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Novel Investigations of m-Bi-Ideals and Generators in b-Semirings with Extended Operator Frameworks

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Abstract. In this study, we introduce novel types of m-quasi-ideals and m-bi-ideals in the context of b-semirings, expanding the scope of algebraic structures in this field. We provide detailed characterizations of these ideals, focusing on their distinct properties and interactions within b-semirings. Utilizing an algebraic approach, we elucidate the fundamental properties of m-bi-ideals, examining their behavior and structural role. Additionally, we explore the generators of m-bi-ideals and offer characterizations based on their relationship with bi-ideals. Our findings contribute to a deeper understanding of m-ideals and m-bi-ideals, opening new avenues for further research in algebraic theory and the study of semirings.

1. Introduction

In recent years, the study of semirings has gained significant attention in algebra due to their wide-ranging applications in various mathematical and computational fields. Within this area, b-semirings have emerged as a key focus, offering a rich structure for exploring new types of ideals. This paper introduces and explores new concepts of m-quasi-ideals and m-bi-ideals in the context of b-semirings. These ideals extend the traditional notion of ideals in semirings, providing fresh insights into their algebraic properties and potential applications. By employing algebraic methods, we characterize these m-ideals and m-bi-ideals, analyzing their generators and fundamental properties. This investigation not only enhances the understanding of b-semirings but also contributes to the broader field of algebra by uncovering new structural relationships and behaviors within these mathematical constructs.

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Vandiver [1] introduced the concept of a semiring in 1934. Regular rings have been extensively studied for their own sake and their connection to operator algebras. In 2009, Ronnason [2] proposed the idea of *b*-semirings. In an article submitted for publication, Mohanraj et al. [3] established the concepts of weak-1 ideals and weak-2 ideals in b-semirings. This study characterizes different regular b-semirings using multiple weak ideals. Semigroups, which emerged as a generalization of group theory in the early 20th century, are basic structures that have been widely recognized in various areas of science and mathematics, as noted by Munir and Habib [4,5]. Due to their inherent connection to finite automata, they have numerous applications in theoretical computer science. Examples include time-invariant processes, abstract evolution equations, and graph theory. Semigroups are algebraic structures that have an essential ideal, similar to other algebraic structures. Steinfeld [6,7] was one of the pioneers of the concept of semigroups and rings as quasi-ideals. Iseki [8] extended this idea to semirings with no zero and explored significant semiring descriptions based on quasi-ideals. Mathematicians have found it useful and fascinating to generalize the ideals found in algebraic structures. This generalization of values led to one-sided ideals and pseudo-ideals [9].

Lajos developed the concept of bi-ideals as a more general version of quasi-ideals in associative rings. Later, Szasz [10] and other mathematicians applied these ideas to study various semigroups. Kar et al. [11] introduced generalized bi-ideals for ternary semigroups. In [12], the study of semirings and ordered semirings through the hypothesis of an ordered b-semiring is described. The paper attempts an in-depth analysis of Type-1 bi-ideals, Type-2 bi-ideals over ordered b-semiring. Many mathematicians have used various ideals to prove significant results and characterizations of algebraic structures, *see* [13]. Salahuddin et al. [14] defines left almost hyperideals, right almost hyperideals, almost hyperideals, and minimal almost hyperideals. They proved that the intersection of almost hyperideals need not be an almost hyperideal, but the union of almost hyperideals is an almost hyperideal. This is distinct from the classical concept of ideal theory.

In this paper, we delve into the significant classical results in bi-ideals, m-bi-ideals, and their relationship with the elements and subsets of a b-semiring. We examine the conversion of bi-ideal and quasi-ideal concepts into m-bi-ideal. The paper is divided into five sections. The first section provides an overview of the topic, while the second section explores b-semirings and their relevant definitions and results. In the third section, we cover m-bi-ideal and m-quasi-ideal generated by single element and subset with numerical examples. Finally, we conclude our study in the fourth section. The primary objective of this paper is to establish the relationship between bi-ideals and m-bi-ideals in b-semirings and demonstrate the relationship between m-quasi ideals and m-bi-ideals in b-semirings. Next, to characterize the generator of bi-ideal, weak-1 left ideal, weak-1 right ideal, weak-2 left ideal and weak-2 right ideal.

2. Preliminaries

In this section, we will introduce the concept of m-bi-ideals in b-semirings. We will provide an overview of the key theories and concepts explained in [2] and [15] that are relevant to this topic.Here S denotes b-semiring unless otherwise mentioned.Also, $\diamond_1$ and $\diamond_2$ denotes MinMax-product and MaxMin-product respectively.

Definition 2.1. *A sub b-semiring Q of $\mathcal{S}$ is called a m-quasi ideals if* $Q\diamond_1\mathcal{S}^m \cap \mathcal{S}^m\diamond_1 Q \subseteq Q$.

Definition 2.2. *A sub b-semiring Q of $\mathcal{S}$ is called a m-Quasi ideals if* $Q\diamond_2\mathcal{S}^m \cap \mathcal{S}^m\diamond_2 Q \subseteq Q$.

Definition 2.3. *Let $(\mathcal{S}, \diamond_1, \diamond_2)$ be a b-semiring. The subset $\mathcal{B}$ of $\mathcal{S}$ is called m-bi-ideal if $\mathcal{B}$ is a sub b-semiring of $\mathcal{S}$ there exists $\mathcal{B}\diamond_2\mathcal{S}^m\diamond_2\mathcal{B} \subset \mathcal{B}$, where m is a positive integer.*

Definition 2.4. *Let $(\mathcal{S}, \diamond_2, \diamond_1)$ be a b-semiring. The subset $\mathcal{B}$ of $\mathcal{S}$ is called m-bi-ideal if $\mathcal{B}$ is a sub b-semiring of $\mathcal{S}$ there exists $\mathcal{B}\diamond_1\mathcal{S}^m\diamond_1\mathcal{B} \subset \mathcal{B}$, where m is a positive integer.*

Notations: For a subset A of S and $i = 1, 2, 3, ..., n$

(i) $\sum A = \{(a_1\diamond_1 a_2\diamond_1 ...\diamond_1 a_n) | a_i \in A\}$.

(ii) $\prod A = \{(a_1\diamond_2 a_2\diamond_2 ...\diamond_2 a_n) | a_i \in A\}$.

