International Journal of Analysis and Applications

Inertial Bilevel Variational Monotone Inclusion Problem in Banach Spaces

Ikechukwu Godwin Ezugorie[1], **Godwin Chidi Ugwunnadi**[2,4,*], **Eric Uwadiegwu Ofoedu**[3], **Maggie Aphane**[4]

[1]*Department of Mathematics, Enugu State University of Science and Technology, Enugu, Nigeria*
[2]*Department of Mathematics, University of Eswatini, Private 4, Kwaluseni M201, Eswatini*
[3]*Department of Mathematics, Nnamdi Azikiwe University, Awka, Nigeria*
[4]*Department of Mathematics and Applied Mathematics, Sefako Makgato Health Science University, P.O. Box 94, Pretoria 0204, South Africa*

[*]*Corresponding author:* ugwunnadi4u@yahoo.com

Abstract. In this paper, we study accelerated Halpern-type iterative method for finding zero solution of sum of two monotone operators which solves variational inequality problem of inverse strongly monotone mapping in 2-uniformly convex and uniformly smooth real Banach spaces. The strong convergence of our proposed method is establish under some standard conditions imposed on parameters. Our theorems generalize many recently announced results in the literature.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$. Let E^* denotes the dual of E, and $\langle x, f\rangle$, the value of $f \in E^*$ at $x \in E$.

Definition 1.1. *A mapping* $A : D(A) \subseteq E \to R(A) \subseteq E$ $\Big($*where* $D(A)$ *and* $R(A)$ *are the domain of* A *and the range of* A*, respectively*$\Big)$ *is called*

(a) γ*-strongly monotone if there exists a constant* $\gamma > 0$ *such that*

$$\langle Ax - Ay, x - y\rangle \geq \gamma\|x - y\|^2,\ \forall x, y \in D(A).$$

Received: Jun. 17, 2024.

2020 *Mathematics Subject Classification.* 47H09, 47J25.

Key words and phrases. variational inclusion problem; monotone and inverse mappings; Banach spaces; strong convergence.

https://doi.org/10.28924/2291-8639-22-2024-210
ISSN: 2291-8639

(b) *α-inverse strongly monotone (see [2–4]) if there exists a positive real number α such that*

$$\langle x - y, Ax - Ay \rangle \geq \alpha ||Ax - Ay||^2$$

for all $x, y \in C$.

(c) *the variational inequality problem on C that is defined as follows:*

$$\text{find } x^* \in C \text{ such that } \langle x - x^*, Ax^* \rangle \geq 0 \text{ for all } x \in C. \tag{1.1}$$

The set of solutions of the variational inequality problem is denoted by $VI(C, A)$.

(d) *Lipschitz continuous if there exists a constant $L > 0$ such that $\forall x, y \in D(A)$,*

$$||Ax - Ay|| \leq L||x - y||.$$

A multi-valued operator $A : E \to 2^{E^*}$ with graph $G(A) = \{(x, x^*) : x^* \in Ax\}$ is said to be monotone if for any $x, y \in D(A)$, (where $D(A)$ denote domain of A), $x^* \in Ax$ and $y^* \in Ay$

$$\langle x - y, x^* - y^* \rangle \geq 0.$$

A monotone operator B is said to be maximal if for any monotone operator $S : E \to 2^{E^*}$ such that $G(B) \subseteq G(S)$, we have that $B = S$. Let E be a reflexive, strictly convex and smooth real Banach space and $B : E \to 2^{E^*}$ be a maximal monotone operator, then for each $r > 0$ and $x \in E$, there corresponds a unique element $x_r \in E$ such that

$$Jx \in Jx_r + rBx_r.$$

This unique element x_r is define as the resolvent of B, denoted by $J_r^B x$. In other words, $J_r^B = (J + rB)^{-1}J$ for all $r > 0$. It is easy to show that $B^{-1}0 = F(J_r^B)$ for all $r > 0$, where $F(J_r^B)$ denotes the set of all fixed points of J_r^B. Also, the Yosida approximation of B can be defined for each $r > 0$ by $A_r = \frac{J - JJ_r^B}{r}$ (see [11]) for more details. Consider an inclusion problem of finding:

$$z \in E \text{ such that } 0 \in (A + B)z. \tag{1.2}$$

where $B : E \to 2^{E^*}$ be a maximal monotone operator and $A : E \to E^*$ be a Lipschitz continuous monotone operator. This problem is called monotone inclusion problem(1.2) became a problem of contemporary interest due to its vast applicability in solving many important problems such as convex minimization problem, variational inequality inequality, Nash equilibrium problem in non-cooperative games, image restoration, and signal processing, (See, for example, [1,16–18,23–26,34] and the references therein). Many iterative algorithms for solving problem (1.2) have been introduced in the setting of real Hilbert spaces H, among which are the well-known forward-backward splitting algorithm of Passty [24], Lions and Mercier [25] and Peaceman-Rachford algorithm [33] plus many others. The forward-backward algorithm generates a sequence $\{x_n\}_{n=1}^{\infty}$ given by:

$$x_1 \in H, \ x_{n+1} = (I + \lambda_n B)^{-1}(I - \lambda_n A)x_n, \ n \in \mathbb{N} \tag{1.3}$$

where $\forall n \in \mathbb{N}$, $\lambda_n > 0$ and $D(B) \subset D(A)$. The algorithm (1.3) has been extensively studied by many authors due to nonexpansive nature of the resolvent operator $(I + \lambda_n B)^{-1}$ in the backward step (1.3). To obtain strong convergence, Takahashi et al. [30] studied a modified Halpern-type algorithm in real Hilbert spaces. They proved that the sequence generated by their algorithm converges strongly to solution of (1.2). Also, Kitkuan et al. [32] proved a strong convergence theorem by introducing and studying a viscosity type algorithm approximating of solutions of problem (1.2) in the setting of real Hilbert spaces.

Due to the known slow convergence properties of iterative methods involving monotone operators, the inertial technique has been successfully employed to accelerate the convergence process of the algorithm (1.3) and its modifications. For example, Lorenz and Pock [27] proved weak convergence of sequence of iterate generated by an inertial version of the algorithm (1.3) for a solution of (1.2) in the setting of real Hilbert spaces. Also, Cholamjiak et al. [28] proved strong convergence of an inertial Halpern-type version of algorithm (1.3) in a real Hilbert space. In 2021, Adamu et al. [29] introduced and studied a three-step modified inertial viscosity-type of algorithm in the setting of real Hilbert spaces and established strong convergence result.
It is worth of mentioning that all the results obtained by the authors mentioned above were established in the setting of real Hilbert spaces. Most real-life problems, however, do not reside only in Hilbert spaces. In 2019, Shehu [13] proved the following theorem which extends the inclusion problem (1.2) involving monotone operators to real Banach spaces:

Theorem 1.1. *Let E be uniformly smooth and 2-uniformly convex real Banach space. Let $A : E \to E^*$ be a monotone and L-Lipschitz continuous mapping and $B : E \to 2^{E^*}$ be a maximal monotone mapping. Suppose $(A+B)^{-1}(0)$ is nonempty and that the normalized duality mapping J on E is weakly sequentially continuous. Let $\{x_n\}$ be a sequence defined by:*

$$\begin{cases} x_1 \in E; \\ y_n = (J + \lambda_n B)^{-1}(Jx_n - \lambda_n A x_n), \\ x_{n+1} = J^{-1}(Jy_n - \lambda_n(Ay_n - Ax_n)), \quad n \geq 1, \end{cases} \tag{1.4}$$

where λ_n satisfies the following condition: $0 < a < \lambda_n < \frac{1}{\sqrt{2}\mu k L}$, where μ is the 2-uniform convexity constant of E; k is the 2-uniform smoothness constant of E^* and L is the Lipschitz constant of A. Then, the sequence $\{x_n\}$ converges weakly to a solution of problem (1.2).

