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On Null Vertex in Bipolar Fuzzy Graphs

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Abstract. We present a novel vertex in Bipolar fuzzy graph, null vertex, which is distinct from boundary vertex and interior vertex and also attempt a study on null vertex in bipolar fuzzy closed helm graph CH_n.

1. Introduction

Euler developed the idea of graph theory whereas Rosenfeld created fuzzy graph (FG) theory [6]. Graph theory deals with the study of graphs, which consist of vertices and edges. The idea of fuzzy sets put forward by Zadeh initiated explosive developments in research [10]. Fuzzy set theory deals with uncertainty and imprecision in the description of sets. FGs are mathematical representation that combines graph theory with fuzzy set theory. Bipolar fuzzy (BF) sets were introduced to represent uncertainty and ambiguity in a more nuanced way than traditional fuzzy sets. In FGs, vertices and edges have membership values in [0,1]. Bipolar fuzzy graphs (BFG), on the other hand, use bipolar membership values, which can take values from the set [-1,1]. The definition of BFG is introduced in [1]. The concept of interior vertex (I-vertex) and boundary vertex (B-vertex) in graphs, FGs and BFGs are discussed in [3,4,5,9]. Null vertex, a vertex distinct from B-vertices and I-vertices in graphs and FGs are discussed in [7,8]. We introduce null vertex in BFGs and initiate a study on null vertex in BF closed helm graph CH_n.

2. Preliminaries

Definition 2.1. [1] A BFG is $G = (\chi, \psi)$, where, $\chi = (\mu_\chi^+, \mu_\chi^-)$ is a BF set on V, $\psi = (\mu_\psi^+, \mu_\psi^-)$ is a BF set on $E \subseteq V \times V$,

$$\mu_\psi^+(\epsilon, \omega) \leq min\{\mu_\chi^+(\epsilon), \mu_\chi^+(\omega)\}$$

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$$\mu^-_\psi(\epsilon,\omega) \geq max\{\mu^-_\chi(\epsilon), \mu^-_\chi(\omega)\}, \quad \forall \quad \epsilon,\omega \in V.$$

Definition 2.2. [2] In a BFG $G = (\chi, \psi)$, a path is a sequence of vertices $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, such that for some $y_i = (\epsilon_{i-1}, \epsilon_i)$, that satisfies (i), (ii) or (iii).
(i) $\mu^+(y_i) > 0$, $\mu^-(y_i) < 0$ (ii) $\mu^+(y_i) > 0$, $\mu^-(y_i) = 0$ (iii) $\mu^+(y_i) = 0$, $\mu^-(y_i) < 0$.

Definition 2.3. [2] A BFG $G = (\chi, \psi)$ is connected if any two vertices are joined by a path.

Definition 2.4. [2] In a BFG $G = (\chi, \psi)$, the μ^+ strength of connectedness, $CONN^+_G(\epsilon,\omega)$, between ϵ, ω is the maximum of the strength of all paths between them. The μ^- strength of connectedness, $CONN^-_G(\epsilon,\omega)$, is the minimum of the strength of all paths between them. An arc (ϵ,ω) of G is a strong arc if $\mu^+(\epsilon,\omega) \geq CONN^+_{G-(\epsilon,\omega)}(\epsilon,\omega)$ and $\mu^-(\epsilon,\omega) \leq CONN^-_{G-(\epsilon,\omega)}(\epsilon,\omega)$.

Definition 2.5. Two vertices ϵ and ω of the BFG $G = (\chi, \psi)$ are neighbours if $\mu^+(\epsilon,\omega) > 0$, $\mu^-(\epsilon,\omega) < 0$. If an arc (ϵ,ω) of $G = (\chi,\psi)$ is strong, then ω is called a strong neighbour of ϵ. A vertex ω is a BF end vertex of G if it has only one strong neighbour.

