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Wijsman and Wijsman Randomly Triple Ideal Convergence Sequences of Sets in Probabilistic Metric Spaces

M.H.M. Rashid*, **Rabaa Al-Maita**

Department of Mathematics& Statistics, Faculty of Science P.O.Box(7), Mutah University, Alkarak, Jordan

**Corresponding author:* malik_okasha@Yahoo.com

Abstract. Many authors have extended the concept of convergence from number sequences to sequences of sets. In this paper, we focus on two notable adaptations: Wijsman convergence and randomly ideal convergence. We introduce and analyze several new types of convergence for sequences of sets: $\mathscr{I}_{W_3}^{\psi}$-convergence, $\mathscr{I}_{W_3}^{*,\psi}$-convergence, $\mathscr{I}_{W_3}^{\psi}$-Cauchy, $\mathscr{I}_{W_3}^{*,\psi}$-Cauchy, $(\mathscr{I}_{W_3}^{\psi}, \mathscr{I}_{W}^{\psi})$-convergence, and $(\mathscr{I}_{W_3}^{*,\psi}, \mathscr{I}_{W}^{*,\psi})$-convergence. These new concepts expand the framework of convergence and provide a deeper understanding of the behavior of set sequences under various conditions. Through rigorous analysis, we demonstrate the relationships between these new forms of convergence and their classical counterparts, highlighting their theoretical significance and potential applications in mathematical analysis and related fields. Our findings offer a comprehensive exploration of these advanced convergence concepts, paving the way for further research and development in this area.

1. Introduction

Fast [11] and Steinhaus [26] independently introduced the concept of statistical convergence for sequences of real numbers in the same year 1951, and since then several generalizations and applications of this notion have been investigated by various authors, including [1–4,6,19,20,22,25]. One of its interesting generalizations is $\mathscr{I}$-convergence, which was provided by Kostyrko et al. [14]. Balcerzak et al. [3] recently researched $\mathscr{I}$-convergence for function sequences. The concept of statistical convergence has applications in many fields of mathematics, including number theory by Erdos and Tenenbaum [8], statistics and probability theory by Fridy and Khan [13] and Ghosal [10], approximation theory by Gadjiev and Orhan [9], Hopfield neural network by Martinez et al. [21], and optimization by Pehlivan and Mamedov [24].

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Some authors have expanded the concept of convergence of point sequences to convergence of set sequences. The idea of Wijsman convergence in [35,36] is one of these expansions discussed in this study. Nuray and Rhoades proposed statistical convergence definitions for set sequences in [22]. Ulusu and Nuray [33,34] investigated the concept of Wijsman lacunary statistical convergence of set sequences. Nuray et al. [23] investigated the relationship between the concepts of Wijsman Cesáro summability and Wijsman lacunary convergence of double sequences of sets. This notion is used in a variety of ways in [7, 29–32]. Kisi and Nuray [17] proposed a new convergence notion for set sequences termed Wijsman $\mathscr{I}$-convergence. Dundar and Pancaroglu [7] recently extended this approach to Wijsman regularly ideal convergence of double sequences of sets. In [5], The authors expanded those concepts to triple sequences of sets and studied some relationship between them. We present the ideas of Wijsman convergence and Randomly ideal convergence for triple set sequences in this paper. In addition, we study some of these ideas' features and examine their relationship.

The paper is organized as follows: In section 2, we present and investigate the notions of $\mathscr{I}_{W_3}$-convergence, $\mathscr{I}^*_{W_3}$-convergence, $\mathscr{I}_{W_3}$-Cauchy, and $\mathscr{I}^*_{W_3}$-Cauchy, and build a diagram to explain the relationships between them. Part four investigates and demonstrates the concepts of randomly $(\mathscr{I}^{\psi}_{W_3}, \mathscr{I}^{\psi}_{W})$-convergence and randomly $(\mathscr{I}^{*,\psi}_{W_3}, \mathscr{I}^{*,\psi}_{W})$-convergence.

Throughout of this paper, $\mathscr{I}_3 \subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ denotes a strongly admissible ideal, and $(X, \psi, *)$ denotes a Menger probabilistic metric space and $\wp, \wp_{mnk}$ are any non-empty closed subsets of X. Furthermore, $\mathbb{N}$ and $\mathbb{R}$ denote the set of all positive integers and the set of all real numbers, respectively. Next, we recall some definitions and notions which are useful for the development of this paper.

Definition 1.1. [27] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-norm if $([0,1], *)$ is a topological monoid with unit 1 such that $\kappa_1 * \kappa_2 \le \kappa_3 * \kappa_4$ whenever $\kappa_1 \le \kappa_3, \kappa_2 \le \kappa_4$ for all $\kappa_1, \kappa_2, \kappa_3, \kappa_4 \in [0,1]$.

Let $\mathscr{D}$ denote the set of all distribution functions and $\mathscr{D}^+ = \{\psi : \psi \in \mathscr{D}, \psi(t) = 0 \text{ for all } t \le 0\}$. Let τ_a be the specific distribution function defined by

$$\tau_a(t) = \begin{cases} 1, & \text{if } t > a; \\ 0, & \text{if } t \le a. \end{cases}$$

Definition 1.2. [27] A function $\psi : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function if it is non-decreasing and left-continuous with $\inf_{t\in\mathbb{R}} \psi(t) = 0$ and $\sup_{t\in\mathbb{R}} \psi(t) = 1$.

Definition 1.3. [15] A Menger probabilistic metric space (or random metric spaces) is a triple $(X, \psi, *)$, where X is a nonempty set, $*$ is a continuous t-norm, and ψ is a mapping from $X \times X$ into $\mathscr{D}^+$ such that, if $\psi_{\xi,\zeta}$ denotes the value of ψ at a point $(\xi, \zeta) \in X \times X$, the following conditions hold: for all $\xi, \zeta, \eta \in X$,

(PM1) $\psi_{\xi,\zeta}(t) = \tau_0(t)$ for all $t > 0$ if and only if $\xi = \zeta$;

(PM2) $\psi_{\xi,\zeta}(t) = \psi_{\zeta,\xi}(t)$ for all $t > 0$;
(PM3) $\psi_{\xi,\zeta}(t+s) \geq \psi_{\xi,\eta}(t) * \psi_{\eta,\zeta}(s)$ for all $t, s > 0$ and $\xi, \zeta, \eta \in X$.

