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Solution of an Algebraic Linear System of Equations Using Fixed Point Results in C^*-Algebra Valued Extended Branciari S_b-Metric Spaces

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Abstract. This study explores the realm of metric spaces, advancing beyond conventional boundaries by introducing two innovative types of metrics known as generalized Branciari-type metrics. Through exacting examination and exemplification, we shed light on the intricacies of these newly defined metric spaces and their extended versions. By drawing parallels with established theorems such as Banach and Kannan, we unveil corollaries that establish necessary symmetric conditions for the existence and uniqueness of fixed points concerning self-operators within these spaces. The inclusion of illustrative examples not only bolsters our theoretical framework but also underscores the practical relevance of our findings. Furthermore, we utilize our research to address real-world applications, showcasing how our results can be employed to determine the existence of unique solutions for algebraic systems of linear equations, thereby bridging the theoretical and applied aspects of mathematical exploration. Through these interventions, our study significantly contributes to the comprehensive understanding and utilization of all the properties in metric spaces within diverse mathematical contexts.

1. Introduction

Despite its history dating back more than a century, fixed point theory remains a fascinating area of study, particularly due to the role of straightness in its applications. The appeal of fixed point results is seen in their wide range of applications, often leveraging symmetric properties. The Banach contraction principle, discovered by Banach in 1922, is essentially the primary finding on fixed points for mappings of contractive types. This outcome has been demonstrated to be

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an extremely helpful instrument for ensuring the existence and uniqueness of solutions to many types of difficult problems that arise in a variety of fields, both inside and outside of mathematics. The traditional Banach contraction concept has been expanded upon and developed in a variety of ways, with recent advancements emphasizing the significance of straightness in enhancing the robustness and applicability of these results (see [11–15, 17, 23, 25, 28–33]).

Abbreviations:

uCA	unital C^*-algebra
avMS	C^*-algebra valued metric space
bMS	$b-$metric space
avbMS	C^*-algebra valued $b-$metric space
ebMS	extended $b-$metric space
avebMS	C^*-algebra valued extended $b-$metric space
SMS	$S-$metric space
avSMS	C^*-algebra valued $S-$metric space
eSMS	extended $S-$metric space
aveSMS	C^*-algebra valued extended $S-$metric space
SbMS	S_b-metric space
eSbMS	extended S_b-metric space
aveSbMS	C^*-algebra valued extended S_b-metric space
BSbMS	Branciari S_b-metric space
avBSbMS	C^*-algebra valued Branciari S_b-metric space
aveBSbMS	C^*-algebra valued extended Branciari S_b-metric space

By swapping out the range set $\mathbb{R}$ for a uCA, Ma et al. [3] introduced a class of avMS, a more broad class of metric spaces in 2014. He then used these classes to demonstrate various fixed point results, emphasizing the role of symmetry in these spaces. There are many generalizations of the theory in the context of Banach algebra (see, for more, [4–8]). To broaden the scope of applicability, Czerwik [10] developed the idea of bMS as a notional advancement of metric spaces and demonstrated fixed point findings as a balanced counterpart to the Banach contraction theorem. In fact, a sizeable body of work has already been written about the theory of fixed points in bMS for single-valued and multivalued mappings, showcasing the significance of the properties in these theoretical advancements (see [3, 9, 10, 18, 19, 25, 27]).

Kamran [26], on the other hand, developed ebMS as a generalization of bMS, incorporating symmetry into the structure. Many studies subsequently validated fixed point results in these generalized metric spaces, including their existence and uniqueness. A year later, Ma et al. [16] introduced another generalization of avMS, called avbMS, and demonstrated various fixed point results, highlighting symmetric properties. Sedghi et al. in [18] defined a new metric dependent on three variables and introduced SMS. Consequently, Ege et al. [20] introduced avSMS. Combining the definitions of bMS and SMS, Rohen et al. [21, 23] introduced more generalized SbMS and proved many contraction-type theorems, emphasizing symmetry. In 2018, Kalaivani et al. [24] proposed future work and defined avSbMS, further exploring symmetric aspects in metric space theory.

Our research builds upon the foundations laid by Kalaivani et al. [24] and Roy [25], who examined the fixed point problems in avSMS and BSbMS respectively, with a particular focus on the extension of metric spaces. We extend these spaces by framing new postulates to derive avBSbMS and aveBSbMS. Subsequently, we establish conditions that guarantee the existence and uniqueness of a symmetric fixed point for self-operators. We substantiate our conclusions with illustrative examples. We deduced some corollaries and proved two main theorems, which are analogous to the very famous Banach and Kannan type theorems, incorporating symmetry. Further, we explore a method to check the unique existence of solutions of systems of algebraic linear equations. In conclusion, our work introduces innovative techniques for addressing fixed point problems, challenging the traditional assumption of the range set in a generalized metric structure. Our methodology has implications for enhancing effectiveness and enabling mathematical models to better represent real-world scenarios, leading to more insightful decision-making across various scientific domains.

2. Preliminaries

Let us denote C by a unital C^*-algebra (see [1, 2]) with linear involution $*$ and unit element g such that for all $c_1, c_2 \in \mathsf{C}$, $(c_1c_2)^* = c_1^*c_2^*$, and $c^{**} = c$. An element $c \in \mathsf{C}$ is positive, if $c \succeq \theta$, where θ is a zero element in C. Let $c \in \mathsf{C}_h = \{c \in \mathsf{C} : c = c^*\}$, then a partial ordering $\succeq$, we can define on C_h by $c_1 \preceq c_2$ if and only if $c_2 - c_1 \succeq \theta$. We denote the set $\{c \in \mathsf{C} : c \succeq \theta\}$ by C_+ and $|c| = (c^*c)^{\frac{1}{2}}$, C_c will denote the set $\{c_1 \in \mathsf{C} : c_1c_2 = c_2c_1, \text{ for all } c_2 \in \mathsf{C}\}$ and $C_g = \{c_1 \in \mathsf{C} : c_1c_2 = c_2c_1, \text{ for all } c_2 \in \mathsf{C} \text{ and } c_1 \succeq g\}$.

Lemma 2.1. *[1] In a unital C^*-algebra C with a zero element θ and a unit element g, we have followings*

- $(L1)$ $c \preceq g$ *if and only if* $\|c\| \leq 1$, *for all* $c \in \mathsf{C}$.
- $(L2)$ $g - c$ *is invertible such that* $\|c(g-c)^{-1}\| < 1$, *for all* $c \in \mathsf{C}_+$ *with* $\|c\| \leq \frac{1}{2}$.
- $(L3)$ $c_1c_2 \succeq \theta$, *for all* $c_1, c_2 \in \mathsf{C}_+$ *with* $c_1c_2 = c_2c_1$.
- $(L4)$ *for* $c_1, c_2 \in \mathsf{C}_+$ *with* $c_1 \preceq c_2$ *implies* $c^*c_1c \preceq c^*c_2c$, *for all* $c \in \mathsf{C}$.
- $(L5)$ *for* $c \in \mathsf{C}_c$ *if* $g - c \in \mathsf{C}_1$ *is invertible, then* $(g-c)^{-1}c_1 \succeq (g-c)^{-1}c_2$, *for all* $c_1, c_2 \in \mathsf{C}$ *with* $c_1 \succeq c_2 \succeq \theta$.

Now we recall the definitions of some previously known metric spaces.

Definition 2.1. *[10] A pair (V, d) with non empty set V and $d : V \times V \longrightarrow \mathbb{R}$ satisfying*

$(bM1)$ $d(\vartheta_1, \vartheta_2) \geq 0$ *and* $d(\vartheta_1, \vartheta_2) = 0$ *if and only if* $\vartheta_1 = \vartheta_2$,
$(bM2)$ $d(\vartheta_1, \vartheta_2) = d(\vartheta_2, \vartheta_1)$,
$(bM3)$ $d(\vartheta_1, \vartheta_2) \leq b[d(\vartheta_1, u) + d(u, \vartheta_2)]$,

for all $\vartheta_1, \vartheta_2 \in V$ *and for all* $u \in V \setminus \{\vartheta_1, \vartheta_2\}$ *with* $b \geq 1$, *is called a bMS.*

Definition 2.2. *[26] A pair* (V, d) *with non empty set* V, $e : V \times V \longrightarrow [1, +\infty)$ *and* $d : V \times V \longrightarrow \mathbb{R}$ *satisfying*

$(ebM1)$ $d(\vartheta_1, \vartheta_2) \geq 0$ *and* $d(\vartheta_1, \vartheta_2) = 0$ *if and only if* $\vartheta_1 = \vartheta_2$,
$(ebM2)$ $d(\vartheta_1, \vartheta_2) = d(\vartheta_2, \vartheta_1)$,
$(ebM3)$ $d(\vartheta_1, \vartheta_2) \leq e(\vartheta_1, \vartheta_2)[d(\vartheta_1, u) + d(u, \vartheta_2)]$,

for all $\vartheta_1, \vartheta_2 \in V$ *and for all* $u \in V \setminus \{\vartheta_1, \vartheta_2\}$ *with* $b \geq 1$, *is called an ebMS.*

Definition 2.3. *[16] A triplet* (V, C, d) *with non empty set* V *and* $d : V \times V \longrightarrow \mathsf{C}$ *satisfying*

$(avbM1)$ $d(\vartheta_1, \vartheta_2) \geq \theta$ *and* $d(\vartheta_1, \vartheta_2) = \theta$ *if and only if* $\vartheta_1 = \vartheta_2$,
$(avbM2)$ $d(\vartheta_1, \vartheta_2) = d(\vartheta_2, \vartheta_1)$,
$(avbM3)$ $d(\vartheta_1, \vartheta_2) \leq b[d(\vartheta_1, u) + d(u, \vartheta_2)]$,

for all $\vartheta_1, \vartheta_2 \in V$ *and for all* $u \in V \setminus \{\vartheta_1, \vartheta_2\}$ *with* $b \in \mathsf{C}_g$, *is called an avbMS.*

Definition 2.4. *[27] A triplet* (V, C, d) *with non empty set* V, $e : V \times V \longrightarrow \mathsf{C}_g$ *and* $d : V \times V \longrightarrow \mathsf{C}$ *satisfying*

