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Some Entire Topological Indices of Specific Graph Families

Buthinah A. Bin Dehaish*, **Anwar Saleh**, **Sarah Alraddadi**

Department of Mathematics and Statistics, Faculty of Science, University of Jeddah, Jeddah 21589, Saudi Arabia

**Corresponding author:* bbindehaish@uj.edu.sa.

Abstract. What distinguishes entire topological indices from other topological indices is that their formulas include information about both the edges and vertices, not just the connections between vertices. This provides more comprehensive and detailed picture of the graph's structure. In our article, we study and analyze some entire Zagreb indices by investigating their behavior for four families of graphs; subdivision graphs, central graphs, corona products and m bridge graphs over path, cycle and complete graphs. We explore the properties of these graph structures by deriving explicit formulae for the first, second and modified first entire Zagreb indices for each family. Our results provide detailed information on the structural properties stored by the first, second and modified entire Zagreb indices. These different graph families show the way for future research and potential applications in fields such as chemical modeling and network investigation.

1. Introduction

A graph comprises both edges and a set of vertices that is not empty. In this study, we specifically examine finite graphs that are undirected, without any loops or multiple edges between the same pair of vertices. The number of vertices n and edges m in the graph G are referred to as the order and size, respectively. The degree of any vertex and edge in G is denoted by d_x. The line graph $L(G)$ is defined as the graph where each vertex represents an edge of G. Two vertices are adjacent in $L(G)$ if and only if the corresponding edges in G share a common vertex. The path, cycle and complete graphs with n vertices are known as P_n, C_n and K_n, respectively. For a more comprehensive understanding of the symbols and explanations, please refer to [1].

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A topological index is a numerical value linked to the chemical structure, aiming to correlate the structure with various physicochemical properties, chemical reactivity, or biological activity. In molecular modeling, these indices play a crucial role in understanding the structural features and predicting the properties or activities of molecules.

The concept of topological indices was first introduced by Harold Wiener when he discovered the initial topological index, which is called the Wiener index [2] in 1947 for searching boiling points. One of the initial topological indices introduced is the Zagreb index, which was first introduced by Gutman and Trinajstić [3], where they investigated how the total energy of π-electron depends on the structure of molecules. The first and the second Zagreb indices for a molecular graph are defined as follows:

$$M_1(G) = \sum_{x \in V(G)} d_x^2,$$

$$M_2(G) = \sum_{xy \in E(G)} d_x d_y.$$

For the latest research on Zagreb indices, we direct the reader to recent studies [4–15].

In 2018, Alwardi, A., et al. [16] introduced the definitions of the first and second entire Zagreb topological indices as shown below:

$$M_1^{\mathcal{E}}(G) = \sum_{x \in V \cup E} d_x^2,$$

$$M_2^{\mathcal{E}}(G) = \sum_{\substack{x\, adjacent\, to\, y \\ or\, x\, incident\, to\, y}} d_x d_y.$$

The entire Zagreb indices have been receiving significant attention from numerous authors, as shown by [17–25].

In this study, we have established implicit expressions for the subdivision and central graphs pertaining to the first, second and modified first entire Zagreb indices. Recently, in [30], the same authors of this paper have introduced the modified first entire Zagreb index as

$$MM_1^{\mathcal{E}}(G) = \sum_{\substack{x\, adjacent\, to\, y \\ or\, x\, incident\, to\, y}} (d_x + d_y).$$

2. Entire Zagreb Indices of Some Derived Graphs

The derived graphs are those graphs which can be obtained by some particular operations from a given graph. By researching the relationships between a graph and its derived graph, one can acquire information about one based on the information on the other. In this section, we will study three types of derived graphs namely the subdivision graph, central graph and the corona product.

2.1. **Entire topological indices of the subdivision graph.** The subdivision of a graph G, denoted by $S(G)$, is obtained by inserting an additional vertex into each edge of G, [26]. Subdivision graphs are used to obtain several mathematical and chemical properties of more complex graphs from more basic graphs. The subdivision graph of the cycle graph is illustrated in Figure1.

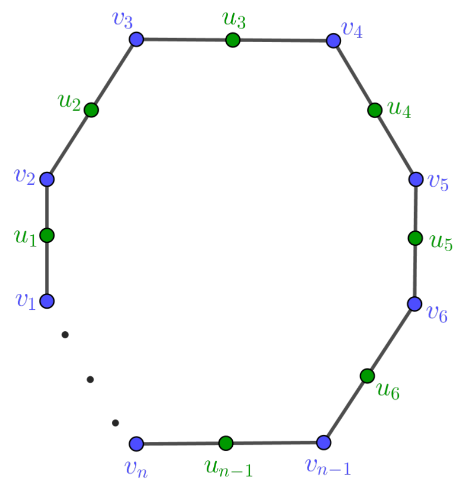

FIGURE 1. Subdivision of the cycle $S(C(G))$.

In Figure 1, the new vertices that are added to the cycle graph are green.

Proposition 2.1. *For the path P_n and the cycle C_n, we have*

i. $M_1^{\mathcal{E}}(S(P_n)) = 16n - 24.$
ii. $M_2^{\mathcal{E}}(S(P_n)) = 32n - 54.$
iii. $MM_1^{\mathcal{E}}(S(P_n)) = 32n - 46.$
iv. $MM_1^{\mathcal{E}}(S(C_n)) = M_2^{\mathcal{E}}(S(C_n)) = 2M_1^{\mathcal{E}}(S(C_n)) = 32n.$

Proposition 2.2. *Let $S(K_n)$ be the subdivision of the complete graph K_n. Then,*

i. $M_1^{\mathcal{E}}(S(K_n)) = n^4 - 2n^3 + 3n^2 - 2n.$
ii. $M_2^{\mathcal{E}}(S(K_n)) = \dfrac{n^5 - 2n^4 + 8n^3 - 14n^2 + 7n}{2}.$
iii. $MM_1^{\mathcal{E}}(S(K_n)) = n^4 + n^3 - n^2 - n.$

Proof. i. There are n vertices of degree $(n-1)$, $\left(\dfrac{n(n-1)}{2}\right)$ vertices of degree two and $n(n-1)$ edges of degree $(n-1)$. We have

$$M_1^{\mathcal{E}}(S(k_n)) = n^4 - 2n^3 + 3n^2 - 2n.$$

ii. For the second entire Zagreb index, we have

$$M_2^{\mathcal{E}}(G) = \sum_{uv\in E(G)} d_u d_v + \sum_{ef\in E(L(G))} d_e d_f + \sum_{v \text{ incident to } e} d_v d_e,$$

to calculate the first part we use the partition in Table 1 and, we get

$$\sum_{uv\in E(G)} d_u d_v = (2n-2)(n^2-n).$$

Table 1. The partition of the edges in the subdivision of complete graph.