He [19] further proved that the sequence $\{x_n\}$ generated by

$$\begin{cases} x_1 \in E; \\ y_n = (J + \lambda_n B)^{-1}(Jx_n - \lambda_n A x_n), \\ w_n = J^{-1}[Jy_n - \lambda_n(Ay_n - Ax_n)], \\ x_{n+1} = J^{-1}[\alpha_n J x_1 + (1-\alpha_n) J w_n)], \quad \forall n \geq 1, \end{cases} \tag{1.5}$$

converges strongly to the solution of problem (1.2), where $\{\alpha_n\}$ is a real sequence in $(0,1)$ such that $\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty.$
Kimura and Nakajo [22] proved the following strong convergence theorem:

Theorem 1.2. *Let C be a nonempty closed and convex subset of a uniformly smooth and 2-uniformly convex real Banach space E. Let $A : C \to E^*$ be an α-inverse strongly monotone mapping and $B : E \to 2^{E^*}$ be a maximal monotone mapping. Suppose the solution set $(A + B)^{-1}(0)$ is nonempty. Let u, $\{x_n\}$ in C be a sequence defined by:*

$$\begin{cases} x_1 \in C; \\ x_{n+1} = \Pi_C(J + \lambda_n B)^{-1}(\gamma_n Ju + (1 - \gamma_n)(Jx_n - \lambda_n Ax_n)), \quad \forall n \geq 1, \end{cases} \tag{1.6}$$

where Π_C is the generalized projection, $\lambda_n \in (0, \infty)$ and $\gamma_n \in [0, 1]$ such that $\lim_{n\to\infty} \gamma_n = 0$ and $\sum_{n=1}^{\infty} \gamma_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to a solution of problem (1.2).

Recently, Adamu et al. [20] introduced the following modified inertial Halpern-type Forward-Backward algorithm involving monotone operators in the setting of 2-uniformly convex and uniformly smooth real Banach spaces.

$$\begin{cases} w_1, v \in E; \\ y_n = J^{-1}[Jw_n + \mu_n(Jw_n - Jw_{n-1})], \\ z_n = (J + \lambda_n B)^{-1}(Jy_n - \lambda_n Ay_n), \\ w_{n+1} = J^{-1}[a_n Jv + b_n Jy_n + c_n Jz_n)], \quad \forall n \geq 1, \end{cases} \tag{1.7}$$

and

$$\mu_n \leq \bar{\mu}_n = \begin{cases} \min\left\{\mu, \frac{\vartheta_n}{\|Jw_n - Jw_{n-1}\|}, \frac{\vartheta_n}{\phi(w_n, w_{n-1})}\right\} & if \;\; w_n \neq w_{n-1}, \\ \\ \mu, \quad otherwise \end{cases} \tag{1.8}$$

where $\mu \in (0, 1)$, $\vartheta_n \in (0, 1)$ such that $\sum_{n=1}^{\infty} \vartheta_n < \infty$, $\{a_n\}, \{b_n\}, \{c_n\} \subset (0, 1)$ with $a_n + b_n + c_n = 1$ and $\lim_{n\to\infty} a_n = 0$. They proved that the sequence $\{w_n\}$ generated by the algorithm (1.7) converges strongly to the solution of the inclusion problem (1.2).

Motivated by the aforementioned results, we introduce and study inertial bilevel variational monotone inclusion problem in the setting of 2-uniformly convex and uniformly smooth real Banach spaces and establish strong convergence of the iterative sequence generated by our algorithm. It is of interest to mention that the following condition $\frac{\vartheta_n}{\phi(w_n, w_{n-1})}$ on inertial term in [20] is relaxed in our result.

2. Preliminaries

Let E be a real Banach space and $U = \{x \in E : \|x\| = 1\}$ a unit sphere in E. E is said to be smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists for all $x, y \in U$. It is said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$.

We denote by J the normalized duality mapping from E to 2^{E^*} (E^* is the dual space of E) defined by

$$J = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},\ x \in E,$$

where $\langle .,.\rangle$ denotes the duality pairing between element of E and that of E^*. If E is a Hilbert space, then $J = I$, where I is the identity mapping. The space E is said to be uniformly convex if for all $\epsilon \in (0, 2]$, $\delta_E(\epsilon) > 0$ where δ_E is the modulus of convexity of E defined for all $\epsilon \in [0, 2]$ by

$$\delta_E(\epsilon) = \inf\Big\{1 - \|\frac{x+y}{2}\| : x, y \in B_E, \|x-y\| \geq \epsilon\Big\}.$$

The space E is said to be 2-uniformly convex if there exists $c > 0$ such that for all $\epsilon \in [0, 2]$, $\delta_E(\epsilon) \geq c\epsilon^2$. It is obvious that every 2-uniformly convex Banach space is uniformly convex. It is known that all Hilbert spaces are uniformly smooth and 2-uniformly convex. It is also known that all the Lebesgue spaces L_p are uniformly smooth and 2-uniformly convex whenever $1 < p \leq 2$ (see [9]).

The normalized duality mapping J has the following properties (see, Takahashi [10]):

(a) If E is reflexive and strictly convex with the strictly convex dual space E^*, then J is single-valued, one-to-one and onto mapping. In this case, $J^{-1} : E^* \to E$ and we have $J^{-1} = J^*$, where J^* is the normalized duality mapping on E^*;

(b) if E is uniformly smooth, then J is norm-to-norm uniformly continuous on bounded subsets of E.

Lemma 2.1. *Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$, then the space E is q-uniformly smooth if and only if its dual E^* is p-uniformly convex.*

Lemma 2.2. [21] *If E is a 2-uniformly convex and uniform smooth real Banach space. Then J^{-1} is Lipschtzian from E^* into E, i.e. there exists constant $L^* > 0$ such that for all $x^*, y^* \in E^*$ the following holds*

$$\|J^{-1}x^* - J^{-1}y^*\| \leq L^*\|x^* - y^*\|.$$

Let E be a real smooth Banach space. The following functional is studied in Alber [6], Kamimura and Takahashi [7] and Reich [8]

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \tag{2.2}$$

for all $x, y \in E$. It is obvious from the definition of ϕ that for all $x, y \in E$, $(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2$. The following Lemma was proved in Kamimura and Takahashi [7].

Lemma 2.3. *Let E be a uniformly convex, smooth real Banach space, and let $\{x_n\}$, $\{y_n\}$ be sequences in E. If $\{x_n\}$ or $\{y_n\}$ is bounded and $\lim_{n\to\infty} \phi(x_n, y_n) = 0$, then $\lim_{n\to\infty} \|x_n - y_n\| = 0$.*

Let E be a reflexive, strictly convex, smooth real Banach space and C a nonempty, closed convex subset of E. From Alber [6], we see that for $x \in E$, there exists a unique element $x_0 \in C$ $\Big($ denoted by $\Pi_C(x)\Big)$ which is called the generalized projection from E onto C such that for all $y \in C$,

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

If E is a Hilbert space, then Π_C coincides with the metric projection from E onto C. The following lemmas are well known, see [6] and [7].

Lemma 2.4. *Let E be a smooth real Banach space, C a nonempty, closed convex subset of E and $x_0 \in E$, then*

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x)$$

if and only if

$$\langle y - x_0, Jx_0 - Jx \rangle \geq 0, \quad \forall y \in C.$$

Lemma 2.5. *Let E be a real Banach space. The following are equivalent*

(1) *E is 2-uniformly smooth;*
(2) *There exists a constant $k > 0$ such that for all $x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x) \rangle + 2k^2\|y\|^2.$$

The constant k is called the 2-uniform smoothness constant of the space. In Hilbert spaces H, $k = \frac{1}{\sqrt{2}}$.

The following Lemma that gives some identities of functional ϕ can be found in [6].

Lemma 2.6. *Let E be a uniformly convex, smooth real Banach space, then the following hold:*

(i) $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle, \quad \forall\, x, y, z \in E.$
(ii) $\phi(x, y) + \phi(y, x) = 2\langle x - y, Jx - Jy \rangle, \quad \forall\, x, y, z \in E.$

Let C be a nonempty, closed and convex subset of a uniformly convex real Banach space E. Let the functional $V : E \times E^* \to \mathbb{R}$ be defined for all $x \in E, x^* \in E^*$ by

$$V(x, x^*) := \|x\|_E^2 - 2\langle x, x^* \rangle + \|x^*\|_{E^*}^2 \tag{2.3}$$

which satisfies the following conditions (see [6]), for all $x \in E, x^* \in E^*$,

$$V(x, x^*) = \phi(x, J^{-1}x^*),$$

and for all $x \in E, x^* \in E^*$

$$V(x, x^*) + 2\langle J^{-1}x^* - x, y^* \rangle \leq V(x, x^* + y^*), \quad \forall\, x \in E, x^*, y^* \in E^*. \tag{2.4}$$

The following Lemmas are essential in the study of our result.