Definition 1.4. [27] Let $(X, \psi, *)$ be a random metric space and H be a non-empty subset of X. Then for any $\xi \in X$, we define the distance from ξ to $\wp$ by

$$\psi_{\xi,\wp}(t) = \sup_{\zeta \in \wp} \psi_{\xi,\zeta}(t) \text{ for } t > 0.$$

Definition 1.5. [5] A non trivial ideal $\mathscr{I}_3$ of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ is said to be strongly admissible if $\{i\}\times\mathbb{N}\times\mathbb{N}$, $\mathbb{N}\times\{i\}\times\mathbb{N}$ and $\mathbb{N}\times\mathbb{N}\times\{i\}$ belong to $\mathscr{I}_3$ for each $i \in \mathbb{N}$. It is clear that a strongly admissible ideal is an admissible ideal.

If $\mathscr{I}_3^0 = \{A \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N} : (\exists m(A) \in \mathbb{N})\,(i, j, k \geq m(A)) \Rightarrow (i, j, k) \notin A\}$. Then, $\mathscr{I}_3^0$ is a non-trivial strongly admissible ideal and we can see that $\mathscr{I}_3$ is a strongly admissible ideal if and only if $\mathscr{I}_3^0 \subset \mathscr{I}_3$.

Definition 1.6. [5] An admissible ideal $\mathscr{I}_3 \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ satisfies the property (AP3) if for every countable family of mutually disjoint sets $\{\wp_1, \wp_2, \cdots\}$ belong to $\mathscr{I}_3$, there exits a countable family of sets $\{\mho_1, \mho_2, \cdots\}$ such that $\wp_j \triangle \mho_j \in \mathscr{I}_3$ is included in the finite union of rows and columns in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ for each $j \in \mathbb{N}$ and $G = \bigcup_{j=1}^{\infty} \mho_j \in \mathscr{I}_3$, consequently $\mho_j \in \mathscr{I}_3$ for each $j \in \mathbb{N}$.

Remark 1.7. Note that if $\mathscr{I}$ is the ideal $\mathscr{I}_0$ then $\mathscr{I}$-convergence coincides with the usual convergence and if we take $\mathscr{I}_d = \{A \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \delta_3(A) = 0\}$ then $\mathscr{I}_d$-convergence becomes statistical convergence.

2. Wijsman $\mathscr{I}_3$-convergence of triple sequences

In this part, we present and investigate the concepts of $\mathscr{I}_{W_3}$-convergence, $\mathscr{I}^*_{W_3}$-convergence, $\mathscr{I}_{W_3}$-Cauchy, and $\mathscr{I}^*_{W_3}$-Cauchy, as well as their relationships.

Definition 2.1. Let $(X, \psi, *)$ be a random metric spaces. A triple sequence of sets $\{\wp_{mnk}\}$ is Wijsman convergent to $\wp$ with respect to probabilistic metric ψ if for each $\xi \in X$,

$$\lim_{m,n,k\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t) \text{ for all } t > 0.$$

or equivalently,

$$\lim_{m,n,k\to\infty} \psi_{\xi,\wp_{mnk}-\wp}(t) = 1 \text{ for all } t > 0.$$

In this case, we write $\psi\text{-}\lim_{m,n,k\to\infty} \wp_{mnk} = \wp$.

Definition 2.2. Let $(X, \psi, *)$ be a probabilistic metric space. A triple sequence of sets $\{\wp_{mnk}\}$ is $\mathscr{I}_{W_3}$-convergent to $\wp$ if for every $\varepsilon > 0$, $\lambda \in (0, 1)$ and each $\xi \in X$,

$$\left\{(m, n, k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) \leq 1 - \lambda\right\} \in \mathscr{I}_3.$$

In this case, we write $\mathscr{I}^{\psi}_{W_3}\text{-}\lim_{m,n,k\to\infty} \wp_{mnk} = \wp$.

Definition 2.3. Let $(X, \psi, *)$ be a random metric space. A triple sequence of sets $\{\wp_{mnk}\}$ is $\mathscr{I}^{*}_{W_3}$-convergent to $\wp$ if there exits a set $\mathscr{A}_3 \in \mathscr{F}(\mathscr{I}_3)$, this means $B = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \setminus \mathscr{A}_3 \in \mathscr{I}_3$ such that ψ-$\lim_{m,n,k} \wp_{mnk} = \wp$ and $(m, n, k) \in \mathscr{A}_3$. In this case we write $\mathscr{I}^{*,\psi}_{W_3}$-$\lim \wp_{mnk} = \wp$.

Theorem 2.4. *Let $(X, \psi, *)$ be a random metric space. If a triple sequence of sets $\{\wp_{mnk}\}$ is $\mathscr{I}^{*,\psi}_{W_3}$-convergent, then it is $\mathscr{I}^{\psi}_{W_3}$ -convergent.*

Proof. Suppose that $\mathscr{I}^{*,\psi}_{W_3}$-$\lim_{m,n,k} \wp_{mnk} = \wp$. Then there exists a set $\mathscr{A}_3 \in \mathscr{F}(\mathscr{I}_3)$ (i.e. $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \setminus \mathscr{A}_3 = B \in \mathscr{I}_3$) such that for each $\xi \in X$, ψ-$\lim_{m,n,k} \wp_{mnk} = \wp$ and $(m, n, k) \in \mathscr{A}_3$. Let $\varepsilon > 0, \lambda \in (0, 1)$, then there exists $n_0 \in \mathbb{N}$ such that for each $\xi \in X$, $\psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) > 1 - \lambda$ for all $(m, n, k) \in \mathscr{A}_3$ and $m, n, k \geq n_0$. Then, for each $\varepsilon > 0, \lambda \in (0, 1)$, and $x \in X$, we have that

$$\begin{aligned} E(\lambda, x) &= \{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) \leq 1 - \lambda\} \subset B \cup (\mathscr{A}_3 \cap K)\,, \text{ where} \\ K &= (\{1, 2, \cdots, n_0 - 1\} \times \mathbb{N} \times \mathbb{N}) \cup (\mathbb{N} \times \{1, 2, \cdots, n_0 - 1\} \times \mathbb{N}) \cup (\mathbb{N} \times \mathbb{N} \times \{1, 2, \cdots, n_0 - 1\})\,. \end{aligned}$$

Since $\wp \cup (\mathscr{A}_3 \cap K) \in \mathscr{I}_3$, we get $E(\lambda, x) \in \mathscr{I}_3$ and consequently, $\mathscr{I}^{\psi}_{W_3}$-$\lim_{m,n,k} \wp_{mnk} = \wp$. ■

Theorem 2.5. *Let $(X, \psi, *)$ be a random metric space. If the ideal $\mathscr{I}_3$ has the property (AP3), then $\mathscr{I}^{\psi}_{W_3}$-convergent implies $\mathscr{I}^{*,\psi}_{W_3}$-convergence of triple sequences of sets.*

