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New Soliton Solutions to the Space-Time Fractional Boussinesq Equation Using a Reliable Method

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Abstract. Fractional partial Differential Equations (FPDEs) play an essential role in interpreting a broad variety of events in various scientific disciplines, including physics, engineering, biology, and economics. One remarkable feature of PDEs is the presence and investigation of traveling wave solutions, which are solutions that propagate with a constant speed and preserve their shape. The main purpose of this work is to obtain the traveling wave solutions of the space-time fractional Boussinesq equation using a relatively new mechanism which is the improved modified extended tanh-function approach. Fractional derivatives based on Jumarie's modified Riemann–Liouville are utilized to deal with the fractional derivatives which appear in the fractional Boussinesq problem. Some periodic and solitary wave solutions are presented in the form of trigonometric, hyperbolic, complex, and rational functions. In addition, the effectiveness of the employed methodology is compared with the performances of other techniques such as the fractional sub-equation approach, $(\frac{G'}{G^2})$-expansion process, and the modified Kudryashov method.

1. Introduction

The exploration of traveling wave solutions in fractional partial differential equations is a captivating area of research with profound and broad repercussions in multiple disciplines in science. In numerous disciplines of science and engineering, FPDEs have become extremely useful mathematical tools for simulating and developing findings from real-world occurrences. FPDEs utilize fractional-order derivatives to measure non-local and memory-dependent impacts, in contrast to normal partial differential equations, which represent the dynamics of systems with integer-order derivatives. Due to this special characteristic of FPDEs, FPDEs are especially appropriate for explaining complicated and sophisticated processes characterized by long-range interactions,

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anomalous diffusion, and fractal behavior. FPDEs have been found beneficial for a variety of practical applications, involving physics, engineering, chemistry, biology, finance, etc. FPDEs, for illustration, have been utilized throughout physics to successfully simulate the dynamics of fluids flow, the behavior of viscoelastic materials, and the propagation of electromagnetic waves and water waves. Furthermore, FPDEs have been nicely employed in biology to investigate disease transmission, population growth, and the movement of nutrients in cellular tissues.

Due to their fundamental characteristics and adaptability, traveling wave solutions have been successfully used in modeling extensive applications in a variety of real-life scenarios. For instance, traveling waves are utilized in telecommunications to send signals over far distances with a least amount of distortion and loss. Current telecommunications networks depend on optical fibers because they enable data to be transmitted rapidly through the use of traveling waves. Further, traveling wave characteristics have been effectively utilized in a variety of medical imaging technologies, including magnetic resonance imaging (MRI) and ultrasonic waves. In oceans and seas, traveling waves play a significant role in the movement of ships, migration of fishes, etc. The numerous ways that traveling wave solutions are being used in electrical power systems, medical imaging, and telecommunications indicate how important they are to the development of technical advancements and the enhancement of many different aspects of humanity.

This paragraph especially provides some essential principles for fractional calculus, involving fractional derivatives, their history and definitions. The idea of fractional calculus came from the concept of 1/2-derivative of a given function. In particular, fractional derivatives were emerged in 1695 when L'Hopital wrote to Leibniz to ask about the 1/2-derivative [1]. Subsequently, scientists started investigating certain forms of derivatives, such as fractional, complex, and irrational derivatives. Indeed, some valuable concepts for fractional derivatives were nicely established over the previous decades. However, when a certain function's derivative is calculated using these definitions, they may produce different results. Lapalce, for illustration, employed integrals to develop an advantageous definition of a fractional derivative of functions in 1812 [1]. Liouville discovered the first Liouville definition in 1832 [2]. It relies on an algorithm for differentiating an exponential function [2]. In his significant contribution, which is available in [3], Riemann proposed his practical definition for fractional derivatives. Furthermore, a useful definition of a fractional derivative for a given function was effectively developed in the late 19th century by Riemann-Liouville. Finally, Khalil et al. [4] introduced a new definition for fractional derivatives called conformable fractional derivative.

In hydrodynamics, the Boussinesq equations are employed to characterize how waves move in nonlinear surfaces [5,6]. The Boussinesq equation describes how long waves with relatively small amplitudes move on the surface of nonlinear strings and non-dimensional nonlinear lattices [7]. In the process of investigating long waves in shallow water, this equation was successfully developed. A variety of models addressing unconfined flow of groundwater and subsurface drainage difficulties depend on the Boussinesq equations. Due to their ability to accurately represent particle transport in complicated processes and diverse mediums, fractional differential equations have garnered an extensive amount of attention recently. Water transmission in diverse porous surfaces can be investigated by applying the fractional Boussinesq equation [8]. El-Wakil and Abulwafa [8] developed the space-time fractional Boussinesq equation using the fractional variational concepts and the semi-inverse approach. Then, the authors utilized the fractional sub-equation technique to present the solution of this problem in the form of hyper-geometric, triangle and rational functions. Yaslan and Girgin [7] used the simplified $tan(\frac{\Phi(\xi)}{2})$-expansion approach to find some traveling wave solutions for the proposed model. These solutions were presented in terms of rational, trigonometric and exponential functions. Chen et al. [9] utilized the $(\frac{G'}{G^2})$-expansion approach to extract three types of traveling wave solutions for the conformable space–time fractional Boussinesq equation. Some exact traveling wave solutions of the Boussinesq equation were constructed via the repeated homogeneous balance strategy in [10]. Furthermore, Hosseini and Ansari [11] presented some solutions for the nonlinear Boussinesq equations using the well-established modified Kudryashov method. Akbar et al. [12] utilized sine-Gordon expansion strategy to analyse some traveling wave solutions for the considered model. More information about other models can be found in refs. [13–20].

The motivation of this paper comes from the lack of studies about the traveling wave solutions of the space-time fractional Boussinesq equation. This equation has not been studied adequately in terms of its significance and solutions. A special motivation for investigating this kind of problems is that the abundance of traveling wave solutions provided by the used technique. The principal objective of the present research is to employ the improved modified extended tanh-function methodology to derive traveling wave solutions for the space-time fractional Boussinesq equation which is given by the following form:

$$D_t^{2\alpha}\Psi(x,t) + bD_x^{2\alpha}\Psi(x,t) + r_1 D_x^{2\alpha}\Psi^2(x,t) + r_2 D_x^{4\alpha}\Psi(x,t) = 0, \quad 0 < \alpha \leq 1, \quad t > 0, \tag{1.1}$$

where $\Psi(x,t)$ denotes the vertical deflection, D_t^{α} represents the fractional derivative, and b, r_1 and r_2 are constants. The Jumarie's modified Riemann–Liouville definition [16] is used to deal with the fractional derivatives which appear in Eq. (1.1). We attempt to extract various types of traveling wave solutions in terms of rational, trigonometric and exponential functions. In addition, we plot some selected solutions under some specific values of parameters. Yang and Hon [21] shew that under certain restrictions, the improved modified extended tanh-function approach generates twenty-two distinct possible solutions including kink-shaped and bell-like solitary wave solutions, rational solutions, triangular type solutions, periodic solutions, hyperbolic type solutions, and exponential type solutions. The performance of the used technique is successfully compared with other techniques' performances. The improved modified extended tanh-function methodology demonstrates that it is effective and productive in the extraction of novel soliton solutions.

The structure of this article is as follows. Section 2 summarizes the used approach. In Section 3, we present the obtained traveling wave solutions for the proposed problem. We also present some 2D and 3D figures for some selected solutions in this part. Moreover, Section 4 discusses the results and compares them with other results obtained in other articles. Finally, we conclude the main results of this article in Section 5.

2. Improved modified extended tanh-function technique

This section is devoted to describe the used method as shown in [21]. Firstly, we begin with a nonlinear fractional order partial differential equation given as follows:

$$\Xi(\Psi, D_t^{\alpha}\Psi, D_x^{\alpha}\Psi, D_t^{\alpha}D_x^{\alpha}\Psi, D_{tt}^{2\alpha}\Psi, D_{xx}^{2\alpha}\Psi, ...) = 0, \tag{2.1}$$

where Ξ is a polynomial in $\Psi(x,t)$ and α is a fractional order. We then use the fractional complex transformation

$$\Psi = \Psi(x,t) = v(\eta), \quad \eta = \frac{lx^{\alpha}}{\Gamma(1+\alpha)} + \frac{kt^{\alpha}}{\Gamma(1+\alpha)}, \tag{2.2}$$

to convert Eq. (2.1) into the following equation:

$$\Xi(v, v', v'', v''', ...) = 0, \tag{2.3}$$

where $v' = \frac{dv}{d\eta}$. The used method offers the travelling wave solution of Eq. (2.3) in the following form

$$v(\eta) = \sum_{j=0}^{J} a_j H(\eta)^j + \sum_{j=1}^{J} b_j H(\eta)^j, \tag{2.4}$$

where

$$H'(\eta) = \sqrt{\sum_{j=0}^{4} c_j H^j(\eta)}, \tag{2.5}$$

where the constant $c_j \ \forall\ j$ are given under some conditions. The value of J is obtained using the balance technique between the nonlinear term and the highest derivative. Substituting Eq. (2.5) along with Eq. (2.4) into Eq. (2.3) gives a polynomial. Taking the power-like terms and equating them to zero gives a system of equations. Solving these equations using Maple or Mathematica software gives the solutions of a_j and b_j which can be used in Eq. (2.4) to find the traveling wave solutions. The forms of the solutions of the function H are given in Appendix 5.