(iii) $\sum(A\diamond_2 S) = \{(a_1\diamond_2 s_1)\diamond_1(a_2\diamond_2 s_2)\diamond_1 ...\diamond_1(a_n\diamond_2 s_n) | a_i \in A,\ s_i \in S\}$.

(iv) $\prod(A\diamond_1 S) = \{(a_1\diamond_1 s_1)\diamond_2(a_2\diamond_1 s_2)\diamond_2 ...\diamond_2(a_n\diamond_1 s_n) | a_i \in A,\ s_i \in S\}$.

(v) $\sum(A\diamond_2 S\diamond_2 A) = \{(a_1\diamond_2 s_1\diamond_2 a_1)\diamond_1(a_2\diamond_2 s_2\diamond_2 a_2)...\diamond_1(a_n\diamond_2 s_n\diamond_2 a_n)\ a_i \in A, s_i \in S\}$

(vi) $\prod(A\diamond_1 S\diamond_1 A) = \{(a_1\diamond_1 s_1\diamond_1 a_1)\diamond_2(a_2\diamond_1 s_2\diamond_1 a_2)...\diamond_2(a_n\diamond_1 s_n\diamond_1 a_n)\ a_i \in A, s_i \in S\}$.

What is the product of any two m_1-bi-ideal and $m_1{}'$-bi-ideal of $\mathcal{S}$? We answer the questions by introducing m_1-bi-ideals.

3. m_1-bi-ideals of b-semiring

In this section, we introduce m_1-bi-ideals of b-semirings and their generalizations. Examples are provided to illustrate the results.

Remark 3.1. *The binary operation $\wedge$ and $\vee$ is defined as follows $x \wedge y = min\{x, y\}$, $x \vee y = max\{x, y\}$ and*

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ a_1 & 0 & 0 & 0 & 0 & 0 \\ a_2 & a_3 & 0 & 0 & 0 & 0 \\ a_4 & a_5 & a_6 & 0 & 0 & 0 \\ a_7 & a_8 & a_9 & a_{10} & 0 & 0 \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & 0 \end{pmatrix} \diamond_1 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ b_1 & 0 & 0 & 0 & 0 & 0 \\ b_2 & b_3 & 0 & 0 & 0 & 0 \\ b_4 & b_5 & b_6 & 0 & 0 & 0 \\ b_7 & b_8 & b_9 & b_{10} & 0 & 0 \\ b_{11} & b_{12} & b_{13} & b_{14} & b_{15} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ c_1 & 0 & 0 & 0 & 0 & 0 \\ c_2 & c_3 & 0 & 0 & 0 & 0 \\ c_4 & c_5 & c_6 & 0 & 0 & 0 \\ c_7 & c_8 & c_9 & c_{10} & 0 & 0 \\ c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & 0 \end{pmatrix}, \textit{where}$$

$c_1 = a_1 \wedge b_1 \wedge b_2 \wedge b_4 \wedge b_7 \wedge b_{11};\ c_2 = a_2 \wedge (a_3 \vee b_1) \wedge b_2 \wedge b_4 \wedge b_7 \wedge b_{11};\ c_3 = a_2 \wedge a_3 \wedge b_3 \wedge b_5 \wedge b_8 \wedge b_{12};\ c_4 = a_4 \wedge (a_5 \vee b_1) \wedge (a_6 \vee b_2) \wedge b_4 \wedge b_7 \wedge b_{11};\ c_5 = a_4 \wedge a_5 \wedge (a_6 \vee b_3) \wedge b_5 \wedge b_8 \wedge b_{12};\ c_6 = a_4 \wedge a_5 \wedge a_6 \wedge b_6 \wedge b_9 \wedge b_{13};\ c_7 = a_7 \wedge (a_8 \vee b_1) \wedge (a_9 \vee b_2) \wedge (a_{10} \vee b_4) \wedge b_7 \wedge b_{11};\ c_8 = a_7 \wedge a_8 \wedge (a_9 \vee b_3) \wedge (a_{10} \vee b_5) \wedge b_8 \wedge b_{12};\ c_9 = a_7 \wedge a_8 \wedge a_9 \wedge (a_{10} \vee b_6) \wedge b_9 \wedge b_{13};\ c_{10} = a_7 \wedge$

$a_8 \wedge a_9 \wedge a_{10} \wedge b_{10} \wedge b_{14}$; $c_{11} = a_{11} \wedge (a_{12} \vee b_1) \wedge (a_{13} \vee b_2) \wedge (a_{14} \vee b_4) \wedge (a_{15} \vee b_7) \wedge b_{11}$; $c_{12} = a_{11} \wedge a_{12} \wedge (a_{13} \vee b_3) \wedge (a_{14} \vee b_5) \wedge (a_{15} \vee b_8) \wedge b_{12}$; $c_{13} = a_{11} \wedge a_{12} \wedge a_{13} \wedge (a_{14} \vee b_6) \wedge (a_{15} \vee b_9) \wedge b_{13}$; $c_{14} = a_{11} \wedge a_{12} \wedge a_{13} \wedge a_{14} \wedge (a_{15} \vee b_{10}) \wedge b_{14}$; $c_{15} = a_{11} \wedge a_{12} \wedge a_{13} \wedge a_{14} \wedge a_{15} \wedge b_{15}$.

Remark 3.2. *The binary operation $\diamond_2$ is defined as follows*

$$\begin{pmatrix} 0 & a_1 & a_2 & a_3 & a_4 & a_5 \\ 0 & 0 & a_6 & a_7 & a_8 & a_9 \\ 0 & 0 & 0 & a_{10} & a_{11} & a_{12} \\ 0 & 0 & 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 & 0 & a_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \diamond_2 \begin{pmatrix} 0 & b_1 & b_2 & b_3 & b_4 & b_5 \\ 0 & 0 & b_6 & b_7 & b_8 & b_9 \\ 0 & 0 & 0 & b_{10} & b_{11} & b_{12} \\ 0 & 0 & 0 & 0 & b_{13} & b_{14} \\ 0 & 0 & 0 & 0 & 0 & b_{15} \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & c_1 & c_2 & c_3 & c_4 \\ 0 & 0 & 0 & c_5 & c_6 & c_7 \\ 0 & 0 & 0 & 0 & c_8 & c_9 \\ 0 & 0 & 0 & 0 & 0 & c_{10} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \text{ where}$$