Proof. Suppose that $\mathscr{I}_3$ possesses property (AP3). Now, let $\mathscr{I}^{\psi}_{W_3}$-$\lim_{m,n,k} \wp_{mnk} = \wp$. Then, for each $\varepsilon > 0, \lambda \in (0, 1)$ and $\xi \in X$,

$$\begin{aligned} E(\lambda, x) = E_\lambda &= \{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) \leq 1 - \lambda\} \in \mathscr{I}_3. \text{ Take} \\ E_1 &= \{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) \leq 1\} \text{ and} \end{aligned}$$

$$E_n = E(n, x) = \{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : 1 - \frac{1}{n} \leq \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) < 1 - \frac{1}{n+1}\}$$

for $n \geq 2$ and $n \in \mathbb{N}$. It is clear that $E_i \cap E_k = 1 - \lambda$ for $i \neq k$ and $E_i \in \mathscr{I}_3$ for each $i \in \mathbb{N}$. By the property (AP3) there exits a sequence of sets $\{\mho_k\}_{k\in\mathbb{N}}$ such that $E_k \triangle \mho_k$ is included in finite union of rows and columns in $\mathbb{N} \times \mathbb{N} \times \mathbb{N}$ for each k and $G = \bigcup_{k=1}^{\infty} \mho_k \in \mathscr{I}_3$. Now, we shall prove that for $\mathscr{A}_3 = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \setminus G$ we have ψ-$\lim_{m,n,k} \wp_{mnk} = \wp$, where $(m, n, k) \in \mathscr{A}_3$.
Let $\eta \in (0, 1), \varepsilon > 0$ be given. Take $n \in \mathbb{N}$ such that $\frac{1}{n} < \eta$. Then

$$\{(m, n, k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) \leq 1 - \eta\} \subset \bigcup_{n=1}^{k} E_n.$$

Since $E_k \triangle \mho_k, k = 1, 2, \cdots$ are included in finite union of rows and columns, there exists $n_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{n=1}^{k} E_k\right) \cap \{(m, n, k) : m \geq n_0, n \geq n_0 \wedge k \geq n_0\} = \left(\bigcup_{n=1}^{k} \mho_k\right) \cap \{(m, n, k) : m \geq n_0, n \geq n_0 \wedge k \geq n_0\}.$$

If $m, n, k > n_0$ and $(m, n, k) \notin G$, then $(m, n, k) \notin \bigcup_{n=1}^{k} \mho_k$ and $(m, n, k) \notin \bigcup_{n=1}^{k} E_k$. This implies that $\psi_{\xi,\wp_{mnk}-\wp}(t) > 1 - \frac{1}{n} > 1 - \eta$. Therefore, $\mathscr{I}^{*,\psi}_{W_3}$-$\lim_{m,n,k} \wp_{mnk} = \wp$. ■

Definition 2.6. Let $(X, \psi, *)$ be a random metric space. A triple sequence of sets $\{\wp_{mnk}\}$ is $\mathscr{I}_{W_3}^{\psi}$-Cauchy if for every $\varepsilon > 0, \lambda \in (0,1)$ and each $\xi \in X$, there exits $p = p(\lambda), q = q(\lambda)$ and $r = r(\lambda)$ in $\mathbb{N}$ such that

$$\left\{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp_{pqr}}(\varepsilon) \leq 1-\lambda\right\} \in \mathscr{I}_3.$$

Theorem 2.7. *Let $(X, \psi, *)$ be a random metric space. If a triple sequence of sets $\{\wp_{mnk}\}$ is $\mathscr{I}_{W_3}^{\psi}$-convergent, then it is $\mathscr{I}_{W_3}^{\psi}$-Cauchy.*

Proof. Suppose that $\mathscr{I}_{W_3}^{\psi}\text{-}\lim_{m,n,k} \wp_{mnk} = \wp$. Then, for each $\varepsilon > 0, \lambda \in (0,1)$, and each $\xi \in X$, we have

$$G(x,\lambda) = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) \leq 1-\lambda\} \in \mathscr{I}_3.$$

This implies that

$$G^c(x,\lambda) = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) > 1-\lambda\} \notin \mathscr{I}_3.$$

Choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1-\lambda$. Since $\mathscr{I}_3$ is a strongly admissible ideal, then for all $p,q,r \in \mathbb{N}$ such that $(p,q,r) \in G^c(x,\lambda)$,

$$\psi_{\xi,\wp_{mnk}-\wp_{pqr}}(2\varepsilon) \geq \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) * \psi_{\wp_{pqr}-\wp,x}(\varepsilon) > (1-\eta) * (1-\eta) > 1-\lambda.$$

Hence

$$\{(n,m,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp_{pqr}}(2\varepsilon) \leq 1-\lambda\} \in \mathscr{I}_3$$

and so $\{\wp_{mnk}\}$ is $\mathscr{I}_{W_3}^{\psi}$-Cauchy. ∎

Definition 2.8. Let $(X, \psi, *)$ be a random metric space. A triple sequence of sets $\{\wp_{mnk}\}$ is $\mathscr{I}_{W_3}^{*,\psi}$-Cauchy if there exits a set $\mathscr{A}_3 \in \mathscr{F}(\mathscr{I}_3)$, this means that $B = \mathbb{N} \times \mathbb{N} \times \mathbb{N} \setminus \mathscr{A}_3 \in \mathscr{I}_3$ such that for each $\xi \in X$ and $(m,n,k), (p,q,r) \in \mathscr{A}_3$, $\lim_{m,n,k,p,q,r} \psi_{\xi,\wp_{mnk}-\wp_{pqr}}(t) = 1$ for all $t > 0$.

Theorem 2.9. *Let $(X, \psi, *)$ be a random metric space. If a triple sequence of sets $\wp_{mnk}$ is $\mathscr{I}_{W_3}^{*,\psi}$-Cauchy, then it is $\mathscr{I}_{W_3}^{\psi}$-Cauchy.*

Proof. Let $\wp_{mnk}$ be $\mathscr{I}_{W_3}^{*,\psi}$-Cauchy triple sequence, then by the definition, there exist a set $\mathscr{A}_3 \in \mathscr{F}(\mathscr{I}_3)$ (i.e., $\mathbb{N} \times \mathbb{N} \times \mathbb{N} \setminus \mathscr{A}_3 \in \mathscr{I}_3$) such that for each $\varepsilon > 0, \lambda \in (0,1)$ and for each $\xi \in X$, $\psi_{\xi,\wp_{mnk}-\wp_{pqr}}(\varepsilon) > 1-\lambda$ for all $(m,n,k), (p,q,r) \in \mathscr{A}_3$, $m,n,k,p,q,r > N = N(x,\lambda)$ and $N \in \mathbb{N}$. Then, for each $\varepsilon > 0, \lambda \in (0,1)$ and for each $\xi \in X$, we have

$$G(\lambda,x) = \{(m,n,k) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp_{pqr}}(\varepsilon) \leq 1-\lambda\} \subset B \cup (\mathscr{A}_3 \cap T_3),$$

where

$$T_3 = (\{1,2,\cdots,N-1\} \times \mathbb{N} \times \mathbb{N}) \cup (\mathbb{N} \times \{1,2,\cdots,N-1\} \times \mathbb{N}) \cup (\mathbb{N} \times \mathbb{N} \times \{1,2,\cdots,N-1\}).$$