3. Traveling wave solutions

In this part, we apply the considered technique to find some traveling wave solutions for Eq. (1.1). We first insert Eq. (2.2) into Eq. (1.1) with the use of the Jumarie's modified Riemann–Liouville definition [16] to have the following equation

$$k^2 v''(\eta) + bl^2 v''(\eta) + r_1 l^2 (v^2)''(\eta) = 0, \tag{3.1}$$

which can be integrated twice and simplify to yield

$$(k^2 + bl^2)v(\eta) + r_1 l^2 v^2(\eta) + r_2 l^4 v''(\eta) = 0, \tag{3.2}$$

where the integration constants are taken by zero. Balancing the nonlinear term with the highest derivative gives $J = 2$. Hence, Eq. (2.4) becomes

$$v(\eta) = a_0 + a_1 H(\eta) + a_2 H^2(\eta) + \frac{b_1}{H(\eta)} + \frac{b_2}{H^2(\eta)}. \tag{3.3}$$

Inserting Eq. (3.3) into Eq. (3.2) and equating the power-like terms give a system of equations whose solutions are given in some cases given as follows.
First case: when $c_0 = c_1 = c_3 = 0$. Then, we have

$$a_0 = \frac{-b + \sqrt{b^2 + 16c_2k^2r_2}}{2r_1}, \quad a_1 = b_1 = b_2 = 0, \quad a_2 = \frac{3c_4(-b + \sqrt{b^2 + 16c_2k^2r_2})}{4c_2r_1},$$
$$l = \mp\sqrt{\frac{b - \sqrt{b^2 + 16c_2k^2r_2}}{8(c_2r_2)}}.$$

Hence, the corresponding traveling wave solutions can be written in the following forms:

$$v_{1,2}(\eta) = \frac{-b + \sqrt{b^2 + 16c_2k^2r_2}}{2r_1} - \frac{3(-b + \sqrt{b^2 + 16c_2k^2r_2})}{4r_1} sech^2(\sqrt{c_2}\eta), \quad c_2 > 0, \; c_4 < 0,$$

$$v_{3,4}(\eta) = \frac{-b + \sqrt{b^2 + 16c_2k^2r_2}}{2r_1} - \frac{3(-b + \sqrt{b^2 + 16c_2k^2r_2})}{4r_1} sec^2(\sqrt{-c_2}\eta), \quad c_2 < 0, \; c_4 > 0,$$

where

$$\eta = \mp\frac{\sqrt{\frac{b - \sqrt{b^2+16c_2k^2r_2}}{c_2r_2}}}{2\sqrt{2}} \frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{kt^\alpha}{\Gamma(1+\alpha)}.$$

Second case: when $c_1 = c_3 = 0$, we have several families of solutions given as follows.
The first family :

$$a_0 = \frac{1}{4\sqrt{c_2^2 + 12c_0c_4}\, r_1}\left(b(c_2 - \sqrt{c_2^2 + 12c_0c_4}) - (c_2^2 + 12c_0c_4 - c_2\sqrt{c_2^2 + 12c_0c_4})r_2\sqrt{\frac{b^2 - 16\sqrt{c_2^2 + 12c_0c_4}\,k^2r_2}{(c_2^2 + 12c_0c_4)r_2^2}}\right),$$

$$a_1 = b_1 = 0, \quad a_2 = \frac{3c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}} + r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}\,k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4r_1},$$

$$b_2 = \frac{3c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}} + r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}\,k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4r_1}, \quad l = \mp\frac{\sqrt{-\frac{\frac{b}{\sqrt{c_2^2+12c_0c_4}} + r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}\,k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}}{r_2}}}{2\sqrt{2}}.$$

Hence, the corresponding traveling wave solutions can be written in the following forms:

$$v_{5,6}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(b(c_2-\sqrt{c_2^2+12c_0c_4})-(c_2^2+12c_0c_4-c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$

$$-\frac{3c_2c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4r_1}\left(tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)\right)-\frac{3c_4c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2r_1tanh^2\sqrt{\frac{-c_2}{2}}\eta}$$

$$c_2<0,\ c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{7,8}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(b(c_2-\sqrt{c_2^2+12c_0c_4})-(c_2^2+12c_0c_4-c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$

$$+\frac{3c_2c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4r_1}\left(tan^2\left(\sqrt{\frac{c_2}{2}}\eta\right)\right)-\frac{3c_4c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2r_1tan^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)},$$

$$c_2>0,\ c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{9,10}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(b(c_2-\sqrt{c_2^2+12c_0c_4})-(c_2^2+12c_0c_4-c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$

$$-\frac{3c_2p^2c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4(2p^2-1)r_1}cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta-\frac{3c_4(2p^2-1)c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2p^2r_1cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta}$$

$$c_2>0,\ c_4<0, c_0=\frac{c_2^2p^2(1-p^2)}{c_4(2p^2-1)^2},$$

$$v_{11,12}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(b(c_2-\sqrt{c_2^2+12c_0c_4})-(c_2^2+12c_0c_4-c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$

$$-\frac{3p^2c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4(2-p^2)r_1}dn^2\sqrt{\frac{c_2}{2-p^2}}\eta-\frac{3c_4(2-p^2)c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4p^2r_1dn^2\sqrt{\frac{c_2}{2-p^2}}\eta}$$

$$c_2>0,\ c_4<0, c_0=\frac{c_2^2(1-p^2)}{c_4(2-p^2)^2},$$

$$v_{13,14}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(b(c_2-\sqrt{c_2^2+12c_0c_4})-(c_2^2+12c_0c_4-c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$

$$-\frac{3c_2p^2c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4(p^2+1)r_1}sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta-\frac{3c_4(p^2+1)c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2p^2r_1sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta}$$

$$c_2<0,\ c_4>0, c_0=\frac{c_2^2p^2}{c_4(p^2+1)^2}.$$

Here, p denotes a modulus.

$$\eta=\mp\sqrt{-\frac{\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}}{8r_2}}\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{kt^\alpha}{\Gamma(1+\alpha)}.$$

The second family is given by

$$a_0=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(b(c_2-\sqrt{c_2^2+12c_0c_4})(c_2^2+12c_0c_4-c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right),$$

$$a_1=b_1=0,\quad a_2=\frac{3c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4r_1},\quad b_2=\frac{3c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4r_1},$$

$$l=\mp\sqrt{-\frac{b}{8\sqrt{c_2^2+12c_0c_4}r_2}+\frac{1}{8}\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}}.$$

The traveling wave solutions, hence, are given by

$$v_{15,16}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(b(c_2-\sqrt{c_2^2+12c_0c_4})(c_2^2+12c_0c_4-c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$
$$-\frac{3c_2c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4r_1}\left(tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)\right)-\frac{3c_4c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2r_1tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)},$$
$$c_2<0,\ c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{17,18}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(b(c_2-\sqrt{c_2^2+12c_0c_4})(c_2^2+12c_0c_4-c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$
$$+\frac{3c_2c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4r_1}\left(tan^2\left(\sqrt{\frac{c_2}{2}}\eta\right)\right)+\frac{3c_4c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2r_1tan^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)},$$
$$c_2>0,\ c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{19,20}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(b(c_2-\sqrt{c_2^2+12c_0c_4})(c_2^2+12c_0c_4-c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$
$$-\frac{3c_2p^2c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4(2p^2-1)r_1}cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta$$
$$-\frac{3c_4(2p^2-1)c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2p^2r_1cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta},\ \ c_2>0,\ c_4<0, c_0=\frac{c_2^2p^2(1-p^2)}{c_4(2p^2-1)^2},$$

$$v_{21,22}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(b(c_2-\sqrt{c_2^2+12c_0c_4})(c_2^2+12c_0c_4-c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$
$$-\frac{3p^2c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4(2-p^2)r_1}dn^2\sqrt{\frac{c_2}{2-p^2}}\eta-\frac{3c_4(2-p^2)c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4p^2r_1dn^2\sqrt{\frac{c_2}{2-p^2}}\eta}$$
$$c_2>0,\ c_4<0, c_0=\frac{c_2^2(1-p^2)}{c_4(2-p^2)^2},$$

$$v_{23,24}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(b(c_2-\sqrt{c_2^2+12c_0c_4})(c_2^2+12c_0c_4-c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$
$$-\frac{3c_2p^2c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4(p^2+1)r_1}sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta-\frac{3c_4(p^2+1)c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2p^2r_1sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta}$$
$$c_2<0,\ c_4>0, c_0=\frac{c_2^2p^2}{c_4(p^2+1)^2},$$

where p denotes a modulus.

$$\eta=\mp\sqrt{-\frac{b}{8\sqrt{c_2^2+12c_0c_4}r_2}+\frac{1}{8}\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}}\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{kt^\alpha}{\Gamma(1+\alpha)}.$$