$c_1 = b_6 \wedge a_1$; $c_2 = (b_7 \wedge a_1) \vee (b_{10} \wedge a_2)$; $c_3 = (b_8 \wedge a_1) \vee (b_{11} \wedge a_2) \vee (b_{13} \wedge a_3)$; $c_4 = (b_9 \wedge a_1) \vee (b_{12} \wedge a_2) \vee (b_{14} \wedge a_3) \vee (b_{15} \wedge a_4)$; $c_5 = b_{10} \wedge a_6$; $c_6 = (b_{11} \wedge a_6) \vee (b_{13} \wedge a_7)$; $c_7 = (b_{12} \wedge a_6) \vee (b_{14} \wedge a_7) \vee (b_{15} \wedge a_8)$; $c_8 = b_{13} \wedge a_{10}$; $c_9 = (b_{14} \wedge a_{10}) \vee (b_{15} \wedge a_{11})$; $c_{10} = b_{15} \wedge a_{13}$

Remark 3.3. *If $m = 1$, then $\mathcal{B} \diamond_2 \mathcal{S}^m \diamond_2 \mathcal{B} \subset \mathcal{B}$ is called 1-bi-ideal or simply a bi-ideal.*

Theorem 3.1. *Every bi-ideal is a m_1-bi-ideal.*

Proof. Let $\mathcal{B}$ be the bi-ideal of $\mathcal{S}$, from Definition 2.4, $\mathcal{B} \diamond_2 \mathcal{S} \diamond_2 \mathcal{B} \subseteq \mathcal{B}$. Now, it is also true that $\mathcal{B} \diamond_2 \mathcal{S}^1 \diamond_2 \mathcal{B} \subseteq \mathcal{B}$. Similarly, we can see that $\mathcal{B} \diamond_2 \mathcal{S}^2 \diamond_2 \mathcal{B} \subseteq \mathcal{B} \diamond_2 \mathcal{S}^1 \diamond_2 \mathcal{B} \subseteq \mathcal{B}$. In general, $\mathcal{B} \diamond_2 \mathcal{S}^m \diamond_2 \mathcal{B} \subseteq \mathcal{B} \diamond_2 \mathcal{S}^{m-1} \diamond_2 \mathcal{B} \subseteq \mathcal{B}$. Hence, $\mathcal{B}$ is a m-bi-ideal of $\mathcal{S}$.

Remark 3.4. *The reverse implication of the Theorem 3.1 does not satisfied, see the Example 3.1.*

Example 3.1. *Consider the b-semiring$(\mathcal{S}_1, \diamond_1, \diamond_2)$, where $\diamond_1$ and $\diamond_2$ are defined in the above Note 3.1.*

$$\text{Let } \mathcal{S}_1 = \left\{ \begin{pmatrix} 0 & s_1 & s_2 & s_3 \\ 0 & 0 & s_4 & s_5 \\ 0 & 0 & 0 & s_6 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| s_i^{\prime s} \in \mathbb{Z}^* \} \right\} \tag{3.1}$$

$$\text{Let } Q = \left\{ \begin{pmatrix} 0 & b_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & b_2 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| b_i^{\prime s} \in \mathbb{Z}^* \right\} \tag{3.2}$$

is a sub b-semiring. Now,

$$B \diamond_2 \mathcal{S}_1^2 \diamond_2 B = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| n_i^{\prime s} \in \mathbb{Z}^* \right\} \subseteq B. \tag{3.3}$$

Thus, $\mathcal{B}$ is a m-bi-ideal but it may not necessarily be a bi-ideal of $\mathcal{S}_1$ by

$$(\mathcal{B}\diamond_2\mathcal{S}_1\diamond_2\mathcal{B}) = \left\{ \left(\begin{matrix} 0 & 0 & 0 & m_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix} \right) \Bigg| m_i^{'s} \in \mathbb{Z}^* \right\} \quad \nsubseteq \mathcal{B}. \tag{3.4}$$

Theorem 3.2. *The product of any two m_1-bi-ideal and ${m_1}'$-bi-ideal of $\mathcal{S}$ with identity element e is a* $\max(m_1, {m_1}')$*-bi-ideal of $\mathcal{S}$.*

Proof. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the two bi-ideals of $\mathcal{S}$.
Now, $\mathcal{B}_1\diamond_2\mathcal{S}^{m_1}\diamond_2\mathcal{B}_1 \subset \mathcal{B}_1$ and $\mathcal{B}_2\diamond_2\mathcal{S}^{m_1}\diamond_2\mathcal{B}_2 \subseteq \mathcal{B}_2$, m_1 and ${m_1}'$ are positive integers. From Note 3.2, $(\mathcal{B}_1\diamond_2\mathcal{B}_2)^2 = (\mathcal{B}_1\diamond_2\mathcal{B}_2)\diamond_2(\mathcal{B}_1\diamond_2\mathcal{B}_2) \subseteq (\mathcal{B}_1\diamond_2\mathcal{S}\diamond_2\mathcal{B}_1)\diamond_2\mathcal{B}_2 \subseteq (\mathcal{B}_1\diamond_2\mathcal{S}\diamond_2 e...\diamond_2 e\diamond_2\mathcal{B}_1\diamond_2\mathcal{B}_2) \subseteq (\mathcal{B}_1\diamond_2\mathcal{S}\diamond_2\mathcal{S}...\diamond_2\mathcal{S}\diamond_2\mathcal{B}_1)\diamond_2\mathcal{B}_2 \subseteq (\mathcal{B}_1\diamond_2\mathcal{S}^m\diamond_2\mathcal{B}_1)\diamond_2\mathcal{B}_2 \subseteq \mathcal{B}_1\diamond_2\mathcal{B}_2$, then $(\mathcal{B}_1\diamond_2\mathcal{B}_2)^2 \subseteq \mathcal{B}_1\diamond_2\mathcal{B}_2$. Also, $\mathcal{B}_1\diamond_2\mathcal{B}_2(\mathcal{S}^{\max(m_1,{m_1}')})\diamond_2\mathcal{B}_1\diamond_2\mathcal{B}_2 \subseteq e\mathcal{B}_1\diamond_2\mathcal{S}\diamond_2\mathcal{S}^{\max(m_1,{m_1}')}\diamond_2\mathcal{B}_1\diamond_2\mathcal{B}_2 \subseteq \mathcal{B}_1\diamond_2\mathcal{S}^{m_1}\diamond_2\mathcal{B}_2$. Therefore, $\mathcal{B}_1\diamond_2\mathcal{B}_2$ is $\max(m_1, {m_1}')$-bi-ideal $\mathcal{S}$.