Lemma 2.7. [12] *Suppose that E is 2-uniformly convex real Banach space, then there exists $\mu \geq 1$ such that $\forall x, y \in E$*

$$\mu\|x - y\|^2 \leq \phi(x, y). \tag{2.5}$$

Lemma 2.8 ([11]). *Let the multivalued operator $B : E \to 2^{E^*}$ be maximal monotone and $A : E \to E^*$ be a Lipschitz continuous and singled-valued monotone mapping, then the mapping $A + B$ is a maximal monotone.*

Lemma 2.9 ([13]). *Let the multivalued operator $B : E \to 2^{E^*}$ be maximal monotone and $A : E \to E^*$ be a single-valued mapping. Define a mapping*

$$T_\lambda x = J_\lambda^B \circ J^{-1}(J - \lambda A)(x), \quad x \in E, \lambda > 0. \tag{2.6}$$

Then $F(T_\lambda) = (A + B)^{-1}(0)$, where $F(T_\lambda)$ denotes the set of all fixed points of T_λ.

Lemma 2.10 ([15]). *Let $\{a_n\}$ be a sequence of real numbers such that there exists a subsequence $\{n_i\}$ of $\{n\}$ such that $a_{n_i} \le a_{n_i+1}$ for all $i \in \mathbb{N}$. Then there exists a non-decreasing sequence $\{m_k\} \subset \mathbb{N}$ such that $m_k \to \infty$ and the following properties are satisfied by all (sufficiently large) numbers $k \in \mathbb{N}$:*

$$a_{m_k} \le a_{m_k+1}$$

$$a_k \le a_{m_k+1}$$

In fact, $m_k = \max\{j \le k : a_j < a_{j+1}\}$.

Lemma 2.11 ([14]). *Let $\{a_n\}$ be a sequence of non-negative real numbers satisfying the following relation:*

$$a_{n+1} \le (1 - \alpha_n)a_n + \alpha_n\delta_n + \gamma_n, \ n \ge 0,$$

where (i) $\{\alpha_n\} \subset [0, 1]$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, (ii) $\limsup_{n\to\infty} \delta_n \le 0$ and (iii) $\gamma_n \ge 0 (n \ge 0)$. Then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

We begin this section with the following assumptions under which the strong convergence results are established:

Assumption 3.1. *Let C be a nonempty closed and convex subset of 2-uniformly convex and uniformly smooth real Banach space E with dual E^*. Suppose the following conditions are satisfied:*

- (C1) *$A : E \to E^*$ is a monotone and L-Lipschitz continuous operator with $L > 0$, and $B : E \to 2^{E^*}$ maximal monotone operators such that $(A + B)^{-1}0 \neq \emptyset$.*
- (C2) *$F : E \to E^*$ is $\delta-$inverse strongly monotone such that $VI(C, F) \neq \emptyset$; $\|Fu\| \le \|Fu - Fv\|$ for all $u \in C$, $v \in VI(C, F)$ and $\delta > 2L^*$, where L^* is a Lipschitz constant for J^{-1}; J is the duality mapping from E into E^*.*
- (C3) *$\{\gamma_n\}$ is a sequence in $\left(0, \frac{\mu}{2}\right)$, where μ is as defined in (2.5), $\gamma_n = \circ(\theta_n)$, where $\{\theta_n\}_{n=1}^{\infty}$ is a sequence in $(0, 1)$ such that $\lim_{n\to\infty}\theta_n = 0$ and $\sum_{n=1}^{\infty} \theta_n = \infty$.*
- (C4) $\Gamma := \{z \in (A + B)^{-1}(0) : \langle F(x), x - z\rangle \ge 0, \quad \forall\, x \in (A + B)^{-1}(0)\} \neq \emptyset.$

Algorithm 3.1. **Initialization:** *Let $\alpha > 0$ and choose $x_0, x_1 \in E$ to be arbitrary.*

Iterative Steps: *Calculate x_{n+1} as follows:*

Step 1. *Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose α_n such that*

$$\alpha_n \leq \bar{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\gamma_n}{\|Jx_n - Jx_{n-1}\|}\right\}, & if \;\; x_n \neq x_{n-1}, \\ \\ \alpha, & otherwise \end{cases} \tag{3.1}$$

Step 2. Compute

$$\begin{cases} u_n = J^{-1}(Jx_n + \alpha_n(Jx_n - Jx_{n-1})) \\ v_n = J^B_{\lambda_n} \circ J^{-1}(Ju_n - \lambda_n Au_n) \\ w_n = J^{-1}(Jv_n - \lambda_n(Av_n - Au_n)) \\ y_n = \Pi_C J^{-1}(Jw_n - \beta_n Fw_n) \\ x_{n+1} = J^{-1}(\theta_n Jx_0 + (1-\theta_n)Jy_n). \end{cases} \tag{3.2}$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ satisfies the condition: $\lambda_n \in (0, (\mu/(2L^*L^2))^{0.5})$ and $0 < a < \beta_n < b < \delta/2L^*$, where $[a, b] \subset (0, 1)$, μ, δ, L^* and L are positive constants defined in (C2) and (C3).

Theorem 3.1. *Let C be a nonempty closed and convex subset of 2-uniformly convex and uniformly smooth real Banach space E with dual E^*. Suppose conditions (C1) to (C4) are satisfied. Let $\{x_n\}_{n=1}^{\infty}$ be sequence generated by Algorithm 3.1, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to some point in Γ.*

Proof. Let $p \in \Gamma$, then we know from Lemma 2.6(i) that

$$\phi(p, x_n) = \phi(p, u_n) + \phi(u_n, x_n) + 2\langle p - u_n, Ju_n - Jx_n\rangle.$$

This implies that

$$\phi(p, u_n) = \phi(p, x_n) - \phi(u_n, x_n) + 2\langle p - u_n, Jx_n - Ju_n\rangle. \tag{3.3}$$

But from the Algorithm 3.1, Lemma 2.7 and the fact that $ab \leq \frac{a^2+b^2}{2}, a, b \in \mathbb{R}$, we have that

$$\begin{aligned} \langle p - u_n, Jx_n - Ju_n\rangle &= \|p - u_n\|\|Jx_n - Ju_n\| \\ &\leq \frac{1}{2}\|Jx_n - Ju_n\|[\|p - u_n\|^2 + 1] \\ &= \frac{\alpha_n}{2}\|Jx_n - Jx_{n-1}\|[\|p - u_n\|^2 + 1] \\ &\leq \frac{\alpha_n}{2}\|Jx_n - Jx_{n-1}\|[(\|p - x_n\| + \|x_n - u_n\|)^2 + 1] \\ &\leq \frac{\alpha_n}{2}\|Jx_n - Jx_{n-1}\|[2(\|p - x_n\|^2 + \|x_n - u_n\|^2) + 1] \\ &= \alpha_n\|Jx_n - Jx_{n-1}\|[\|p - x_n\|^2 + \|u_n - x_n\|^2] + \frac{\alpha_n}{2}\|Jx_n - Jx_{n-1}\| \\ &\leq \frac{\gamma_n}{\mu}\phi(p, x_n) + \frac{\gamma_n}{\mu}\phi(u_n, x_n) + \frac{\gamma_n}{2}. \end{aligned} \tag{3.4}$$

Thus, from (3.3) and (3.4), we obtain

$$\phi(p,u_n) \leq (1+\frac{2\gamma_n}{\mu})\phi(p,x_n) - (1-\frac{2\gamma_n}{\mu})\phi(u_n,x_n) + \gamma_n \tag{3.5}$$

Using Lemma 2.6(i), we get

$$\phi(p,v_n) = \phi(p,u_n) - \phi(v_n,u_n) + 2\langle p - v_n, Ju_n - Jv_n\rangle. \tag{3.6}$$