The third family:

$$a_0=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(-b(c_2-\sqrt{c_2^2+12c_0c_4})+(c_2^2+12c_0c_4+c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right),$$
$$a_1=b_1=0,\ \ a_2=\frac{3c_4\left(-\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4r_1},\ \ b_2=\frac{3c_0\left(-\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4r_1},$$

$$l = \mp \sqrt{\frac{b}{8\sqrt{c_2^2 + 12c_0c_4}r_2} - \frac{1}{8}\sqrt{\frac{b^2 + 16\sqrt{c_2^2 + 12c_0c_4}k^2r_2}{(c_2^2 + 12c_0c_4)r_2^2}}}.$$

As a result, the traveling wave solutions of this family are shown as follows:

$$v_{25,26}(\eta) = \frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(-b(c_2 - \sqrt{c_2^2 + 12c_0c_4}) + (c_2^2 + 12c_0c_4 + c_2\sqrt{c_2^2 + 12c_0c_4})r_2\sqrt{\frac{b^2 + 16\sqrt{c_2^2 + 12c_0c_4}k^2r_2}{(c_2^2 + 12c_0c_4)r_2^2}}\right)$$
$$- \frac{3c_2c_4\left(-\frac{b}{\sqrt{c_2^2+12c_0c_4}} + r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4r_1}\left(tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)\right) - \frac{3c_4c_0\left(-\frac{b}{\sqrt{c_2^2+12c_0c_4}} + r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2r_1 tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)},$$
$$c_2 < 0,\ c_4 > 0, c_0 = \frac{c_2^2}{4c_4},$$

$$v_{27,28}(\eta) = \frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(-b(c_2 - \sqrt{c_2^2 + 12c_0c_4}) + (c_2^2 + 12c_0c_4 + c_2\sqrt{c_2^2 + 12c_0c_4})r_2\sqrt{\frac{b^2 + 16\sqrt{c_2^2 + 12c_0c_4}k^2r_2}{(c_2^2 + 12c_0c_4)r_2^2}}\right)$$
$$+ \frac{3c_2c_4\left(-\frac{b}{\sqrt{c_2^2+12c_0c_4}} + r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4r_1}\left(tan^2\left(\sqrt{\frac{c_2}{2}}\eta\right)\right) + \frac{3c_4c_0\left(-\frac{b}{\sqrt{c_2^2+12c_0c_4}} + r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2r_1 tan^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)},$$
$$c_2 > 0,\ c_4 > 0, c_0 = \frac{c_2^2}{4c_4},$$

$$v_{29,30}(\eta) = \frac{1}{4\sqrt{c_2^2 + 12c_0c_4}r_1}\left(-b(c_2 - \sqrt{c_2^2 + 12c_0c_4}) + (c_2^2 + 12c_0c_4 + c_2\sqrt{c_2^2 + 12c_0c_4})r_2\sqrt{\frac{b^2 + 16\sqrt{c_2^2 + 12c_0c_4}k^2r_2}{(c_2^2 + 12c_0c_4)r_2^2}}\right)$$
$$- \frac{3c_2p^2c_4\left(-\frac{b}{\sqrt{c_2^2+12c_0c_4}} + r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4(2p^2-1)r_1}cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta$$
$$- \frac{3c_4(2p^2-1)c_0\left(-\frac{b}{\sqrt{c_2^2+12c_0c_4}} + r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2p^2r_1cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta},\ c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2p^2(1-p^2)}{c_4(2p^2-1)^2},$$

$$v_{31,32}(\eta) = \frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(-b(c_2 - \sqrt{c_2^2 + 12c_0c_4}) + (c_2^2 + 12c_0c_4 + c_2\sqrt{c_2^2 + 12c_0c_4})r_2\sqrt{\frac{b^2 + 16\sqrt{c_2^2 + 12c_0c_4}k^2r_2}{(c_2^2 + 12c_0c_4)r_2^2}}\right)$$
$$- \frac{3p^2c_4\left(-\frac{b}{\sqrt{c_2^2+12c_0c_4}} + r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4(2-p^2)r_1}dn^2\sqrt{\frac{c_2}{2-p^2}}\eta - \frac{3c_4(2-p^2)c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}} - r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4p^2r_1dn^2\sqrt{\frac{c_2}{2-p^2}}\eta}$$
$$c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2(1-p^2)}{c_4(2-p^2)^2},$$

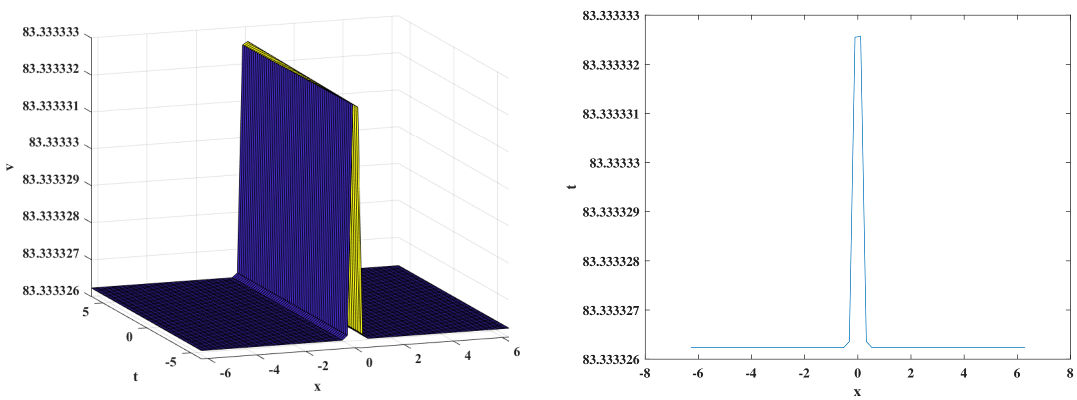
$$v_{33,34}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(-b(c_2-\sqrt{c_2^2+12c_0c_4})+(c_2^2+12c_0c_4+c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$

$$-\frac{3c_2p^2c_4\left(-\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_4(p^2+1)r_1}sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta-\frac{3c_4(p^2+1)c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}-r_2\sqrt{\frac{b^2-16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2p^2r_1sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta}$$

$$c_2<0,\ c_4>0, c_0=\frac{c_2^2p^2}{c_4(p^2+1)^2},$$

where p represents a modulus.

$$\eta=\mp\sqrt{\frac{b}{8\sqrt{c_2^2+12c_0c_4}r_2}-\frac{1}{8}\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}}\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{kt^\alpha}{\Gamma(1+\alpha)}.$$



(A) A traveling wave solution for v_{25}.

(B) A 2D figure for v_{25} when $t=1$.

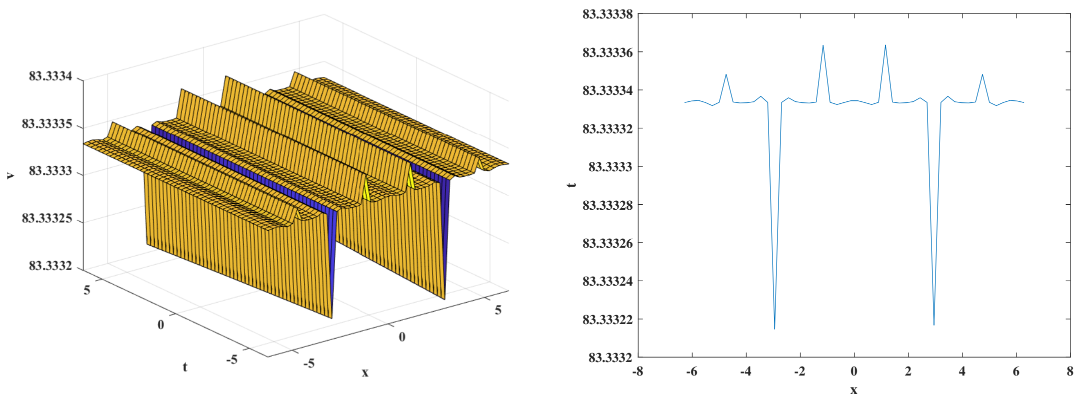
FIGURE 1. 3D and 2D solutions for v_{25} under $r_1=0.3$, $r_2=30$, $c_2=-0.7, c_4=1$, $b=50$, $k=0.001$ and $2\pi\leq t,x\leq 2\pi$.

The fourth family :

$$a_0=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(-b(c_2+\sqrt{c_2^2+12c_0c_4})-(c_2^2+12c_0c_4+c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right),$$

$$a_1=b_1=0,\ \ a_2=-\frac{3c_4\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4r_1},\ \ b_2=-\frac{3c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4r_1},$$

$$l=\mp\sqrt{\frac{b}{8\sqrt{c_2^2+12c_0c_4}r_2}+\frac{1}{8}\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}}.$$



(A) A periodic traveling wave solution for v_{27}. (B) A 2D figure for v_{27} when $t = 1$.