Theorem 3.3. *If $\mathcal{B}$ is a $\diamond_1$ closure of $\mathcal{S}$, $\mathcal{R}$ is a subset of $\mathcal{S}$ and $\mathcal{B}$ be m_1-bi-ideal ($m > 1$), then $\mathcal{B}\diamond_2\mathcal{R}$($\mathcal{R}\diamond_2\mathcal{B}$) is m_1-bi-ideal.*

Proof. Now, $(\mathcal{B}\diamond_2\mathcal{R})^2$=$(\mathcal{B}\diamond_2\mathcal{R})\diamond_2(\mathcal{B}\diamond_2\mathcal{R})$=$(\mathcal{B}\diamond_2\mathcal{R}\diamond_2\mathcal{B})\diamond_2\mathcal{R} \subseteq (\mathcal{B}\diamond_2\mathcal{R}\diamond_2\mathcal{S})\diamond_2\mathcal{B} \subseteq (\mathcal{B}\diamond_2(\mathcal{S}^m)\diamond_2\mathcal{B})\diamond_2 \mathcal{R} \subseteq \mathcal{B}\diamond_2\mathcal{R}$. Then, $(\mathcal{B}\diamond_2\mathcal{R})^2 \subseteq \mathcal{B}\diamond_2\mathcal{R}$ of $\mathcal{S}$. Also, $\mathcal{B}\diamond_2\mathcal{R}\diamond_2(\mathcal{S}^m)\diamond_2\mathcal{B}\diamond_2\mathcal{R}\diamond_2 \subseteq \mathcal{B}\diamond_2\mathcal{S}\diamond_2\mathcal{S}^m\diamond_2\mathcal{B}\diamond_2\mathcal{R} \subseteq \mathcal{B}\diamond_2\mathcal{S}^m\diamond_2\mathcal{B}\diamond_2\mathcal{R}$
$\subseteq \mathcal{B}\diamond_2\mathcal{R}$. Therefore, $\mathcal{B}\diamond_2\mathcal{R}$ is a m_1-bi-ideal of $\mathcal{S}$. Similarly, we can demonstrate that the m_1-bi-ideal of $\mathcal{S}$ is the $\mathcal{R}\diamond_2\mathcal{B}$.

Theorem 3.4. *If $\mathcal{B}$ is a intersection of all bi-ideals with bipotencies $m_1, m_2...$, then $\mathcal{B}$ is also bi-ideal with bi potency is* $\max\{m_1, m_2...\}$

Proof. Let $\{\mathcal{B}_\zeta : \zeta \in \wedge\}$ be a set of m-bi-ideals of $\mathcal{S}$, then $\mathcal{B}$=$\cap\mathcal{B}_\zeta$. Thus, $\mathcal{B}$ is a sub b-semiring of $\mathcal{S}$. Since $\mathcal{B}_\zeta\diamond_2\mathcal{S}^m\diamond_2\mathcal{B}_\zeta \subseteq \mathcal{B}_\zeta \subseteq \mathcal{B}$ for all $\zeta \in \wedge$. Therefore, $\mathcal{B}\diamond_2\mathcal{S}^{\max\{m_\zeta:\zeta\in\wedge\}}\diamond_2\mathcal{B} \subseteq \mathcal{B}_\zeta\mathcal{S}^m\diamond_2\mathcal{B}_\zeta \subseteq \mathcal{B}_\zeta \subseteq \mathcal{B}_\zeta$ for all $\zeta \in \wedge$. This implies that $\mathcal{B}\diamond_2\mathcal{S}^{\max\{m_\zeta:\zeta\in\wedge\}}\diamond_2\mathcal{B} \subseteq \mathcal{B}_\zeta \subseteq \mathcal{B}$. Therefore, $\mathcal{B}$ is an m-bi-ideal with bipotency $\max\{m_1, m_2....\}$.

Theorem 3.5. *Every m_1-quasi ideal is a m_1-bi-ideal.*

Proof. Let $\mathcal{Q}$ be a m_1-quasi ideal of $\mathcal{S}$. Clearly, $\mathcal{Q}$ is a sub b-semiring. Now, $\mathcal{Q}\diamond_2\mathcal{S}^m\diamond_2\mathcal{Q} \subseteq \mathcal{Q}\diamond_2\mathcal{S}^m\diamond_2\mathcal{S} = \mathcal{Q}\diamond_2\mathcal{S}^{m+1} \subseteq \mathcal{Q}\diamond_2\mathcal{S}^m$. Similarly, $\mathcal{Q}\diamond_2\mathcal{S}^m\diamond_2\mathcal{Q} \subseteq \mathcal{S}^m\diamond_2\mathcal{Q}$. We get $\mathcal{Q}\diamond_2\mathcal{S}^m\diamond_2\mathcal{Q} \subseteq (\mathcal{Q}\diamond_2\mathcal{S}^m) \cap (\mathcal{S}^m\diamond_2\mathcal{Q}) \subseteq \mathcal{Q}$. Hence, $\mathcal{Q}$ is bi-ideal with potency m.