Since $B \cup (\mathscr{A}_3 \cap T_3) \in \mathscr{I}_3$, so we have $G(\lambda,x) \in \mathscr{I}_3$. Therefore, $\wp_{mnk}$ is $\mathscr{I}_{W_3}^{\psi}$-Cauchy triple sequence. ∎

Theorem 2.10. *Let $(X, \psi, *)$ be a random metric space. If a triple sequence of sets $\wp_{mnk}$ is $\mathscr{I}_{W_3}^{*,\psi}$-convergent, then it is $\mathscr{I}_{W_3}$-Cauchy.*

Proof. Let $\mathscr{I}_{W_3}^{*,\psi}$-$\lim_{m,n,k} \wp_{mnk} = \wp$. So there exists a set $\mathscr{A}_3 \in \mathscr{F}(\mathscr{I}_3)$ (i.e., $B = \mathbb{N}\times\mathbb{N}\times\mathbb{N} \setminus \mathscr{A}_3 \in \mathscr{I}_3$) such that for each $\xi \in X$, we have

$$\psi\text{-} \lim_{\substack{m,n,k\to\infty \\ (m,n,k)\in\mathscr{A}_3}} \wp_{mnk} = \wp.$$

Let $\lambda \in (0,1)$, $\varepsilon > 0$ be given. Then there exists $n_0 \in \mathbb{N}$ such that for each $\xi \in X$, $\psi_{\wp_{mnk}-\wp}(\varepsilon) > 1-\lambda$ for all $(m,n,k) \in \mathscr{A}_3$ and $m,n,k \geq n_0$. Choose $\eta \in (0,1)$ such that $(1-\eta) * (1-\eta) > 1-\lambda$. Then

$$\psi_{\xi,\wp_{mnk}-\wp_{pqr}}(2\varepsilon) \geq \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) * \psi_{\xi,\wp_{pqr}-\wp}(\varepsilon) > (1-\eta) * (1-\eta) > 1-\lambda.$$

Therefore, for each $\xi \in X$ and $(m,n,k),(p,q,r) \in \mathscr{A}_3$, we have

$$\lim_{m,n,k,p,q,r\to\infty} \psi_{\xi,\wp_{mnk}-\wp_{pqr}}(\varepsilon) = 1.$$

This implies that $\wp_{mnk}$ is $\mathscr{I}_{W_3}^{*,\psi}$-Cauchy triple sequence and by the Theorem 2.9, $\wp_{mnk}$ is $\mathscr{I}_{W_3}^{\psi}$-Cauchy triple sequence. ■

From the proof of Theorem 2.10, we have

Corollary 2.11. *Let $(X, \psi, *)$ be a random metric space. If a triple sequence of sets $\{\wp_{mnk}\}$ is $\mathscr{I}_{W_3}^{*,\psi}$-convergent, then it is $\mathscr{I}_{W_3}^{*,\psi}$-Cauchy.*

Theorem 2.12. *Let $\{P_i\}_{i\in\mathbb{N}}$ be a countable collection of subsets of $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $\{P_i\}_{i\in\mathbb{N}} \in \mathscr{F}(\mathscr{I}_3)$ for each i, where $\mathscr{F}(\mathscr{I}_3)$ is a filter associate with a strongly admissible ideal $\mathscr{I}_3$ with the property (AP3). Then there exists a set $P \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $P \in \mathscr{F}(\mathscr{I}_3)$ and the set $P \setminus P_i$ is finite for all i.*

Proof. Let $\mathscr{A}_1 = \mathbb{N}\times\mathbb{N}\times\mathbb{N} \setminus P_1$, $\mathscr{A}_m = (\mathbb{N}\times\mathbb{N}\times\mathbb{N} \setminus P_m) \setminus (\mathscr{A}_1 \cup \mathscr{A}_2 \cup \cdots \cup \mathscr{A}_{m-1})$, $(m = 2,3,\cdots)$. It is easy to see that $\mathscr{A}_i \in \mathscr{I}_3$ for each i and $\mathscr{A}_i \cup \mathscr{A}_j = \emptyset$, when $i \neq j$. Then by (AP3) property of $\mathscr{I}_3$ we conclude that there exists a countable family of sets $\{B_1, B_2, \cdots\}$ such that $\mathscr{A}_j \triangle B_j \in \mathscr{I}_3^0$, i.e., $\mathscr{A}_j \triangle B_j$ is included in finite union of rows and columns in $\mathbb{N}\times\mathbb{N}\times\mathbb{N}$ for each j and $B = \bigcup_{j=1}^{\infty} B_j \in \mathscr{I}_3$. Put $P = \mathbb{N}\times\mathbb{N}\times\mathbb{N} \setminus B$. It is clear that $P \in \mathscr{F}(\mathscr{I}_3)$.

Now we prove that the set $P \setminus P_i$ is finite for each i. Assume that there exists a $j_0 \in \mathbb{N}$ such that $P \setminus P_{j_0}$ has infinitely many elements. Since each $\mathscr{A}_j \triangle B_j$ $(j = 1,2,3,\cdots,j_0)$ are included in finite union of rows and columns, there exists $m_0, n_0, k_0 \in \mathbb{N}$ such that

$$\left(\bigcup_{j=1}^{j_0} B_j\right) \cap C_{m_0n_0k_0} = \left(\bigcup_{j=1}^{j_0} \mathscr{A}_j\right) \cap C_{m_0n_0k_0} \tag{2.1}$$

where $C_{m_0n_0k_0} = \{(m,n,k) : m \geq m_0, n \geq n_0, k \geq k_0\}$. If $m \geq m_0, n \geq n_0, k \geq k_0$ and $(m,n,k) \notin B$, then $(m,n,k) \notin \bigcup_{j=1}^{j_0} B_j$ and so by (2.1), $(m,n,k) \notin \bigcup_{j=1}^{j_0} \mathscr{A}_j$.
Since $\mathscr{A}_{j_0} = \left(\mathbb{N}\times\mathbb{N}\times\mathbb{N} \setminus P_{j_0}\right) \setminus \bigcup_{j=1}^{j_0} \mathscr{A}_j$ and $(m,,n,k) \notin \mathscr{A}_{j_0}$, $(m,n,k) \notin \bigcup_{j=1}^{j_0} \mathscr{A}_j$ we have $(m,n,k) \in P_{j_0}$ for $m \geq m_0, n \geq n_0$ and $k \geq k_0$. Therefore, for all $m \geq m_0, n \geq n_0$ and $k \geq k_0$ we get $(m,n,k) \in P$

and $(m,n,k) \in P_{j_0}$. This shows that the set $P \setminus P_{j_0}$ has a finite number of elements. This contradicts our assumption that the set $P \setminus P_{j_0}$is an infinite set. ■