And using (2.4), Lemma 2.2, (3.5) and (3.6), we obtain

$$\begin{aligned}
\phi(p,w_n) &= \phi(p, J^{-1}(Jv_n - \lambda_n(Av_n - Au_n)))\\
&= V(p, Jv_n - \lambda_n(Av_n - Au_n))\\
&\leq V(p, Jv_n) - 2\lambda_n\langle J^{-1}(Jv_n - \lambda_n(Av_n - Au_n)) - p, Av_n - Au_n\rangle\\
&= \phi(p,v_n) - 2\lambda_n\langle J^{-1}(Jv_n - \lambda_n(Av_n - Au_n)) - J^{-1}(Jv_n), Av_n - Au_n\rangle\\
&\quad +2\lambda_n\langle v_n - p, Av_n - Au_n\rangle\\
&\leq \left(1+\frac{2\gamma_n}{\mu}\right)\phi(p,x_n) - \left(1-\frac{2\gamma_n}{\mu}\right)\phi(u_n,x_n) + \gamma_n - \phi(v_n,u_n)\\
&\quad +2\lambda_n\|J^{-1}(Jv_n) - J^{-1}(Jv_n - \lambda_n(Av_n - Au_n))\|\|Av_n - Au_n\|\\
&\quad +2\langle p - v_n, Ju_n - Jv_n + \lambda_n(Av_n - Au_n)\rangle\\
&\leq \left(1+\frac{2\gamma_n}{\mu}\right)\phi(p,x_n) - \left(1-\frac{2\gamma_n}{\mu}\right)\phi(u_n,x_n) + \gamma_n - \phi(v_n,u_n)\\
&\quad +2\lambda_n^2 L^* L^2\|v_n - u_n\|^2 + 2\langle p - v_n, Ju_n - Jv_n + \lambda_n(Av_n - Au_n)\rangle\\
&\leq \left(1+\frac{2\gamma_n}{\mu}\right)\phi(p,x_n) - \left(1-\frac{2\gamma_n}{\mu}\right)\phi(u_n,x_n) + \gamma_n - \left(1-\frac{2\lambda_n^2 L^* L^2}{\mu}\right)\phi(v_n,u_n)\\
&\quad +2\langle p - v_n, Ju_n - Jv_n + \lambda_n(Av_n - Au_n)\rangle
\end{aligned} \tag{3.7}$$

By definition of $v_n := J^B_{\lambda_n} \circ J^{-1}(Ju_n - \lambda_n Au_n)$, where $J^B_{\lambda_n} = (J + \lambda_n B)^{-1}J$, then $(Ju_n - \lambda_n Au_n) \in (Jv_n + \lambda_n Bv_n)$. Since B is maximal monotone, then there exists say $z_n \in Bv_n$ such that

$$Ju_n - \lambda_n Au_n = Jv_n + \lambda_n z_n$$

which implies

$$z_n = \frac{1}{\lambda_n}(Ju_n - Jv_n - \lambda_n Au_n), \tag{3.8}$$

and since $0 \in (A+B)(p)$, $Av_n + z_n \in (A+B)v_n$ and $A+B$ is monotone, then

$$\langle p - v_n, Av_n + z_n\rangle \leq 0. \tag{3.9}$$

Combining (3.8) and (3.9), we get

$$\langle p - v_n, Ju_n - Jv_n + \lambda_n(Av_n - Au_n)\rangle \leq 0. \tag{3.10}$$

Thus, from (3.7) and (3.10), we get

$$\begin{aligned}\phi(p,w_n) &\leq \Big(1+\frac{2\gamma_n}{\mu}\Big)\phi(p,x_n)-\Big(1-\frac{2\gamma_n}{\mu}\Big)\phi(u_n,x_n)+\gamma_n \\ &\quad -\Big(1-\frac{2\lambda_n^2L^*L^2}{\mu}\Big)\phi(v_n,u_n). \end{aligned} \tag{3.11}$$

Next, we obtain from (3.2) using properties of the functional ϕ and V that

$$\begin{aligned}\phi(p,y_n) &= \phi(p,\Pi_C J^{-1}(Jw_n-\beta_nFw_n)) \\ &\leq \phi(p,J^{-1}(Jw_n-\beta_nFw_n)) \\ &= V(p,Jw_n-\beta_nFw_n) \\ &\leq V(p,(Jw_n-\beta_nFw_n)+\beta_nFw_n)-2\langle J^{-1}(Jw_n-\beta_nFw_n)-p,\beta_nFw_n\rangle \\ &= V(p,Jw_n)-2\langle J^{-1}(Jw_n-\beta_nFw_n)-p,\beta_nFw_n\rangle \\ &= \phi(p,w_n)-2\beta_n\langle w_n-p,Fw_n\rangle+2\beta_n\langle J^{-1}(Jw_n-\beta_nFw_n)-w_n,Fw_n\rangle \\ &= \phi(p,w_n)-2\beta_n\langle w_n-p,Fw_n-Fp\rangle-2\beta_n\langle w_n-p,Fp\rangle \\ &\quad +2\beta_n\langle J^{-1}(Jw_n-\beta_nFw_n)-J^{-1}(Jw_n),Fw_n\rangle \\ &\leq \phi(p,w_n)-2\delta\beta_n\|Fw_n-Fp\|^2+2\beta_n\|J^{-1}(Jw_n-\beta_nFw_n)-J^{-1}(Jw_n)\|\|Fw_n\| \\ &\leq \phi(p,w_n)-2\delta\beta_n\|Fw_n-Fp\|^2+2\beta_nL^*\|J^{-1}(Jw_n-\beta_nFw_n)-J^{-1}(Jw_n)\|\|Fw_n\| \\ &= \phi(p,w_n)-2\delta\beta_n\|Fw_n-Fp\|^2+4L^*\beta_n\|(Jw_n-\beta_nFw_n)-Jw_n\|\|Fw_n\| \\ &= \phi(p,w_n)-2\delta\beta_n\|Fw_n-Fp\|^2+4L^*\beta_n^2\|Fw_n\|^2 \\ &\leq \phi(p,w_n)-2\delta\beta_n\|Fw_n-Fp\|^2+4L^*\beta_n^2\|Fw_n-Fp\|^2 \\ &= \phi(p,w_n)-2\beta_n(\delta-2L^*\beta_n)\|Fw_n-Fp\|^2. \end{aligned} \tag{3.12}$$

Combining (3.11) and (3.12), we obtain

$$\begin{aligned}\phi(p,y_n) &\leq \Big(1+\frac{2\gamma_n}{\mu}\Big)\phi(p,x_n)-\Big(1-\frac{2\gamma_n}{\mu}\Big)\phi(u_n,x_n)+\gamma_n-\Big(1-\frac{2\lambda_n^2L^*L^2}{\mu}\Big)\phi(v_n,u_n) \\ &- 2\beta_n(\delta-2L^*\beta_n)\|Fw_n-Fp\|^2. \end{aligned}$$

Thus, taking $\epsilon\in(0,\frac{\mu}{2})$, then from (C3) there exists a natural $N\in\mathbb{N}$ such that for all $n\geq N$, $\frac{2\gamma_n}{\mu}<\theta_n\epsilon$, therefore, for all $n\geq N$,

$$\begin{aligned}\phi(p,x_{n+1}) &= \phi(p,J^{-1}(\theta_nJx_0+(1-\theta_n)Jy_n)) \\ &\leq \theta_n\phi(p,x_0)+(1-\theta_n)\phi(p,y_n) \\ &\leq \theta_n\phi(p,x_0)+(1-\theta_n)\Big(1+\frac{2\gamma_n}{\mu}\Big)\phi(p,x_n)-(1-\theta_n)\Big(1-\frac{2\gamma_n}{\mu}\Big)\phi(u_n,x_n)+\gamma_n \\ &- (1-\theta_n)\Big(1-\frac{2\lambda_n^2L^*L^2}{\mu}\Big)\phi(v_n,u_n)-2\beta_n(1-\theta_n)(\delta-2L^*\beta_n)\|Fw_n-Fp\|^2 \qquad (3.13) \\ &\leq \Big\{1-\theta_n(1-\epsilon)\Big\}\phi(p,x_n)+\theta_n(1-\epsilon)\Big[\frac{\phi(p,x_0)}{1-\epsilon}+\frac{\epsilon\mu}{2(1-\epsilon)}\Big]. \end{aligned}$$

Thus, by induction we obtain that $\forall\, n \geq N$,

$$\phi(p, x_n) \leq \max\left\{\phi(p, x_N), \frac{2\phi(p, x_0) + \epsilon\mu}{2(1-\epsilon)}\right\}.$$

Therefore, $\left\{\phi(p, x_n)\right\}$is bounded. So, by Lemma 2.7 it follows that $\{x_n\}$ is bounded and hence $\{y_n\}$,$\{w_n\}$, $\{u_n\}$ and $\{v_n\}$ are all bounded. But from (3.13) we get that

$$\begin{aligned}(1-\theta_n)\Big[(1-\frac{2\gamma_n}{\mu})\phi(u_n, x_n) \quad &+ \quad \Big(1-\frac{2\lambda_n^2 L^* L^2}{\mu}\Big)\phi(v_n, u_n) + 2\beta_n(\delta - 2L^*\beta_n)\|Fw_n - Fp\|^2\Big] \\ &\leq \quad (\phi(p, x_n) - \phi(p, x_{n+1})) + \theta_n\Big(\phi(p, u) - \phi(p, x_n) + \frac{\gamma_n}{\theta_n}\Big). \quad (3.14)\end{aligned}$$

Now, we divide the remaining part of the proof into two cases.