Edge type	The number of edges
$E_{n-1,2}$	$n(n-1)$

Also, by using the adjacent edge partition as in Table 2, we have

$$\sum_{ef\in E(L(G))} d_e d_f = (n-1)^2\left(\frac{n^3-2n^2+n}{2}\right).$$

Table 2. The partition of the adjacent edges in the subdivision of complete graph.

$(d_e, d_f), where\ ef \in E(L(G))$	Number of pairs
$(n-1, n-1)$	$\frac{n(n-1)}{2} + \frac{n(n-2)(n-1)}{2}$

And by using Table 3, we get

$$\sum_{v\ incident\ to\ e} d_v d_e = (n-1)^2(n^2-n) + (2n-2)(n^2-n).$$

Table 3. The partition of the vertices incident with the edges in the subdivision of complete graph.

$\mathcal{E}_{d_v,d_e}, where\ v\ incident\ to\ e$	Number of pairs
$\mathcal{E}_{n-1,n-1}$	$n(n-1)$
$\mathcal{E}_{2,n-1}$	$n(n-1)$

Thus, $M_2^{\mathcal{E}}(S(k_n)) = \frac{n^5-2n^4+8n^3-14n^2+7n}{2}$.

iii. Similarly, as mentioned in ii, we get

$$MM_1^{\mathcal{E}}(S(k_n)) = n^4+n^3-n^2-n.$$

□

Proposition 2.3. *Let $S(K_{r,s})$ be the subdivision of complete bipartite graph $K_{r,s}$. Then*

i. $M_1^{\mathcal{E}}(S(K_{r,s})) = sr(s+r+4+s^2+r^2)$.

ii. $M_2^{\mathcal{E}}(S(K_{r,s})) = \frac{rs^4+rs^3+r^4s+r^3s+8rs^2+8r^2s+2r^2s^2}{2}$.

iii. $MM_1^{\mathcal{E}}(S(K_{r,s})) = rs^3 + sr^3 + 4sr^2 + 8rs + 4rs^2$.

Proof. i. We have r vertices of degree s, s vertices of degree r and rs vertices of degree two. Similarly, for the edges we have rs edges of degree s and rs edges of degree r. We get

$$M_1^{\mathcal{E}}(S(K_{r,s})) = sr(s + r + 4 + s^2 + r^2).$$

ii. For the second entire Zagreb index, we have

$$M_2^{\mathcal{E}}(G) = \sum_{uv \in E(G)} d_u d_v + \sum_{ef \in E(L(G))} d_e d_f + \sum_{v \text{ incident to } e} d_v d_e,$$

to calculate the first part we use the partition in Table 4 and, we get

$$\sum_{uv \in E(G)} d_u d_v = 2rs^2 + 2r^2 s.$$

Table 4. The partition of the edges in the subdivision of complete bipartite graph .

Edge type	The number of edges
$E_{s,2}$	rs
$E_{r,2}$	rs

Also, by using the adjacent edge partition as in Table 5, we have

$$\sum_{ef \in E(L(G))} d_e d_f = s^2 r\left(\frac{s(s-1)}{2}\right) + r^2 s\left(\frac{r(r-1)}{2}\right) + r^2 s^2.$$

Table 5. The partition of the adjacent edges in the subdivision of complete bipartite graph.

$(d_e, d_f), \text{where } ef \in E(L(G))$	Number of pairs
(s, s)	$r\left(\frac{s(s-1)}{2}\right)$
(r, r)	$s\left(\frac{r(r-1)}{2}\right)$
(s, r)	rs

And by using Table 6, we get

$$\sum_{v \text{ incident to } e} d_v d_e = rs^3 + 2rs^2 + 2r^2 s + r^3 s.$$

Table 6. The partition of the vertices incident with the edges in complete bipartite graph.

$\mathcal{E}_{d_v,d_e}$, *where v incident to e*	Number of pairs
$\mathcal{E}_{s,s}$	rs
$\mathcal{E}_{2,s}$	rs
$\mathcal{E}_{2,r}$	rs
$\mathcal{E}_{r,r}$	rs

Thus,

$$M_2^{\mathcal{E}}(S(K_{r,s})) = \frac{rs^4 + rs^3 + r^4s + r^3s + 8rs^2 + 8r^2s + 2r^2s^2}{2}.$$

iii. Similarly, as ii, we get

$$MM_1^{\mathcal{E}}(S(K_{r,s})) = rs^3 + sr^3 + 4sr^2 + 8rs + 4rs^2.$$

□

Theorem 2.4. *Let G be any graph with n vertices, m edges, and S(G) its subdivision. Then*

i $M_1^{\mathcal{E}}(S(G)) = M_1(G) + F(G) + 4m.$

ii $M_2^{\mathcal{E}}(S(G)) = 4M_1(G) + M_2(G) + \frac{F(G)}{2} + \frac{1}{2}\sum_{v\in V(G)} d_v^4.$

iii $MM_1^{\mathcal{E}}(S(G)) = 4M_1(G) + F(G) + 8m.$

Proof. i.

$$\begin{aligned} M_1^{\mathcal{E}}(S(G)) &= \sum_{x\in V(S(G))\cup E(S(G))} d_x^2 \\ &= \sum_{v\in V(S(G))} d_v^2 + \sum_{e\in E(S(G))} d_e^2 \\ &= 4m + \sum_{v\in V(G)} d_v^2 + \sum_{uv\in E(G)} (d_u^2 + d_v^2) \\ &= M_1(G) + F(G) + 4m. \end{aligned}$$

ii.