Theorem 2.13. *Let $(X,\psi,*)$ be a random metric space and $\mathscr{I}_3 \subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with property (AP3). Then the concepts $\mathscr{I}^{\psi}_{W_3}$-triple Cauchy sequence and $\mathscr{I}^{*,\psi}_{W_3}$-triple Cauchy sequence coincide.*

Proof. If $\{\wp_{mnk}\}$ is $\mathscr{I}^{*,\psi}_{3}$-triple Cauchy sequence of sets, then it is $\mathscr{I}^{\psi}_{W_3}$-triple Cauchy sequence by Theorem 2.9 (even if $\mathscr{I}_3$ does not have the (AP3) property).
So, we have to prove the converse. Let $\{\wp_{mnk}\}$ be an $\mathscr{I}^{\psi}_{W_3}$-triple Cauchy sequence of sets. Then by definition, there exists an $m_0 = m_0(\lambda), n_0 = n_0(\lambda), k = k_0(\lambda)$ such that

$$A(\lambda) = \left\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp_{m_0n_0k_0}}(\varepsilon) \le 1-\lambda\right\} \in \mathscr{I}_3$$

for every $\lambda \in (0,1), \varepsilon > 0$ and for each $\xi \in X$.
Let $P_i = \left\{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp_{r_is_it_i}}(\varepsilon) > 1-\frac{1}{i}\right\}$, $i = 1,2,\cdots,$ where $r_i = m_0\left(\frac{1}{i}\right), s_i = n_0\left(\frac{1}{i}\right), t_i = k_0\left(\frac{1}{i}\right)$. It is clear that $P_i \in \mathscr{F}(\mathscr{I}_3)$ for $i = 1,2,\cdots$. Since $\mathscr{I}_3$ has the property (AP3), then by Theorem 2.12 there exists a set $P \subset \mathbb{N}\times\mathbb{N}\times\mathbb{N}$ such that $P \in \mathscr{F}(I_3)$, and $P \setminus P_i$ is finite for all i.
Now we prove that

$$\lim_{\substack{m,n,k,r,s,t\to\infty\\(m,n,k),(r,s,t)\in P}} \psi_{\xi,\wp_{mnk}-\wp_{rst}}(\varepsilon) = 1.$$

To prove this, let $\varepsilon > 0$, $\lambda \in (0,1)$ and $w \in \mathbb{N}$ such that $\left(1-\frac{1}{w}\right)*\left(1-\frac{1}{w}\right) > 1-\lambda$. If $(m,n,k),(r,s,t) \in P$, then $P \setminus P_w$ is finite set, so there exists $q = q(w)$ such that $(m,n,k),(r,s,t) \in P$ for all $m,n,k,r,s,t > q(w)$. Therefore, $\psi_{\xi,\wp_{mnk}-\wp_{r_ws_wt_w}}(\varepsilon/2) > 1-\frac{1}{w}$ and $\psi_{z,\wp_{rst}-\wp_{r_ws_wt_w}}(\varepsilon/2) > 1-\frac{1}{w}$ for all $m,n,k,r,s,t > q(w)$. Hence it follows that

$$\psi_{z,\wp_{mnk}-\wp_{rst}}(\varepsilon) \ge \psi_{z,\wp_{mnk}-\wp_{r_ws_wt_w}}(\varepsilon/2) * \psi_{z,\wp_{rst}-\wp_{r_ws_wt_w}}(\varepsilon/2) > \left(1-\frac{1}{w}\right)*\left(1-\frac{1}{w}\right) > 1-\lambda$$

for all $m,n,k,r,s,t > q(w)$.
Thus, for any $\varepsilon > 0$, $\lambda \in (0,1)$ there exists $q = q(w)$ such that $m,n,k,r,s,t > q(w)$ and $m,n,k,r,s,t \in P \in \mathscr{F}(\mathscr{I}_3)$,

$$\psi_{z,\wp_{mnk}-\wp_{rst}}(\varepsilon) > 1-\lambda$$

for every $\xi \in X$. This shows that $\wp_{mnk}$ is an $\mathscr{I}^{*,\psi}_{W_3}$-triple Cauchy sequence in X. ■

The relationships discovered in this section are depicted in the diagram below.

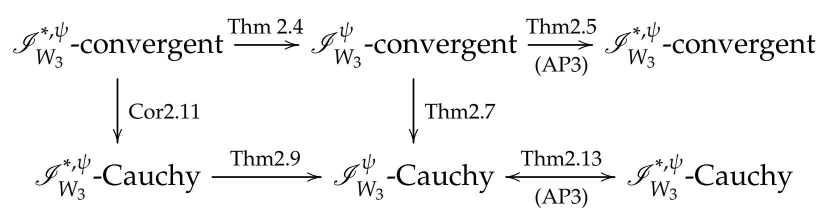

Diagram I

3. Wijsman randomly $\mathscr{I}_3$-convergence of triple sequences

In this part, we present and investigate the concepts of randomly $(\mathscr{I}^{\psi}_{W_3}, \mathscr{I}^{\psi}_{W})$-convergence and randomly $(\mathscr{I}^{*,\psi}_{W_3}, \mathscr{I}^{*,\psi}_{W})$-convergence, as well as their relationships.

Theorem 3.1. *Let $(X, \psi, *)$ be a random metric space and let $\{\wp_{mnk}\}$ be a triple sequence of sets. Then, for each $\xi \in X$ and $t > 0$,*

$$\lim_{m,n,k\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t) \ \textit{ implies } \mathscr{I}_3 - \lim_{m,n,k\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t).$$

Proof. Suppose that $\lim_{m,n,k\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t)$ for all $t > 0$. Then, for every $\varepsilon > 0, \lambda \in (0,1)$ and each $x \in X$ there exists $n_0 = n_0(\lambda, x) \in \mathbb{N}$ such that $\psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) > 1 - \lambda$ for all $m, n, k > n_0$. Hence, for each $\xi \in X$ we have

$$K(\lambda) = \{(m,n,k) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) \leq 1 - \lambda\} \subset M_3 \in \mathscr{I}_3,$$

where

$$M_3 = (\{1,2,\cdots,n_0\}\times\mathbb{N}\times\mathbb{N}) \cup (\mathbb{N}\times\{1,2,\cdots,n_0\}\times\mathbb{N}) \cup (\mathbb{N}\times\mathbb{N}\times\{1,2,\cdots,n_0\})$$

and then, we have that $K(\lambda) \in \mathscr{I}_3$. ∎

Definition 3.2. Let $(X, \psi, *)$ be a random metric space. A triple sequence $\{\wp_{mnk}\}$ is said to be Wijsman randomly convergent ($\mathscr{R}^{\psi}(\mathscr{W}_3, \mathscr{W})$-convergent) if it is convergent in Pringsheim's sense with respect to probabilistic metric ψ and fore each $\xi \in X$ and $t > 0$ the limits $\lim_{m\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t), n, k \in \mathbb{N}$, $\lim_{n\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t), m, k \in \mathbb{N}$ and $\lim_{k\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t), m, n \in \mathbb{N}$ exist .