Definition 2.6. [9] In a connected BFG G, let $\epsilon, \omega \in V(G)$. For $i = 1, 2, \ldots,$ let $\mathbf{P} = \{P_i : P_i$ is a $\epsilon - \omega$ path$\}$. For any path P, $L^+(P) = \sum_{i=1}^n \mu^+(\epsilon_{i-1}, \epsilon_i)$, $L^-(P) = \sum_{i=1}^n \mu^-(\epsilon_{i-1}, \epsilon_i)$. The sum distance between ϵ and ω is $d_s(\epsilon,\omega) = \left(d^+_s(\epsilon,\omega), d^-_s(\epsilon,\omega)\right)$ where, $d^+_s(\epsilon,\omega) = min\{L^+(P_i) : P_i \in \mathbf{P}\}$, $d^-_s(\epsilon,\omega) = max\{L^-(Pi) : Pi \in \mathbf{P}\}$.

3. Main Results

Definition 3.1. [9] In a BFG G, let $\epsilon, \omega \in V(G)$. Then, ω is a B-vertex of ϵ if, for all neighbours θ of ω, $d^+_s(\epsilon,\omega) \geq d^+_s(\epsilon,\theta)$, $d^-_s(\epsilon,\omega) \leq d^-_s(\epsilon,\theta)$. ω is a B-vertex of G if some vertex of G has ω as a B-vertex.

Definition 3.2. [9] In a BFG G, a vertex θ lies between two other vertices ϵ, ω where, $\epsilon \neq \theta \neq \omega$, if $d^+_s(\epsilon,\omega) = d^+_s(\epsilon,\theta) + d^+_s(\theta,\omega)$, $d^-_s(\epsilon,\omega) = d^-_s(\epsilon,\theta) + d^-_s(\theta,\omega)$. A vertex θ is an I-vertex of G, if for each vertex ϵ, there exists a vertex ω, where, $\epsilon \neq \theta \neq \omega$ such that θ lies between ϵ and ω. A B-vertex of a BFG is not a I-vertex.

Definition 3.3. In a BFG, a vertex that is neither a B-vertex nor an I-vertex is called a null vertex.

Proposition 3.1. For a connected BFG $G = (\chi, \psi)$, a BF end vertex is a B-vertex.

Proof. Let $G = (\chi, \psi)$ be a connected BFG with vertices $\omega_1, \omega_2, \ldots, \omega_n$. Consider a BF end vertex ω_1. Then, ω_1 has exactly one neighbour say, ω_2. Then, $d^+_s(\omega_i,\omega_1) \geq d^+_s(\omega_i,\omega_2)$, $d^-_s(\omega_i,\omega_1) \leq d^-_s(\omega_i,\omega_2)$, $1 < i \leq n$. ie, ω_1 is a B-vertex of ω_i. □

Theorem 3.1. A BF path graph P_n has two B-vertices and $(n-2)$ I-vertices.

Proof. Consider P_n with vertices $\omega_1, \omega_2, \ldots, \omega_n$. Suppose ω_1, ω_n are BF end vertices. Then, ω_1, ω_n are B-vertices. Also, ω_j, $j = 2, 3, \ldots (n-1)$ are I-vertices because, for every ω_i, there exists ω_k, $i \neq j \neq k$ such that $d^+_s(\omega_i,\omega_k) = d^+_s(\omega_i,\omega_j) + d^+_s(\omega_j,\omega_k)$, $d^-_s(\omega_i,\omega_k) = d^-_s(\omega_i,\omega_j) + d^-_s(\omega_j,\omega_k)$, $1 \leq i,k \leq n$. □

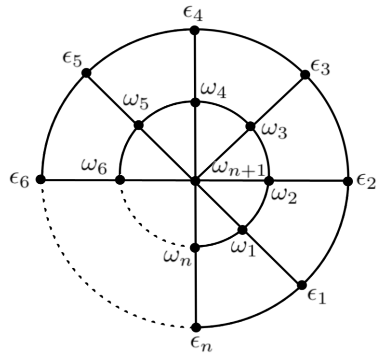

FIGURE 1. BF Closed helm graph CH_n

Theorem 3.2. A null vertex ω_{n+1} exists in the BF closed helm graph $CH_n, n \geq 3$ (Figure 1) with $2n+1$ vertices $\epsilon_i, \omega_i, i = 1, 2, \ldots n$, and ω_{n+1}, the apex vertex joining ϵ_i, ω_i,