$$M_2^{\mathcal{E}}(S(G)) = \sum_{\substack{x\,adjacent\ to\ y \\ or\ x\ incident\ to\ y}} d_x d_y$$

$$\begin{aligned} &= 2\sum_{v\in V(G)} d_v^2 + \sum_{uv\in E(G)} d_u d_v + \sum_{v\in V(G)} d_v^2\left(\frac{d_v(d_v-1)}{2}\right) + \sum_{v\in V(G)} d_v^3 \\ &+ 2\sum_{uv\in E(G)} (d_u + d_v) \\ &= 2M_1(G) + 2M_1(G) + M_2(G) + \frac{1}{2}\sum_{v\in V(G)} d_v^4 - \frac{1}{2}\sum_{v\in V(G)} d_v^3 + \sum_{v\in V(G)} d_v^3 \\ &= 4M_1(G) + M_2(G) + \frac{F(G)}{2} + \frac{1}{2}\sum_{v\in V(G)} d_v^4. \end{aligned}$$

iii. $$MM_1^{\mathcal{E}}(S(G)) = \sum_{\substack{x\, adjacent\, to\, y \\ or\, x\, incident\, to\, y}} (d_x + d_y)$$

$$= \sum_{v\in V(G)} d_v(d_v+2) + \sum_{uv\in E(G)} (d_u+d_v) + \sum_{v\in V(G)} 2d_v\left(\frac{d_v(d_v-1)}{2}\right)$$

$$+ 2\sum_{v\in V(G)} d_v^2 + \sum_{uv\in E(G)} [(d_u+2)+(d_v+2)]$$

$$= M_1(G) + 2M_1(G) + \sum_{v\in V(G)} d_v^2 + 2\sum_{v\in V(G)} d_v + \sum_{v\in V(G)} (d_v^3 - d_v^2) + \sum_{uv\in E(G)} 4$$

$$+ \sum_{uv\in E(G)} (d_u+d_v)$$

$$= M_1(G) + 2M_1(G) + M_1(G) + 4m + F(G) - M_1(G) + 4m + M_1(G)$$

$$= 4M_1(G) + F(G) + 8m.$$

□

2.2. **Entire topological indices of the central graph.** The central graph of a graph G is obtained by subdividing each edge of G exactly once and joining all the nonadjacent vertices of G, also the central graph of G is denoted by $C(G)$ [27]. For example the central of the path P_n is illustrated in Figure 2.

The central graph C(G) can be used to analyze the connectivity and closeness centrality of vertices (nodes) in a network. Beyond network analysis, C(G) holds promise for unlocking new solutions in graph algorithms and even combinatorial optimization a powerful technique for tackling complex problems.

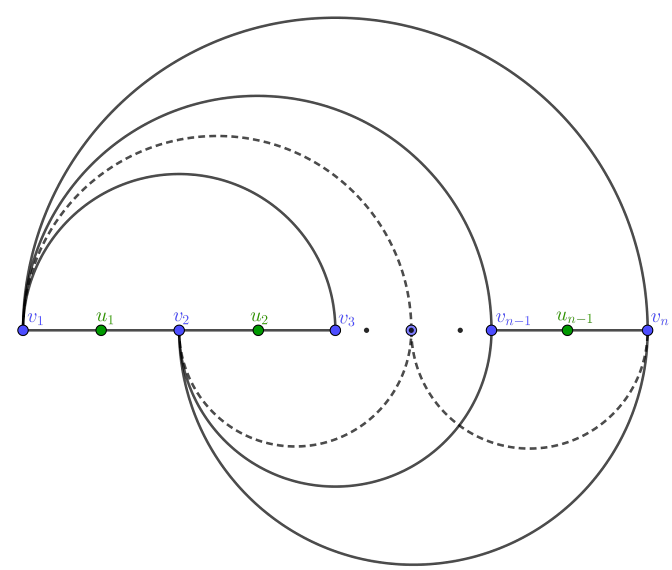

FIGURE 2. Central graph of the path $C(P_n)$.

In Figure 2 the green vertices represent the new vertices in the central of path graph.

Theorem 2.5. *Let $C(P_n)$ be a central of path graph. Then*

i. $M_1^{\mathcal{E}}(C(P_n)) = 2n^4 - 11n^3 + 28n^2 - 29n + 10.$

ii. $M_2^{\mathcal{E}}(C(P_n)) = \dfrac{4n^5 - 31n^4 + 115n^3 - 221n^2 + 229n - 104}{2}.$

iii. $MM_1^{\mathcal{E}}(C(P_n)) = 2n^4 - 8n^3 + 20n^2 - 16n + 2.$

Proof. i. We have n vertices of degree $(n-1)$ and $(n-1)$ vertices of degree two. Similarly, for the edges we have $2(n-1)$ edges of degree $(n-1)$ and $(n-1)(n-2)/2$ edges of degree $(2n-4)$,
Thus,

$$M_1^{\mathcal{E}}(C(P_n)) = 2n^4 - 11n^3 + 28n^2 - 29n + 10.$$

ii. For the second entire Zagreb index, we have

$$M_2^{\mathcal{E}}(G) = \sum_{uv \in E(G)} d_u d_v + \sum_{ef \in E(L(G))} d_e d_f + \sum_{v \text{ incident to } e} d_v d_e,$$

to calculate the first part, we use the partition in Table 7 and, we get
$\sum_{uv \in E(G)} d_u d_v = (2n-2)^2 + (n-1)^2 \left(\dfrac{n^2 - 3n + 2}{2} \right).$

Table 7. The partition of the edges in the central of path graph.

Edge type	The number of edges
$E_{n-1,2}$	$2n-2$
$E_{n-1,n-1}$	$\dfrac{n^2-3n+2}{2}$

Also, by using the adjacent edge partition as in Table 8, we have
$\sum_{ef \in E(L(G))} d_e d_f = 2n^5 - 18n^4 + 70n^3 - 143n^2 + 152n - 67.$

Table 8. The partition of the adjacent edges in the central of path graph.

(d_e, d_f), *where* $ef \in E(L(G))$	Number of pairs
$(n-1, n-1)$	$2n-3$
$(n-1, 2n-4)$	$(2n-4) + (n-2)(2n-6)$
$(2n-4, 2n-4)$	$(n-3)(n-2) + (n-2)\left(\dfrac{(n-4)(n-3)}{2}\right)$

According to the partition in Table 9, we get

$$\sum_{v \text{ incident to } e} d_v d_e = 2n^4 - 10n^3 + 24n^2 - 26n + 10.$$

Table 9. The partition of the vertices incident with the edges in the central of path graph.