FIGURE 2. 3D and 2D periodic traveling wave solutions for v_{27} under $r_1 = 0.3$, $r_2 = 30$, $c_2 = 0.7, c_4 = 1$, $b = 50$, $k = 0.0001$ and $2\pi \le t, x \le 2\pi$.

Hence, the traveling wave solutions can be written in the following forms:

$$v_{35,36}(\eta) = \frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(-b(c_2+\sqrt{c_2^2+12c_0c_4})-(c_2^2+12c_0c_4+c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$
$$+\frac{3c_2\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4r_1}\left(tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)\right)+\frac{3c_4c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2r_1tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)},$$
$$c_2<0,\ c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{37,38}(\eta) = \frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(-b(c_2+\sqrt{c_2^2+12c_0c_4})-(c_2^2+12c_0c_4+c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$
$$-\frac{3c_2\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4r_1}\left(tan^2\left(\sqrt{\frac{c_2}{2}}\eta\right)\right)+\frac{3c_4c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2r_1tan^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)},$$
$$c_2>0,\ c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{39,40}(\eta) = \frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(-b(c_2+\sqrt{c_2^2+12c_0c_4})-(c_2^2+12c_0c_4+c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$
$$+\frac{3c_2p^2\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4(2p^2-1)r_1}cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta+\frac{3c_4(2p^2-1)c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2p^2r_1cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta}$$
$$c_2>0,\ c_4<0, c_0=\frac{c_2^2p^2(1-p^2)}{c_4(2p^2-1)^2},$$

$$v_{41,42}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(-b(c_2+\sqrt{c_2^2+12c_0c_4})-(c_2^2+12c_0c_4+c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$
$$+\frac{3p^2\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4(2-p^2r_1}dn^2\sqrt{\frac{c_2}{2-p^2}}\eta+\frac{3c_4(2-p^2)c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4p^2r_1dn^2\sqrt{\frac{c_2}{2-p^2}}\eta}$$
$$c_2>0,\ c_4<0, c_0=\frac{c_2^2(1-p^2)}{c_4(2-p^2)^2},$$

$$v_{43,44}(\eta)=\frac{1}{4\sqrt{c_2^2+12c_0c_4}r_1}\left(-b(c_2+\sqrt{c_2^2+12c_0c_4})-(c_2^2+12c_0c_4+c_2\sqrt{c_2^2+12c_0c_4})r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)$$
$$+\frac{3c_2p^2\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4(p^2+1)r_1}sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta+\frac{3c_4(p^2+1)c_0\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}}+r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4c_2p^2r_1sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta}$$
$$c_2<0,\ c_4>0, c_0=\frac{c_2^2p^2}{c_4(p^2+1)^2}.$$

Here, p is a modulus.

$$\eta=\mp\sqrt{\frac{b}{8\sqrt{c_2^2+12c_0c_4}r_2}+\frac{1}{8}\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}}\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{kt^\alpha}{\Gamma(1+\alpha)}.$$

The fifth family is

$$a_0=\frac{bc_2-b\sqrt{c_2^2-3c_0c_4}+r_2\left(-c_2^2+3c_0c_4+\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}},$$

$$a_1=b_1=b_2=0,\quad a_2=\frac{3bc_4+3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}},$$

$$l=\mp\sqrt{\frac{-b-\sqrt{(c_2^2-3c_0c_4)r_2^2}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2-3c_0c_4)r_2^2}}}.$$

Consequently, the traveling wave solutions are shown as follows:

$$v_{45,46}(\eta)=\frac{bc_2-b\sqrt{c_2^2-3c_0c_4}+r_2\left(-c_2^2+3c_0c_4+\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$-\frac{3bc_4+3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}\left(\frac{c_2}{c_4}tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)\right),$$

$$c_2 < 0,\ c_4 > 0, c_0 = \frac{c_2^2}{4c_4},$$

$$v_{47,48}(\eta) = \frac{bc_2 - b\sqrt{c_2^2-3c_0c_4} + r_2\left(-c_2^2+3c_0c_4+\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$+\frac{3bc_4+3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}\left(\frac{c_2}{c_4}tan^2\left(\sqrt{\frac{c_2}{2}}\eta\right)\right),$$
$$c_2 > 0,\ c_4 > 0, c_0 = \frac{c_2^2}{4c_4},$$

$$v_{49,50}(\eta) = \frac{bc_2 - b\sqrt{c_2^2-3c_0c_4} + r_2\left(-c_2^2+3c_0c_4+\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$-\frac{3bc_4+3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}\frac{c_2p^2}{c_4(2p^2-1)}cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta,$$
$$c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2p^2(1-p^2)}{c_4(2p^2-1)^2},$$

$$v_{51,52}(\eta) = \frac{bc_2 - b\sqrt{c_2^2-3c_0c_4} + r_2\left(-c_2^2+3c_0c_4+\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$-\frac{3bc_4+3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}\frac{p^2}{c_4(2-p^2)}dn^2\sqrt{\frac{c_2}{2-p^2}}\eta,$$
$$c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2(1-p^2)}{c_4(2-p^2)^2},$$

$$v_{53,54}(\eta) = \frac{bc_2 - b\sqrt{c_2^2-3c_0c_4} + r_2\left(-c_2^2+3c_0c_4+\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$-\frac{3bc_4+3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}\frac{c_2p^2}{c_4(p^2+1)}sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta,$$
$$c_2 < 0,\ c_4 > 0, c_0 = \frac{c_2^2p^2}{c_4(p^2+1)^2},$$

where p denotes a modulus.

$$\eta = \mp\sqrt{\frac{-b-\sqrt{(c_2^2-3c_0c_4)r_2^2}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2-3c_0c_4)r_2^2}}}\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{kt^\alpha}{\Gamma(1+\alpha)}.$$

Next, the sixth family is given by

$$a_0 = \frac{bc_2 - b\sqrt{c_2^2-3c_0c_4}+r_2\left(-c_2^2+3c_0c_4+\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}},$$

$$a_1 = b_1 = a_2 = 0,\quad b_2 = \frac{3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}},$$

$$l = \mp\sqrt{\frac{-b-\sqrt{(c_2^2-3c_0c_4)r_2^2}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2-3c_0c_4)r_2^2}}}.$$

As a result, the traveling wave solutions of this family are obtained as follows:

$$v_{55,56}(\eta) = \frac{bc_2 - b\sqrt{c_2^2-3c_0c_4}+r_2\left(-c_2^2+3c_0c_4+\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$-\frac{c_4\left(3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2r_1tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)\sqrt{c_2^2-3c_0c_4}},$$
$$c_2<0,\ c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{57,58}(\eta) = \frac{bc_2 - b\sqrt{c_2^2-3c_0c_4}+r_2\left(-c_2^2+3c_0c_4+\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$-\frac{c_4\left(3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2r_1tan^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)\sqrt{c_2^2-3c_0c_4}},$$
$$c_2>0,\ c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{59,60}(\eta) = \frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(-c_2^2 + 3c_0c_4 + \sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}}$$

$$- \frac{c_4(2p^2-1)\left(3bc_0 + 3c_0r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2p^2r_1cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta\sqrt{c_2^2 - 3c_0c_4}}$$

$$c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2p^2(1-p^2)}{c_4(2p^2-1)^2},$$

$$v_{61,62}(\eta) = \frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(-c_2^2 + 3c_0c_4 + \sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}}$$

$$- \frac{c_4(2-p^2)\left(3bc_0 + 3c_0r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4p^2r_1dn^2\sqrt{\frac{c_2}{2-p^2}}\eta\sqrt{c_2^2 - 3c_0c_4}}$$

$$c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2(1-p^2)}{c_4(2-p^2)^2},$$

$$v_{63,64}(\eta) = \frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(-c_2^2 + 3c_0c_4 + \sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}}$$

$$- \frac{c_4(p^2+1)\left(3bc_0 + 3c_0r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2p^2r_1\sqrt{c_2^2 - 3c_0c_4}sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta}$$

$$c_2 < 0,\ c_4 > 0, c_0 = \frac{c_2^2p^2}{c_4(p^2+1)^2},$$

where p denotes a modulus.

$$\eta = \mp\sqrt{\frac{-b - \sqrt{(c_2^2 - 3c_0c_4)r_2^2}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}}\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{kt^\alpha}{\Gamma(1+\alpha)}.$$

The seventh family is given by

$$a_0 = \frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(c_2^2 + 3c_0c_4 - c_2\sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}},$$

$$a_1 = b_1 = b_2 = 0, \quad a_2 = -\frac{-3bc_4 + 3c_4 r_2 \sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}},$$

$$l = \mp\sqrt{\frac{-b + \sqrt{(c_2^2 - 3c_0c_4)r_2^2}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}}.$$