Case 1. Assume that $\{\phi(p, x_n)\}_{n=1}^{\infty}$ is non-increasing real sequence of numbers. Since $\{\phi(p, x_n)\}_{n=1}^{\infty}$ is bounded, then the limit exists. So, $\lim\limits_{n\to\infty}(\phi(p, x_n) - \phi(p, x_{n+1})) = 0$. therefore, from (3.14), we obtain

$$\lim_{n\to\infty}\phi(u_n, x_n) = \lim_{n\to\infty}\phi(v_n, u_n) = \lim_{n\to\infty}\|Fw_n - Fp\| = 0. \quad (3.15)$$

Applying Lemma 2.3, we obtain from (3.15) that

$$\lim_{n\to\infty}\|u_n - x_n\| = \lim_{n\to\infty}\|v_n - u_n\| = 0. \quad (3.16)$$

Also, from (2.4), we get

$$\begin{aligned}\phi(w_n, y_n) \quad &= \quad \phi(w_n, J^{-1}(Jw_n - \beta_n Fw_n)) \\ &= \quad V(w_n, Jw_n - \beta_n Fw_n) \\ &\leq \quad V(w_n, (Jw_n - \beta_n Fw_n) + \beta_n Fw_n) - 2\langle J^{-1}(Jw_n - \beta_n Fw_n) - w_n, \beta_n Fw_n\rangle \\ &= \quad \phi(w_n, w_n) - 2\langle J^{-1}(Jw_n - \beta_n Fw_n) - w_n, \beta_n Fw_n\rangle \\ &= \quad 2\beta_n\langle J^{-1}(Jw_n - \beta_n Fw_n) - J^{-1}(Jw_n), Fw_n\rangle \\ &\leq \quad 2\beta_n\|J^{-1}(Jw_n - \beta_n Fw_n) - J^{-1}(Jw_n)\|\|Fw_n\| \\ &\leq \quad 2L^*\beta_n^2\|Fw_n\|^2 \\ &\leq \quad 2L^*\beta_n^2\|Fw_n - Fp\|^2. \quad (3.17)\end{aligned}$$

Using (3.15) into (3.17), we get

$$\lim_{n\to\infty}\phi(w_n, y_n) = 0. \quad (3.18)$$

Consequently, we get from (3.18) that

$$\lim_{n\to\infty}\|w_n - y_n\| = 0. \quad (3.19)$$

Since J is uniformly norm-to-norm continuous on set, we get from (3.16) and (3.19) that

$$\lim_{n\to\infty}\|Ju_n - Jx_n\| = \lim_{n\to\infty}\|Jv_n - Ju_n\| = \lim_{n\to\infty}\|Jw_n - Jy_n\| = 0. \quad (3.20)$$

By (3.2), we get

$$\begin{aligned}\|Jw_n - Jv_n\| &= \lambda_n\|Av_n - Au_n\| \\ &\leq \lambda_n L\|v_n - u_n\|.\end{aligned}$$

It follows from (3.20) that

$$\lim_{n\to\infty}\|Jw_n - Jv_n\| = 0. \tag{3.21}$$

Since

$$\|Jx_n - Jy_n\| \leq \|Jx_n - Ju_n\| + \|Ju_n - Jv_n\| + \|Jv_n - Jy_n\|,$$

then by (3.20) and (3.21)

$$\lim_{n\to\infty}\|Jx_n - Jy_n\| = 0. \tag{3.22}$$

Moreover, from the iteration, we get that

$$\|Jx_{n+1} - Jy_n\| = \alpha_n\|Jx_0 - Jy_n\| \leq \alpha_n K \to 0 \text{ as } n \to \infty, \tag{3.23}$$

for some $K > 0$. Also, by (3.21) and (3.22) we obtain

$$\|Jx_{n+1} - Jx_n\| \leq \|Jx_{n+1} - Jy_n\| + \|Jy_n - Jx_n\| \to 0 \text{ as } n \to \infty. \tag{3.24}$$

Since J^{-1} is norm-to-norm uniformly continuous on bounded subset of E^*, we get from (3.22) and (3.24) that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0 = \lim_{n\to\infty}\|x_n - y_n\|. \tag{3.25}$$

Furthermore, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\{x_{n_i}\}$ converges weakly to a point x^* in E as $i \to \infty$. By (3.25), we get $y_{n_i} \rightharpoonup x^*$ as $i \to \infty$. We now show that $x^* \in (A+B)^{-1}(0)$. Let $(y,z) \in \text{Graph}(A+B)$, that is $z \in (A+B)y$ which means $z - Ay \in By$. Also, we know from $v_{n_i} = (J + \lambda_{n_i}B)^{-1}J \circ J^{-1}(Ju_{n_i} - \lambda_{n_i}Au_{n_i})$ that

$$(J - \lambda_{n_i}A)u_{n_i} \in (J + \lambda_{n_i}B)v_{n_i}$$

and

$$\frac{1}{\lambda_{n_i}}(Ju_{n_i} - Jv_{n_i} - \lambda_{n_i}Au_{n_i}) \in Bv_{n_i}.$$

Since B is maximal monotone, we get

$$\langle y - v_{n_i}, z - Ay - \frac{1}{\lambda_{n_i}}(Ju_{n_i} - Jv_{n_i} - \lambda_{n_i}Au_{n_i})\rangle \geq 0. \tag{3.26}$$

Thus,

$$\begin{aligned}\langle y - v_{n_i}, z\rangle &\geq \langle y - v_{n_i}, Ay + \frac{1}{\lambda_{n_i}}(Ju_{n_i} - Jv_{n_i} - \lambda_{n_i}Au_{n_i})\rangle \\ &= \langle y - v_{n_i}, Ay - Au_{n_i}\rangle + \langle y - v_{n_i}, \frac{1}{\lambda_{n_i}}(Ju_{n_i} - Jv_{n_i}\rangle \\ &= \langle y - v_{n_i}, Ay - Av_{n_i}\rangle + \langle y - v_{n_i}, Av_{n_i} - Au_{n_i}\rangle + \langle y - v_{n_i}, \frac{1}{\lambda_{n_i}}(Ju_{n_i} - Jv_{n_i}\rangle \\ &\geq \langle y - v_{n_i}, Av_{n_i} - Au_{n_i}\rangle + \langle y - v_{n_i}, \frac{1}{\lambda_{n_i}}(Ju_{n_i} - Jv_{n_i}\rangle.\end{aligned} \tag{3.27}$$

We know that $\lim_{n\to\infty}\|v_n - u_n\| = 0$. since A is Lipschitz continuous then $\lim_{n\to\infty}\|Av_n - Au_n\| = 0$. Also, since $v_{n_i} \to x^*$ as $i \to \infty$, then it follows from (3.27) that

$$\langle y - x^*, z\rangle \geq 0. \tag{3.28}$$

Furthermore, since $A + B$ is maximal monotone, we get that $0 \in (A + B)x^*$. Thus, $x^* \in (A + B)^{-1}(0)$. Since p is the unique solution of the problem F in Γ, then

$$\limsup_{i\to\infty}\langle F(p), p - x_{n_i}\rangle = \langle F(p), x^* - p\rangle \geq 0.$$