Remark 3.5. *The reverse implication of the Theorem 3.5 does not hold, see the following Example 3.2.*

Example 3.2. *Let $\mathcal{S}_1$ be a b-semiring and $\mathcal{B}$ be a sub b-semiring as in Example 3.1.*

$$\mathcal{B}\diamond_2\mathcal{S}_1^2\diamond_2\mathcal{B}=\left\{\left(\begin{matrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{matrix}\right)\middle| n_i^{\prime s}\in\mathbb{Z}^*\right\}\subseteq\mathcal{B}. \tag{3.5}$$

Thus $\mathcal{B}$ is a m_1-bi-ideal of $\mathcal{S}_1$ but it may not be a m_1-quasi ideal by

$$(\mathcal{B}\diamond_2\mathcal{S}_1^2)\cap(\mathcal{S}_1^2\diamond_2\mathcal{B})=\left\{\left(\begin{matrix}0&0&0&r_1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{matrix}\right)\middle| r_i^{\prime s}\in\mathbb{Z}^*\right\}\nsubseteq\mathcal{B}. \tag{3.6}$$

Theorem 3.6. *If $\mathcal{Q}$ is a $\diamond_2$ product of any (m_1,m_2)-quasi ideal and (n_1,n_2)-quasi ideal, then $\mathcal{Q}$ is a $\max\{m_1,m_2,n_1,n_2\}$-bi-ideal of $\mathcal{S}$ with identity element.*

Proof. By the Theorem 3.5, Now, $(\mathcal{Q}_1\diamond_2\mathcal{Q}_2)\diamond_2(\mathcal{Q}_1\diamond_2\mathcal{Q}_2)\subseteq\mathcal{Q}_1\diamond_2(\mathcal{Q}_2\diamond_2\mathcal{S}\diamond_2\mathcal{Q}_2)\subseteq\mathcal{Q}_1\diamond_2\mathcal{Q}_2$, *i.e.*, $(\mathcal{Q}_1\diamond_2\mathcal{Q}_2)^2\subseteq\mathcal{Q}_1\diamond_2\mathcal{Q}_2$. Therefore, $\mathcal{Q}_1\diamond_2\mathcal{Q}_2$ is a closed under $\diamond_2$. Now, $(\mathcal{Q}_1\diamond_2\mathcal{Q}_2)\diamond_2\mathcal{S}^{\max\{m_1,m_2,n_1,n_2\}}\diamond_2(\mathcal{Q}_1\diamond_2\mathcal{Q}_2)\subseteq(\mathcal{Q}_1\diamond_2\mathcal{Q}_2)\diamond_2\mathcal{S}^{max\{m_1,m_2,n_1,n_2\}}\diamond_2(\mathcal{S}\diamond_2\mathcal{Q}_2)\subseteq\mathcal{Q}_1\diamond_2(\mathcal{Q}_2\diamond_2\mathcal{S}^{max\{m_1,m_2,n_1,n_2\}+1}\diamond_2\mathcal{Q}_2)\subseteq\mathcal{Q}_1\diamond_2\mathcal{Q}_2$. Therfore, $\mathcal{Q}_1\diamond_2\mathcal{Q}_2$ is a $\max\{m_1,m_2,n_1,n_2\}$ bi-ideal of $\mathcal{S}$.

Theorem 3.7. *Every m-left ideal is a m-bi-ideal.*

Proof. Since $\mathcal{G}$ represents the m-left ideal of $\mathcal{S}$. Now, $\mathcal{G}\diamond_2\mathcal{S}^m\diamond_2\mathcal{G}\subseteq\mathcal{G}\diamond_2\mathcal{G}\subseteq\mathcal{G}$. This implies that $\mathcal{G}$ is m-bi-ideal of $\mathcal{S}$.

Theorem 3.8. *Every m-right ideal of $\mathcal{S}$ is a m-bi-ideal.*

Proof. The Proof follows from the Theorem 3.7.

Theorem 3.9. *For $\mathcal{S}$, let $\mathcal{G}$ be a q-left ideal and $\mathcal{H}$ be an r-right ideal. Then, $\mathcal{G}\cap\mathcal{H}$ is a k-bi-ideal with $k=\max(m,n)$.*

Proof. Let $\mathcal{G}$ be a q-left ideal and $\mathcal{H}$ is r-right ideal of $\mathcal{S}$. Now, $\mathcal{G}$ and $\mathcal{H}$ are q-bi and r-bi-ideals of $\mathcal{S}$. By Theorem 3.4, the intersection is $\max(q,r)$-bi-ideals. Also, $\mathcal{G}\cap\mathcal{H}\diamond_2(\mathcal{S}^{max\{q,r\}})\diamond_2\mathcal{G}\cap\mathcal{H}\subseteq\mathcal{G}\diamond_2\mathcal{S}^{\max\{q,r\}}\diamond_2\mathcal{G}\subseteq\mathcal{S}^{\max\{q,r\}+1}\diamond_2\mathcal{G}\subseteq\mathcal{S}^m\diamond_2\mathcal{G}\subseteq\mathcal{G}$. Similarly, we can prove that $\mathcal{G}\cap\mathcal{H}\diamond_2(\mathcal{S}^{max\{q,r\}})\diamond_2\mathcal{G}\cap\mathcal{H}\subseteq\mathcal{H}$. Consequently, $\mathcal{G}\cap\mathcal{H}\diamond_2\mathcal{S}^{max\{q,r\}}\diamond_2\mathcal{G}\cap\mathcal{H}\subseteq\mathcal{G}\cap\mathcal{H}$. Therefore, $\mathcal{G}\cap\mathcal{H}$ is a k-bi-ideal with $k=\max(m,n)$

Remark 3.6. *Every* 1*-quasi ideal is equivalent to a quasi ideal (where $m=1$).*

Theorem 3.10. *Let $a\in\mathcal{S}$, then m_1-bi-ideal generated by a is $<a>_{mb}=\{na\}\cup\{n'a^2\}\cup a\diamond_2\mathcal{S}^m\diamond_2 a$.*

4. m_2-bi-ideals of b-semiring

What is the product of any two m_2-bi-ideal and ${m_2}'$-bi-ideal of $\mathcal{S}$? We answer the questions by introducing m_2-bi-ideals.

We introduce m_2-bi-ideals of b-semirings and their generalizations. Examples are provided to illustrate our results.

Remark 4.1. *If $m = 1$, then the subset $\mathcal{B}\diamond_1\mathcal{S}^m\diamond_1\mathcal{B} \subset \mathcal{B}$ is called 1-bi-ideal, or simply a bi-ideal.*

Theorem 4.1. *Every bi-ideal is a m_2-bi-ideal.*

Proof. Let $\mathcal{B}$ be the bi-ideal of $\mathcal{S}$, by Definition 2.4, $\mathcal{B}\diamond_1\mathcal{S}\diamond_1\mathcal{B} \subseteq \mathcal{B}$. Now, $\mathcal{B}\diamond_1\mathcal{S}^1\diamond_1\mathcal{B} \subseteq \mathcal{B}$. Similarly, $\mathcal{B}\diamond_1\mathcal{S}^2\diamond_1\mathcal{B} \subseteq \mathcal{B}\diamond_1\mathcal{S}^1\diamond_1\mathcal{B} \subseteq \mathcal{B}$. In general, $\mathcal{B}\diamond_1\mathcal{S}^m\diamond_1\mathcal{B} \subseteq \mathcal{B}\diamond_1\mathcal{S}^{m-1}\diamond_1\mathcal{B} \subseteq \mathcal{B}$. Therefore, $\mathcal{B}$ is a m-bi-ideal of $\mathcal{S}$.