Hence, from the seventh family the traveling wave solutions can be found in the following forms:

$$\begin{aligned} v_{65,66}(\eta) = &\frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(c_2^2 + 3c_0c_4 - c_2\sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}} \\ &+ \frac{c_2\left(-3bc_4 + 3c_4 r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4 r_1\sqrt{c_2^2 - 3c_0c_4}}\left(tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)\right), \\ &c_2 < 0,\ c_4 > 0, c_0 = \frac{c_2^2}{4c_4}, \end{aligned}$$

$$\begin{aligned} v_{67,68}(\eta) = &\frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(c_2^2 + 3c_0c_4 - c_2\sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}} \\ &- \frac{c_2\left(-3bc_4 + 3c_4 r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4 r_1\sqrt{c_2^2 - 3c_0c_4}}\left(tan^2\left(\sqrt{\frac{c_2}{2}}\eta\right)\right), \\ &c_2 > 0,\ c_4 > 0, c_0 = \frac{c_2^2}{4c_4}, \end{aligned}$$

$$\begin{aligned} v_{69,70}(\eta) = &\frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(c_2^2 + 3c_0c_4 - c_2\sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}} \\ &+ \frac{c_2 p^2\left(-3bc_4 + 3c_4 r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4(2p^2-1)r_1\sqrt{c_2^2 - 3c_0c_4}} cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta \\ &c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2 p^2(1-p^2)}{c_4(2p^2-1)^2}, \end{aligned}$$

$$v_{71,72}(\eta) = \frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(c_2^2 + 3c_0c_4 - c_2\sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}}$$
$$+ \frac{p^2\left(-3bc_4 + 3c_4r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4(2-p^2)r_1\sqrt{c_2^2 - 3c_0c_4}} dn^2\sqrt{\frac{c_2}{2-p^2}}\eta$$
$$c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2(1-p^2)}{c_4(2-p^2)^2},$$

$$v_{73,74}(\eta) = \frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(c_2^2 + 3c_0c_4 - c_2\sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}}$$
$$+ \frac{c_2p^2\left(-3bc_4 + 3c_4r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4(p^2+1)r_1\sqrt{c_2^2 - 3c_0c_4}} sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta$$
$$c_2 < 0,\ c_4 > 0, c_0 = \frac{c_2^2p^2}{c_4(p^2+1)^2},$$

where p denotes a modulus.

$$\eta = \mp\sqrt{\frac{-b + \sqrt{(c_2^2 - 3c_0c_4)r_2^2}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}}\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{kt^\alpha}{\Gamma(1+\alpha)}.$$

The eighth family is shown as follows:

$$a_0 = \frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(c_2^2 - 3c_0c_4 - c_2\sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}},$$

$$a_1 = b_1 = a_2 = 0, \quad b_2 = -\frac{-3bc_0 + 3c_0r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}},$$

$$l = \mp\sqrt{\frac{-b + \sqrt{(c_2^2 - 3c_0c_4)r_2^2}\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}}.$$

Therefore, the traveling wave solutions of this family can be introduced as follows:

$$v_{75,76}(\eta)=\frac{bc_2-b\sqrt{c_2^2-3c_0c_4}+r_2\left(c_2^2-3c_0c_4-c_2\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$+\frac{c_4\left(-3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2r_1\sqrt{c_2^2-3c_0c_4}\left(tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)\right)},$$
$$c_2<0,\ c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{77,78}(\eta)=\frac{bc_2-b\sqrt{c_2^2-3c_0c_4}+r_2\left(c_2^2-3c_0c_4-c_2\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$+\frac{c_4\left(-3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2r_1\sqrt{c_2^2-3c_0c_4}\left(tan^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)\right)},$$
$$c_2>0,\ c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{79,80}(\eta)=\frac{bc_2-b\sqrt{c_2^2-3c_0c_4}+r_2\left(c_2^2-3c_0c_4-c_2\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$+\frac{c_4(2p^2-1)\left(-3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2p^2r_1\sqrt{c_2^2-3c_0c_4}cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta}$$
$$c_2>0,\ c_4<0, c_0=\frac{c_2^2p^2(1-p^2)}{c_4(2p^2-1)^2},$$

$$v_{81,82}(\eta)=\frac{bc_2-b\sqrt{c_2^2-3c_0c_4}+r_2\left(c_2^2-3c_0c_4-c_2\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$+\frac{c_4(2-p^2)\left(-3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4p^2r_1\sqrt{c_2^2-3c_0c_4}dn^2\sqrt{\frac{c_2}{2-p^2}}\eta}$$

$$c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2(1-p^2)}{c_4(2-p^2)^2},$$

$$v_{83,84}(\eta) = \frac{bc_2 - b\sqrt{c_2^2-3c_0c_4} + r_2\left(c_2^2 - 3c_0c_4 - c_2\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$

$$+\frac{c_4(p^2+1)\left(-3bc_0 + 3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2p^2r_1\sqrt{c_2^2-3c_0c_4}sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta}$$

$$c_2 < 0,\ c_4 > 0, c_0 = \frac{c_2^2p^2}{c_4(p^2+1)^2},$$

where p represents a modulus.

$$\eta = \mp\sqrt{\frac{-b + \sqrt{(c_2^2-3c_0c_4)r_2^2}\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2-3c_0c_4)r_2^2}}}\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{kt^\alpha}{\Gamma(1+\alpha)}.$$

The ninth family is given by

$$a_0 = \frac{-bc_2 - b\sqrt{c_2^2-3c_0c_4} + r_2\left(c_2^2 - 3c_0c_4 + c_2\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}},$$

$$a_1 = b_1 = b_2 = 0, \quad a_2 = \frac{-3bc_4 + 3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}},$$

$$l = \mp\sqrt{\frac{b - \sqrt{(c_2^2-3c_0c_4)r_2^2}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2-3c_0c_4)r_2^2}}}.$$

Thus, the traveling wave solutions are given by

$$v_{85,86}(\eta) = \frac{-bc_2 - b\sqrt{c_2^2-3c_0c_4} + r_2\left(c_2^2 - 3c_0c_4 + c_2\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$

$$+\frac{3c_2\left(\frac{b}{\sqrt{c_2^2+12c_0c_4}} + r_2\sqrt{\frac{b^2+16\sqrt{c_2^2+12c_0c_4}k^2r_2}{(c_2^2+12c_0c_4)r_2^2}}\right)}{4r_1}\left(tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)\right),$$

$$c_2 < 0,\ c_4 > 0, c_0 = \frac{c_2^2}{4c_4},$$

$$v_{87,88}(\eta)=\frac{-bc_2-b\sqrt{c_2^2-3c_0c_4}+r_2\left(c_2^2-3c_0c_4+c_2\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$+\frac{c_2\left(-3bc_4+3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4r_1\sqrt{c_2^2-3c_0c_4}}\left(tan^2\left(\sqrt{\frac{c_2}{2}}\eta\right)\right),$$
$$c_2>0,\ c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{89,90}(\eta)=\frac{-bc_2-b\sqrt{c_2^2-3c_0c_4}+r_2\left(c_2^2-3c_0c_4+c_2\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$-\frac{c_2p^2\left(-3bc_4+3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4(2p^2-1)r_1\sqrt{c_2^2-3c_0c_4}}cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta$$
$$c_2>0,\ c_4<0, c_0=\frac{c_2^2p^2(1-p^2)}{c_4(2p^2-1)^2},$$

$$v_{91,92}(\eta)=\frac{-bc_2-b\sqrt{c_2^2-3c_0c_4}+r_2\left(c_2^2-3c_0c_4+c_2\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$-\frac{p^2\left(-3bc_4+3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4(2-p^2)r_1\sqrt{c_2^2-3c_0c_4}}dn^2\sqrt{\frac{c_2}{2-p^2}}\eta$$
$$c_2>0,\ c_4<0, c_0=\frac{c_2^2(1-p^2)}{c_4(2-p^2)^2},$$

$$v_{93,94}(\eta)=\frac{-bc_2-b\sqrt{c_2^2-3c_0c_4}+r_2\left(c_2^2-3c_0c_4+c_2\sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$-\frac{c_2p^2\left(-3bc_4+3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4(p^2+1)r_1\sqrt{c_2^2-3c_0c_4}}sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta$$
$$c_2<0,\ c_4>0, c_0=\frac{c_2^2p^2}{c_4(p^2+1)^2}.$$

Here, p is a modulus.

$$\eta = \mp\sqrt{\frac{b - \sqrt{(c_2^2-3c_0c_4)r_2^2}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2-3c_0c_4)r_2^2}}}\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{kt^\alpha}{\Gamma(1+\alpha)}.$$

Next, the tenth family is

$$a_0 = \frac{bc_2 - b\sqrt{c_2^2-3c_0c_4} + r_2\left(c_2^2-3c_0c_4 - \sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}},$$

$$a_1 = a_2 = b_1 = 0, \quad b_2 = \frac{-3bc_0 + 3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}},$$

$$l = \mp\sqrt{\frac{b - \sqrt{(c_2^2-3c_0c_4)r_2^2}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2-3c_0c_4)r_2^2}}}.$$