We now show that $p = \Pi_\Gamma x_0$. but by Lemma 2.4, we obtain

$$\langle Jp - Jx_0, p - x^*\rangle \leq 0, \ \forall x^* \in \Gamma. \tag{3.29}$$

Hence, $p = \Pi_\Gamma x_0$. Next, we show that

$$\limsup_{n\to\infty}\langle Jx_0 - Jp, x_{n+1} - p\rangle \leq 0.$$

Using (3.25) and the fact that $x_{n_i} \rightharpoonup x^*$,we get

$$\begin{aligned}\limsup_{n\to\infty}\langle Ju - Jp, x_{n+1} - p\rangle &= \limsup_{i\to\infty}\langle Jx_0 - Jp, x_{n_i+1} - x_{n_i}\rangle + \limsup_{i\to\infty}\langle Jx_0 - Jp, x_{n_i} - p\rangle \\ &= \langle Jx_0 - Jp, x^* - p\rangle.\end{aligned}$$

Therefore, we obtain that

$$\limsup_{n\to\infty}\langle Ju - Jp, x_{n+1} - p\rangle = \langle Jp - Jx_0, p - x^*\rangle \leq 0.$$

Hence,

$$\begin{aligned}\phi(p, x_{n+1}) &= \phi(p, J^{-1}(\theta_n Jx_0 + (1 - \theta_n)Jy_n)) \\ &= V(p, \theta_n Jx_0 + (1 - \theta_n)Jy_n) \\ &= V(p, \theta_n Jx_0 + (1 - \theta_n)Jy_n - \theta_n(Jx_0 - Jp)) + 2\theta_n\langle Jx_0 - Jp, x_{n+1} - p\rangle \\ &= V(p, \theta_n Jp + (1 - \theta_n)Jy_n) + 2\theta_n\langle Jx_0 - Jp, x_{n+1} - p\rangle \\ &\leq \theta_n V(p, Jp) + (1 - \theta_n)V(p, Jy_n) + 2\theta_n\langle Jx_0 - Jp, x_{n+1} - p\rangle \\ &\leq (1 - \theta_n)V(p, y_n) + 2\theta_n\langle Jx_0 - Jp, x_{n+1} - p\rangle \\ &= (1 - \theta_n)\phi(p, x_n) + 2\theta_n\langle Jx_0 - Jp, x_{n+1} - p\rangle \end{aligned} \tag{3.30}$$

By Lemma 2.11, we obtain $\phi(p, x_n) \to 0$ as $n \to \infty$. Also, by Lemma 2.7, we get $x_n \to p$.

Case 2. Let $\{\phi(p, x_n)\}_{n=1}^{\infty}$ be a sequence of non-decreasing real numbers. Then, by Lemma 2.10, we set $\Upsilon_n := \phi(p, x_n)$ and let $r : \mathbb{N} \to \mathbb{N}$ be a mapping for all $n \geq n_0$ (for some n_0 large enough), defined by

$$r(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Then, r is a non-decreasing sequence such that $r(n) \to \infty$ as $n \to \infty$. Thus

$$0 \leq \Upsilon_{r(n)} \leq \Upsilon_{r(n)+1}, \ \forall\, n \geq n_0$$

which means that $\phi(p, x_{r(n)}) \leq \phi(p, x_{r(n)+1})$, for all $n \geq n_0$. Since $\{\phi(p, x_{r(n)})\}$ is bounded, therefore $\lim_{n\to\infty} \phi(p, x_{r(n)})$ exists. Thus following the same line of action as in Case 1, we obtain that $\lim_{n\to\infty} \phi(x_{r(n)+1}, x_{r(n)}) = 0$ and for any $\{x_{r(n)}\}$ which is bounded, there exists a subsequence of $\{x_{r(n)}\}$, still denoted by $\{x_{r(n)}\}$ such that $x_{r(n)}$ converges weakly to z as $n \to \infty$, we then obtain $z \in \Gamma$.. Furthermore, for any $p = \Pi_\Gamma x_0$, we have

$$\limsup_{n\to\infty} \langle Jx_0 - Jp, x_{r(n)+1} - p\rangle \leq 0. \tag{3.31}$$

Also, like in (3.30), we get

$$\phi(p, x_{r(n)+1}) \leq (1 - \theta_{r(n)})\phi(p, x_{r(n)}) + 2\theta_{r(n)}\langle Jx_0 - Jp, x_{r(n)+1} - p\rangle. \tag{3.32}$$

Using $\Upsilon_{r(n)} \leq \Upsilon_{r(n)+1}$ and since $\theta_{r(n)} > 0$ we obtain

$$\phi(p, x_{r(n)}) \leq 2\langle Jx_0 - Jp, x_{r(n)+1} - p\rangle$$

which implies by (3.31)

$$\limsup_{n\to\infty} \phi(p, x_{r(n)}) \leq 0.$$

Thus

$$\lim_{n\to\infty} \phi(p, x_{r(n)}) = 0.$$

Now by (3.32), we have

$$\lim_{n\to\infty} \phi(p, x_{r(n)+1}) = 0.$$

Therefore, by Lemma 2.10, we get

$$\phi(p, x_n) \leq \phi(p, x_{r(n)+1}) \to 0 \text{ as } n \to \infty$$

which gives that $\lim_{n\to\infty} \phi(p, x_n) = 0$, which implies that $\lim_{n\to\infty} \|p - x_n\| = 0$. Thus $x_n \to p := \Pi_\Gamma x_0$. This completes the proof.

□

Remark 3.2 Our result is more general than some related results in the literature and, hence, might be applied for a wider class of mappings. For example, we next present the advantage of our method compared with the recent results

4. Application

In this section, we study the problem of solving convex minimization problem.

4.1. **Application to convex minimization problem.** Consider the structured nonsmooth convex minimization problem:

$$f(x^*) + g(x^*) = \min_{x\in E}\{f(x) + g(x)\} \tag{4.1}$$

where $f : E \to \mathbb{R}$ is a convex smooth function and $g : E \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower-semicontinuous function. Solving (4.1) is equivalent to finding $x^* \in E$ such that

$$0 \in \nabla f(x^*) + \partial g(x^*)$$

where ∇f is the gradient of f and ∇g is the subdifferential of g. Since ∇f is $(1/L)$-Lipschitz continuous, then it is L-inverse strongly monotone and ∂g is maximal monotone. Setting $A = \nabla f$, $B = \partial g$ and $F = I - x_0$, where I is an identity mapping and x_0 is an arbitrary fixed vector in E, then F is 1-Lipschitz continuous, so it is 1-inverse monotone. Hence, from Algorithm 3.1, we obtain the following Algorithm:

Algorithm 4.1. **Initialization:** *Let $\alpha > 0$ and choose $x_0, x_1 \in E$ to be arbitrary.*
Iterative Steps: *Calculate x_{n+1} as follows:*

Step 1. *Given the iterates x_{n-1} and x_n for each $n \geq 1$, choose α_n such that*

$$\alpha_n \leq \bar{\alpha}_n = \begin{cases} \min\left\{\alpha, \frac{\gamma_n}{\|Jx_n - Jx_{n-1}\|}\right\}, & if \;\; x_n \neq x_{n-1}, \\ \\ \alpha, & otherwise \end{cases}$$

Step 2. Compute

$$\begin{cases} u_n = J^{-1}(Jx_n + \alpha_n(Jx_n - Jx_{n-1}) \\ v_n = J_{\lambda_n}^{\partial g} \circ J^{-1}(Ju_n - \lambda_n \nabla f(u_n)) \\ w_n = J^{-1}(Jv_n - \lambda_n(\nabla f(v_n) - \nabla f(u_n)) \\ y_n = \Pi_C J^{-1}((1-\beta_n)w_n + \beta_n x_0) \\ x_{n+1} = J^{-1}(\theta_n Jx_0 + (1-\theta_n)Jy_n). \end{cases}$$

where $\{\lambda_n\}$ and $\{\beta_n\}$ satisfies the condition: $\lambda_n \in (0, (\mu/(2L^*L^2))^{0.5})$ and $0 < a < \beta_n < b < \delta/2L^*$, where $[a, b] \subset (0, 1)$, μ, δ, L^* and L are positive constants defined in (C2) and (C3).