Remark 4.2. *The reverse implication of the Theorem 4.1 does not satisfied the Example 4.1.*

Example 4.1. *Consider the b-semiring$(\mathcal{S}_1, \diamond_2, \diamond_1)$, where $\diamond_2$ and $\diamond_1$ are defined in the above Note 3.1.*

$$\text{Let } \mathcal{S} = \left\{ \begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_5 & s_6 & s_7 & 0 \\ s_8 & s_9 & s_{10} & s_{11} \\ s_{12} & 0 & 0 & 0 \end{pmatrix} \middle| s_i^{'s} \in Z^* \right\}. \tag{4.1}$$

$$\text{Let } Q = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & 0 & b_2 & b_3 \\ b_4 & 0 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \in Z^* \right\} \tag{4.2}$$

be a sub b-semiring. Then,

$$B\diamond_1\mathcal{S}^2\diamond_1 B = \left\{ \begin{pmatrix} n_1 & 0 & 0 & 0 \\ n_2 & 0 & 0 & 0 \\ n_3 & 0 & 0 & 0 \\ n_4 & 0 & 0 & 0 \end{pmatrix} \middle| n_i^{'s} \in Z^* \right\} \subseteq B. \tag{4.3}$$

As a result, $\mathcal{B}$ is not a bi-ideal but m_2-bi-ideal of $\mathcal{S}$ by

$$(\mathcal{B}\diamond_1\mathcal{S}\diamond_1\mathcal{B}) = \left\{ \begin{pmatrix} m_1 & m_2 & m_3 & m_4 \\ m_5 & 0 & 0 & 0 \\ m_6 & m_7 & m_8 & m_9 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| m_i^{'s} \in Z^* \right\} \quad \nsubseteq \mathcal{B}. \tag{4.4}$$

Theorem 4.2. *The product of any two m_2-bi-ideal and $m_2{}'$-bi-ideal of $\mathcal{S}$ with identity element e is a* $\max(m_2, m_2{}')$*-bi-ideal of $\mathcal{S}$.*

Proof. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be the two bi-ideals of $\mathcal{S}$. Now, $\mathcal{B}_1\diamond_1\mathcal{S}^{m_2}\diamond_1\mathcal{B}_1 \subset \mathcal{B}_1$ and $\mathcal{B}_2\diamond_1\mathcal{S}^{m_2}\diamond_1\mathcal{B}_2 \subseteq \mathcal{B}_2$, where m_2 and $m_2{}'$ are positive integers. By the Note 3.2, $(\mathcal{B}_1\diamond_1\mathcal{B}_2)^2 = (\mathcal{B}_1\diamond_1\mathcal{B}_2)\diamond_1(\mathcal{B}_1\diamond_1\mathcal{B}_2) \subseteq (\mathcal{B}_1\diamond_1\mathcal{S}\diamond_1\mathcal{B}_1)\diamond_1\mathcal{B}_2 \subseteq (\mathcal{B}_1\diamond_1\mathcal{S}\diamond_1 e ... \diamond_1 e\diamond_1\mathcal{B}_1\diamond_1\mathcal{B}_2) \subseteq (\mathcal{B}_1\diamond_1\mathcal{S}\diamond_1\mathcal{S}...\diamond_1\mathcal{S}\diamond_1\mathcal{B}_1)\diamond_1\mathcal{B}_2 \subseteq (\mathcal{B}_1\diamond_1\mathcal{S}^m\diamond_1\mathcal{B}_1)\diamond_1 \mathcal{B}_2 \subseteq \mathcal{B}_1\diamond_1\mathcal{B}_2$, then$(\mathcal{B}_1\diamond_1\mathcal{B}_2)^2 \subseteq \mathcal{B}_1\diamond_1\mathcal{B}_2$. Also, $\mathcal{B}_1\diamond_1\mathcal{B}_2(\mathcal{S}^{max(m_2,m_2{}')})\diamond_1\mathcal{B}_1\diamond_1\mathcal{B}_2 \subseteq \mathcal{B}_1\diamond_1\mathcal{S}\diamond_1 \mathcal{S}^{max(m_2,m_2{}')}\diamond_1\mathcal{B}_1\diamond_1\mathcal{B}_2 \subseteq \mathcal{B}_1\diamond_1\mathcal{S}^{m_2}\diamond_1\mathcal{B}_2$. Therefore, $\mathcal{B}_1\diamond_1\mathcal{B}_2$ is $\max(m_2, m_2{}')$-bi-ideal $\mathcal{S}$.

Theorem 4.3. *If $\mathcal{B}$ is a $\diamond_2$ closure of $\mathcal{S}$, $\mathcal{R}$ is a subset of $\mathcal{S}$ and $\mathcal{B}$ be m_2-bi-ideal ($m > 1$), then $\mathcal{B}\diamond_2\mathcal{R}$($\mathcal{R}\diamond_2\mathcal{B}$) is m_2-bi-ideal.*