$(avebM1)$ $d(\vartheta_1, \vartheta_2) \geq \theta$ *and* $d(\vartheta_1, \vartheta_2) = \theta$ *if and only if* $\vartheta_1 = \vartheta_2$,
$(avebM2)$ $d(\vartheta_1, \vartheta_2) = d(\vartheta_2, \vartheta_1)$,
$(avebM3)$ $d(\vartheta_1, \vartheta_2) \leq e(\vartheta_1, \vartheta_2)[d(\vartheta_1, u) + d(u, \vartheta_2)]$,

for all $\vartheta_1, \vartheta_2 \in V$ *and for all* $u \in V \setminus \{\vartheta_1, \vartheta_2\}$, *is called an avebMS.*

Definition 2.5. *[18] A pair* (V, d) *with non empty set* V *and* $d : V \times V \times V \longrightarrow \mathbb{R}$ *satisfying*

$(SM1)$ $d(\vartheta_1, \vartheta_2, \vartheta_3) \geq 0$,
$(SM2)$ $d(\vartheta_1, \vartheta_2, \vartheta_3) = 0$ *if and only if* $\vartheta_1 = \vartheta_2 = \vartheta_3$,
$(SM3)$ $d(\vartheta_1, \vartheta_2, \vartheta_3) \leq d(\vartheta_1, \vartheta_1, u) + d(\vartheta_2, \vartheta_2, u) + d(\vartheta_3, \vartheta_3, u)$,

for all $\vartheta_1, \vartheta_2, \vartheta_3, u \in V$, *is called a SMS.*

Definition 2.6. *[20] A triplet* (V, C, d) *with non empty set* V *and* $d : V \times V \times V \longrightarrow \mathsf{C}$ *satisfying*

$(avSM1)$ $d(\vartheta_1, \vartheta_2, \vartheta_3) \geq \theta$,
$(avSM2)$ $d(\vartheta_1, \vartheta_2, \vartheta_3) = \theta$ *if and only if* $\vartheta_1 = \vartheta_2 = \vartheta_3$,
$(avSM3)$ $d(\vartheta_1, \vartheta_2, \vartheta_3) \leq d(\vartheta_1, \vartheta_1, u) + d(\vartheta_2, \vartheta_2, u) + d(\vartheta_3, \vartheta_3, u)$,

for all $\vartheta_1, \vartheta_2, \vartheta_3, u \in V$, *is called an avSMS.*

Definition 2.7. *[21,23] A pair (V,d) with non empty set V and $d : V\times V\times V \longrightarrow \mathbb{R}$ satisfying*

(S_bM1) $d(\vartheta_1,\vartheta_2,\vartheta_3)\geq 0$ *and* $d(\vartheta_1,\vartheta_2,\vartheta_3)=0$ *if and only if* $\vartheta_1=\vartheta_2=\vartheta_3$,
(S_bM2) $d(\vartheta_1,\vartheta_1,\vartheta_2)=d(\vartheta_2,\vartheta_2,\vartheta_1)$,
(S_bM3) $d(\vartheta_1,\vartheta_2,\vartheta_3)\leq s_b[d(\vartheta_1,\vartheta_1,u)+d(\vartheta_2,\vartheta_2,u)+d(\vartheta_3,\vartheta_3,u)]$,

for all $\vartheta_1,\vartheta_2,\vartheta_3,u\in V$ *and* $s_b\geq 1$, *is called a SbMS.*

Definition 2.8. *[28] A pair (V,d) with non empty set V, $e : V\times V\times V \longrightarrow [1,+\infty)$ and $d : V\times V\times V \longrightarrow \mathbb{R}$ satisfying*

(eS_bM1) $d(\vartheta_1,\vartheta_2,\vartheta_3)\geq 0$,
(eS_bM2) $d(\vartheta_1,\vartheta_2,\vartheta_3)=0$ *if and only if* $\vartheta_1=\vartheta_2=\vartheta_3$,
(eS_bM3) $d(\vartheta_1,\vartheta_2,\vartheta_3)\leq e(\vartheta_1,\vartheta_2,\vartheta_3)[d(\vartheta_1,\vartheta_1,u)+d(\vartheta_2,\vartheta_2,u)+d(\vartheta_3,\vartheta_3,u)]$,

for all $\vartheta_1,\vartheta_2,\vartheta_3,u\in V$, *is called an eSbMS.*

Definition 2.9. *[24] A triplet (V,C,d) with non empty set V and $d : V\times V\times V \longrightarrow \mathsf{C}$ satisfying*

(avS_bM1) $d(\vartheta_1,\vartheta_2,\vartheta_3)\geq \theta$,
(avS_bM2) $d(\vartheta_1,\vartheta_2,\vartheta_3)=\theta$ *if and only if* $\vartheta_1=\vartheta_2=\vartheta_3$,
(avS_bM3) $d(\vartheta_1,\vartheta_2,\vartheta_3)\leq s_b[d(\vartheta_1,\vartheta_1,u)+d(\vartheta_2,\vartheta_2,u)+d(\vartheta_3,\vartheta_3,u)]$,

for all $\vartheta_1,\vartheta_2,\vartheta_3,u\in V$ *and* $s_b\in\mathsf{C}_g$, *is called an avSbMS.*

Definition 2.10. *[25] A pair (V,d) with non empty set V and $d : V\times V\times V \longrightarrow \mathbb{R}$ satisfying*

(BS_bM1) $d(\vartheta_1,\vartheta_2,\vartheta_3)\geq 0$,
(BS_bM2) $d(\vartheta_1,\vartheta_2,\vartheta_3)=0$ *if and only if* $\vartheta_1=\vartheta_2=\vartheta_3$,
(BS_bM3) $d(\vartheta_1,\vartheta_2,\vartheta_2)\leq s_b[d(\vartheta_1,\vartheta_1,u)+d(\vartheta_2,\vartheta_2,u)+d(\vartheta_3,\vartheta_3,w)+d(u,u,w)]$,

for all $\vartheta_1,\vartheta_2,\vartheta_3\in V$ *and for all* $u,w\in V\setminus\{\vartheta_1,\vartheta_2,\vartheta_3\}$ *with* $u\neq w$ *and* $s_b\geq 1$, *is called a BSbMS.*

3. Main Result

In response to the findings made previously, we extend the classes of metric spaces by introducing new classes of metric spaces, namely aveBSbMS and a particular case, avBSbMS. Also, we make use of them to demonstrate some fixed point results. We also provide a few instances that highlight the usefulness of our main finding.

Definition 3.1. *A triplet (V,C,d) with non empty set V and $d : V\times V\times V \longrightarrow \mathsf{C}$ satisfying*

$(avBS_bM1)$ $d(\vartheta_1,\vartheta_2,\vartheta_3)\geq \theta$,
$(avBS_bM2)$ $d(\vartheta_1,\vartheta_2,\vartheta_3)=\theta$ *if and only if* $\vartheta_1=\vartheta_2=\vartheta_3$,
$(avBS_bM3)$ $d(\vartheta_1,\vartheta_2,\vartheta_2)\leq S_b[d(\vartheta_1,\vartheta_1,u)+d(\vartheta_2,\vartheta_2,u)+d(\vartheta_3,\vartheta_3,w)+d(u,u,w)]$,

for all $\vartheta_1,\vartheta_2,\vartheta_3\in V$ *and for all* $u,w\in V\setminus\{\vartheta_1,\vartheta_2,\vartheta_3\}$ *with* $u\neq w$ *and* $S_b\in\mathsf{C}_g$, *is called an avBSbMS.*

Definition 3.2. *A triplet (V, C, d) with non empty set V, $E : V \times V \times V \longrightarrow \mathsf{C}_g$ and $d : V \times V \times V \longrightarrow \mathsf{C}$ satisfying*

$(aveBS_bM1)$ $d(\vartheta_1, \vartheta_2, \vartheta_3) \succeq \theta$,

$(aveBS_bM2)$ $d(\vartheta_1, \vartheta_2, \vartheta_3) = \theta$ *if and only if* $\vartheta_1 = \vartheta_2 = \vartheta_3$,

$(aveBS_bM3)$ $d(\vartheta_1, \vartheta_2, \vartheta_2) \preceq E(\vartheta_1, \vartheta_2, \vartheta_2)[d(\vartheta_1, \vartheta_1, u) + d(\vartheta_2, \vartheta_2, u) + d(\vartheta_3, \vartheta_3, w) + d(u, u, w)]$,

for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$ and for all $u, w \in V \setminus \{\vartheta_1, \vartheta_2, \vartheta_3\}$ with $u \neq w$, is called an aveBSbMS.

Definition 3.3. *An avBSbMS (or aveBSbMS) (V, C, d) is called symmetric if*

$$d(\vartheta_1, \vartheta_1, \vartheta_2) = d(\vartheta_2, \vartheta_2, \vartheta_1), \text{ for all } \vartheta_1, \vartheta_2 \in V.$$

The following remarks give us relations between an aveBSbMS, an avBSbMS, and an avSbMS.

Remark 3.1. *Note that, if $E(\vartheta_1, \vartheta_2, \vartheta_2) = S_b \succeq g$, then an aveBSbMS will become an avBSbMS.*

Remark 3.2. *From Definition 2.9 of an avSbMS (V, C, d), we have*

$$\begin{aligned} d(\vartheta_1, \vartheta_2, \vartheta_3) &\preceq s_b[d(\vartheta_1, \vartheta_1, u) + d(\vartheta_2, \vartheta_2, u) + d(\vartheta_3, \vartheta_3, u)] \\ &\preceq s_b[d(\vartheta_1, \vartheta_1, u) + d(\vartheta_2, \vartheta_2, u) + s_b[2d(\vartheta_3, \vartheta_3, w) + d(u, u, w)]] \\ &\preceq 2s_b^2[d(\vartheta_1, \vartheta_1, u) + d(\vartheta_2, \vartheta_2, u) + d(\vartheta_3, \vartheta_3, w) + d(u, u, w)]. \end{aligned}$$

So, for $S_b = 2s_b^2$, (V, C, d) become an avBSbMS.

Remark 3.3. *For $E(\vartheta_1, \vartheta_2, \vartheta_3) = 2e(\vartheta_1, \vartheta_2, \vartheta_3)^2$, an aveSbMS is an aveBSbMS.*

Now we give some topological notions in our generalized metric structures.