Theorem 4.1. *Let C be a nonempty closed and convex subset of 2-uniformly convex and uniformly smooth real Banach space E with dual E^*. Suppose that $f : E \to \mathbb{R}$ is a convex smooth function and $g : E \to \mathbb{R} \cup \{+\infty\}$ is a proper convex and lower-semicontinuous function such that ∇f is $(1/L)$-Lipschitz continuous, ∂g is maximal monotone and $S := \min_{x\in E}\{f(x) + g(x)\} \neq \emptyset$. Let $\{x_n\}_{n=1}^{\infty}$ be sequence generated by Algorithm 4.1, then $\{x_n\}_{n=1}^{\infty}$ converges strongly to some point in S.*

4.2. **Numerical examples.** Next, we present some numerical examples to illustrate the efficiency and performance of the proposed algorithm. We compare the performance of our method Algorithm 3.1 with (1.4), (1.5) and (1.7).

Example 4.1. *Let* $E = (l_2(\mathbb{R}), \|.\|_{l_2})$*, where* $l_2(\mathbb{R}) := \{x = (x_1, x_2, x_3, \dots),\ x_i \in \mathbb{R} : \sum_{i=1}^{\infty} |x_i|^2 < \infty\}$ *and* $\|x\|_{l_2} := \left(\sum_{i=1}^{\infty} |x_i|^2\right)^{\frac{1}{2}}$, $\forall x \in l_2(\mathbb{R})$*. Now, define the operator* $B : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ *by* $Bx = (3x_1, 3x_2, 3x_3, \dots)$, $\forall x \in l_2(\mathbb{R})$*. Then, B is a maximal linear operator on* $l_2(\mathbb{R})$ *with resolvent* $J^B_{\lambda_n} = \frac{x}{1+3\lambda_n}$ $\forall x \in l_2(\mathbb{R})$*. Let* $C = \{x \in l_2(\mathbb{R}) : |x_i| \leq \frac{1}{i},\ i = 1, 2, 3, \dots\}$*. Thus, we have explicit formula for* P_C*. Now, define the operator* $A : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ *by*

$$Ax = \left(\|x\| + \frac{1}{\|x\| + \alpha}\right)x,$$

for some $\alpha > 0$*. Then, A is monotone on* $l_2(\mathbb{R})$ *(see [31]). Furthermore, define the mapping* $S : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ *by* $Sx = (0, x_1, x_2, \dots)$*, and* $F : l_2(\mathbb{R}) \to l_2(\mathbb{R})$ *by* $Fx = x - x_0$*. Then, F is inverse strongly monotone and Lipschitz continuous.*

For this example, we take $\varepsilon = 10^{-8}$ *as the stopping criterion and choose the starting points as follows:*

Case 1: *Take* $x_1 = (1, \frac{1}{2}, \frac{1}{3}, \cdots)$ *and* $x_0 = (\frac{1}{2}, \frac{1}{5}, \frac{1}{10}, \cdots)$.

Case 2: *Take* $x_1 = (1, \frac{1}{2}, \frac{1}{7}, \cdots)$ *and* $x_0 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{10}, \cdots)$.

Case 3: *Take* $x_1 = (1, \frac{1}{4}, \frac{1}{8}, \cdots)$ *and* $x_0 = (\frac{1}{2}, \frac{1}{6}, \frac{1}{9}, \cdots)$.

Case 4: *Take* $x_1 = (\frac{1}{2}, \frac{1}{3}, \frac{1}{8}, \cdots)$ *and* $x_0 = (1, \frac{1}{4}, \frac{1}{9}, \cdots)$.

The numerical results reported in Table 1 and Figure 1.

Table 2. Numerical results for Example 4.1 with $\varepsilon = 10^{-8}$**.**

Cases		*Algorithm* 3.1	*Algorithm* (1.7)	*Algorithm* (1.5)	*Algorithm* (1.4)
1	*CPU Iter.*	*0.7635* *10545*	*5.3401* *47956*	*2.3226* *32202*	*5.5915* *54678*
2	*CPU Iter.*	*0.4805* *10545*	*3.5037* *47956*	*1.4088* *32202*	*3.5795* *54373*
3	*CPU Iter.*	*0.4764* *10545*	*3.3908* *47956*	*1.3816* *32202*	*3.4018* *53967*
4	*CPU Iter.*	*0.4843* *10545*	*3.2722* *47956*	*1.4390* *32202*	*3.268* *53694*

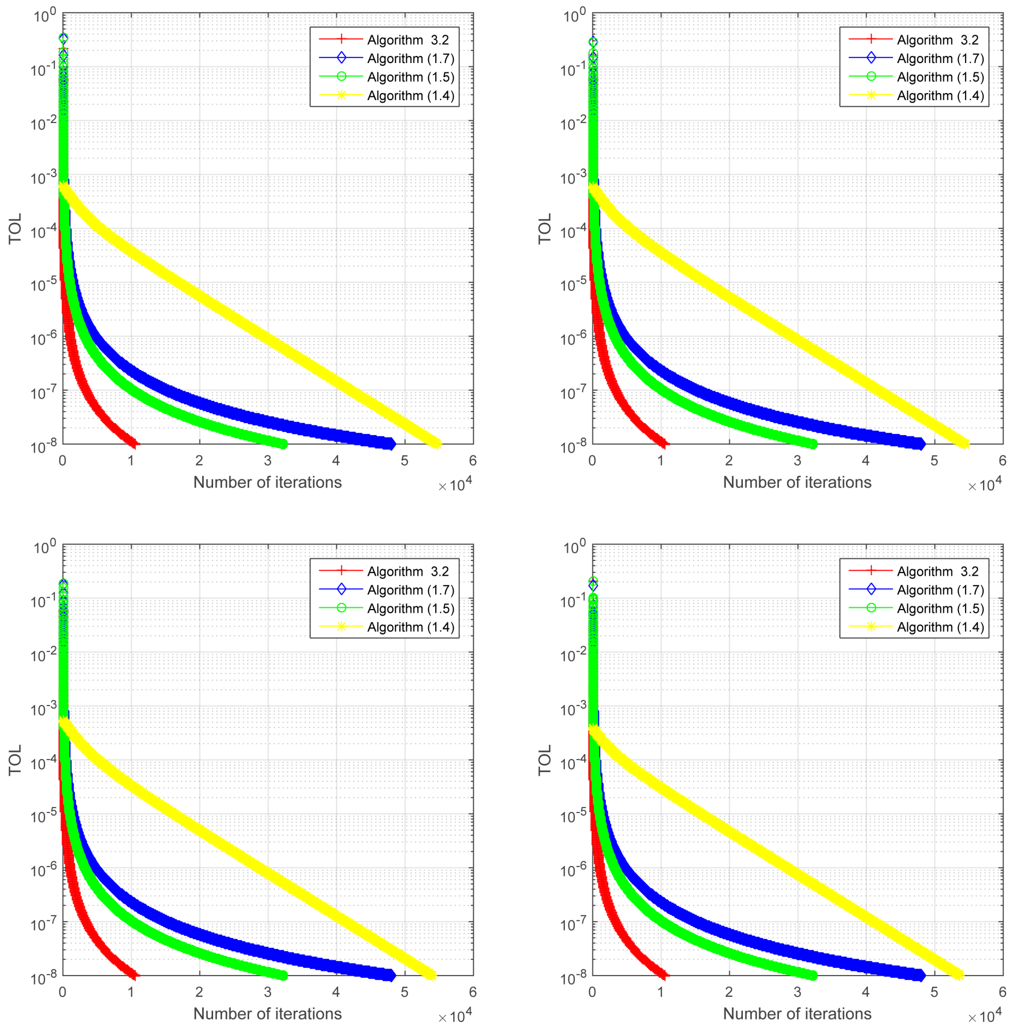

FIGURE 1. The behavior of TOL_n with $\varepsilon = 10^{-8}$ for Example 4.1: Top Left: **Case 1**; Top Right: **Case 2**; Bottom Left: **Case 3**; Bottom Right: **Case 4**.