We can see that if $\{\wp_{mnk}\}$ is Wijsman randomly convergent to H, the limits

$$\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty}\lim_{k\to\infty} \psi_{\xi,\wp_{mnk}}(t) &= \psi_{\xi,\wp}(t)\\
\lim_{m\to\infty}\lim_{k\to\infty}\lim_{n\to\infty} \psi_{\xi,\wp_{mnk}}(t) &= \psi_{\xi,\wp}(t)\\
\lim_{n\to\infty}\lim_{m\to\infty}\lim_{k\to\infty} \psi_{\xi,\wp_{mnk}}(t) &= \psi_{\xi,\wp}(t)\\
\lim_{n\to\infty}\lim_{k\to\infty}\lim_{m\to\infty} \psi_{\xi,\wp_{mnk}}(t) &= \psi_{\xi,\wp}(t)\\
\lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} \psi_{\xi,\wp_{mnk}}(t) &= \psi_{\xi,\wp}(t)\\
\lim_{k\to\infty}\lim_{n\to\infty}\lim_{m\to\infty} \psi_{\xi,\wp_{mnk}}(t) &= \psi_{\xi,\wp}(t)
\end{aligned}$$

exist for all $t > 0$. In this case we write $\mathscr{R}^{\psi}(\mathscr{W}_3, \mathscr{W})$-$\lim_{m,n,k\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t)$ or $\wp_{mnk} \xrightarrow{\mathscr{R}^{\psi}(\mathscr{W}_3,\mathscr{W})} H$.

Definition 3.3. Let $(X, \psi, *)$ be a random metric space. A triple sequence $\{\wp_{mnk}\}$ is said to be randomly $(\mathscr{I}^{\psi}_{W_3}, \mathscr{I}^{\psi}_{W})$-convergent ($\mathscr{R}^{\psi}(\mathscr{I}^{\psi}_{W_3}, \mathscr{I}^{\psi}_{W})$-convergent) if it is $\mathscr{I}^{\psi}_{W_3}$-convergent in Pringsheim's sense with respect to probabilistic metric ψ and for every $\varepsilon > 0$, $\lambda \in (0,1)$ and each $\xi \in X$, the following hold:

$$\Big\{k \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-\mathscr{A}_n}(\varepsilon) \leq 1-\lambda\Big\} \in \mathscr{I}, \text{ for some } \mathscr{A}_n \in X \text{ and each } n \in \mathbb{N},$$
$$\Big\{n \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-B_m}(\varepsilon) \leq 1-\lambda\Big\} \in \mathscr{I}, \text{ for some } B_m \in X \text{ and each } m \in \mathbb{N},$$
$$\Big\{m \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-C_k}(\varepsilon) \leq 1-\lambda\Big\} \in \mathscr{I}, \text{ for some } C_k \in X \text{ and each } k \in \mathbb{N}.$$

If $\{\wp_{mnk}\}$ is $\mathscr{R}^\psi(\mathscr{I}^\psi_{W_3}, \mathscr{I}^\psi_W)$-convergent to H, then we write $\mathscr{R}^\psi(\mathscr{I}^\psi_{W_3}, \mathscr{I}^\psi_W)\text{-}\lim_{m,n,k\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t)$ or $\wp_{mnk} \xrightarrow{\mathscr{R}^\psi(\mathscr{I}^\psi_{W_3},\mathscr{I}^\psi_W)} \wp$.

Theorem 3.4. *Let $(X,\psi,*)$ be a random metric space. If a triple sequence $\{\wp_{mnk}\}$ is $\mathscr{R}^\psi(\mathscr{W}_3,\mathscr{W})$-convergent, then $\{\wp_{mnk}\}$ is $\mathscr{R}^\psi(\mathscr{I}^\psi_{W_3}, \mathscr{I}^\psi_W)$-convergent.*

Proof. Let $\{\wp_{mnk}\}$ be $\mathscr{R}^\psi(\mathscr{W}_3,\mathscr{W})$-convergent to $\wp$. Then, $\{\wp_{mnk}\}$ is convergent to H in Pringsheim's sense with respect to probabilistic metric ψ and for each $\xi \in X$ the limits $\lim_{m\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t), n,k \in \mathbb{N}$, $\lim_{n\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t), m,k \in \mathbb{N}$ and $\lim_{k\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t), m,n \in \mathbb{N}$ exist. By the Theorem 3.1, we get that $\{\wp_{mnk}\}$ is $\mathscr{I}^\psi_{W_3}$-convergent. Besides, for each $\varepsilon > 0, \lambda \in (0,1)$ and each $\xi \in X$, there exist $m = m_0(\lambda,x), n = n_0(\lambda,x), k = k_0(\lambda,x)$ such that $\psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) > 1-\lambda$ for each $m,n \in \mathbb{N}$ and $k > k_0$, $\psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) > 1-\lambda$ for each $n,k \in \mathbb{N}$ and $m > m_0$ and $\psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) > 1-\lambda$ for each $m,k \in \mathbb{N}$ and $n > n_0$. Now, since $\mathscr{I}$ is an admissible ideal, for every $\varepsilon > 0, \lambda \in (0,1)$ and each $\xi \in X$ we have $\{n \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) \leq 1-\lambda\} \subset \{1,2,\cdots,n_0\} \in \mathscr{I}$, $\{m \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) \leq 1-\lambda\} \subset \{1,2,\cdots,m_0\} \in \mathscr{I}$ and $\{k \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-\wp}(\varepsilon) \leq 1-\lambda\} \subset \{1,2,\cdots,k_0\} \in \mathscr{I}$. Therefore, $\{\wp_{mnk}\}$ is $\mathscr{R}^\psi(\mathscr{I}^\psi_{W_3}, \mathscr{I}^\psi_W)$-convergent. ■