Definition 3.4. *For any sequence $\{\vartheta_n\}$ in an aveBSbMS (or avBSbMS) (V, C, d), we say*

(i) *$\{\vartheta_n\}$ converges to $\vartheta \in V$, if for all $\varepsilon \succ \theta$ there exist $n_\varepsilon \in \mathbb{N}$ with $d(\vartheta_n, \vartheta_n, \vartheta) \preceq \varepsilon$, for all $n \geq n_\varepsilon$.*

(ii) *if $\{\vartheta_n\}$ is convergent to $\vartheta \in V$, then we write $\lim\limits_{n \to +\infty} d(\vartheta_n, \vartheta_n, \vartheta) = \theta$ or $\lim\limits_{n \to +\infty} \vartheta_n = \vartheta$.*

(iii) *$\{\vartheta_n\}$ is Cauchy, if for all $\varepsilon \succ \theta$, there exist $n_\varepsilon \in \mathbb{N}$ with $d(\vartheta_m, \vartheta_m, \vartheta_n) \preceq \varepsilon$, for all $n > m \geq n_\varepsilon$. That is if $\lim\limits_{n,m \to +\infty} d(\vartheta_m, \vartheta_m, \vartheta_n) = \theta$.*

Definition 3.5. *If every Cauchy sequence in an aveBSbMS (or avBSbMS) (V, C, d) is convergent, then we say (V, C, d) is complete.*

Lemma 3.1. *If a convergent sequence $\{\vartheta_n\}$ in a symmetric complete aveBSbMS (V, C, d) is Cauchy. Then, $\{\vartheta_n\}$ will converge uniquely.*

Proof. Let $\varepsilon \succ \theta$ and $\{\vartheta_n\}$ converges to both ϑ' and ϑ''. Since $\{\vartheta_n\}$ is a Cauchy, there exist $n_1 \in \mathbb{N}$ such that $d(\vartheta_m, \vartheta_m, \vartheta_n) \preceq \frac{\varepsilon}{3E(\vartheta', \vartheta', \vartheta'')}$, for all $n > m \geq n_1$. Again, $\{\vartheta_n\}$ converges to both $\vartheta', \vartheta'' \in V$, then there exists $n_2, n_3 \in \mathbb{N}$ such that $d(\vartheta_n, \vartheta_n, \vartheta') \leq \frac{\varepsilon}{6E(\vartheta',\vartheta',\vartheta'')}$, for all $n \geq n_2$ and $d(\vartheta_n, \vartheta_n, \vartheta'') \leq \frac{\varepsilon}{6E(\vartheta',\vartheta',\vartheta'')}$, for all $n \geq n_3$. Let $n_0 = \max\{n_1, n_2, n_3\}$. Then,

$$\begin{aligned} d(\vartheta', \vartheta', \vartheta'') &\leq E(\vartheta', \vartheta', \vartheta'')[2d(\vartheta', \vartheta', \vartheta_m) + d(\vartheta'', \vartheta'', \vartheta_n) + d(\vartheta_m, \vartheta_m, \vartheta_n)] \\ &\leq E(\vartheta', \vartheta', \vartheta'')\left[\frac{2\varepsilon}{6E(\vartheta', \vartheta', \vartheta'')} + \frac{\varepsilon}{6E(\vartheta', \vartheta', \vartheta'')} + \frac{\varepsilon}{3E(\vartheta', \vartheta', \vartheta'')}\right] \\ &= \frac{5}{6}\varepsilon \\ &< \varepsilon, \text{ for all } n \geq n_0. \end{aligned}$$

As ε is arbitrary, $d(\vartheta', \vartheta', \vartheta'') = \theta$ implies $\vartheta' = \vartheta''$. Hence, $\{\vartheta_n\}$ converges uniquely. □

Lemma 3.2. *If a convergent sequence $\{\vartheta_n\}$ in a symmetric complete avBSbMS (V, C, d) is Cauchy. Then, $\{\vartheta_n\}$ will converge uniquely.*

Proof. Letting $E(\vartheta_1, \vartheta_2, \vartheta_3) = S_b$, for all $\vartheta_1, \vartheta_2, \vartheta_3 \in V$, Lemma 3.2 is a particular case of Lemma 3.1. □

Lemma 3.3. *Let $\{\vartheta_n\}$ be a Cauchy sequence in a symmetric complete aveBSbMS (V, C, d) converging to unique $\vartheta \in V$. Then,*

$$\frac{1}{m}d(\vartheta, \vartheta, \vartheta') \leq \liminf_n d(\vartheta_n, \vartheta_n, \vartheta') \leq \limsup_n d(\vartheta_n, \vartheta_n, \vartheta') \leq Md(\vartheta, \vartheta, \vartheta'), \text{ for all } \vartheta' \in V. \tag{3.1}$$

Where $m = \inf_{s_1,s_2,s_3 \in \{\vartheta_n\}} E(s_1, s_2, s_3)$ *and* $M = \sup_{s_1,s_2,s_3 \in \{\vartheta_n\}} E(s_1, s_2, s_3)$.

Proof. Let $\vartheta \neq \vartheta'$, otherwise, we will get the equality. Now for ϑ_n, where $n \in \mathbb{N}$ so that $\vartheta, \vartheta' \notin \{\vartheta_n, \vartheta_{n+1}, \ldots\}$, we have

$$\begin{aligned} d(\vartheta, \vartheta, \vartheta') &\leq E(\vartheta, \vartheta, \vartheta')\left[2d(\vartheta, \vartheta, \vartheta_{n+1}) + d(\vartheta', \vartheta', \vartheta_n) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_n)\right] \\ &\leq \inf_{s_1,s_2,s_3 \in \{\vartheta_n\}} E(s_1, s_2, s_3) \liminf_n d(\vartheta', \vartheta', \vartheta_n) \\ &= m \liminf_n d(\vartheta_n, \vartheta_n, \vartheta') \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} d(\vartheta_n, \vartheta_n, \vartheta') &\leq E(\vartheta_n, \vartheta_n, \vartheta')\left[2d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta', \vartheta', \vartheta) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta)\right] \\ \Rightarrow \limsup_n d(\vartheta_n, \vartheta_n, \vartheta') &\leq \sup_{s_1,s_2,s_3 \in \{\vartheta_n\}} E(s_1, s_2, s_3) d(\vartheta', \vartheta', \vartheta) \\ &= Md(\vartheta, \vartheta, \vartheta'). \end{aligned} \tag{3.3}$$

Combining (3.2) and (3.3), we have our inequality (3.1). □

Definition 3.6. *For any $V_1, V_2 \subset V$ in an aveBSbMS (V, C, d), we define*

i) $d(V_1, V_1, V_2) = \inf\{d(\vartheta_1, \vartheta_1, \vartheta_2) : \vartheta_1 \in V_1, \vartheta_2 \in V_2\}$ *by distance between V_1 and V_2.*
ii) $d(\vartheta_1, \vartheta_1, V_2) = \inf\{d(\vartheta_1, \vartheta_1, \vartheta_2) : \vartheta_2 \in V_2\}$ *by distance between $\vartheta_1 \in V_1$ and V_2.*

iii) $\delta(V_1) = \sup\{d(\vartheta_1, \vartheta_1, \vartheta_2) : \vartheta_1, \vartheta_2 \in V_1\}$ *by diameter of* V_1 *and* $\delta(V_1) \prec +\infty$ *by* V_1 *is bounded if there is some* $\varepsilon(\succ \theta) \in \mathsf{C}$ *such that* $d(\vartheta_1, \vartheta_1, \vartheta_2) \prec \varepsilon$, *for all* $\vartheta_1, \vartheta_2 \in V_1$.

iv) $\mathsf{O}_d^{\mathsf{C}^*}(\vartheta_1, \varepsilon) = \{\vartheta_2 \in V : d(\vartheta_2, \vartheta_2, \vartheta_1) \prec \varepsilon\}$ *by an open ball of radius* $\varepsilon(\succ \theta) \in \mathsf{C}$ *centred at* $\vartheta_1 \in V$.

v) $\mathsf{B}_d^{\mathsf{C}^*}[\vartheta_1, \varepsilon] = \{\vartheta_2 \in V : d(\vartheta_2, \vartheta_2, \vartheta_1) \preceq \varepsilon\}$ *by a closed ball of radius* $\varepsilon(\succ \theta) \in \mathsf{C}$ *centred at* $\vartheta_1 \in V$.

vi) $\tau_d^{\mathsf{C}^*} = \{\emptyset\} \cup \{V' \subset V : \mathsf{O}_d^{\mathsf{C}^*}(\vartheta_1, \varepsilon) \subset V'$, *for some* $\varepsilon(\succ \theta) \in \mathsf{C}$ *and* $\vartheta_1 \in V'\}$.

Then, $\tau_d^{\mathsf{C}^*}$ *is a topology on* (V, C, d).

The following is a numerical example of an aveBSbMS but not an avBSbMS and not an aveSMS.

Example 3.1. *Let* $V = \{0, 1, 2, 3, \ldots\}, \mathsf{C} = M_2(\mathbb{R})$ *and* $d : V \times V \times V \longrightarrow \mathsf{C}, E : V \times V \times V \longrightarrow \mathsf{C}_g$ *be given by*

$$d(\vartheta_1, \vartheta_2, \vartheta_3) = \begin{bmatrix} M' & 0 \\ 0 & M' \end{bmatrix}, \text{ for all } \vartheta_1, \vartheta_2, \vartheta_3 \in V, \text{where } M' = |\vartheta_1 - \vartheta_2| + |\vartheta_3|$$

and

$$E(\vartheta_1, \vartheta_2, \vartheta_3) = \begin{bmatrix} M' + 1 & 0 \\ 0 & M' + 1 \end{bmatrix}, \text{ for all } \vartheta_1, \vartheta_2, \vartheta_3 \in V.$$

Then (V, C, d) *is an aveBSbMS.*

Remark 3.4. *Since*

$$d(\vartheta_1, \vartheta_1, u) + d(\vartheta_2, \vartheta_2, u) + d(\vartheta_3, \vartheta_3, w) + d(u, u, w) = \begin{bmatrix} 2(|u| + |w|) & 0 \\ 0 & 2(|u| + |w|) \end{bmatrix},$$

there does not exist any fixed $S_b \succeq I_2$ *in* $\mathsf{C}_g = \{c_1 \in M_2(\mathbb{R}) : c_1 c_2 = c_2 c_1$, *for all* $c_2 \in M_2(\mathbb{R})$ *and* $c_1 \succeq I_2\}$ *so that*

$$d(\vartheta_1, \vartheta_2, \vartheta_3) \preceq S_b[d(\vartheta_1, \vartheta_1, u) + d(\vartheta_2, \vartheta_2, u) + d(\vartheta_3, \vartheta_3, w) + d(u, u, w)].$$

Hence, in the Example 3.1, $(V, M_2(\mathbb{R}), d)$ *is not an avBSbMS. Consequently not an avSbMS and not an avSMS. Also note that* $(V, M_2(\mathbb{R}), d)$ *is not an aveSbMS and not an aveSMS.*

The following is a numerical example of avBSbMS but not an avSbMS and not an avSMS.