Therefore, the traveling wave solutions are written in the following forms:

$$v_{95,96}(\eta) = \frac{bc_2 - b\sqrt{c_2^2-3c_0c_4} + r_2\left(c_2^2-3c_0c_4 - \sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$- \frac{c_4\left(-3bc_0 + 3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2r_1\sqrt{c_2^2-3c_0c_4}tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)},$$
$$c_2 < 0,\ c_4 > 0, c_0 = \frac{c_2^2}{4c_4},$$

$$v_{97,98}(\eta) = \frac{bc_2 - b\sqrt{c_2^2-3c_0c_4} + r_2\left(c_2^2-3c_0c_4 - \sqrt{c_2^2-3c_0c_4}\right)\sqrt{\frac{b^2-16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$- \frac{c_4\left(-3bc_0 + 3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2r_1\sqrt{c_2^2-3c_0c_4}tan^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)},$$
$$c_2 > 0,\ c_4 > 0, c_0 = \frac{c_2^2}{4c_4},$$

$$v_{99,100}(\eta) = \frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(c_2^2 - 3c_0c_4 - \sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}{(c_2^2 - 3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}}$$

$$- \frac{c_4(2p^2 - 1)\left(-3bc_0 + 3c_0r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 + 16k^2\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}{(c_2^2 - 3c_0c_4)r_2^2}}\right)}{4c_2p^2r_1\sqrt{c_2^2 - 3c_0c_4}cn^2\sqrt{\frac{c_2}{2p^2 - 1}}\eta}$$

$$c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2p^2(1 - p^2)}{c_4(2p^2 - 1)^2},$$

$$v_{101,102}(\eta) = \frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(c_2^2 - 3c_0c_4 - \sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}{(c_2^2 - 3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}}$$

$$- \frac{c_4(2 - p^2)\left(-3bc_0 + 3c_0r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 + 16k^2\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}{(c_2^2 - 3c_0c_4)r_2^2}}\right)}{4p^2r_1\sqrt{c_2^2 - 3c_0c_4}dn^2\sqrt{\frac{c_2}{2 - p^2}}\eta}$$

$$c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2(1 - p^2)}{c_4(2 - p^2)^2},$$

$$v_{103,104}(\eta) = \frac{bc_2 - b\sqrt{c_2^2 - 3c_0c_4} + r_2\left(c_2^2 - 3c_0c_4 - \sqrt{c_2^2 - 3c_0c_4}\right)\sqrt{\frac{b^2 - 16k^2\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}{(c_2^2 - 3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}}$$

$$- \frac{c_4(p^2 + 1)\left(-3bc_0 + 3c_0r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2 + 16k^2\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}{(c_2^2 - 3c_0c_4)r_2^2}}\right)}{4c_2p^2r_1\sqrt{c_2^2 - 3c_0c_4}sn^2\sqrt{\frac{-c_2}{p^2 + 1}}\eta}$$

$$c_2 < 0,\ c_4 > 0, c_0 = \frac{c_2^2p^2}{c_4(p^2 + 1)^2},$$

where p denotes a modulus.

$$\eta = \mp\sqrt{\frac{b - \sqrt{(c_2^2 - 3c_0c_4)r_2^2}\sqrt{\frac{b^2 + 16k^2\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}{(c_2^2 - 3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}}\frac{x^\alpha}{\Gamma(1 + \alpha)} + \frac{kt^\alpha}{\Gamma(1 + \alpha)}.$$

Now, the eleventh family is shown as follows:

$$a_0 = \frac{-bc_2 + b\sqrt{c_2^2 - 3c_0c_4} - r_2\left(c_2^2 - 3c_0c_4 + c_2\sqrt{(c_2^2 - 3c_0c_4)}\right)\sqrt{\frac{b^2 + 16k^2\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}{(c_2^2 - 3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}},$$

$$a_1 = b_1 = b_2 = 0, \quad a_2 = -\frac{3bc_4 + 3c_4 r_2 \sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}},$$

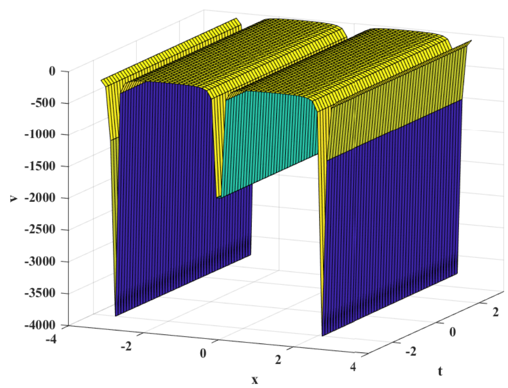
$$l = \mp\sqrt{\frac{b+\sqrt{(c_2^2-3c_0c_4)r_2^2}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2-3c_0c_4)r_2^2}}}.$$

Next, the traveling wave solutions of this family are given by

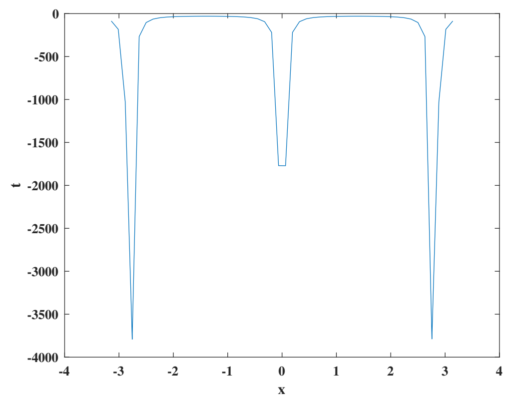
$$v_{105,106}(\eta) = \frac{-bc_2 + b\sqrt{c_2^2-3c_0c_4} - r_2\left(c_2^2 - 3c_0c_4 + c_2\sqrt{(c_2^2-3c_0c_4)}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$+\frac{c_2\left(3bc_4 + 3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4r_1\sqrt{c_2^2-3c_0c_4}}\left(tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)\right),$$
$$c_2 < 0,\ c_4 > 0, c_0 = \frac{c_2^2}{4c_4},$$

$$v_{107,108}(\eta) = \frac{-bc_2 + b\sqrt{c_2^2-3c_0c_4} - r_2\left(c_2^2 - 3c_0c_4 + c_2\sqrt{(c_2^2-3c_0c_4)}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$-\frac{c_2\left(3bc_4 + 3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4r_1\sqrt{c_2^2-3c_0c_4}}\left(tan^2\left(\sqrt{\frac{c_2}{2}}\eta\right)\right),$$
$$c_2 > 0,\ c_4 > 0, c_0 = \frac{c_2^2}{4c_4},$$

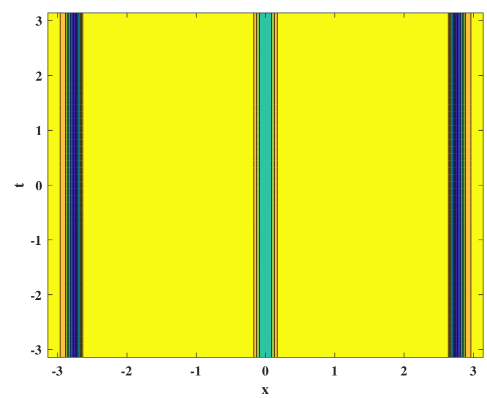
$$v_{109,110}(\eta) = \frac{-bc_2 + b\sqrt{c_2^2-3c_0c_4} - r_2\left(c_2^2 - 3c_0c_4 + c_2\sqrt{(c_2^2-3c_0c_4)}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$
$$+\frac{c_2p^2\left(3bc_4 + 3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4(2p^2-1)r_1\sqrt{c_2^2-3c_0c_4}}cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta$$
$$c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2p^2(1-p^2)}{c_4(2p^2-1)^2},$$



(A) A periodic traveling wave solution for v_{107}.



(B) A 2D figure for v_{107} when $t = 1$.



(C) A contour of the solution v_{107}.

FIGURE 3. 3D and 2D periodic traveling wave solutions for v_{107} under $r_1 = 1$, $r_2 = 0.6$, $c_2 = 200, c_4 = 0.1$, $b = 60$, $k = 0.000001$ and $\pi \leq t, x \leq \pi$.

$$v_{111,112}(\eta) = \frac{-bc_2 + b\sqrt{c_2^2 - 3c_0c_4} - r_2\left(c_2^2 - 3c_0c_4 + c_2\sqrt{(c_2^2 - 3c_0c_4)}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}}$$
$$+\frac{p^2\left(3bc_4 + 3c_4r_2\sqrt{c_2^2 - 3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4(2-p^2)r_1\sqrt{c_2^2 - 3c_0c_4}}dn^2\sqrt{\frac{c_2}{2-p^2}}\eta$$
$$c_2 > 0,\ c_4 < 0, c_0 = \frac{c_2^2(1-p^2)}{c_4(2-p^2)^2},$$

$$v_{113,114}(\eta) = \frac{-bc_2 + b\sqrt{c_2^2 - 3c_0c_4} - r_2\left(c_2^2 - 3c_0c_4 + c_2\sqrt{(c_2^2 - 3c_0c_4)}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2 - 3c_0c_4}}$$

$$+\frac{c_2p^2\left(3bc_4+3c_4r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_4(p^2+1)r_1\sqrt{c_2^2-3c_0c_4}}sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta$$

$$c_2<0,\, c_4>0, c_0=\frac{c_2^2p^2}{c_4(p^2+1)^2},$$

where p is a modulus.