$\mathcal{E}_{d_v,d_e}$, *where v incident to e*	Number of pairs
$\mathcal{E}_{n-1,2n-4}$	$(2n-4)+(n-3)(n-2)$
$\mathcal{E}_{2,n-1}$	$2n-2$
$\mathcal{E}_{n-1,n-1}$	$2n-2$

Thus,

$$M_2^{\mathcal{E}}(C(P_n)) = \frac{4n^5 - 31n^4 + 115n^3 - 221n^2 + 229n - 104}{2}.$$

iii. Similaly, as ii, we get

$$MM_1^{\mathcal{E}}(C(P_n)) = 2n^4 - 8n^3 + 20n^2 - 16n + 2.$$

□

Theorem 2.6. *Let $C(C_n)$ be a central of cycle graph, we have*

i. $M_1^{\mathcal{E}}(C(C_n)) = 2n^4 - 11n^3 + 26n^2 - 17n.$
ii. $M_2^{\mathcal{E}}(C(C_n)) = \dfrac{4n^5 - 31n^4 + 107n^3 - 165n^2 + 109n}{2}.$
iii. $MM_1^{\mathcal{E}}(C(C_n)) = 2n^4 - 8n^3 + 18n^2 - 4n.$

Proof. i. We have n vertices of degree $(n-1)$, n vertices of degree two, $2n$ edges of degree $(n-1)$ and $\dfrac{n(n-3)}{2}$ of degree $(2n-4)$.
we get,

$$M_1^{\mathcal{E}}(C(C_n)) = 2n^4 - 11n^3 + 26n^2 - 17n.$$

ii. For the second entire Zagreb index, we have

$$M_2^{\mathcal{E}}(G) = \sum_{uv \in E(G)} d_u d_v + \sum_{ef \in E(L(G))} d_e d_f + \sum_{v \text{ incident to } e} d_v d_e,$$

to compute the initial segment, we utilize the division shown in the Table 10 and, we get

$$\sum_{uv\in E(G)} d_u d_v = \frac{n^4-5n^3+15n^2-11n}{2}.$$

Table 10. The partition of the edges in the central of cycle graph.

Edge type	The number of edges
$E_{n-1,2}$	$2n$
$E_{n-1,n-1}$	$\frac{n(n-3)}{2}$

Also, by using the adjacent edge partition as in Table 11, we have

$$\sum_{ef\in E(L(G))} d_e d_f = 2n^5-18n^4+66n^3-112n^2+74n.$$

Table 11. The partition of the adjacent edges in the central of cycle graph.

(d_e, d_f), where $ef \in E(L(G))$	Number of pairs
$(n-1, n-1)$	$2n$
$(2n-4, 2n-4)$	$n\left(\frac{(n-3)(n-4)}{2}\right)$
$(n-1, 2n-4)$	$n(2n-6)$

And by using Table 12, we get

$$\sum_{v \text{ incident to } e} d_v d_e = 2n(n-1)^2+(n-1)(2n-4)(n^2-3n)+2n(2n-2).$$

Table 12. The partition of the vertices incident with the edges in the central of cycle graph.

$\mathcal{E}_{d_v,d_e}$, where v incident to e	Number of pairs
$\mathcal{E}_{n-1,n-1}$	$2n$
$\mathcal{E}_{n-1,2n-4}$	$n(n-3)$
$\mathcal{E}_{2,n-1}$	$2n$

Thus,

$$M_2^{\mathcal{E}}(C(C_n)) = \frac{4n^5-31n^4+107n^3-165n^2+109n}{2}.$$

iii. Straightforwardly, as ii, we get

$$MM_1^{\mathcal{E}}(C(C_n)) = 2n^4-8n^3+18n^2-4n.$$

□

Theorem 2.7. *Let be G any graph with n vertices, m edges and $C(G)$ is the central graph of G. Then*

i. $M_1^{\mathcal{E}}(C(G)) = 2n^4 - 9n^3 - 2n^2m + 14n^2 + 12nm - 7n - 10m.$

ii. $$M_2^{\mathcal{E}}(C(G)) = \frac{50m + 4n^5 - 23n^4 - 8mn^3 + 53n^3 + 50mn^2 - 57n^2 - 84mn + 23n + n^2M_1(G)}{2} + \frac{9M_1(G) - 6nM_1(G)}{2}.$$

iii. $MM_1^{\mathcal{E}}(C(G)) = 12mn - 2m - 6n^3 + 6n^2 - 2n - 2mn^2 + 2n^4.$

Proof. From the definition of the central graph, we observe that there are two types of vertices; m vertices of degree two and n vertices of degree $n-1$, where n is the number of vertices in G. In the same way for edges there are two types of edges according to their degrees; $\frac{n(n-1)-2m}{2}$ edges of degree $2n-4$ and $2m$ of degree $n-1$. Then

$$\begin{aligned}
M_1^{\mathcal{E}}(C(G)) &= \sum_{x\in V(C(G))\cup E(C(G))} d_x^2 \\
&= \sum_{v\in V(C(G))} d_v^2 + \sum_{e\in E(C(G))} d_e^2 \\
&= 4m + n(n-1)^2 + 2m(n-1)^2 + (2n-4)^2\left(\frac{n(n-1)-2m}{2}\right) \\
&= 2n^4 - 9n^3 - 2n^2m + 14n^2 + 12nm - 7n - 10m.
\end{aligned}$$

ii. $$M_2^{\mathcal{E}}(G) = \sum_{\substack{x\, adjacent\ to\ y \\ or\ x\ incident\ to\ y}} d_x d_y$$

$$\begin{aligned}
&= 2m(2n-2) + (n-1)^2\left(\frac{n(n-1)-2m}{2}\right) + m(n-1)^2 + (n-1)^2\left(\sum_{v\in V(G)}\sum_{i=1}^{d_v-1} d_v - i\right) \\
&+ (n-1)(2n-4)\left(\sum_{v\in V(G)}((n-1)d_v - d_v^2)\right) + (2n-4)^2\left(\sum_{v\in V(G)}\sum_{i=1}^{n-d_v-2}(n-1-d_v-i)\right) \\
&+ 2m(2n-2) + (n-1)(2n-4)\left(\sum_{v\in V(G)}(n-1-d_v)\right) + (n-1)^2\sum_{v\in V(G)} d_v \\
&= 2m(2n-2) + (n-1)^2\left(\frac{n(n-1)-2m}{2}\right) + m(n-1)^2 + (n-1)^2\left(\sum_{v\in V(G)}\frac{d_v(d_v-1)}{2}\right) \\
&+ (n-1)(2n-4)(2m(n-1) - M_1(G)) + (2n-4)^2\left(\sum_{v\in V(G)}\frac{(n-d_v-2)(n-d_v-1)}{2}\right) \\
&+ 2m(2n-2) + (n-1)(2n-4)(n^2-n-2m) + 2m(n-1)^2 \\
&= 2m(2n-2) + (n-1)^2\left(\frac{n(n-1)-2m}{2}\right) + m(n-1)^2 + (n-1)^2\left(\frac{1}{2}M_1(G) - m\right)
\end{aligned}$$