Example 3.2. *Let* $V = \{1, 2, 3, \ldots\}, \mathsf{C} = M_2(\mathbb{R})$ *and* $d : V \times V \times V \longrightarrow \mathsf{C}$ *is given by*

$$d(\vartheta_1, \vartheta_2, \vartheta_3) = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix}, \text{ for all } \vartheta_1, \vartheta_2, \vartheta_3 \in V,$$

$$\text{where } M = \begin{cases} 10 & if \quad \vartheta_1 = \vartheta_2 = 1 \text{ and } \vartheta_3 = 2, \\ \frac{1}{2(\vartheta_3 + 1)} & if \quad \vartheta_1 = \vartheta_2 = 1 \text{ and } \vartheta_3 \geq 3, \\ \frac{1}{\vartheta_3 + 2} & if \quad \vartheta_1 = \vartheta_2 = 2 \text{ and } \vartheta_3 \geq 3, \\ 5 & otherwise. \end{cases}$$

Then, (V, C, d) *is an avBSbMS for* $S_b = 4I_2$ *but not an avSbMS. Consequently not an avSMS.*

Now we are ready to present our main theorem analogous to Banach in the context of avBSbMS.

Theorem 3.1. *In a symmetric complete avBSbMS* (V, C, d)*, suppose* $H : V \longrightarrow V$ *satisfies*

$$d(H\vartheta_1, H\vartheta_1, H\vartheta_2) \preceq c^* d(\vartheta_1, \vartheta_1, \vartheta_2)c\ ;\ \text{for all } \vartheta_1, \vartheta_2, \vartheta_3 \in V, \tag{3.4}$$

where $c \in \mathsf{C}$ *with* $\|c\| < 1$*. Then* H *possesses a unique* $\vartheta \in V$ *such that* $H\vartheta = \vartheta$.

Proof. If $c = \theta$, then H is a constant map. So, we can assume $c \neq \theta$. For $\vartheta_0 \in V$ define a sequence $\{\vartheta_n\} \subseteq V$ by $\vartheta_n = H\vartheta_{n-1}$, for all $n \in \mathbb{N}$. If $\vartheta_{n+1} = \vartheta_n$, for some $n \in \mathbb{N}$, then ϑ_n is a fixed point. So, we can assume $\vartheta_{n+1} \neq \vartheta_n$, for all $n \in \mathbb{N}$. Now from (3.4), we have

$$\begin{aligned} d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) = d(H\vartheta_{n-1}, H\vartheta_{n-1}, H\vartheta_n) &\preceq c^* d(\vartheta_{n-1}, \vartheta_{n-1}, \vartheta_n)c \\ &= c^* d(H\vartheta_{n-2}, H\vartheta_{n-2}, H\vartheta_{n-1})c \\ &\preceq (c^*)^2 d(\vartheta_{n-2}, \vartheta_{n-2}, \vartheta_{n-1})(c)^2, \text{ by (L4) of 2.1} \\ &\ \vdots \\ &\preceq (c^*)^n d(\vartheta_0, \vartheta_0, \vartheta_1)(c)^n, \text{ for any } n \in \mathbb{N}. \end{aligned}$$

Again, as (V, C, d) is an avBSbMS there exists $S_b(\succeq g) \in \mathsf{C}$. So, for any $n, m \in \mathbb{N}$ with $m \geq 2$

$$\begin{aligned} &d(\vartheta_n, \vartheta_n, \vartheta_{n+m}) \\ &\preceq S_b[2d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+m}\vartheta_{n+m}, \vartheta_{n+2}) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2})] \\ &= 2S_b d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + S_b d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2}) + S_b d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+m}) \\ &\preceq 2S_b d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + S_b d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2}) + (S_b)^2[2d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+3}) \\ &\quad + d(\vartheta_{n+m}, \vartheta_{n+m}, \vartheta_{n+4}) + d(\vartheta_{n+3}, \vartheta_{n+3}, \vartheta_{n+4})] \\ &= 2S_b d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + S_b d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2}) + 2(S_b)^2 d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+3}) \\ &\quad + (S_b)^2 d(\vartheta_{n+3}, \vartheta_{n+3}, \vartheta_{n+4}) + (S_b)^2 d(\vartheta_{n+4}, \vartheta_{n+4}, \vartheta_{n+m}) \\ &\preceq 2S_b[d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2})] + 2(S_b)^2[d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+3}) \\ &\quad + d(\vartheta_{n+3}, \vartheta_{n+3}, \vartheta_{n+4})] + (S_b)^2 d(\vartheta_{n+4}, \vartheta_{n+4}, \vartheta_{n+m}) \\ &\preceq 2S_b[d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2})] + 2(S_b)^2[d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+3}) \\ &\quad + d(\vartheta_{n+3}, \vartheta_{n+3}, \vartheta_{n+4})] + 2(S_b)^3[d(\vartheta_{n+4}, \vartheta_{n+4}, \vartheta_{n+5}) + d(\vartheta_{n+5}, \vartheta_{n+5}, \vartheta_{n+6})] \\ &\quad + (S_b)^3 d(\vartheta_{n+6}, \vartheta_{n+6}, \vartheta_{n+m}) \\ &\preceq 2S_b[d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2})] + 2(S_b)^2[d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+3}) \\ &\quad + d(\vartheta_{n+3}, \vartheta_{n+3}, \vartheta_{n+4})] + 2(S_b)^3[d(\vartheta_{n+4}, \vartheta_{n+4}, \vartheta_{n+5}) + d(\vartheta_{n+5}, \vartheta_{n+5}, \vartheta_{n+6})] \\ &\quad + 2(S_b)^4[d(\vartheta_{n+6}, \vartheta_{n+6}, \vartheta_{n+7}) + d(\vartheta_{n+7}, \vartheta_{n+7}, \vartheta_{n+8})] + (S_b)^4 d(\vartheta_{n+8}, \vartheta_{n+8}, \vartheta_{n+m}) \\ &\ \vdots \\ &\preceq 2S_b[d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2})] + 2(S_b)^2[d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+3}) \\ &\quad + d(\vartheta_{n+3}, \vartheta_{n+3}, \vartheta_{n+4})] + 2(S_b)^3[d(\vartheta_{n+4}, \vartheta_{n+4}, \vartheta_{n+5}) + d(\vartheta_{n+5}, \vartheta_{n+5}, \vartheta_{n+6})] \end{aligned}$$

$$
\begin{aligned}
&+2(S_b)^4[d(\vartheta_{n+6},\vartheta_{n+6},\vartheta_{n+7})+d(\vartheta_{n+7},\vartheta_{n+7},\vartheta_{n+8})]+\cdots\\
&+2(S_b)^{\frac{m-1}{2}}[d(\vartheta_{n+m-3},\vartheta_{n+m-3},\vartheta_{n+m-2})+d(\vartheta_{n+m-2},\vartheta_{n+m-2},\vartheta_{n+m-1})]\\
&+(S_b)^{\frac{m-1}{2}}d(\vartheta_{n+m-1},\vartheta_{n+m-1},\vartheta_{n+m})\\
=\;& 2\sum_{j=0}^{\frac{m-3}{2}}(S_b)^{j+1}[d(\vartheta_{n+2j},\vartheta_{n+2j},\vartheta_{n+2j+1})+d(\vartheta_{n+2j+1},\vartheta_{n+2j+1},\vartheta_{n+2j+2})]\\
&+(S_b)^{\frac{m-1}{2}}d(\vartheta_{n+m-1},\vartheta_{n+m-1},\vartheta_{n+m})\\
\preceq\;& 2\sum_{j=0}^{\frac{m-3}{2}}(S_b)^{j+1}(c^*)^{n+2j}[d(\vartheta_0,\vartheta_0,\vartheta_1)+(c^*)d(\vartheta_0,\vartheta_0,\vartheta_1)(c)](c)^{n+2j}\\
&+(S_b)^{\frac{m-1}{2}}(c^*)^{n+m-1}d(\vartheta_0,\vartheta_0,\vartheta_1)(c)^{n+m-1}.
\end{aligned}
$$

