x) = 0$.

Theorem 5.1. *(Green's Theorem:) Let $f(x)$ be any real valued function defined on a random walk $\{T, p(x,y)\}$, with the function $\varphi(x)$ as defined above. Assume $\varphi(s)p(s,s^{\sim})$ is a constant C for any state $s, |s| \geq 1$, where $s^{\sim}$ is the unique state such that $|s^{\sim}| = n-1$ and $s^{\sim} \sim s$. Then $\frac{1}{C}\sum_{|x|<n} \varphi(x)\Delta f(x) = \sum_{|s|=n} f(s) - \sum_{|s|=n-1}[deg(s) - 1]f(s)$.*

Proof. We know that $\sum_{|x|\leq n} \Delta_c f(x) = 0$, for $n \geq 1$.

$$\sum_{|x|<n} \Delta_c f(x) + \sum_{|x|=n} \Delta_c f(x) = 0 \tag{5.1}$$

for each s, $|s| = n$ there is only one state $s^{\sim}, s^{\sim} \sim s$, with $|s^{\sim}| = n-1$

$$\begin{aligned} \Delta_c f(s) &= \sum_{s^{\sim}\sim s} \varphi(s)p(s,s^{\sim})[f(s^{\sim}) - f(s)] \\ &= C[f(s^{\sim}) - f(s)] \end{aligned}$$

Now, if we take the sum

$$\sum_{|s|=n} \Delta_c f(s) = C\sum_{|s|=n}[f(s^{\sim}) - f(s)]$$

$$= C\sum_{|s|=n} f(s^{\sim}) - C\sum_{|s|=n} f(s) \tag{5.2}$$

Substituting (5.2) in (5.1),

$$\sum_{|x|<n} \Delta_c f(x) + C\sum_{|s|=n} f(s^{\sim}) - C\sum_{|s|=n} f(s) = 0$$

$$\sum_{|x|<n} \Delta_c f(x) = C\sum_{|s|=n} f(s) - C\sum_{|s|=n} f(s^{\sim})$$

For each $s^{\sim}$, there are $deg(s^{\sim}) - 1$ neighbours with $|s| = n$. Consequently, when s runs through all the vertices in ∂B_n, $s^{\sim}$ runs $deg(s^{\sim}) - 1$ times through each vertex in ∂B_{n-1}. Hence we write

$$\sum_{|s|=n} f(s^{\sim}) = \sum_{|s|=n-1} (deg(s) - 1)f(s).$$

Therefore,

$$\frac{1}{C}\sum_{|x|<n} \varphi(x)\Delta f(x) = \sum_{|s|=n} f(s) - \sum_{|s|=n-1} [deg(s) - 1]f(s).$$

□

Remark: If T is a homogeneous tree of degree q, then $\varphi \equiv 1$, $C = \frac{1}{q}$ and $deg(s) = q$, so that, we have the following:

Corollary 5.1. *Let $f(x)$ be a real-valued function on a homogeneous tree of order q with a fixed vertex e, then for any $n \geq 1$,*

$$q \sum_{|x|<n} \Delta f(x) = \sum_{|s|=n} f(s) - (q-1) \sum_{|s|=n-1} f(s)$$

Theorem 5.2. *(Version 1) : If $h(x)$ is a median function on a homogeneous tree of order $q \geq 2$ then*

$$h(e) = \frac{1}{q(q-1)^{n-1}} \sum_{|s|=n} h(s).$$

Proof. From Corollary 5.1,

$$q \sum_{|x|<n} \Delta h(x) = \sum_{|s|=n} h(s) - (q-1) \sum_{|s|=n-1} h(s)$$

Since $\Delta h(x) = 0$, we have

$$0 = \sum_{|s|=n} h(s) - (q-1) \sum_{|s|=n-1} h(s)$$

$$\sum_{|s|=n} h(s) = (q-1) \sum_{|s|=n-1} h(s)$$

A repetition of this calculation for $(n-1)$ times, we get

$$\sum_{|s|=n} h(s) = (q-1)^{n-1} \sum_{|s|=1} h(s) \tag{5.3}$$

Since $h(x)$ is median at e, we have

$$h(e) = \frac{1}{q} \sum_{|s|=1} h(s) \tag{5.4}$$

From (5.3)

$$\sum_{|s|=1} h(s) = \frac{1}{(q-1)^{n-1}} \sum_{|s|=n} h(s) \tag{5.5}$$

Substitute (5.5) in (5.4) , $h(e) = \frac{1}{q(q-1)^{n-1}} \sum_{|s|=n} h(s)$. Hence the theorem is proved. □

Theorem 5.3. *(Version 2) : If $h(x)$ is a median function on a homogeneous tree of order $q > 2$ then*

$$h(e) = \frac{q-2}{q[(q-1)^{n}-1]} \sum_{|s|=1}^{|s|=n} h(s).$$

Proof.

$$\sum_{|s|=1}^{|s|=n} h(s) = \sum_{|m|=1}^{|m|=n} \sum_{|s|=m} h(s)$$

From Corollary 5.1

$$\begin{aligned} h(e) &= \frac{1}{q(q-1)^{m-1}} \sum_{|s|=m} h(s) \\ \sum_{|s|=m} h(s) &= q(q-1)^{m-1} h(e) \end{aligned}$$

We get,

$$\begin{aligned} \sum_{|s|=1}^{|s|=n} h(s) &= \sum_{|m|=1}^{|m|=n} q(q-1)^{m-1} h(e) \\ &= q[(q-1)^{1-1} + (q-1)^{2-1} + .. \\ &\quad . \ . + (q-1)^{n-1}] h(e) \\ &= q[1 + (q-1)^{1} + (q-1)^{2} + ... \\ &\quad . \ . + (q-1)^{n-1}] h(e) \\ &= q\left[\frac{1-(q-1)^{n}}{1-(q-1)}\right] h(e) \\ &= q\left[\frac{1-(q-1)^{n}}{(2-q)}\right] h(e) \\ \sum_{|s|=1}^{|s|=n} h(s) &= q\left[\frac{(q-1)^{n}-1}{(q-2)}\right] h(e) \end{aligned}$$

Which implies,

$$h(e) = \frac{(q-2)}{q[(q-1)^{n}-1]} \sum_{|s|=1}^{|s|=n} h(s).$$

Hence the theorem is proved. □

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