$$+ (n-1)(2n-4)(2m(n-1) - M_1(G)) + \frac{1}{2}(2n-4)^2(n^3 + M_1(G) + 6m - 4mn - 3n^2 + 2n)$$
$$+ 2m(2n-2) + (n-1)(2n-4)(n^2 - n - 2m) + 2m(n-1)^2$$
$$= \frac{50m + 4n^5 - 23n^4 - 8mn^3 + 53n^3 + 50mn^2 - 57n^2 - 84mn + 23n + n^2M_1(G)}{2}$$
$$+ \frac{9M_1(G) - 6nM_1(G)}{2}.$$

iii. $$MM_1^{\mathcal{E}}(G) = \sum_{\substack{x\, adjacent\, to\, y \\ or\, x\, incident\, to\, y}} (d_x + d_y)$$
$$= 2m(n+1) + (2n-2)\left(\frac{n(n-1)-2m}{2}\right) + m(2n-2) + (2n-2)\left(\sum_{v\in V(G)} \sum_{i=1}^{d_v-1} d_v - i\right)$$
$$+ (3n-5)\left(\sum_{v\in V(G)} (n-1)d_v - d_v^2\right) + (4n-8)\left(\sum_{v\in V(G)} \sum_{i=1}^{n-d_v-2} n-1-d_v-i\right)$$
$$+ 2m(n+1) + (3n-5)\left(\sum_{v\in V(G)} n-1-d_v\right) + (2n-2)\sum_{v\in V(G)} d_v$$
$$= 2m(n+1) + (2n-2)\left(\frac{n(n-1)-2m}{2}\right) + m(2n-2) + (2n-2)\left(\sum_{v\in V(G)} \frac{d_v(d_v-1)}{2}\right)$$
$$+ (3n-5)(2mn - 2m - M_1(G)) + (4n-8)\left(\sum_{v\in V(G)} \frac{(n-d_v-2)(n-d_v-1)}{2}\right)$$
$$+ (2mn + 2m) + (3n-5)(n^2 - n - 2m) + 2m(2n-2)$$
$$= 2m(n+1) + (2n-2)\left(\frac{n(n-1)-2m}{2}\right) + m(2n-2) + (2n-2)(\frac{1}{2}M_1(G) - m)$$
$$+ (3n-5)(2mn - 2m - M_1(G)) + \frac{1}{2}(4n-8)(n^3 + M_1(G) + 6m - 4mn - 3n^2 + 2n)$$
$$+ (2mn + 2m) + (3n-5)(n^2 - n - 2m) + 2m(2n-2)$$
$$= 12mn - 2m - 6n^3 + 6n^2 - 2n - 2mn^2 + 2n^4.$$

□

2.3. **Entire topological indices of the corona product of two graphs.** The corona product of two graphs G_1 and G_2, is a graph denoted by $G_1 \circ G_2$ which is constructed by taking $|n_1|$ copies of G_2 and joining each vertex of the *ith* copy with vertex $u \in V(G_1)$. The corona product of two graphs can represent networks with hierarchical structures, like molecules with atoms and surrounding bonds, or transportation systems with stations and connecting routes.

Theorem 2.8. *The corona product $P_n \circ P_m$ of two graphs P_n and P_m with $\left|V(P_n)\right| = n$, $\left|V(P_m)\right| = m$. Then*

i. $M_1^{\mathcal{E}}(P_n \circ P_m) = 5nm^2 + 4m^2 + 37mn - 32n - 20m + 2m^3 - 28.$

ii. $M_2^{\mathcal{E}}(P_n \circ P_m) = \dfrac{-12m^3 - 96m^2 - 202m + m^4n + 15m^3n + 87m^2n + 191mn - 284n + 12}{2}.$

iii. $MM_1^{\mathcal{E}}(P_n \circ P_m) = -8m^2 - 48m + 73mn - 60n + 14m^2n + m^3n - 28.$

Proof. i. There are $(n-2)$ vertices of degree $(m+2)$, two vertices of degree $(m+1)$, $2n$ vertices of degree two and $n(m-2)$ vertices of degree three.

For the edges, there are two edges of degree $(2m+1)$, $(n-3)$ edges of degree $(2m+2)$, four edges of degree $(m+1)$, $2(m-2)$ edges of degree $(m+2)$, $2n$ edges of degree three and $n(m-3)$ edges of degree four.

ii. Regarding the second entire Zagreb index, we have

$$M_2^{\mathcal{E}}(G) = \sum_{uv\in E(G)} d_u d_v + \sum_{ef\in E(L(G))} d_e d_f + \sum_{v \text{ incident to } e} d_v d_e,$$

by utilizing the partition specified in the Table 13, we calculate the first part,

$$\sum_{uv\in E(G)} d_u d_v = -m^2 - 12m + 4m^2n + 17mn - 15n - 4.$$

Table 13. The partition of the edges in the corona product.

Edge type	The number of edges
$E_{m+1,m+2}$	2
$E_{m+2,m+2}$	$n-3$
$E_{m+1,2}$	4
$E_{m+1,3}$	$2m-4$
$E_{2,3}$	$2n$
$E_{3,3}$	$nm-3n$
$E_{m+2,2}$	$2n-4$
$E_{m+2,3}$	$(m-2)(n-2)$

Also, by using the adjacent edge partition as in Table 14, we have

$$\sum_{ef\in E(L(G))} d_e d_f = \frac{-12m^3 - 70m^2 + m^4n + 13m^3n + 55m^2n + 63mn - 90m - 158n + 48}{2}.$$

Table 14. The partition of the adjacent edges in the corona product.