Let $d(\vartheta_0,\vartheta_0,\vartheta_1)=c'\in \mathsf{C}$, then

$d(\vartheta_n,\vartheta_n,\vartheta_{n+m})$

$$
\begin{aligned}
\preceq\;& 2\sum_{j=0}^{\frac{m-3}{2}}(S_b)^{j+1}(c^*)^{n+2j}[c'+(c^*)c'(c)](c)^{n+2j}+(S_b)^{\frac{m-1}{2}}(c^*)^{n+m-1}c'(c)^{n+m-1}\\
=\;& 2\sum_{j=0}^{\frac{m-3}{2}}\left((S_b)^{\frac{j+1}{2}}c'^{\frac12}(c)^{n+2j}\right)^*\left((S_b)^{\frac{j+1}{2}}c'^{\frac12}(c)^{n+2j}\right)\\
&+2\sum_{j=0}^{\frac{m-3}{2}}\left((S_b)^{\frac{j+1}{2}}c'^{\frac12}(c)^{n+2j+1}\right)^*\left((S_b)^{\frac{j+1}{2}}c'^{\frac12}(c)^{n+2j+1}\right)\\
&+\left((S_b)^{\frac{m-1}{4}}c'^{\frac12}(c)^{n+m-1}\right)^*\left((S_b)^{\frac{m-1}{4}}c'^{\frac12}(c)^{n+m-1}\right)\\
=\;& 2\sum_{j=0}^{\frac{m-3}{2}}\left|(S_b)^{\frac{j+1}{2}}c'^{\frac12}(c)^{n+2j}\right|^2+2\sum_{j=0}^{\frac{m-3}{2}}\left|(S_b)^{\frac{j+1}{2}}c'^{\frac12}(c)^{n+2j+1}\right|^2\\
&+\left|(S_b)^{\frac{m-1}{4}}c'^{\frac12}(c)^{n+m-1}\right|^2\\
\preceq\;& 2\|c'\|\sum_{j=0}^{\frac{m-3}{2}}\|S_b\|^{j+1}\|c\|^{2n+4j}g+2\|c'\|\sum_{j=0}^{\frac{m-3}{2}}\|S_b\|^{j+1}\|c\|^{2n+4j+2}g\\
&+\|c'\|\|S_b\|^{\frac{m-1}{2}}\|c\|^{2n+2m-2}g\\
=\;& 2\|c'\|\|S_b\|^{\frac{m-1}{2}}\left(\sum_{j=0}^{\frac{m-3}{2}}\frac{\|c\|^{2n+4j}}{\|S_b\|^{\frac{m-2j-3}{2}}}g+\sum_{j=0}^{\frac{m-3}{2}}\frac{\|c\|^{2n+4j+2}}{\|S_b\|^{\frac{m-2j-3}{2}}}g+\frac12\|c\|^{2n+2m-2}g\right)\\
=\;& 2\|c'\|\|S_b\|^{\frac{m-1}{2}}\left(\frac{\|c\|^{2n}}{\|S_b\|^{\frac{m-3}{2}}\left(1-\frac{\|c\|^4}{\|S_b\|}\right)}g+\frac{\|c\|^{2n+2}}{\|S_b\|^{\frac{m-3}{2}}\left(1-\frac{\|c\|^4}{\|S_b\|}\right)}g\;+\frac12\|c\|^{2n+2m-2}g\right)
\end{aligned}
$$

$$= 2\|c'\|\|S_b\|^2\left(\frac{\|c\|^{2n}}{\|S_b\|-\|c\|^4}g+\frac{\|c\|^{2n+2}}{\|S_b\|-\|c\|^4}g+\frac{1}{2}\|c\|^{2n+2m-2}g\right).$$

Since $\|c\| < 1$,

$$\lim_{n,m\to+\infty} d(\vartheta_n,\vartheta_n,\vartheta_{n+m}) = \theta.$$

This shows that $\{\vartheta_n\}$ is a Cauchy sequence in a symmetric complete avBSbMS (V,C,d). By Lemma 3.1, $\vartheta\in V$ exists uniquely such that $\lim\limits_{n\to+\infty} d(\vartheta_{n,m},\vartheta)=\theta$.
Again, $d(\vartheta,\vartheta,H\vartheta)=\lim\limits_{n\to+\infty} d(\vartheta_{n+1},\vartheta_{n+1},H\vartheta)\preceq\lim\limits_{n\to+\infty} c^*d(\vartheta_n,\vartheta_n,\vartheta)c=\theta$ implies $H\vartheta=\vartheta$, consequently ϑ is a unique fixed point of H. □

The following remark gives us a method to deal with the operator H^n.

Remark 3.5. *If H satisfies inequality (3.4) for some $c\in\mathsf{C}$ with $\|c\|<1$, then H^n also satisfies inequality (3.4) for $c^n\in\mathsf{C}$ with $\|c^n\|<1$. In this case, by Theorem 3.1, H^n has a unique fixed point. Consequently, H has a unique fixed point.*

Now we have an illustration of Theorem 3.1.

Example 3.3. *Let (V,C,d) be same as in Example 3.2. Define $H:V\longrightarrow V$ by*

$$H\vartheta=\begin{cases}4, & \vartheta=1\\ 5, & \text{otherwise.}\end{cases}$$

Then, for any $c\in\mathsf{C}$ with $\|c\|<1$, H^2 satisfies inequality (3.4) of Theorem 3.1. By, Remark 3.5, 5 is a unique point in V such that $H5=5$.

The result below is analogous to Kannan in the context of an avBSbMS.

Theorem 3.2. *In a symmetric complete avBSbMS (V,C,d), suppose $H:V\longrightarrow V$ satisfies*

$$d(H\vartheta_1,H\vartheta_1,H\vartheta_2)\preceq c[d(\vartheta_1,\vartheta_1,H\vartheta_1)+d(\vartheta_2,\vartheta_2,H\vartheta_2)]\ ;\ \text{for all } \vartheta_1,\vartheta_2,\vartheta_3\in V, \tag{3.5}$$

where $c\in\mathsf{C}$ with $\|c\|<\frac{1}{2}$. Then, H possesses a unique $\vartheta\in V$ such that $H\vartheta=\vartheta$.

Proof. If $c=\theta$, then H is a constant map. So, we can assume $c\neq\theta$. For $\vartheta_0\in V$, define a sequence $\{\vartheta_n\}\subseteq V$ by $\vartheta_n=H\vartheta_{n-1}$, for all $n\in\mathbb{N}$. If $\vartheta_{n+1}=\vartheta_n$, for some $n\in\mathbb{N}$, then ϑ_n is a fixed point. So, we can assume $\vartheta_{n+1}\neq\vartheta_n$, for all $n\in\mathbb{N}$. Now from (3.5), we have

$$\begin{aligned}
d(\vartheta_n,\vartheta_n,\vartheta_{n+1})=d(H\vartheta_{n-1},H\vartheta_{n-1},H\vartheta_n) &\preceq c[d(\vartheta_{n-1},\vartheta_{n-1},H\vartheta_{n-1})+d(\vartheta_n,\vartheta_n,H\vartheta_n)]\\
&= c[d(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)+d(\vartheta_n,\vartheta_n,\vartheta_{n+1})]\\
\Rightarrow (g-c)d(\vartheta_n,\vartheta_n,\vartheta_{n+1}) &\preceq cd(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)\\
\Rightarrow d(\vartheta_n,\vartheta_n,\vartheta_{n+1}) &\preceq (g-c)^{-1}cd(\vartheta_{n-1},\vartheta_{n-1},\vartheta_n)\\
&\preceq \left((g-c)^{-1}c\right)^2 d(\vartheta_{n-2},\vartheta_{n-2},\vartheta_{n-1}),\ \text{by (L5) of 2.1}\\
&\ \ \vdots\\
&\preceq \left((g-c)^{-1}c\right)^n d(\vartheta_0,\vartheta_0,\vartheta_1),\ \text{for any } n\in\mathbb{N}.
\end{aligned}$$

Let $(g-c)^{-1}c = \hat{c},\ d(\vartheta_0, \vartheta_0, \vartheta_1) = c' \in \mathsf{C}$. By (L2) of Lemma 3.1, we have $\|\hat{c}\| < 1$ and

$$d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) \preceq (\hat{c})^n d(\vartheta_0, \vartheta_0, \vartheta_1), \text{ for any } n \in \mathbb{N}.$$

Similarly, proceeding as of Theorem 3.1, for any $n, m \in \mathbb{N}$ with $m \geq 2$, we have

$$\begin{aligned}
&d(\vartheta_n, \vartheta_n, \vartheta_{n+m}) \\
&\preceq 2\sum_{j=0}^{\frac{m-3}{2}} (S_b)^{j+1}[(\hat{c})^{n+2j}c' + (\hat{c})^{n+2j+1}c'] + (S_b)^{\frac{m-1}{2}}(\hat{c})^{n+m-1}c' \\
&= 2(S_b)^{\frac{m-1}{2}}\left[\sum_{j=0}^{\frac{m-3}{2}}\left[\frac{(\hat{c})^{n+2j}}{(S_b)^{\frac{m-2j-3}{2}}} + \frac{(\hat{c})^{n+2j+1}}{(S_b)^{\frac{m-2j-3}{2}}}\right] + \frac{1}{2}(\hat{c})^{n+m-1}\right]c' \\
&\leq 2\|c'\|\|S_b\|^{\frac{m-1}{2}}\left[\sum_{j=0}^{\frac{m-3}{2}}\left[\frac{\|\hat{c}\|^{n+2j}}{\|S_b\|^{\frac{m-2j-3}{2}}} + \frac{\|\hat{c}\|^{n+2j+1}}{\|S_b\|^{\frac{m-2j-3}{2}}}\right] + \frac{1}{2}\|\hat{c}\|^{n+m-1}\right](g) \\
&= 2\|c'\|\|S_b\|^{\frac{m-1}{2}}\left[\frac{\|\hat{c}\|^n}{\|S_b\|^{\frac{m-3}{2}}\left(1-\frac{\|\hat{c}\|^2}{\|S_b\|}\right)} + \frac{\|\hat{c}\|^{n+1}}{\|S_b\|^{\frac{m-3}{2}}\left(1-\frac{\|\hat{c}\|^2}{\|S_b\|}\right)} + \frac{1}{2}\|\hat{c}\|^{n+m-1}\right]g \\
&= 2\|c'\|\frac{\|S_b\|^2}{\|S_b\|-\|\hat{c}\|^2}\left[\|\hat{c}\|^n + \|\hat{c}\|^{n+1} + \frac{1}{2}\|\hat{c}\|^{n+m-1}\right]g.
\end{aligned}$$

Which implies $\lim_{n,m\to+\infty} d(\vartheta_n, \vartheta_n, \vartheta_{n+m}) = \theta$, since $\|\hat{c}\| < 1$.

This shows that $\{\vartheta_n\}$ is a Cauchy sequence in a symmetric complete avBSbMS (V, C, d). By Lemma 3.1, $\vartheta \in V$ exists uniquely so that $\lim_{n\to+\infty} d(\vartheta_n, \vartheta_n, \vartheta) = \theta$.
Again, $d(\vartheta, \vartheta, H\vartheta) = \lim_{n\to+\infty} d(\vartheta_{n+1}, \vartheta_{n+1}, H\vartheta) \preceq \lim_{n\to+\infty} \hat{c}d(\vartheta_n, \vartheta_n, \vartheta) = \theta$ implies $H\vartheta = \vartheta$, consequently ϑ is a unique fixed point of H. □

We now present some consequent results in the context of an aveBSbMS.