$$\eta=\mp\sqrt{\frac{b+\sqrt{(c_2^2-3c_0c_4)r_2^2}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2-3c_0c_4)r_2^2}}}\frac{x^\alpha}{\Gamma(1+\alpha)}+\frac{kt^\alpha}{\Gamma(1+\alpha)}.$$

The twelfth family is written as

$$a_0=\frac{-bc_2+b\sqrt{c_2^2-3c_0c_4}-r_2\left(c_2^2-3c_0c_4+c_2\sqrt{(c_2^2-3c_0c_4)}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}},$$

$$a_1=a_2=b_1=0,\quad b_2=-\frac{3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}},$$

$$l=\mp\sqrt{\frac{b+\sqrt{(c_2^2-3c_0c_4)r_2^2}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2-3c_0c_4)r_2^2}}}.$$

Hence, the traveling wave solutions of this family are shown as follows.

$$v_{115,116}(\eta)=\frac{-bc_2+b\sqrt{c_2^2-3c_0c_4}-r_2\left(c_2^2-3c_0c_4+c_2\sqrt{(c_2^2-3c_0c_4)}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$

$$+\frac{c_4\left(3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2r_1\sqrt{c_2^2-3c_0c_4}tanh^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)},$$

$$c_2<0,\, c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{117,118}(\eta)=\frac{-bc_2+b\sqrt{c_2^2-3c_0c_4}-r_2\left(c_2^2-3c_0c_4+c_2\sqrt{(c_2^2-3c_0c_4)}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$

$$+\frac{c_4\left(3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2r_1\sqrt{c_2^2-3c_0c_4}tan^2\left(\sqrt{\frac{-c_2}{2}}\eta\right)},$$

$$c_2>0,\ c_4>0, c_0=\frac{c_2^2}{4c_4},$$

$$v_{119,120}(\eta)=\frac{-bc_2+b\sqrt{c_2^2-3c_0c_4}-r_2\left(c_2^2-3c_0c_4+c_2\sqrt{(c_2^2-3c_0c_4)}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$

$$+\frac{c_4(2p^2-1)\left(3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2p^2r_1\sqrt{c_2^2-3c_0c_4}cn^2\sqrt{\frac{c_2}{2p^2-1}}\eta}$$

$$c_2>0,\ c_4<0, c_0=\frac{c_2^2p^2(1-p^2)}{c_4(2p^2-1)^2},$$

$$v_{121,122}(\eta)=\frac{-bc_2+b\sqrt{c_2^2-3c_0c_4}-r_2\left(c_2^2-3c_0c_4+c_2\sqrt{(c_2^2-3c_0c_4)}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$

$$+\frac{c_4(2-p^2)\left(3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4p^2r_1\sqrt{c_2^2-3c_0c_4}dn^2\sqrt{\frac{c_2}{2-p^2}}\eta}$$

$$c_2>0,\ c_4<0, c_0=\frac{c_2^2(1-p^2)}{c_4(2-p^2)^2},$$

$$v_{123,124}(\eta)=\frac{-bc_2+b\sqrt{c_2^2-3c_0c_4}-r_2\left(c_2^2-3c_0c_4+c_2\sqrt{(c_2^2-3c_0c_4)}\right)\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{4r_1\sqrt{c_2^2-3c_0c_4}}$$

$$+\frac{c_4(p^2+1)\left(3bc_0+3c_0r_2\sqrt{c_2^2-3c_0c_4}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}\right)}{4c_2p^2r_1\sqrt{c_2^2-3c_0c_4}sn^2\sqrt{\frac{-c_2}{p^2+1}}\eta}$$

$$c_2<0,\ c_4>0, c_0=\frac{c_2^2p^2}{c_4(p^2+1)^2},$$

where p denotes a modulus.

$$\eta = \mp\sqrt{\frac{b + \sqrt{(c_2^2 - 3c_0c_4)r_2^2}\sqrt{\frac{b^2+16k^2\sqrt{(c_2^2-3c_0c_4)r_2^2}}{(c_2^2-3c_0c_4)r_2^2}}}{8\sqrt{(c_2^2 - 3c_0c_4)r_2^2}}}\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{kt^\alpha}{\Gamma(1+\alpha)}.$$

Third case: $c_0 = c_1 = c_4 = 0$. Then, we have two families given as follows. The first family of this case is given by

$$a_0 = \frac{-b + \sqrt{b^2 + 4c_2k^2r_2}}{2r_1}, \quad a_2 = b_1 = b_2 = 0, \quad a_1 = \frac{3c_3(-b + \sqrt{b^2 + 4c_2k^2r_2})}{4c_2r_1},$$
$$l = \mp\sqrt{\frac{b - \sqrt{b^2 + 4c_2k^2r_2}}{2c_2r_2}}.$$

Thus, the traveling wave solutions can be written in the following forms:

$$v_{125,126}(\eta) = \frac{-b + \sqrt{b^2 + 4c_2k^2r_2}}{2r_1} - \frac{3(-b + \sqrt{b^2 + 4c_2k^2r_2})}{4r_1}\left(sech^2\left(\frac{\sqrt{c_2}\eta}{2}\right)\right), c_2 > 0,$$

$$v_{127,128}(\eta) = \frac{-b + \sqrt{b^2 + 4c_2k^2r_2}}{2r_1} - \frac{3(-b + \sqrt{b^2 + 4c_2k^2r_2})}{4r_1}\left(sec^2\left(\frac{\sqrt{-c_2}\eta}{2}\right)\right), c_2 < 0,$$

where

$$\eta = \mp\sqrt{\frac{b - \sqrt{b^2 + 4c_2k^2r_2}}{2c_2r_2}}\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{kt^\alpha}{\Gamma(1+\alpha)}.$$

The second family is

$$a_0 = \frac{-b + \sqrt{b^2 + 4c_2k^2r_2}}{2r_1}, \quad a_2 = b_1 = b_2 = 0, \quad a_1 = -\frac{3c_3(b + \sqrt{b^2 + 4c_2k^2r_2})}{4c_2r_1},$$
$$l = \mp\sqrt{\frac{b + \sqrt{b^2 + 4c_2k^2r_2}}{2c_2r_2}}$$

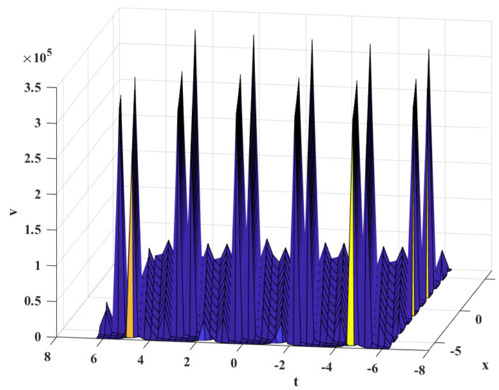
The traveling wave solutions, therefore, can be expressed in the following forms:

$$v_{129,130}(\eta) = \frac{-b + \sqrt{b^2 + 4c_2k^2r_2}}{2r_1} + \frac{3(b + \sqrt{b^2 + 4c_2k^2r_2})}{4r_1}\left(sech^2\left(\frac{\sqrt{c_2}\eta}{2}\right)\right), c_2 > 0,$$

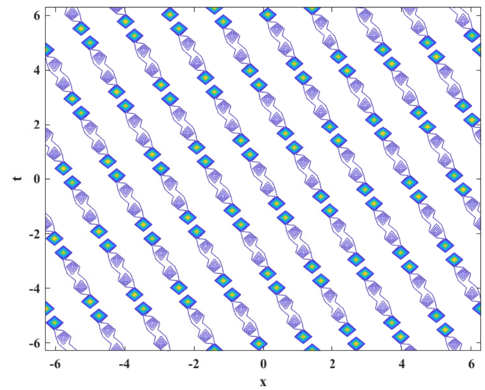
$$v_{131,132}(\eta) = \frac{-b + \sqrt{b^2 + 4c_2k^2r_2}}{2r_1} + \frac{3(b + \sqrt{b^2 + 4c_2k^2r_2})}{4r_1}\left(sec^2\left(\frac{\sqrt{-c_2}\eta}{2}\right)\right), c_2 < 0,$$

where

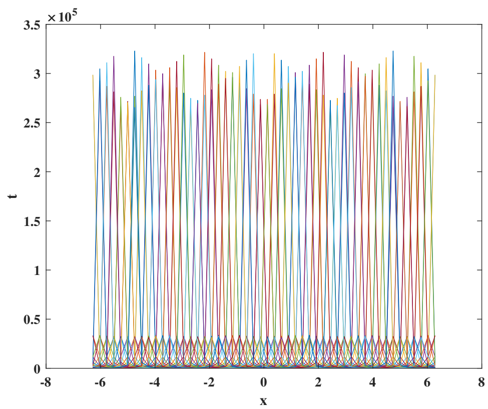
$$\eta = \mp\sqrt{\frac{b + \sqrt{b^2 + 4c_2k^2r_2}}{2c_2r_2}}\frac{x^\alpha}{\Gamma(1+\alpha)} + \frac{kt^\alpha}{\Gamma(1+\alpha)}.$$



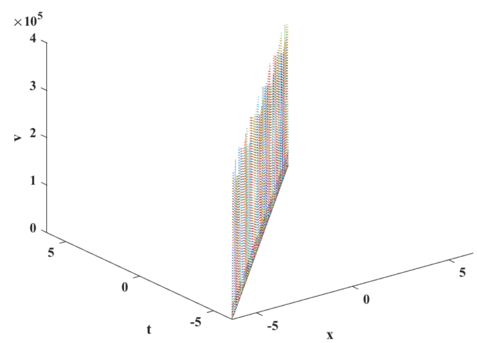
(A) A periodic traveling wave solution for v_{131}.