Definition 3.5. Let $(X,\psi,*)$ be a random metric space. A triple sequence $\{\wp_{mnk}\}$ is said to be randomly $(\mathscr{I}^{*,\psi}_{W_3}, \mathscr{I}^{*,\psi}_W)$-convergent ($\mathscr{R}^\psi(\mathscr{I}^{*,\psi}_{W_3}, \mathscr{I}^{*,\psi}_W)$-convergent) if there exist the sets $A \in \mathscr{F}(\mathscr{I}_3), \mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3 \in \mathscr{F}(\mathscr{I})$ (i.e., $\mathbb{N}\times\mathbb{N}\times\mathbb{N} \setminus A \in \mathscr{I}_3, \mathbb{N} \setminus \mathscr{A}_i \in \mathscr{I}, i = 1,2,3$) $\lim_{m,n,k\to\infty} \psi_{\xi,\wp_{mnk}}(t)$, where $m,n,k \in A$, $\lim_{m\to\infty} \psi_{\xi,\wp_{mnk}}(t)$, where $m \in \mathscr{A}_1$, $\lim_{n\to\infty} \psi_{\xi,\wp_{mnk}}(t)$, where $n \in \mathscr{A}_2$ and $\lim_{k\to\infty} \psi_{\xi,\wp_{mnk}}(t)$, where $k \in \mathscr{A}_3$ exist for fixed $m \in \mathbb{N}, n \in \mathbb{N}$ and $k \in \mathbb{N}$, respectively.

If $\mathscr{R}^\psi(\mathscr{I}^{*,\psi}_{W_3}, \mathscr{I}^{*,\psi}_W)$-convergent to H, then for each $\xi \in X$ the limits

$$\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty}\lim_{k\to\infty} \psi_{\xi,\wp_{mnk}}(t) &= \psi_{\xi,\wp}(t)\\
\lim_{m\to\infty}\lim_{k\to\infty}\lim_{n\to\infty} \psi_{\xi,\wp_{mnk}}(t) &= \psi_{\xi,\wp}(t)\\
\lim_{n\to\infty}\lim_{m\to\infty}\lim_{k\to\infty} \psi_{\xi,\wp_{mnk}}(t) &= \psi_{\xi,\wp}(t)\\
\lim_{n\to\infty}\lim_{k\to\infty}\lim_{m\to\infty} \psi_{\xi,\wp_{mnk}}(t) &= \psi_{\xi,\wp}(t)\\
\lim_{k\to\infty}\lim_{m\to\infty}\lim_{n\to\infty} \psi_{\xi,\wp_{mnk}}(t) &= \psi_{\xi,\wp}(t)\\
\lim_{k\to\infty}\lim_{n\to\infty}\lim_{m\to\infty} \psi_{\xi,\wp_{mnk}}(t) &= \psi_{\xi,\wp}(t)
\end{aligned}$$

exist for all $t > 0$ and are equal to H. In this case, we write $\mathscr{R}^{\psi}(\mathscr{I}^{*,\psi}_{W_3}, \mathscr{I}^{*,\psi}_{W})$-$\lim_{m,n,k\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{\xi,\wp}(t)$ for all $t > 0$ or $\wp_{mnk} \xrightarrow{\mathscr{R}^{\psi}(\mathscr{I}^{*,\psi}_{W_3}, \mathscr{I}^{*,\psi}_{W})} H$.

Theorem 3.6. *Let $(X,\psi,*)$ be a random metric space. If a triple sequence $\{\wp_{mnk}\}$ is $\mathscr{R}^{\psi}(\mathscr{I}^{*,\psi}_{W_3}, \mathscr{I}^{*,\psi}_{W})$-convergent, then $\{\wp_{mnk}\}$ is $\mathscr{R}^{\psi}(\mathscr{I}^{\psi}_{W_3}, \mathscr{I}^{\psi}_{W})$-convergent.*

Proof. Suppose that $\{\wp_{mnk}\}$ is $\mathscr{R}^{\psi}(\mathscr{I}^{*,\psi}_{W_3}, \mathscr{I}^{*,\psi}_{W})$-convergent. Then, it is $\mathscr{I}^{*,\psi}_{W_3}$-convergent and it follows then by Theorem 2.4 that $\{\wp_{mnk}\}$ is $\mathscr{I}^{\psi}_{W_3}$-convergent. In addition, there exist $\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3 \in \mathscr{F}(\mathscr{I})$ such that for every $\varepsilon > 0, \lambda \in (0,1)$ and each $\xi \in X$, there exists $k_0 \in \mathbb{N}$ such that for all $k \geq k_0$, $k \in \mathscr{A}_1$ $\psi_{\xi,\wp_{mnk}-B_m}(\varepsilon) > 1-\lambda$ for some $B_m \in X$ and each $m \in \mathbb{N}$, also there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$, $m \in \mathscr{A}_2$, $\psi_{\wp_{x,mnk}-C_j}(\varepsilon) > 1-\lambda$ for some $C_j \in X$ for all $j \in \mathbb{N}$, moreover, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $n \in \mathscr{A}_3$, $\psi_{\xi,\wp_{mnk}-D_n}(\varepsilon) > 1-\lambda$ for some $D_n \in X$ for all $n \in \mathbb{N}$. Consequently, for every $\varepsilon > 0, \lambda \in (0,1)$ and each $\xi \in X$, we have

$$Q(\lambda) = \{m \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-B_m}(\varepsilon) > 1-\lambda\} \subset W_1 \cup \{1,2,\cdots,m_0-1\}(n,k \in \mathbb{N})$$
$$S(\lambda) = \{n \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-C_n}(\varepsilon) > 1-\lambda\} \subset W_2 \cup \{1,2,\cdots,n_0-1\}(m,k \in \mathbb{N})$$
$$V(\lambda) = \{k \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-D_k}(\varepsilon) > 1-\lambda\} \subset W_3 \cup \{1,2,\cdots,k_0-1\}(n,m \in \mathbb{N})$$

for $W_1, W_2, W_3 \in \mathscr{I}$. Since $\mathscr{I}$ is an admissible ideal, we $W_1 \cup \{1,2,\cdots,m_0-1\} \in \mathscr{I}$, $W_2 \cup \{1,2,\cdots,n_0-1\} \in \mathscr{I}$ and $W_3 \cup \{1,2,\cdots,k_0-1\} \in \mathscr{I}$. Hence, we get $Q(\lambda) \in \mathscr{I}, S(\lambda) \in \mathscr{I}$ and $V(\lambda) \in \mathscr{I}$ Consequently, this proves that $\{\wp_{mnk}\}$ is $\mathscr{R}^{\psi}(\mathscr{I}^{\psi}_{W_3}, \mathscr{I}^{\psi}_{W})$-convergent. ■