Theorem 3.3. *In a symmetric complete aveBSbMS (V, C, d), suppose $H : V \longrightarrow V$, $E : V \times V \times V \longrightarrow \mathsf{C}_g$ satisfies*

$$d(H\vartheta_1, H\vartheta_1, H\vartheta_2) \preceq c^* d(\vartheta_1, \vartheta_1, \vartheta_2)c, \tag{3.6}$$

where $c \in \mathsf{C}$ with $\|c\| < 1$. Also, for an arbitrary $\vartheta_0 \in V$ with $\vartheta_n = H\vartheta_{n-1}$, for all $n \in \mathbb{N}$,

$$\lim_{n,m\to+\infty} cE(\vartheta_n, \vartheta_n, \vartheta_m) = \lim_{n,m\to+\infty} cE_{n,n,m} \preceq g.$$

Then, H possesses a unique $\vartheta \in V$ such that $H\vartheta = \vartheta$.

Proof. Proceeding similar to Theorem 3.1, from (3.6), we have

$$d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) \preceq (c^*)^n d(\vartheta_0, \vartheta_0, \vartheta_1)(c)^n, \text{ for any } n \in \mathbb{N}.$$

Again, as (V, C, d) is an aveBSbMS there exists $E : V \times V \times V \longrightarrow \mathsf{C}_g$. So, for any $n, m \in \mathbb{N}$ with $m \geq 2$

$$
\begin{aligned}
& d(\vartheta_n, \vartheta_n, \vartheta_{n+m}) \\
\leq\ & E(\vartheta_n, \vartheta_n, \vartheta_{n+m})[2d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+m}, \vartheta_{n+m}, \vartheta_{n+2}) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2})] \\
=\ & E_{n,n,n+m}[2d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2})] + E_{n,n,n+m} d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+m}) \\
\leq\ & 2E_{n,n,n+m}[d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2})] \\
& + E_{n,n,n+m} E_{n+2,n+2,n+m}[2d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+3}) + d(\vartheta_{n+m}, \vartheta_{n+m}, \vartheta_{n+4}) + d(\vartheta_{n+3}, \vartheta_{n+3}, \vartheta_{n+4})] \\
=\ & 2E_{n,n,n+m}[d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2})] + \prod_{k=0}^{1} E_{n+2k,n+2k,n+m}[2d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+3}) \\
& + d(\vartheta_{n+3}, \vartheta_{n+3}, \vartheta_{n+4})] + E_{n,n,n+m} E_{n+2,n+2,n+m} d(\vartheta_{n+4}, \vartheta_{n+4}, \vartheta_{n+m}) \\
\leq\ & 2E_{n,n,n+m}[d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2})] + 2\prod_{k=0}^{1} E_{n+2k,n+2k,n+m}[d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+3}) \\
& + d(\vartheta_{n+3}, \vartheta_{n+3}, \vartheta_{n+4})] + E_{n,n,n+m} E_{n+2,n+2,n+m} E_{n+4,n+4,n+m}[2d(\vartheta_{n+4}, \vartheta_{n+4}, \vartheta_{n+5}) \\
& + d(\vartheta_{n+m}, \vartheta_{n+m}, \vartheta_{n+6}) + d(\vartheta_{n+5}, \vartheta_{n+5}, \vartheta_{n+6})] \\
=\ & 2\prod_{k=0}^{0} E_{n+2k,n+2k,n+m}[d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2})] \\
& + 2\prod_{k=0}^{1} E_{n+2k,n+2k,n+m}[d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+3}) + d(\vartheta_{n+3}, \vartheta_{n+3}, \vartheta_{n+4})] \\
& + \prod_{k=0}^{2} E_{n+2k,n+2k,n+m}[2d(\vartheta_{n+4}, \vartheta_{n+4}, \vartheta_{n+5}) + d(\vartheta_{n+5}, \vartheta_{n+5}, \vartheta_{n+6})] \\
& + \prod_{k=0}^{2} E_{n+2k,n+2k,n+m} d(\vartheta_{n+6}, \vartheta_{n+6}, \vartheta_{n+m})] \\
\vdots\ & \\
\leq\ & 2\prod_{k=0}^{0} E_{n+2k,n+2k,n+m}[d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) + d(\vartheta_{n+1}, \vartheta_{n+1}, \vartheta_{n+2})] \\
& + 2\prod_{k=0}^{1} E_{n+2k,n+2k,n+m}[d(\vartheta_{n+2}, \vartheta_{n+2}, \vartheta_{n+3}) + d(\vartheta_{n+3}, \vartheta_{n+3}, \vartheta_{n+4})] \\
& + 2\prod_{k=0}^{2} E_{n+2k,n+2k,n+m}[d(\vartheta_{n+4}, \vartheta_{n+4}, \vartheta_{n+5}) + d(\vartheta_{n+5}, \vartheta_{n+5}, \vartheta_{n+6})] \\
& + 2\prod_{k=0}^{3} E_{n+2k,n+2k,n+m}[d(\vartheta_{n+6}, \vartheta_{n+6}, \vartheta_{n+7}) + d(\vartheta_{n+7}, \vartheta_{n+7}, \vartheta_{n+8})] + \cdots \\
& + \prod_{k=0}^{\frac{m-3}{2}} E_{n+2k,n+2k,n+m}[d(\vartheta_{n+m-3}, \vartheta_{n+m-3}, \vartheta_{n+m-2}) + d(\vartheta_{n+m-2}, \vartheta_{n+m-2}, \vartheta_{n+m-1})]
\end{aligned}
$$

$$+\prod_{k=0}^{\frac{m-3}{2}} E_{n+2k,n+2k,n+m} d(\vartheta_{n+m-1},\vartheta_{n+m-1},\vartheta_{n+m})]$$

$$= 2\sum_{j=0}^{\frac{m-3}{2}}\prod_{k=0}^{j} E_{n+2k,n+2k,n+m}[d(\vartheta_{n+2j},\vartheta_{n+2j},\vartheta_{n+2j+1}) + d(\vartheta_{n+2j+1},\vartheta_{n+2j+1},\vartheta_{n+2j+2})]$$

$$+\prod_{k=0}^{\frac{m-3}{2}} E_{n+2k,n+2k,n+m} d(\vartheta_{n+m-1},\vartheta_{n+m-1},\vartheta_{n+m})]$$

$$\preceq 2\sum_{j=0}^{\frac{m-3}{2}}\prod_{k=0}^{j} E_{n+2k,n+2k,n+m}(c^*)^{n+2j}[d(\vartheta_0,\vartheta_0,\vartheta_1) + (c^*)d(\vartheta_0,\vartheta_0,\vartheta_1)(c)](c)^{n+2j}$$

$$+\prod_{k=0}^{\frac{m-3}{2}} E_{n+2k,n+2k,n+m}(c^*)^{n+m-1} d(\vartheta_0,\vartheta_0,\vartheta_1)(c)^{n+m-1}].$$

Let $d(\vartheta_0,\vartheta_0,\vartheta_1) = c' \in \mathbb{C}$, then

$d(\vartheta_n,\vartheta_n,\vartheta_{n+m})$

$$\preceq 2\sum_{j=0}^{\frac{m-3}{2}}\prod_{k=0}^{j} E_{n+2k,n+2k,n+m}(c^*)^{n+2j}[c' + (c^*)c'(c)](c)^{n+2j}$$

$$+\prod_{k=0}^{\frac{m-3}{2}} E_{n+2k,n+2k,n+m}(c^*)^{n+m-1}c'(c)^{n+m-1}$$

$$= 2\sum_{j=0}^{\frac{m-3}{2}}\left(\left(\prod_{k=0}^{j} E_{n+2k,n+2k,n+m}\right)^{\frac{1}{2}} c'^{\frac{1}{2}}(c)^{n+2j}\right)^*\left(\left(\prod_{k=0}^{j} E_{n+2k,n+2k,n+m}\right)^{\frac{1}{2}} c'^{\frac{1}{2}}(c)^{n+2j}\right)$$

$$+2\sum_{j=0}^{\frac{m-3}{2}}\left(\left(\prod_{k=0}^{j} E_{n+2k,n+2k,n+m}\right)^{\frac{1}{2}} c'^{\frac{1}{2}}(c)^{n+2j+1}\right)^*\left(\left(\prod_{k=0}^{j} E_{n+2k,n+2k,n+m}\right)^{\frac{1}{2}} c'^{\frac{1}{2}}(c)^{n+2j+1}\right)$$

$$+\left(\left(\prod_{k=0}^{\frac{m-3}{2}} E_{n+2k,n+2k,n+m}\right)^{\frac{1}{2}} c'^{\frac{1}{2}}(c)^{n+m-1}\right)^*\left(\left(\prod_{k=0}^{\frac{m-3}{2}} E_{n+2k,n+2k,n+m}\right)^{\frac{1}{2}} c'^{\frac{1}{2}}(c)^{n+m-1}\right)$$

$$= 2\sum_{j=0}^{\frac{m-3}{2}}\left|\left(\prod_{k=0}^{j} E_{n+2k,n+2k,n+m}\right)^{\frac{1}{2}} c'^{\frac{1}{2}}(c)^{n+2j}\right|^2 + 2\sum_{j=0}^{\frac{m-3}{2}}\left|\left(\prod_{k=0}^{j} E_{n+2k,n+2k,n+m}\right)^{\frac{1}{2}} c'^{\frac{1}{2}}(c)^{n+2j+1}\right|^2$$

$$+\left|\left(\prod_{k=0}^{\frac{m-3}{2}} E_{n+2k,n+2k,n+m}\right)^{\frac{1}{2}} c'^{\frac{1}{2}}(c)^{n+m-1}\right|^2$$

$$\leq \quad 2\sum_{j=0}^{\frac{m-3}{2}}\prod_{k=0}^{j}\|E_{n+2k,n+2k,n+m}\|\,\|c'\|\,\|c\|^{2n+4j}g+2\sum_{j=0}^{\frac{m-3}{2}}\|\prod_{k=0}^{j}\|E_{n+2k,n+2k,n+m}\|\,\|c'\|\,\|c\|^{2n+4j+2}g$$
$$+\prod_{k=0}^{\frac{m-3}{2}}\|E_{n+2k,n+2k,n+m}\|\,\|c'\|\,\|c\|^{2n+2m-2}g.$$