$$\mu(\omega_i, \omega_{n+1}) = (p, -p)$$
$$\mu(\epsilon_i, \epsilon_j) = \mu(\omega_i, \omega_j) = \mu(\epsilon_i, \omega_i) = (q, -q), \quad 1 \leq i, j \leq n$$

$$\frac{2p}{n-1} < q < \frac{4p}{n-1}, \quad n \text{ is odd}$$
$$\frac{2p}{n} < q < \frac{4p}{n}, \quad n \text{ is even}$$

Proof. CH_n is created from the helm graph H_n by connecting vertices of degree 1 to form a cycle. CH_n has $2n+1$ vertices ϵ_i, ω_i with $deg(\epsilon_i) = 3, deg(\omega_i) = 4, 1 \leq i \leq n$ and an apex vertex ω_{n+1} with $deg(\omega_{n+1}) = n$.

Case (1) n is odd, $n \geq 3$.

The vertices ϵ_i, for all i are B-vertices of ω_{n+1}. Let $\frac{n-1}{2} = k, \quad \frac{n+1}{2} = m$.

When $i < m, \quad \omega_i$ are B-vertices of $\epsilon_{i+k}, \epsilon_{i+m}$.

When $i = m, \quad \omega_i$ is a B-vertex of $\epsilon_{i+k}, \epsilon_{i-k}$.

When $i > m, \quad \omega_i$ are B-vertices of $\epsilon_{i-k}, \epsilon_{i-m}$.

Consider the vertex ω_{n+1}. The neighbours of ω_{n+1} are $\omega_i, 1 \leq i \leq n$.

$d_s(\epsilon_i, \omega_{n+1}) = ((p+q), -(p+q))$

When $i < m, \quad d_s(\epsilon_i, \omega_{i+k}) = d_s(\epsilon_i, \omega_{i+m}) = (mq, -mq)$

When $i = m, \quad d_s(\epsilon_i, \omega_{i+k}) = d_s(\epsilon_i, \omega_{i-k}) = (mq, -mq)$

When $i > m, \quad d_s(\epsilon_i, \omega_{i-k}) = d_s(\epsilon_i, \omega_{i-m}) = (mq, -mq)$

Given, $p < kq$. Then, $p + q < mq, \quad -(p+q) > -mq$.

For $i < m$ and for the neighbours $\omega_{i+k}, \omega_{i+m}$ of ω_{n+1},

$$\begin{cases} d_s^+(\epsilon_i, \omega_{n+1}) < d_s^+(\epsilon_i, \omega_{i+k}), \quad d_s^+(\epsilon_i, \omega_{n+1}) < d_s^+(\epsilon_i, \omega_{i+m}) \\ d_s^-(\epsilon_i, \omega_{n+1}) > d_s^-(\epsilon_i, \omega_{i+k}), \quad d_s^-(\epsilon_i, \omega_{n+1}) > d_s^-(\epsilon_i, \omega_{i+m}) \end{cases} \tag{3.1}$$

For $i = m$ and for the neighbours $\omega_{i+k}, \omega_{i-k}$ of ω_{n+1}.

$$\begin{cases} d_s^+(\epsilon_i, \omega_{n+1}) < d_s^+(\epsilon_i, \omega_{i+k}), \quad d_s^+(\epsilon_i, \omega_{n+1}) < d_s^+(\epsilon_i, \omega_{i-k}) \\ d_s^-(\epsilon_i, \omega_{n+1}) > d_s^-(\epsilon_i, \omega_{i+k}), \quad d_s^-(\epsilon_i, \omega_{n+1}) > d_s^-(\epsilon_i, \omega_{i-k}) \end{cases} \tag{3.2}$$