(d_e, d_f), *where* $ef \in E(L(G))$	Number of pairs
$(2m+1, m+1)$	4
$(2m+1, m+2)$	$2m-4$
$(m+1, m+1)$	2
$(m+2, m+2)$	$(m-2)(m-3)+(n-2)$
$(2m+1, 2m+2)$	2
$(2m+2, 2m+2)$	$n-4$
$(2m+1, m+2)$	4
$(2m+1, m+3)$	$2m-4$
$(m+3, m+3)$	$(n-2)\left(\frac{(m-2)(m-3)}{2}\right)$
$(2m+2, m+2)$	$4n-12$
$(2m+2, m+3)$	$(2m-4)(n-3)$
$(m+2, 3)$	$2n$
$(m+1, 3)$	4
$(3, 4)$	$2n$
$(4, 4)$	$nm-4n$
$(m+2, 4)$	$4m-12$
$(m+3, 3)$	$2n-4$
$(m+3, 4)$	$(n-2)(2m-6)$
$(m+2, m+3)$	$(2m-4)(n-2)$
$(m+1, m+2)$	$4m-8$

Additionally, by utilizing Table 15, we have

$$\sum_{v \text{ incident to } e} d_v d_e = 47mn - 48n - 12m^2 - 44m + 12nm^2 + nm^3 - 14.$$

TABLE 15. The partition of the vertices incident with the edges in the corona product.

$\mathcal{E}_{d_v,d_e}$, *where v incident to e*	Number of pairs
$\mathcal{E}_{2,m+2}$	$2n-4$
$\mathcal{E}_{2,m+1}$	4
$\mathcal{E}_{2,3}$	$2n$
$\mathcal{E}_{3,3}$	$2n$
$\mathcal{E}_{3,4}$	$2mn-6n$
$\mathcal{E}_{3,m+2}$	$2m-4$
$\mathcal{E}_{3,m+3}$	$(n-2)(m-2)$
$\mathcal{E}_{m+1,m+1}$	4
$\mathcal{E}_{m+1,m+2}$	$2m-4$
$\mathcal{E}_{m+1,2m+1}$	2
$\mathcal{E}_{m+2,2m+2}$	$2n-6$
$\mathcal{E}_{m+2,m+2}$	$2n-4$
$\mathcal{E}_{m+2,m+3}$	$(n-2)(m-2)$

Thus,

$$M_2^{\mathcal{E}}(P_n \circ P_m) = \frac{-12m^3 - 96m^2 - 202m + m^4n + 15m^3n + 87m^2n + 191mn - 284n + 12}{2}.$$

iii. In the same manner , as ii, we get

$$MM_1^{\mathcal{E}}(P_n \circ P_m) = -8m^2 - 48m + 73mn - 60n + 14m^2n + m^3n - 28.$$

□

Theorem 2.9. *[30] For any graph G with m edges, we have:*

$$MM_1^{\mathcal{E}}(G) = F(G) + 2M_2(G).$$

Proposition 2.10. *[16] For any two graphs G_1 and G_2 with $|V(G_1)| = n_1$, $|V(G_2)| = n_2$ and $|E(G_1)| = m_1$, $|E(G_2)| = m_2$, the first and second Zagreb indices of $G_1 \circ G_2$ are given by*

$$M_2(G_1 \circ G_2) = M_2(G_1) + n_1M_2(G_2) + n_2M_1(G_1) + n_1M_1(G_2) + n_2m_1(n_2+2) + m_2(n_1+4m_1) + n_1n_2(n_2+2m_2).$$

Theorem 2.11. *[28] Let G be a graph with n vertices and m edges. Then*
$F(G_1 \circ G_2) = F(G_1) + n_1F(G_2) + n_1n_2^3 + 3n_2M_1(G_1) + 6n_2^2m_1 + n_1n_2 + 3n_1M_1(G_2) + 6n_1m_2.$

Theorem 2.12. *For any two graphs G_1 and G_2 with $\left|V(G_1)\right| = n_1$, $\left|V(G_2)\right| = n_2$ and $\left|E(G_1)\right| = m_1$, $\left|E(G_2)\right| = m_2$, the modified entire Zagreb index of $G_1 \circ G_2$ is given by*

$$MM_1^{\mathcal{E}}(G_1 \circ G_2) = F(G_1) + n_1F(G_2) + n_1n_2^3 + 5n_2M_1(G_1) + 6n_2^2m_1 + n_1n_2 + 5n_1M_1(G_2) + 7n_1m_2 + 2M_2(G_1) + 2n_1M_2(G_2) + n_2^2m_1n_2 + 2n_2m_1.$$

Proof. By Theorem 2.9, we can write

$MM_1^{\mathcal{E}}(G_1 \circ G_2) = F(G_1 \circ G_2) + 2M_2(G_1 \circ G_2)$.

Using the results of Proposition 2.10 and Theorem 2.11, we get

$$MM_1^{\mathcal{E}}(G_1 \circ G_2) = F(G_1) + n_1F(G_2) + n_1n_2^3 + 5n_2M_1(G_1) + 6n_2^2m_1 + n_1n_2 + 5n_1M_1(G_2) + 7n_1m_2 + 2M_2(G_1) + 2n_1M_2(G_2) + n_2^2m_1n_2 + 2n_2m_1.$$

□

2.4. **Entire topological indices of the m bridge over graphs.** A bridge graph is a graph obtained from the number of graphs $G_1, G_2, G_3....G_m$ by associating the vertices v_i and $v_i + 1$ by an edge for every $i = 1, 2...m-1$, [29]. The m bridge play a crucial role in network analysis and wireless communications. They aid in comprehending network connectivity, facilitating the identification of pathways in wireless communications. Moreover, they are employed to model particular network setups or evaluate the effectiveness of wireless networks. Analyzing communication and structural patterns within networks is of paramount importance. The bridge graph over path and cycle are illustrated in Figure 3, 4.

Theorem 2.13. *Let G_m be a bridge graph over path P_n. Then,*

i. $M_1^{\mathcal{E}}(G_m) = 8mn + 16m - 50$.
ii. $M_2^{\mathcal{E}}(G_m) = 16mn + 55m - 161$.
iii. $MM_1^{\mathcal{E}}(G_m) = 16mn + 22m - 76$.

Proof. i. We have $mn - 2m + 2$ vertices of degree two, m vertices of degree one and m-2 vertices of degree three.