Since $\lim_{n,m\to+\infty} cE(\vartheta_n,\vartheta_n,\vartheta_m)=\lim_{n,m\to+\infty} cE_{n,n,m}\preceq g$, for large $m,n\in\mathbb{N}$, we have

$$\prod_{k=0}^{j}\|E_{n+2k,n+2k,n+m}\|\leq\frac{1}{\|c\|^{j+1}} \tag{3.7}$$

and

$$\begin{aligned}d(\vartheta_n,\vartheta_n,\vartheta_{n+m}) &\preceq 2\|c'\|\sum_{j=0}^{\frac{m-3}{2}}\|c\|^{2n+3j-1}g+2\|c'\|\sum_{j=0}^{\frac{m-3}{2}}\|c\|^{2n+3j+1}g+\|c'\|\,\|c\|^{\frac{4n+3m-3}{2}}g\\ &= 2\|c'\|\left[\frac{\|c\|^{2n-1}}{1-\|c\|^3}+\frac{\|c\|^{2n+1}}{1-\|c\|^3}+\frac{1}{2}\|c\|^{\frac{4n+3m-3}{2}}\right]g.\end{aligned}$$

Again, as $\|c\|<1$, we get

$$\lim_{n,m\to+\infty} d(\vartheta_n,\vartheta_n,\vartheta_{n+m})=\theta.$$

This shows that $\{\vartheta_n\}$ is a Cauchy sequence in a symmetric complete aveBSbMS (V,C,d). By Lemma 3.1, $\vartheta\in V$ exists uniquely such that $\lim_{n\to+\infty} d(\vartheta_n,\vartheta_n,\vartheta)=\theta$. Again, $d(\vartheta,\vartheta,H\vartheta)=\lim_{n\to+\infty} d(\vartheta_{n+1},\vartheta_{n+1},H\vartheta)\preceq\lim_{n\to+\infty} c^*d(\vartheta_n,\vartheta_n,\vartheta)c=\theta$ implies $H\vartheta=\vartheta$, consequently ϑ is a unique fixed point of H. □

The remark below gives us a method to deal with operator H^n.

Remark 3.6. *If H satisfies Theorem 3.3, for some $c\in\mathsf{C}$ with $\|c\|<1$, then H^n also satisfies Theorem 3.3 for $c^n\in\mathsf{C}$ with $\|c^n\|<1$. In this case, H^n has a unique fixed point. Consequently, H has a unique fixed point.*

Now we have an illustration of Theorem 3.3.

Example 3.4. *Let (V,C,d) be same as in Example 3.1. Define $H:V\longrightarrow V$ by*

$$H\vartheta=\left[\frac{\vartheta}{3}\right],\ \textit{for all}\ \ \vartheta\in V.$$

Where $[\vartheta]$ is the greatest integer function.

Then for $c=\begin{bmatrix}\frac{1}{\sqrt{3}} & 0\\ 0 & \frac{1}{\sqrt{3}}\end{bmatrix}\in\mathsf{C}$, H satisfies the inequality (3.6) of Theorem 3.3. Again, for any $r_0\in V, r_1=Hr_0=\left[\frac{r_0}{3}\right], r_2=Hr_1=\left[\frac{\left[\frac{r_0}{3}\right]}{3}\right]=\left[\frac{r_0}{3^2}\right],\dots,r_n=\left[\frac{r}{3^n}\right]$; for all $n\in\mathbb{N}$ gives $cE(\vartheta_n,\vartheta_n,\vartheta_m)=\begin{bmatrix}\frac{1}{\sqrt{3}}\left[\frac{r_0}{3^{m-1}}\right]+\frac{1}{\sqrt{3}} & 0\\ 0 & \frac{1}{\sqrt{3}}\left[\frac{r_0}{3^{m-1}}\right]+\frac{1}{\sqrt{3}}\end{bmatrix}\to\begin{bmatrix}\frac{1}{\sqrt{3}} & 0\\ 0 & \frac{1}{\sqrt{3}}\end{bmatrix}\left(\preceq\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)$ as $n,m\to+\infty$. So, H satisfies Theorem 3.3 and 0 is a unique in V such that $H0=0$.

The following is a consequent result of Theorem 3.2.

Theorem 3.4. *In a symmetric complete aveBSbMS* (V, C, d), *suppose* $H : V \longrightarrow V$ *satisfies*

$$d(H\vartheta_1, H\vartheta_1, H\vartheta_2) \preceq c[d(\vartheta_1, \vartheta_1, H\vartheta_1) + d(\vartheta_2, \vartheta_2, H\vartheta_2)]\ ;\ \text{for all } \vartheta_1, \vartheta_2, \vartheta_3 \in V, \tag{3.8}$$

where $c \in \mathsf{C}$ *with* $\|c\| < \frac{1}{2}$. *Also, for arbitrary* $\vartheta_0 \in V$ *with* $\vartheta_n = H\vartheta_{n-1}$, *for all* $n \in \mathbb{N}$, $\lim_{n,m\to+\infty}(g - c)^{-1}cE(\vartheta_n, \vartheta_n, \vartheta_m) = \lim_{n,m\to+\infty}(g - c)^{-1}cE_{n,n,m} \preceq g$. *Then,* H *possesses a unique* $\vartheta \in V$ *such that* $H\vartheta = \vartheta$.

Proof. Proceeding similar to Theorem 3.2, from (3.8), we have

$$d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) \preceq \left((g - c)^{-1}c\right)^n d(\vartheta_0, \vartheta_0, \vartheta_1), \text{ for any } n \in \mathbb{N}.$$

Let $(g - c)^{-1}c = \hat{c}$, $d(\vartheta_0, \vartheta_0, \vartheta_1) = c' \in \mathsf{C}$. By (L2) of Lemma 3.1, we have $\|\hat{c}\| < 1$ and

$$d(\vartheta_n, \vartheta_n, \vartheta_{n+1}) \preceq (\hat{c})^n d(\vartheta_0, \vartheta_0, \vartheta_1), \text{ for any } n \in \mathbb{N}.$$

Again, proceeding as Theorem 3.3, for any $n, m \in \mathbb{N}$ with $m \geq 2$, we have

$$\begin{aligned} d(\vartheta_n, \vartheta_n, \vartheta_{n+m}) &\preceq 2\sum_{j=0}^{\frac{m-3}{2}} \prod_{k=0}^{j} E_{n+2k,n+2k,n+m}[(\hat{c})^{n+2j}c' + (\hat{c})^{n+2j+1}c'] \\ &\quad + \prod_{k=0}^{\frac{m-3}{2}} E_{n+2k,n+2k,n+m}(\hat{c})^{n+m-1}c' \\ &\preceq 2\sum_{j=0}^{\frac{m-3}{2}} \prod_{k=0}^{j} \|E_{n+2k,n+2k,n+m}\|[\|\hat{c}\|^{n+2j}\|c'\| + \|\hat{c}\|^{n+2j+1}\|c'\|]g \\ &\quad + \prod_{k=0}^{\frac{m-3}{2}} \|E_{n+2k,n+2k,n+m}\|\ \|\hat{c}\|^{n+m-1}\|c'\|g. \end{aligned}$$

Since $\lim_{n,m\to+\infty}(g - c)^{-1}cE(\vartheta_n, \vartheta_n, \vartheta_m) = \lim_{n,m\to+\infty}\hat{c}E_{n,n,m} \preceq g$, for large $m, n \in \mathbb{N}$, we have

$$\prod_{k=0}^{j} \|E_{n+2k,n+2k,n+m}\| \leq \frac{1}{\|\hat{c}\|^{j+1}} \tag{3.9}$$

and

$$\begin{aligned} d(\vartheta_n, \vartheta_n, \vartheta_{n+m}) &\preceq 2\sum_{j=0}^{\frac{m-3}{2}} [\|\hat{c}\|^{n+j-1}\|c'\| + \|\hat{c}\|^{n+j}\|c'\|]g + \|\hat{c}\|^{\frac{2n+m-1}{2}}\|c'\|g \\ &= 2\|c'\|\left[\sum_{j=0}^{\frac{m-3}{2}} [\|\hat{c}\|^{n+j-1} + \|\hat{c}\|^{n+j}] + \frac{1}{2}\|\hat{c}\|^{\frac{2n+m-1}{2}}\right]g \\ &= 2\|c'\|\left[\frac{\|\hat{c}\|^{n-1}}{1 - \|\hat{c}\|} + \frac{\|\hat{c}\|^{n}}{1 - \|\hat{c}\|} + \frac{1}{2}\|\hat{c}\|^{\frac{2n+m-1}{2}}\right]g. \end{aligned}$$

Which implies $\lim_{n,m\to+\infty} d(\vartheta_n, \vartheta_n, \vartheta_{n+m}) = \theta$, since $\|\hat{c}\| < 1$.

This shows that $\{\vartheta_n\}$ is a Cauchy sequence in a symmetric complete aveBSbMS (V, C, d). By Lemma 3.1, $\vartheta \in V$ exists uniquely such that $\lim_{n\to+\infty} d(\vartheta_n, \vartheta_n, \vartheta) = \theta$. Again, $d(\vartheta, \vartheta, H\vartheta) = \lim_{n\to+\infty} d(\vartheta_{n+1}, \vartheta_{n+1}, H\vartheta) \preceq \lim_{n\to+\infty} \hat{c}d(\vartheta_n, \vartheta_n, \vartheta) = \theta$ implies $H\vartheta = \vartheta$, consequently ϑ is a unique fixed point of H. □

4. Application

The presence of a unique solution in a system of algebraic linear equations carries significant implications across various domains due to its practical applicability and theoretical importance. This paper delves into the benefits and consequences of knowing that such a solution exists. Firstly, the certainty offered by a unique solution streamlines decision-making processes. This is particularly crucial in fields where precision is paramount, such as engineering design, financial modeling, and scientific simulations. By eliminating ambiguity, unique solutions enable practitioners to focus their efforts on refining the solution rather than navigating alternative scenarios. Secondly, the efficiency gained from working with unique solutions cannot be overstated. Computational algorithms tailored for unique solutions can be optimized to deliver faster and more resource-efficient results. This optimization is especially relevant in large-scale systems and real-time applications, where computational speed directly impacts operational effectiveness.