Theorem 3.7. *Let $(X,\psi,*)$ be a random metric space. Let $\mathscr{I} \subset 2^{\mathbb{N}}$ be an admissible ideal with the property (AP) and $\mathscr{I}_3 \subset 2^{\mathbb{N}\times\mathbb{N}\times\mathbb{N}}$ be a strongly admissible ideal with the property (AP3). If a triple sequence $\{\wp_{mnk}\}$ is $\mathscr{R}^{\psi}(\mathscr{I}^{\psi}_{W_3}, \mathscr{I}^{\psi}_{W})$-convergent, then $\{\wp_{mnk}\}$ is $\mathscr{R}^{\psi}(\mathscr{I}^{*,\psi}_{W_3}, \mathscr{I}^{*,\psi}_{W})$-convergent.*

Proof. Let $\{\wp_{mnk}\}$ be $\mathscr{R}^{\psi}(\mathscr{I}^{\psi}_{W_3}, \mathscr{I}^{\psi}_{W})$-convergent. Then $\{\wp_{mnk}\}$ is $\mathscr{I}^{\psi}_{W_3}$-convergent and hence it follows by Theorem 2.5 that, $\{\wp_{mnk}\}$ is $\mathscr{I}^{*,\psi}_{W_3}$-convergent. Besides, for each $\varepsilon > 0, \lambda \in (0,1)$ and each $\xi \in X$ we have

$$Q(\lambda) = \{k \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-B_m}(\varepsilon) > 1-\lambda\} \in \mathscr{I}, \text{ for some } B_m \in X \text{ and each } m \in \mathbb{N},$$
$$S(\lambda) = \{m \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-C_n}(\varepsilon) > 1-\lambda\} \in \mathscr{I}, \text{ for some } C_n \in X \text{ and each } n \in \mathbb{N},$$
$$V(\lambda) = \{n \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-D_k}(\varepsilon) > 1-\lambda\} \in \mathscr{I}, \text{ for some } D_k \in X \text{ and each } k \in \mathbb{N}.$$

Now, for each $\xi \in X$ take $E_1 = \{n,k \in \mathbb{N} : \psi_{\xi,\wp_{mnk}-B_m}(\varepsilon) \leq 1\}$ and $E_j = \{k \in \mathbb{N} : 1-\frac{1}{j} \leq \psi_{\xi,\wp_{mnk}-B_m}(\varepsilon) \leq 1-\frac{1}{j+1}\}$ for some $B_m \in X$ and for each $m \in \mathbb{N}$. Clearly, $E_l \cap E_i = \emptyset$ for $l \neq i$ and $E_j \in \mathscr{I}$ for each $j \in \mathbb{N}$. By the property (AP), there is a countable family of sets $P_1, P_2, \cdots$ in $\mathscr{I}$ such that $P_j \triangle E_j$ is a finite set for each $j \in \mathbb{N}$ and $P = \bigcup_{j\in\mathbb{N}} P_j \in \mathscr{I}$. We will show that for some $B_m \in X$ and each $m \in \mathbb{N}$, $\lim_{k\to\infty} \psi_{\xi,\wp_{mnk}}(t) = \psi_{x,B_m}(t)$, where $k \in A$, for some $A = \mathbb{N} \setminus P \in \mathscr{F}(\mathscr{I})$ for each $t > 0$ and each $\xi \in X$. To do this, Let $\eta \in (0,1)$. Take $i \in \mathbb{N}$ such that $\eta > \frac{1}{i}$. Then, for each $\xi \in X$ we have $\{n, k \in \mathbb{N} : \psi_{\xi, \wp_{mnk} - B_m}(t) \leq 1 - \eta\} \subset \bigcup_{r=1}^{i-1} E_r$ for some $B_m \in X$ and each $m \in \mathbb{N}$. Since each $E_r \triangle P_r$ is a finite set for $r \in \{1, 2, \cdots, i-1\}$, there exists $s_0 \in \mathbb{N}$ such that $\bigcup_{r=1}^{i-1} P_r \cap \{s \in \mathbb{N} : s \geq s_0\} = \bigcup_{r=1}^{i-1} E_r \cap \{s \in \mathbb{N} : s \geq s_0\}$. If $s \geq s_0$ and $s \notin P$, then $s \notin \bigcup_{r=1}^{i-1} P_r$ and $s \notin \bigcup_{r=1}^{i-1} E_r$. Therefore, for each $t > 0, \eta \in (0, 1)$ and each $\xi \in X$ we have $\psi_{\xi, \wp_{mnk} - B_m}(\varepsilon) > 1 - \frac{1}{i} > 1 - \eta$ for some $B_m \in X$ and each $m \in \mathbb{N}$. This implies that $\lim_{k \to \infty} \psi_{\xi, \wp_{mnk}}(t) = \psi_{x, B_m}(t)$ for some $k \in A$. Hence, for each $\xi \in X$ we have $\mathscr{I}_W^{*,\psi}$-$\lim_{k \to \infty} \psi_{\xi, \wp_{mnk}}(t) = \psi_{x, B_m}(t)$ for some $B_m \in X$ and each $m \in \mathbb{N}$. By the similar argument, we have the same results for the sets $S(\lambda)$ and $V(\lambda)$. Hence, $\{\wp_{mnk}\}$ is $\mathscr{R}^{\psi}(\mathscr{I}_{W_3}^{*,\psi}, \mathscr{I}_W^{*,\psi})$-convergent. ■

The diagram below depicts the relationships discovered in this section.

$$\begin{array}{ccc} \mathscr{R}^{\psi}(\mathscr{I}_{W_3}^{\psi}, \mathscr{I}_W^{\psi})\text{-convergent} & \xleftarrow{\text{Thm3.4}} & \mathscr{R}^{\psi}(\mathscr{W}_3, \mathscr{W})\text{-convergent} \\ \text{Thm3.6} \uparrow & & \\ \mathscr{R}^{\psi}(\mathscr{I}_{W_3}^{*,\psi}, \mathscr{I}_W^{*,\psi})\text{-convergent} & \xleftarrow[\text{(AP3)}]{\text{Thm3.7}} & \mathscr{R}^{\psi}(\mathscr{I}_{W_3}^{\psi}, \mathscr{I}_W^{\psi})\text{-convergent} \end{array}$$

Diagram II

4. Conclusion

Wijsman $\mathscr{I}_3^{\psi}$-Convergence for triple sequences and Wijsman randomly $\mathscr{I}_3^{\psi}$-Convergence for triple sequences (see Diagrams I and II) have been defined and examined in this study. Furthermore, some intriguing outcomes and relationships between these notions were demonstrated. Future work will provide definitions of $\mathscr{R}^{\psi}(\mathscr{I}_{W_3}^{\psi}, \mathscr{I}_W^{\psi})$-Cauchy and $\mathscr{R}^{\psi}(\mathscr{I}_{W_3}^{*,\psi}, \mathscr{I}_W^{*,\psi})$-Cauchy and then investigate some relationships with the ideas discussed in this paper. On the other hand, these ideas can be applied to the lacunary sequence.

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