Similarly, for the edges we have $mn - 3m + 2$ edges of degree two, m of degree one, m of degree three and $m - 3$ of degree four, Thus,

$$M_1^{\mathcal{E}}(G_m) = 8mn + 16m - 50.$$

ii. For the second entire Zagreb index, we get

$$M_2^{\mathcal{E}}(G) = \sum_{uv \in E(G)} d_ud_v + \sum_{ef \in E(L(G))} d_ed_f + \sum_{v \text{ incident to } e} d_vd_e,$$

to calculate the first part we use the partition in Table 16 and, we get

$$\sum_{uv \in E(G)} d_ud_v = 4(mn - 3m + 2) + 2m + 6m + 9(m - 3).$$

Table 16. The partition of the edges in the bridge graph over path.

Edge type	The number of edges
$E_{2,2}$	$mn-3m+2$
$E_{2,1}$	m
$E_{3,2}$	m
$E_{3,3}$	$m-3$

Also, by using the adjacent edge partition as in Table 17, we have

$$\sum_{ef\in E(L(G))} d_e d_f = 6m+4(mn-4m+2)+2m+18+12(2m-4)+16(m-4).$$

Table 17. The partition of the adjacent edges in the bridge graph over path.

(d_e,d_f), where $ef\in E(L(G))$	Number of pairs
$(2,3)$	m
$(2,2)$	$mn-4m+2$
$(2,1)$	m
$(3,3)$	2
$(3,4)$	$2m-4$
$(4,4)$	$m-4$

And by using Table 18, we get

$$\sum_{v \text{ incident to } e} d_v d_e = 4(2mn-6m+4)+2m+m+6m+9m+12(2m-6).$$

Table 18. The partition of the vertices incident with the edges in the bridge graph over path.

$\mathcal{E}_{d_v,d_e}$, where v incident to e	Number of pairs
$\mathcal{E}_{2,2}$	$2mn-6m+4$
$\mathcal{E}_{2,1}$	m
$\mathcal{E}_{1,1}$	m
$\mathcal{E}_{2,3}$	m
$\mathcal{E}_{3,3}$	m
$\mathcal{E}_{3,4}$	$2m-6$

Thus,

$$M_2^{\mathcal{E}}(G_m) = 16mn+55m-161.$$

iii. Similarly, as ii, we get
$MM_1^{\mathcal{E}}(G_m) = 16mn + 22m - 76.$

□

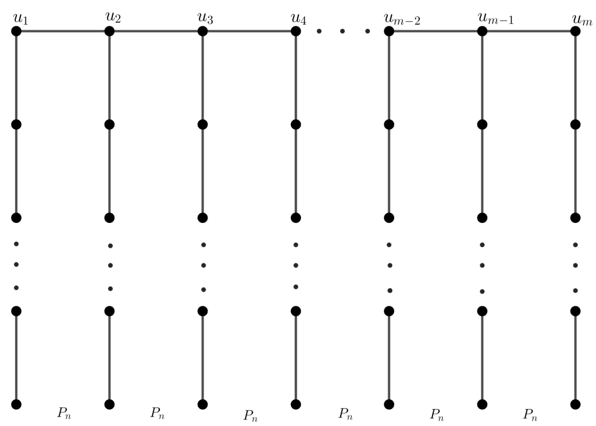

FIGURE 3. Bridge graph over path P_n.

Theorem 2.14. *Let G_m be a bridge graph over cycle C_n. Then,*

i. $M_1^{\mathcal{E}}(G_m) = 8mn + 72m - 100.$
ii. $M_2^{\mathcal{E}}(G_m) = 16mn + 256m - 396.$
iii. $MM_1^{\mathcal{E}}(G_m) = 16mn + 104m - 138.$

Proof. i. There are $mn - m$ vertices of degree two, two vertices of degree three and $m-2$ vertices of degree four.
In the same way, for the edges we have $mn - 2m$ edges of degree two, four edges of degree three, $2m - 4$ edges of degree four, two edges of degree five and $m - 3$ edges of degree six, we get

$$M_1^{\mathcal{E}}(G_m) = 8mn + 72m - 100.$$

ii. For the second entire Zagreb index, we get

$$M_2^{\mathcal{E}}(G) = \sum_{uv \in E(G)} d_u d_v + \sum_{ef \in E(L(G))} d_e d_f + \sum_{v \text{ incident to } e} d_v d_e,$$

to compute the first part, we use the partition in Table 19 and, we get

$$\sum_{uv \in E(G)} d_u d_v = 4mn + 24m - 32.$$

Table 19. The partition of the edges in the bridge graph over cycle.

Edge type	The number of edges
$E_{2,2}$	$m(n-2)$
$E_{3,2}$	4
$E_{3,4}$	4
$E_{4,4}$	$m-3$
$E_{2,4}$	$2(m-2)$

Also, by using the adjacent edge partition as in Table 20, we have
$$\sum_{ef\in E(L(G))} d_e d_f = 4mn + 152m - 254.$$

Table 20. The partition of the adjacent edges in the bridge graph over cycle.

(d_e, d_f), *where* $ef \in E(L(G))$	Number of pairs
$(2,2)$	$m(n-3)$
$(2,3)$	4
$(3,3)$	2
$(2,4)$	$2(m-2)$
$(3,5)$	4
$(4,5)$	4
$(6,5)$	2
$(4,4)$	$m-2$
$(6,4)$	$4(m-3)$
$(6,6)$	$m-4$

By utilizing Table 21, we get

$$\sum_{v \text{ incident to } e} d_v d_e = 8mn + 80m - 110.$$

TABLE 21. The partition of the vertices incident with the edges in the bridge graph over cycle.