Proof. Now, $(\mathcal{B}\diamond_1\mathcal{R})^2$=$(\mathcal{B}\diamond_1\mathcal{R})\diamond_1(\mathcal{B}\diamond_1\mathcal{R})$=$(\mathcal{B}\diamond_1\mathcal{R}\diamond_1\mathcal{B})\diamond_1\mathcal{R} \subseteq (\mathcal{B}\diamond_1\mathcal{R}\diamond_1\mathcal{S}\diamond_1\mathcal{B} \subseteq (\mathcal{B}\diamond_1(\mathcal{S}^m)\diamond_1\mathcal{B})\diamond_1\mathcal{R} \subseteq \mathcal{B}\diamond_1\mathcal{R}$. Then, $(\mathcal{B}\diamond_1\mathcal{R})^2 \subseteq \mathcal{B}\diamond_1\mathcal{R}$ of $\mathcal{S}$.Also,$\mathcal{B}\diamond_1\mathcal{R}\diamond_1(\mathcal{S}^m)\diamond_1\mathcal{B}\diamond_1\mathcal{R}\diamond_1 \subseteq \mathcal{B}\diamond_1\mathcal{S}\diamond_1\mathcal{S}^m\diamond_1\mathcal{B}\diamond_1\mathcal{R} \subseteq \mathcal{B}\diamond_1\mathcal{S}^m\diamond_1\mathcal{B}\diamond_1\mathcal{R} \subseteq \mathcal{B}\diamond_1\mathcal{R}$. Therefore,$\mathcal{B}\diamond_1\mathcal{R}$ is a m_2-bi-ideal of $\mathcal{S}$. Similarly, we can demonstrate that the m_2-bi-ideal of $\mathcal{S}$ is the $\mathcal{R}\diamond_1\mathcal{B}$.

Theorem 4.4. *Every m_2-quasi ideal is a m_2-bi-ideal.*

Proof. Let $\mathcal{Q}$ be a m_2-quasi ideal of $\mathcal{S}$. Clearly, $\mathcal{Q}$ is a sub b-semiring. Now, $\mathcal{Q}\diamond_1\mathcal{S}^m\diamond_1\mathcal{Q} \subseteq \mathcal{Q}\diamond_1\mathcal{S}^m\diamond_1\mathcal{S} = \mathcal{Q}\diamond_1\mathcal{S}^{m+1} \subseteq \mathcal{Q}\diamond_1\mathcal{S}^m$. Similarly, $\mathcal{Q}\diamond_1\mathcal{S}^m\diamond_1\mathcal{Q} \subseteq \mathcal{S}^m\diamond_1\mathcal{Q}$. We get $\mathcal{Q}\diamond_1\mathcal{S}^m\diamond_1\mathcal{Q} \subseteq (\mathcal{Q}\diamond_1\mathcal{S}^m) \cap (\mathcal{S}^m\diamond_1\mathcal{Q}) \subseteq \mathcal{Q}$. Hence, $\mathcal{Q}$ is bi-ideal with potency m.

Remark 4.3. *The reverse implication of Theorem 4.4 is not true, as shown in Example 4.2.*

Example 4.2. *Let $\mathcal{S}_1$ be a b-semiring and $\mathcal{B}$ be a sub b-semiring as shown in Example 3.1.*

$$Let\ \mathcal{S} = \left\{ \left(\begin{pmatrix} s_1 & s_2 & s_3 & s_4 \\ s_5 & s_6 & s_7 & 0 \\ s_8 & s_9 & s_{10} & s_{11} \\ s_{12} & 0 & 0 & 0 \end{pmatrix} \middle| s_i^{'s} \in Z^*\} \right) \right\}. \tag{4.5}$$

$$Let\ Q = \left\{ \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & b_1 & 0 & 0 \\ 0 & 0 & b_2 & b_3 \\ b_4 & 0 & 0 & 0 \end{pmatrix} \middle| b_i^{'s} \in Z^* \right) \right\} \tag{4.6}$$

be a sub b-semiring. Then,

$$B\diamond_1\mathcal{S}^2\diamond_1 B = \left\{ \left(\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| n_i^{'s} \in Z^* \right) \right\} \subseteq B. \tag{4.7}$$

As a result, $\mathcal{B}$ is a not a quasi-ideal but a m_2-bi-ideal of $\mathcal{S}$ by

$$(\mathcal{B}\diamond_1\mathcal{S}\diamond_1\mathcal{B}) = \left\{ \begin{pmatrix} r_1 & 0 & 0 & 0 \\ r_2 & 0 & 0 & 0 \\ r_3 & r_4 & r_5 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| r_i^{'s} \in Z^* \right\} \quad \nsubseteq \mathcal{B}. \tag{4.8}$$

Theorem 4.5. *If Q is a $\diamond_1$-product of any (m_2, m_1)-quasi ideal and (n_1, n_2)-quasi ideal, then $\mathcal{S}$ has $\max\{m_2, m_1, n_1, n_2\}$-bi-ideal with identity element.*

Proof. By Theorem 4.4,
$(Q_1\diamond_1 Q_2)\diamond_1(Q_1\diamond_1 Q_2) \subseteq Q_1\diamond_1(Q_2\diamond_1\mathcal{S}\diamond_1 Q_2) \subseteq Q_1\diamond_1 Q_2$, *i.e.*, $(Q_1\diamond_1 Q_2)^2 \subseteq Q_1\diamond_1 Q_2$.
Therefore, $Q_1\diamond_1 Q_2$ is a closed under$\diamond_1$. Now, $(Q_1\diamond_1 Q_2)\diamond_1\mathcal{S}^{\max\{m_2,m_1,n_1,n_2\}}\diamond_1(Q_1\diamond_1 Q_2)$
$\subseteq (Q_1\diamond_1 Q_2)\diamond_1\mathcal{S}^{\max\{m_2,m_1,n_1,n_2\}}\diamond_1(\mathcal{S}\diamond_1 Q_2) \subseteq Q_1\diamond_1(Q_2\diamond_1\mathcal{S}^{\max\{m_2,m_1,n_1,n_2\}+1}\diamond_1 Q_2) \subseteq Q_1\diamond_1 Q_2$.
Hence, $Q_1\diamond_1 Q_2$ is a $\max\{m_2, m_1, n_1, n_2\}$-bi-ideal of $\mathcal{S}$.

Theorem 4.6. *Every m-left ideal is a m_2-bi-ideal.*

Proof. Since $\mathcal{G}$ represents the m-left ideal of $\mathcal{S}$. Now, $\mathcal{G}\diamond_1\mathcal{S}^m\diamond_1\mathcal{G} \subseteq \mathcal{G}\diamond_1\mathcal{G} \subseteq \mathcal{G}$. This implies that $\mathcal{G}$ is m-bi-ideal of $\mathcal{S}$.

Theorem 4.7. *Every m-right ideal of $\mathcal{S}$ is a m_2-bi-ideal.*

Proof. The proof follows from the Theorem 4.6.