(B) A contour for v_{131}.



(C) A two-dimensional plot for the solution v_{131}.



(D) A three-dimensional analogue of the plot of the solution v_{131}.

Figure 4. Periodic traveling wave solutions and the contour of v_{131} under $r_1 = -r_2 = 1$, $c_2 = -2, c_4 = 1$, $b = 200$, $k = 1$ and $-2\pi \leq t, x \leq 2\pi$.

4. Results and Comparison

In this part, the primary findings of the present study are summarized, namely the traveling wave solutions of the space-time fractional Boussinesq equation. The improved modified extended tanh-function strategy, which is a relatively novel technique, is successfully employed for obtaining these solutions. Several solutions in the form of trigonometric, hyperbolic, complex, and rational functions are presented in this research.

The proposed method is very powerful. We can find more than one-hundred traveling wave solutions in various forms (including kink-shaped and bell-like solitary wave solutions, rational solutions, triangular type solutions, periodic solutions, hyperbolic type solutions, and exponential type solutions) under specific conditions. Some solutions are real and few of them are complex. Complex solutions are not considered in this work. In Fig. 2a, we plot a three dimensions soliton traveling wave solution (v_{25}) under the parameter values $r_1 = 0.3$, $r_2 = 30$, $c_2 = -0.7, c_4 =$

1, $b = 50$, $k = 0.001$ and $2\pi \leq t, x \leq 2\pi$, while Fig. 2b shows this wave in two dimensional plot with the same parameters where $t = 1$. Periodic solutions are obtained by the proposed technique. For instance, Fig. 2a illustrates the periodic solution which is presented by v_{27} under the parameters $r_1 = 0.3$, $r_2 = 30$, $c_2 = 0.7, c_4 = 1$, $b = 50$, $k = 0.0001$ and $2\pi \leq t, x \leq 2\pi$. Moreover, Fig. 2b gives this periodic solution in two dimensional plot when $t = 1$. Furthermore, we plot one more periodic traveling wave solution for solution (v_{107}) in Fig. 3 under the parameters $r_1 = 1$, $r_2 = 0.6$, $c_2 = 200, c_4 = 0.1$, $b = 60$, $k = 0.000001$ and $\pi \leq t, x \leq \pi$. In particular, we present a three dimensional graph for solution (v_{107}) in Fig. 3a and a two dimensional figure for the same solution when $t = 1$ in Fig. 3b. Fig. 3c shows the contour of the solution v_{107}. Furthermore, we illustrate periodic graphs for the traveling wave solution v_{131} in Fig. 4 under the values $r_1 = -r_2 = 1$, $c_2 = -2, c_4 = 1$, $b = 200$, $k = 1$ and $-2\pi \leq t, x \leq 2\pi$. Specifically, Fig. 4a shows the propagation of a three dimensional periodic wave for the solution v_{131}, Fig. 4b illustrates its contour, Fig. 4c demonstrates the propagation of this wave in two dimensional plot when $-2\pi \leq t \leq 2\pi$ and Fig. 4d presents a three-dimensional analogue of the plot of the solution v_{131}.

This study implements the suggested methodology for a number of purposes. To begin with, different types of solutions can be successfully extracted by this technique in numerous forms, including solitary wave solutions in the bell and kink types, triangle type solutions, rational solutions, periodic solutions, exponential type solutions, and hyperbolic type solutions. According to the number of the solutions obtained from Eq. (2.4), there are actually an increasing number of traveling wave solutions for the proposed model using the considered technique. Moreover, using this technique we obtain more than one hundred solutions in various forms such as hyperbolic functions, trigonometric functions, rational functional and complex solutions. In comparison to alternative strategies, the suggested technique is simpler to implement and more practical. For instance, El-Wakil and Abulwafa [8] used the fractional sub-equation approach to find two solutions in the form of hyper-geometric functions, two solutions in the form of fractional triangle functions, and one rational solution. Chen et al. [9] obtained only three types of traveling wave solutions (Jacobian periodic elliptic solutions, periodic solutions, and rational solutions) using the $(\frac{G'}{G^2})$-expansion approach. The $(\frac{G'}{G^2})$-expansion technique is relatively complex [9] while the improved modified extended tanh-function approach is more convenient and powerful from the perspective of the computation procedure. Moreover, Hosseini and Ansari [11] obtained a few number of solutions for the proposed problem using the well-known modified Kudryashov method. As a result, the used method is more practical and productive to obtain more traveling wave solutions for nonlinear fractional partial differential equations.

5. Conclusion

The improved modified extended tanh-function strategy, an approach that is relatively recent, has been implemented in this article to analyze the traveling wave solutions of the space-time fractional Boussinesq equation. More than a hundred traveling wave solutions (such as rational solutions, triangle type solutions, periodic solutions, exponential type solutions, and kink-shaped and bell-like solitary wave solutions) under particular conditions have been analyzed and given. For instance, we have shown two and three dimensional diagrams for the solitary wave solution (v_{25}) under the values v_{25} under $r_1 = 0.3$, $r_2 = 30$, $c_2 = -0.7, c_4 = 1$, $b = 50$, $k = 0.001$ and $2\pi \leq t, x \leq 2\pi$ in Fig. 2. We have evaluated the effectiveness of the employed method in comparison to other strategies and we found that the used technique is more powerful and effective because it gives various types of traveling wave solutions in different forms. Moreover, it is simple in applications.

Appendix

In this section, we illustrate various solution types for Eq. (2.4) as depicted in reference [21].

- The initial scenario: When $c_1 = c_1 = c_3 = 0$, we exhibit a bell-shaped solitary wave solution, a triangular-type solution, and a rational solution for Eq. (2.5) as outlined below:

$$H(\eta) = \sqrt{-\frac{c_2}{c_4}} sech\left(\sqrt{c_2}\eta\right), c_2 > 0, c_4 < 0.$$

$$H(\eta) = \sqrt{-\frac{c_2}{c_4}} sec\left(\sqrt{-a_2}\eta\right), c_2 < 0, c_4 > 0.$$

$$H(\eta) = -\frac{\epsilon}{\sqrt{c_4}\eta}, c_2 = 0, c_4 > 0.$$

- In the second scenario: When $c_0 = c_3 = 0$, we provide a kink-shaped solitary wave solution, a triangular-type solution, and three solutions of the Jacobi elliptic doubly periodic type for Eq. (2.5) as detailed below:

$$H(\eta) = \epsilon\sqrt{-\frac{c_2}{2c_4}} tanh\left(\sqrt{\frac{-c_2}{2}}\eta\right), c_2 < 0, c_4 > 0, c_0 = \frac{c_2^2}{4c_4}$$

$$H(\eta) = \epsilon\sqrt{\frac{c_2}{2c_4}} tan\left(\sqrt{\frac{c_2}{2}}\eta\right), c_2 > 0, c_4 < 0, c_0 = \frac{c_2^2}{4c_4}$$

$$H(\eta) = \sqrt{\frac{-c_2p^2}{c_4(2p^2-1)}} cn\left(\sqrt{\frac{c_2}{2p^2-1}}\eta\right), c_2 > 0, c_4 < 0, c_0 = \frac{c_2^2p^2(1-p^2)}{c_4(2p^2-1)^2}$$

$$H(\eta) = \sqrt{\frac{-p^2}{c_4(2-p^2)}} dn\left(\sqrt{\frac{c_2}{2-p^2}}\eta\right), c_2 > 0, c_4 < 0, c_0 = \frac{c_2^2(1-p^2)}{c_4(2-p^2)^2}$$

$$H(\eta) = \epsilon\sqrt{\frac{-c_2p^2}{c_4(p^2+1)}} sn\left(\sqrt{\frac{-c_2}{p^2+1}}\eta\right), c_2 < 0, c_4 > 0, c_0 = \frac{c_2^2p^2}{c_4(p^2+1)^2}$$

where p denotes a modulus.

- Third scenario: When $c_0 = c_1 = c_4 = 0$, a bell-shaped solitary wave solution, a triangular-type solution, and a rational solution for Eq. (2.5) are provided by

$$H(\eta) = -\frac{c_2}{c_3} sech^2\left(\frac{\sqrt{c_2}}{2}\eta\right), c_2 > 0.$$

$$H(\eta) = -\frac{c_2}{c_3} sec^2\left(\frac{\sqrt{-c_2}}{2}\eta\right), c_2 < 0.$$

$$H(\eta) = \frac{4}{c_3\eta^2}, c_2 = 0.$$

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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