$\mathcal{E}_{d_v,d_e}$, *where v incident to e*	Number of pairs
$\mathcal{E}_{2,2}$	$2mn - 4m$
$\mathcal{E}_{2,3}$	4
$\mathcal{E}_{3,3}$	4
$\mathcal{E}_{3,5}$	2
$\mathcal{E}_{2,4}$	$2(m-2)$
$\mathcal{E}_{4,4}$	$2(m-2)$
$\mathcal{E}_{4,5}$	2
$\mathcal{E}_{4,6}$	$2m - 6$

Thus,

$$M_2^{\mathcal{E}}(G_m) = 16mn + 256m - 396.$$

iii. Likewise, as shown in ii, we obtain

$$MM_1^{\mathcal{E}}(G_m) = 16mn + 104m - 138.$$

□

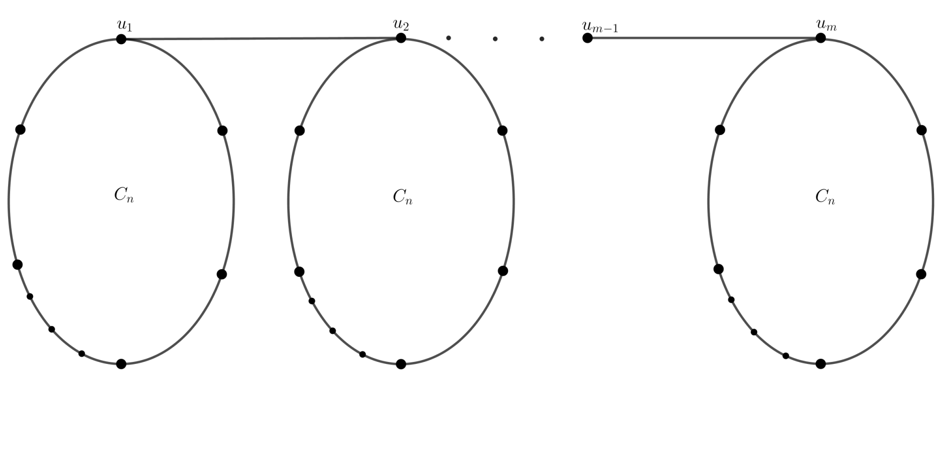

FIGURE 4. Bridge graph over cycle C_n.

Theorem 2.15. *Let G_m be a bridge graph over complete K_n. Then,*

i. $M_1^{\mathcal{E}}(G_m) = 2n^4m - 9n^3m + 26n^2m - 12n^2 - 23nm + 6n + 12m - 10.$

ii. $M_2^{\mathcal{E}}(G_m) = \dfrac{36n^2 - 23n^4m + 85n^3m - 32n^3 - 127n^2m + 127nm - 86n - 34m + 4n^5m + 6}{2}.$

iii. $MM_1^{\mathcal{E}}(G_m) = 2n^4m - 6n^3m + 18n^2m - 12n^2 - 6nm - 6n + 8m - 12.$

Proof. i. We have two vertices of degree n, $m-2$ vertices of degree $n+1$, $mn-m$ vertices of degree $n-1$.

Similarly, for the edges we have two edges of degree $2n-1$, $m-3$ edges of degree $2n$, $2n-2$ edges of degree $2n-3$, $(n-1)(m-2)$ edges of degree $2n-2$ and $\dfrac{m(n^2-3n+2)}{2}$ edges of degree $2n-4$. Thus,

$$M_1^{\mathcal{E}}(G_m) = 2n^4m - 9n^3m + 26n^2m - 12n^2 - 23nm + 6n + 12m - 10.$$

ii. For the second entire Zagreb index, we get

$$M_2^{\mathcal{E}}(G) = \sum_{uv\in E(G)} d_u d_v + \sum_{ef\in E(L(G))} d_e d_f + \sum_{v \text{ incident to } e} d_v d_e,$$

to calculate the first part we use the partition in Table 22 and, we get

$$\sum_{uv\in E(G)} d_u d_v = \frac{-6n^2 + n^4m - 3n^3m + 9n^2m - 5nm + 6m - 10}{2}.$$

Table 22. The partition of the edges in the bridge graph over complete.

Edge type	The number of edges
$E_{n,n+1}$	2
$E_{n+1,n+1}$	$m-3$
$E_{n,n-1}$	$2(n-1)$
$E_{n-1,n-1}$	$\dfrac{m(n^2-3n+2)}{2}$
$E_{n+1,n-1}$	$(n-1)(m-2)$

Also, by using the adjacent edge partition as in Table 23, we have

$$\sum_{ef\in E(L(G))} d_e d_f = 2n^5m - 14n^4m + 52n^3m - 16n^3 + 33n^2 - 90n^2m + 82nm - 45n + 16 - 28m.$$

Table 23. The partition of the adjacent edges in the bridge graph over complete.

(d_e, d_f), where $ef \in E(L(G))$	Number of pairs
$(2n-3, 2n-3)$	$n^2 - 3n + 2$
$(2n-3, 2n-1)$	$2(n-1)$
$(2n-2, 2n-2)$	$\left(\frac{n^2-3n+2}{2}\right)(m-2)$
$(2n-2, 2n-1)$	$2(n-1)$
$(2n-2, 2n)$	$(2n-2)(m-3)$
$(2n-1, 2n)$	2
$(2n, 2n)$	$m-4$
$(2n-4, 2n-4)$	$(mn-m)\left(\frac{n^2-5n+6}{2}\right)$
$(2n-4, 2n-3)$	$2(n-2)(n-1)$
$(2n-2, 2n-4)$	$(n-2)(n-1)(m-2)$

And by using Table 24, we get

$$\sum_{v \text{ incident to } e} d_v d_e = -12n^2 + 2n + 22n^2m - 16nm - 8n^3m + 2n^4m + 8m - 8.$$

Table 24. The partition of the vertices incident with the edges in the bridge graph over complete.

$\mathcal{E}_{d_v,d_e}$, where v incident to e	Number of pairs
$\mathcal{E}_{n,2n-1}$	2
$\mathcal{E}_{n,2n-3}$	$2(n-1)$
$\mathcal{E}_{n+1,2n-1}$	2
$\mathcal{E}_{n+1,2n}$	$2m-6$
$\mathcal{E}_{n+1,2n-2}$	$(n-1)(m-2)$
$\mathcal{E}_{n-1,2n-3}$	$2(n-1)$
$\mathcal{E}_{n-1,2n-4}$	$m(n-2)(n-1)$
$\mathcal{E}_{n-1,2n-2}$	$(n-1)(m-2)$

Thus,

$$M_2^{\mathcal{E}}(G_m) = \frac{36n^2 - 23n^4m + 85n^3m - 32n^3 - 127n^2m + 127nm - 86n - 34m + 4n^5m + 6}{2}.$$

iii. Likewise, as mentioned in ii, we get

$$MM_1^{\mathcal{E}}(G_m) = 2n^4m - 6n^3m + 18n^2m - 12n^2 - 6nm - 6n + 8m - 12.$$

□

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