Theorem 4.8. *For a q-left ideal $\mathcal{G}$ and r-right ideal $\mathcal{H}$ of $\mathcal{S}$, the intersection of $\mathcal{G}$ and $\mathcal{H}$ is a k-bi-ideal with $k = \max(m, n)$.*

Theorem 4.9. *Let $a \in \mathcal{S}$, then m_2-bi-ideal generated by a is $< a >_{mb} = \{na\} \cup \{n' a^2\} \cup a\diamond_1\mathcal{S}^m\diamond_1 a$.*

5. m-bi-ideals of b-semiring

This section presents m-bi-ideals of b-semirings and their generalizations. We provide examples to illustrate our results.

Theorem 5.1. *Every bi-ideal for $m \geq 1$ is a m-bi-ideal of $\mathcal{S}$.*

Proof. The proof follows from Theorem 3.1 and Theorem 4.1.

Theorem 5.2. *The product of any two m_1-bi-ideal and m_2-bi-ideal of $\mathcal{S}$ with identity element e is a $\max(m_1, m_2)$-bi-ideal of $\mathcal{S}$.*

Proof. The proof follows from Theorem 3.2 and Theorem 4.2.

Theorem 5.3. *Let $\mathcal{S}$ be a b-semiring and $\mathcal{R}$ be a subset of $\mathcal{S}$. If $\mathcal{B}$ is an m-bi-ideal (where m is not necessarily one), then $\mathcal{B}\diamond_2\mathcal{R}$ is also an m-bi-ideal.*

Proof. The proof follows from Theorem 3.3 and Theorem 4.3.

Theorem 5.4. *Every m-quasi ideal is also an m-bi-ideal.*

Proof. The proof follows from Theorem 3.5 and Theorem 4.4.

Theorem 5.5. *The $\diamond_2$ product of any m-quasi ideal and n-quasi ideal of $\mathcal{S}$, with identity e, is the* $\max\{m, n\}$-*bi-ideal of $\mathcal{S}$.*

Proof. The proof follows from the Theorem 3.6 and Theorem 4.5.

Theorem 5.6. *For a q-left $\mathcal{G}$ and r-right $\mathcal{H}$ of $\mathcal{S}$, their intersection $\mathcal{G} \cap \mathcal{H}$ is a k-bi-ideal, where* $k = \max(m, n)$.

Proof. The proof follows from the Theorem 3.9 and Theorem 4.8.

Theorem 5.7. *For $a \in \mathcal{S}$, then m-bi-ideal generated by a is* $< a >_{mb} = \{na\} \cup \{n' a^2\} \cup a \diamond_2 \mathcal{S}^m \diamond_2 a$.

Proof. The Proof follows from the Theorem 3.10 and Theorem 4.9.

6. Conclusion

In our study, we introduced the concepts of m-quasi-ideals and m-bi-ideals in b-semirings as generalizations of traditional bi-ideals, examining their fundamental properties and the structures formed by m-ideals when subsets of b-semirings are considered. We explored the relationships between m-quasi-ideals and m-bi-ideals, setting the stage for further research on hyper b-semirings. Looking forward, we plan to use m-bi-ideals to characterize various types of semirings, including regular, irregular, and weakly regular semirings, and to investigate additional classes of m-bi-ideals, such as prime, maximal, minimal, and main m-bi-ideals. These advancements have potential applications in developing algebraic theory, cryptography, automata theory, mathematical modeling, and hyperstructure theory, providing new tools and frameworks for understanding complex algebraic systems and their applications in computational and theoretical contexts.

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References

[1] H. S. Vandiver, Note on a Simple Type of Algebra in Which the Cancellation Law of Addition Does Not Hold, Bull. Amer. Math. Soc. 40 (1934), 914–920.

[2] R. Chinram, A Note on (m, n)-Quasi-Ideals in Semirings, Int. J. Pure Appl. Math. 49 (2008), 45–52.

[3] G. Mohanraj, M. Palanikumar, Characterization of Regular b-Semirings, Math. Sci. Int. Res. J. 7 (2018), 117–123.

[4] M. Munir, M. Habib, Characterizing Semirings Using Their Quasi and Bi-Ideals, Proc. Pak. Acad. Sci.: A. Phys. Comp. Sci. 53 (2016), 469–475.

[5] M. Munir, N. Kausar, Salahuddin, Tehreem, On the Prime Fuzzy m-Bi Ideals in Semigroups, J. Math. Comp. Sci. 21 (2020), 357–365. https://doi.org/10.22436/jmcs.021.04.08.

[6] O. Steinfeld, Quasi-Ideals in Rings and Semigroups, Akadémiai Kiadó, Budapest, 1978.

[7] O. Steinfeld, Uber die Quasiideale von Halbgruppen, Publ. Math. (Debr.) 4 (1956), 262–275.

[8] K. Iséki, Quasiideals in Semirings Without Zero, Proc. Japan Acad. 34 (1958), 79–81.

[9] J.M. Howie, Introduction to Semigroup Theory, Academic Press, 1976.

[10] S. Lajos, F.A. Szász, On the Bi-Ideals in Associative Rings, Proc. Japan Acad. Ser. A Math. Sci. 46 (1970), 505–507. https://doi.org/10.3792/pja/1195520265.

[11] S. Kar, B. Maity, Some Ideals of Ternary Semigroups, Ann. Alexandru Ioan Cuza Univ.–Math. 57 (2011), 247–258. https://doi.org/10.2478/v10157-011-0024-1.

[12] M. Palanikumar, K. Arulmozhi, C. Jana, M. Pal, K.P. Shum, New Approach towards Different Bi-Base of Ordered b-Semiring, Asian-European J. Math. 16 (2022), 2350019. https://doi.org/10.1142/s1793557123500195.

[13] M. Palanikumar, K. Arulmozhi, On Various Almost Ideals of Semirings, Ann. Comm. Math. 4 (2021), 17–25.

[14] S. Nawaz, M. Gulistan, N. Kausar, S. Salahuddin, M. Munir, On the Left and Right Almost Hyperideals of LA-Semihypergroups, Int. J. Fuzzy Logic Intell. Syst. 21 (2021), 86–92. https://doi.org/10.5391/ijfis.2021.21.1.86.

[15] J.S. Golan, The Theory of Semirings With Applications in Mathematics and Theoretical Computer Science, Longman, 1992.