Moreover, the mathematical rigor associated with unique solutions enhances the reliability and interpretability of results. A unique solution signifies a well-posed problem with a well-defined solution space, contributing to the robustness and stability of mathematical models. This robustness translates into practical advantages, such as reduced sensitivity to small perturbations and improved system resilience. The practical applications of unique solutions extend beyond computational efficiency and robustness. In fields like economics, where decision-making hinges on mathematical models, the existence of a unique solution provides a solid foundation for making accurate predictions and informed policy decisions. Similarly, in engineering and physics, unique solutions facilitate the design and analysis of complex systems, ensuring their functionality and performance meet specified criteria.

To show the usefulness of our theorems, we investigate the subsequent system of algebraic linear equations. For this we consider the symmetric complete avBSbMS $(\mathbb{R}^m, M_2(\mathbb{R}), d)$ for $S_b = 4I_2$. Where the metric $d : \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m \longrightarrow \mathsf{C}$ is given by

$$d(u^T, v^T, w^T) = \begin{bmatrix} \sup_i[(u_i - v_i)^2 + (v_1 - w_i)^2] & 0 \\ 0 & \sup_i[(u_i - v_i)^2 + (v_1 - w_i)^2] \end{bmatrix},$$

where $u = (u_i), v = (v_i), w = (w_i) \in \mathbb{R}^m$.

Any system of m linear equations in m unknown variables $u_1, u_2, \ldots, u_m$ can be put in a matrix form as follows:

$$\begin{cases} a_{11}u_1 + a_{12}u_2 + \cdots + a_{1m}u_m + v_1 &= 0 \\ a_{21}u_1 + a_{22}u_2 + \cdots + a_{2m}u_m + v_2 &= 0 \\ \vdots & \vdots \\ a_{m1}u_1 + a_{m2}u_2 + \cdots + a_{mm}u_m + v_m &= 0 \end{cases} ;\ a_{ij}, v_i \in \mathbb{R},\ \text{ for all } 1 \le i, j \le m. \tag{4.1}$$

This can be represented as $Au^T + v^T = o^T$, where $A = (a_{ij}),\ u^T = (u_1, u_2, \ldots, u_m), v^T = (v_1, v_2, \ldots, v_m)$ and $o^T = (0, 0, \ldots, 0)$.
Let $H : \mathbb{R}^m \longrightarrow \mathbb{R}^m$ be given by

$$Hu^T = H\begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{pmatrix} = \begin{pmatrix} (a_{11}+1)u_1 + a_{12}u_2 + \cdots + a_{1m}u_m + v_1 \\ a_{21}u_1 + (a_{22}+1)u_2 + \cdots + a_{2m}u_m + v_2 \\ \vdots \\ a_{m1}u_1 + a_{m2}u_2 + \cdots + (a_{mm}+1)u_m + v_m \end{pmatrix} = (A + I_m)u^T + v^T$$

Then to find a solution $u^T \in \mathbb{R}^m$ of (4.1) is equivalent to find a solution $u^T \in \mathbb{R}^m$ such that $Hu^T = u^T$.

Now we present a theorem consisting of the requirements for the existence of solution of (4.1).

Theorem 4.1. *Any system of m linear equations in m unknown variables given by (4.1) will exhibit a unique solution, if*

$$\sup_i \sum_{j=1}^{m} |b_{ij}| < 1,$$

where $b_{ij} = a_{ij}$, for all $i \neq j$ and $b_{ij} = a_{ij} + 1$, for all $i = j$.

Proof. For arbitrary $u^T = (u_1, u_2, \ldots, u_m), v^T = (v_1, v_2, \ldots, v_m) \in \mathbb{R}^m$, we have

$$\begin{aligned} & d(Hu^T, Hu^T, Hv^T) \\ &= \begin{bmatrix} \sup_i \left[\sum_{j=1}^{m} b_{ij}(u_j - v_j) \right]^2 & 0 \\ 0 & \sup_i \left[\sum_{j=1}^{m} b_{ij}(u_j - v_j) \right]^2 \end{bmatrix} \\ &= \begin{bmatrix} \left[\sup_i \left| \sum_{j=1}^{m} b_{ij}(u_j - v_j) \right| \right]^2 & 0 \\ 0 & \left[\sup_i \left| \sum_{j=1}^{m} b_{ij}(u_j - v_j) \right| \right]^2 \end{bmatrix} \end{aligned}$$

$$\leq \begin{bmatrix} \left[\sup_i \sum_{j=1}^{m} |b_{ij}|\right]^2 \sup_j \left[(u_j - v_j)\right]^2 & 0 \\ 0 & \left[\sup_i \sum_{j=1}^{m} |b_{ij}|\right]^2 \sup_j \left[(u_j - v_j)\right]^2 \end{bmatrix}$$

$$= c^* \begin{bmatrix} \sup_i [(u_i - v_i)^2] & 0 \\ 0 & \sup_i [(u_i - v_i)^2] \end{bmatrix} c.$$

This implies $d(Hu^T, Hu^T, Hv^T) \leq c^* d(u^T, u^T, v^T)c$,

where $c^* = c = \begin{bmatrix} \left[\sup_i \sum_{j=1}^{m} |b_{ij}|\right] & 0 \\ 0 & \left[\sup_i \sum_{j=1}^{m} |b_{ij}|\right] \end{bmatrix}$ with $c \prec I_2$. By Theorem 3.1, the system (4.1) has a unique solution. □

The following is a numerical illustration of Theorem 4.1.

Example 4.1. *Consider a system of linear equations in 4 variables as*

$$\begin{cases} 0.8u_1 + 0.1u_2 + 0.3u_3 + 0.2u_4 &= 6 \\ 0.4u_1 + u_2 + 0.3u_3 + 0.2u_4 &= 1 \\ 0.2u_1 + 0.1u_2 + 0.9u_3 + 0.5u_4 &= 2 \\ 0.1u_1 + 0.2u_2 + 0.3u_3 + 0.7u_4 &= 7 \end{cases} \tag{4.2}$$

Comparing (4.2) with (4.1) for Theorem 4.1, we have $b_{11} = -0.8 + 1 = 0.2, b_{12} = -0.1, b_{13} = -0.3, b_{14} = -0.2, b_{21} = -0.4, b_{22} = -1 + 1 = 0, b_{23} = -0.3, b_{24} = -0.2, b_{31} = -0.2, b_{32} = -0.1, b_{33} = -0.9 + 1 = 0.1, b_{34} = -0.5, b_{41} = -0.1, b_{42} = -0.2, b_{43} = -0.3, b_{44} = -0.7 + 1 = 0.3$ *such that* $\sup_{1\leq i\leq 4} \sum_{j=1}^{4} |b_{ij}| = 0.9 < 1$. *So, the system (4.2) has a unique solution and which is given by* $u_1 \approxeq -6.942, u_2 \approxeq 2.470, u_3 \approxeq 5.831, u_4 \approxeq -12.213$.

We know that linear dependence in a system of equations reduces the effective number of constraints imposed by the system, allowing for a broader range of solutions and ultimately leading to an infinite number of possible solutions. But, checking for linear dependence in a system of equations can indeed be challenging at times, especially for larger systems or when the relationships between equations are not immediately apparent. To address these challenges, computational tools and algorithms are often employed. These tools use mathematical methods such as Gaussian elimination, matrix row reduction, or singular value decomposition to analyze the system and determine linear dependence. While these methods provide accurate results, they may still require expertise and careful interpretation, particularly in more complex scenarios. The example below explains the easiness of our result to check the nonunique existence of solutions.

Example 4.2. *Consider a system of linear equations in 4 variables as*

$$\begin{cases} 0.8u_1 + 0.1u_2 + 0.3u_3 + 0.2u_4 & = & 6 \\ 0.4u_1 + u_2 + 0.3u_3 + 0.2u_4 & = & 1 \\ 0.2u_1 + 0.1u_2 + 0.9u_3 + 0.5u_4 & = & 2 \\ 0.1u_1 + 0.05u_2 + 0.45u_3 + 0.25u_4 & = & 1 \end{cases} \tag{4.3}$$

This system has infinitely many solutions because the third equation is a constant multiple of the fourth equation, indicating a linear dependency among the equations.

Now calculating $\sup\limits_{1\le i\le 4} \sum\limits_{j=1}^{4} |b_{ij}|$, *we see the supremum value is* 1.35*. So, the system (4.3) violates the criteria of Theorem 4.1.*

5. Conclusions and Future Work

In this study, we began by defining novel concepts such as C^*-algebra valued Branciari S_b-metric space and C^*-algebra valued extended Branciari S_b-metric space, emphasizing the role of symmetry. Through a detailed exposition, we demonstrated that the latter serves as a comprehensive generalization encompassing all previously known C^*-algebra valued metric spaces, integrating symmetric properties. Utilizing an illustrative example, we elucidated the broader applicability and scope of C^*-algebra valued extended Branciari S_b-metric spaces. Our exploration extended beyond mere definitions, culminating in the derivation of consequential corollaries and the establishment of two pivotal theorems. These theorems bear resemblance to renowned results like Banach and Kannan theorems, further underlining the significance and relevance of our generalized metric spaces in mathematical discourse, with a special focus on symmetry. Moreover, we delved into practical implications by demonstrating how our theoretical framework can be effectively applied. Specifically, we showcased the utility of our results in verifying the unique existence of solutions within algebraic systems of linear equations, highlighting the tangible impact and applicability of our research in real-world problem-solving contexts.

In future work authors can try to generalize our results which are analogous to the very famous Chatterjea, Ćirić, Reich, Hardy-Roger, etc. type theorems in our generalized symmetric metric spaces.

Open Problem. A variety of fixed point solutions using rational and product terms, such as interpolative contraction, integral type contraction, (α, β, F^*) and $(\alpha, \beta, F^{**})-$weak Geraghty contraction, etc. (for more contractions, see [11]) can be studied because avBSbMS and aveBSbMS are relatively new spaces.

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