5. Conclusion

The approach for solving bilevel variational inclusion problem that uses monotone mappings in the lower-level problem and an inverse strongly monotone mapping in the upper-level case in two uniformly smooth convex real Banach spaces was proposed in the study. Our algorithm is an accelerated Halpern-type iterative method. We prove a strong convergence theorem with some assumptions on parameters.The effectiveness and performance of the suggested iterative strategy are demonstrated using a numerical example. The outcome advances some recent findings in the literature.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] J.A. Abuchu, G.C. Ugwunnadi, O.K. Narain, Inertial Mann-Type Iterative Method for Solving Split Monotone Variational Inclusion Problem With Applications, J. Ind. Manage. Optim. 19 (2023), 3020–3043. https://doi.org/10.3934/jimo.2022075.

[2] F.E. Browder, W.V. Petryshyn, Construction of Fixed Points of Nonlinear Mappings in Hilbert Space, J. Math. Anal. Appl. 20 (1967), 197–228. https://doi.org/10.1016/0022-247x(67)90085-6.

[3] H. Iiduka, W. Takahashi and M. Toyoda, Approximation of Solutions of Variational Inequalities for Monotone Mappings, Panamer. Math. J. 20 (2004), 49–61.

[4] F. Liu, M.Z. Nashed, Regularization of Nonlinear Ill-Posed Variational Inequalities and Convergence Rates, Set-Valued Anal. 6(1998), 313–344. https://doi.org/10.1023/A:1008643727926.

[5] Y. Takahashi, K. Hashimoto, M. Kato, On Sharp Uniform Convexity, Smoothness and Strong Type, Cotype Inequalities, J. Nonlinear Convex Anal. 3 (2002), 267–281.

[6] Y.I. Alber, Metric and Generalized Projection in Banach Space: Properties and Applications, In: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Dekker, New York, pp. 15–50, 1996.

[7] S. Kamimura, W. Takahashi, Strong Convergence of a Proximal-Type Algorithm in a Banach Space, SIAM J. Optim. 13 (2002), 938–945. https://doi.org/10.1137/s105262340139611x.

[8] S. Reich, A weak convergence theorem for the alternating method with Bregman distance, In: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Dekker, New York, pp. 313–318, 1996.

[9] B. Beauzamy, Introduction to Banach Spaces and Their Geometry, North-Holland Mathematics Studies, Vol. 68, North-Holland Publishing Co., Amsterdam, 1985.

[10] W. Takahashi, Nonlinear Functional Analysis, Yokohama Publishers, Yokohama, 2000.

[11] V. Barbu, V. Barbu, Nonlinear Semigroups and Differential Equations in Banach Spaces, Noordhoff International Publishing, Leyden, 1976.

[12] K. Aoyama, F. Kohsaka, Strongly Relatively Nonexpansive Sequences Generated by Firmly Nonexpansive-Like Mappings, Fixed Point Theory Appl 2014 (2014), 95. https://doi.org/10.1186/1687-1812-2014-95.

[13] Y. Shehu, Convergence Results of Forward-Backward Algorithms for Sum of Monotone Operators in Banach Spaces, Results Math. 74 (2019), 138. https://doi.org/10.1007/s00025-019-1061-4.

[14] H.K. Xu, Another Control Condition in an Iterative Method for Nonexpansive Mappings, Bull. Austral. Math. Soc. 65 (2002), 109–113. https://doi.org/10.1017/s0004972700020116.

[15] P.E. Maingé, Strong Convergence of Projected Subgradient Methods for Nonsmooth and Nonstrictly Convex Minimization, Set-Valued Anal. 16 (2008), 899–912. https://doi.org/10.1007/s11228-008-0102-z.

[16] P.L. Combettes, V.R. Wajs, Signal Recovery by Proximal Forward-Backward Splitting, Multiscale Model. Simul. 4 (2005), 1168–1200. https://doi.org/10.1137/050626090.

[17] A. Moudafi, M. Théra, Finding a Zero of The Sum of Two Maximal Monotone Operators, J. Optim. Theory Appl. 94 (1997), 425–448. https://doi.org/10.1023/a:1022643914538.

[18] A. Adamu, D. Kitkuan, A. Padcharoen, et al. Inertial Viscosity-Type Iterative Method for Solving Inclusion Problems With Applications, Math. Comput. Simul. 194 (2022), 445–459. https://doi.org/10.1016/j.matcom.2021.12.007.

[19] B. S. He, A Class of Projection and Contraction Methods for Monotone Variational Inequalities, Appl. Math. Optim. 35 (1997), 69–76. https://doi.org/10.1007/BF02683320.

[20] A. Adamu, D. Kitkuan, P. Kumam, et al. Approximation Method for Monotone Inclusion Problems in Real Banach Spaces With Applications, J. Inequal. Appl. 2022 (2022), 70. https://doi.org/10.1186/s13660-022-02805-0.

[21] C. Diop, T.M.M. Sow, N. Djitte, et al. Constructive Techniques for Zeros of Monotone Mappings in Certain Banach Spaces, SpringerPlus 4 (2015), 383. https://doi.org/10.1186/s40064-015-1169-2.

[22] Y. Kimura, K. Nakajo, Strong Convergence for a Modified Forward-Backward Splitting Method in Banach Spaces, J. Nonlinear Var. Anal. 3 (2019), 5–18. https://doi.org/10.23952/jnva.3.2019.1.02.

[23] D. Kitkuan, P. Kumam, J. Martínez-Moreno, Generalized Halpern-type forward–backward splitting methods for convex minimization problems with application to image restoration problems, Optimization 69 (2019), 1557–1581. https://doi.org/10.1080/02331934.2019.1646742.

[24] G.B. Passty, Ergodic Convergence to a Zero of the Sum of Monotone Operators in Hilbert Space, J. Math. Anal. Appl. 72 (1979), 383–390. https://doi.org/10.1016/0022-247x(79)90234-8.

[25] P.L. Lions, B. Mercier, Splitting Algorithms for the Sum of Two Nonlinear Operators, SIAM J. Numer. Anal. 16 (1979), 964–979. https://doi.org/10.1137/0716071.

[26] G.H.G. Chen, R.T. Rockafellar, Convergence Rates in Forward–Backward Splitting, SIAM J. Optim. 7 (1997), 421–444. https://doi.org/10.1137/s1052623495290179.

[27] D.A. Lorenz, T. Pock, An Inertial Forward-Backward Algorithm for Monotone Inclusions, J. Math. Imaging Vis. 51 (2014), 311–325. https://doi.org/10.1007/s10851-014-0523-2.

[28] W. Cholamjiak, P. Cholamjiak, S. Suantai, An Inertial Forward–backward Splitting Method for Solving Inclusion Problems in Hilbert Spaces, J. Fixed Point Theory Appl. 20 (2018), 42. https://doi.org/10.1007/s11784-018-0526-5.

[29] A. Adamu, J. Deepho, A.H. Ibrahim, et al. Approximation of Zeros of Sum of Monotone Mappings With Applications to Variational Inequality Problem and Image Processing, Nonlinear Funct. Anal. Appl. 26 (2021), 411–432. https://doi.org/10.22771/nfaa.2021.26.02.12.

[30] W. Takahashi, N.C. Wong, J.C. Yao, Two Generalized Strong Convergence Theorems of Halpern's Type in Hilbert Spaces and Applications, Taiwan. J. Math. 16 (2012), 1151–1172. https://doi.org/10.11650/twjm/1500406684.

[31] D.V. Thong, P.T. Vuong, Modified Tseng's Extragradient Methods for Solving Pseudo-Monotone Variational Inequalities, Optimization. 68 (2019), 2207–2226. https://doi.org/10.1080/02331934.2019.1616191.

[32] D. Kitkuan, P. Kumam, J. Martínez-Moreno, et al. Inertial Viscosity Forward–backward Splitting Algorithm for Monotone Inclusions and Its Application to Image Restoration Problems, Int. J. Comput. Math. 97 (2019), 482–497. https://doi.org/10.1080/00207160.2019.1649661.

[33] D.W. Peaceman, H.H. Rachford, Jr., The Numerical Solution of Parabolic and Elliptic Differential Equations, J. Soc. Ind. Appl. Math. 3 (1955), 28–41. https://www.jstor.org/stable/2098834.

[34] G.C. Ugwunnadi, L.Y. Haruna, M.H. Harbau, Accelerated Krasnoselski-Mann Type Algorithm for Hierarchical Fixed Point and Split Monotone Variational Inclusion Problems in Hilbert Spaces, Carpathian Math. Publ. 15 (2023), 158–179. https://doi.org/10.15330/cmp.15.1.158-179.