For $i > m$ and for the neighbours $\omega_{i-k}, \omega_{i-m}$ of ω_{n+1},

$$\begin{cases} d_s^+(\epsilon_i, \omega_{n+1}) < d_s^+(\epsilon_i, \omega_{i-k}), \quad d_s^+(\epsilon_i, \omega_{n+1}) < d_s^+(\epsilon_i, \omega_{i-m}) \\ d_s^-(\epsilon_i, \omega_{n+1}) > d_s^-(\epsilon_i, \omega_{i-k}), \quad d_s^-(\epsilon_i, \omega_{n+1}) > d_s^-(\epsilon_i, \omega_{i-m}) \end{cases} \tag{3.3}$$

From (3.1), (3.2) and (3.3), ω_{n+1} is not a B-vertex of ϵ_i.
$d_s(\omega_i, \omega_{n+1}) = (p, -p)$
When $i < m$, $\quad d_s(\omega_i, \omega_{i+k}) = d_s(\omega_i, \omega_{i+m}) = (kq, -kq)$
When $i = m$, $\quad d_s(\omega_i, \omega_{i+k}) = d_s(\omega_i, \omega_{i-k}) = (kq, -kq)$
When $i > m$, $\quad d_s(\omega_i, \omega_{i-k}) = d_s(\omega_i, \omega_{i-m}) = (kq, -kq)$
Given, $\frac{2p}{n-1} < q$. So, $p < \left(\frac{n-1}{2}\right)q, \quad -p > -\left(\frac{n-1}{2}\right)q$. i.e., $p < kq, \quad -p > -kq$.
For $i < m$ and for the neighbours $\omega_{i+k}, \omega_{i+m}$ of ω_{n+1},

$$\begin{cases} d_s^+(\omega_i, \omega_{n+1}) < d_s^+(\omega_i, \omega_{i+k}), \quad d_s^+(\omega_i, \omega_{n+1}) < d_s^+(\omega_i, \omega_{i+m}) \\ d_s^-(\omega_i, \omega_{n+1}) > d_s^-(\omega_i, \omega_{i+k}), \quad d_s^-(\omega_i, \omega_{n+1}) > d_s^-(\omega_i, \omega_{i+m}) \end{cases}$$

Thus, ω_{n+1} is not a B-vertex of $\omega_i, i < m$.
Similarly, the condition for ω_{n+1} to be a B-vertex of ω_i does not hold for $i = m$ and $i > m$. For $i \neq j$,

$$\begin{cases} d_s^+(\epsilon_i, \omega_{n+1}) + d_s^+(\omega_{n+1}, \omega_j) = 2p + q \\ d_s^-(\epsilon_i, \omega_{n+1}) + d_s^-(\omega_{n+1}, \omega_j) = -(2p + q) \end{cases} \tag{3.4}$$

$$\text{But,} \quad d_s^+(\epsilon_i, \omega_j) < 2p + q, \quad d_s^-(\epsilon_i, \omega_j) > -(2p + q) \tag{3.5}$$

From (3.4), (3.5),

$$\begin{cases} d_s^+(\epsilon_i, \omega_j) \neq d_s^+(\epsilon_i, \omega_{n+1}) + d_s^+(\omega_{n+1}, \omega_j) \\ d_s^-(\epsilon_i, \omega_j) \neq d_s^-(\epsilon_i, \omega_{n+1}) + d_s^-(\omega_{n+1}, \omega_j) \end{cases}$$

i.e., ω_{n+1} does not lie between ϵ_i and ω_j.

$$\begin{cases} d_s^+(\omega_i, \omega_{n+1}) + d_s^+(\omega_{n+1}, \omega_j) = 2p \\ d_s^-(\omega_i, \omega_{n+1}) + d_s^-(\omega_{n+1}, \omega_j) = -2p \end{cases} \tag{3.6}$$

Given, $q < \frac{4p}{n-1}$. So, $(\frac{n-1}{2})q < 2p, -(\frac{n-1}{2})q > -2p$. i.e., $kq < 2p, \quad -kq > -2p$.

$$d_s^+(\omega_i, \omega_j) \leq kq < 2p, \quad d_s^-(\omega_i, \omega_j) > -2p. \tag{3.7}$$

From (3.6), (3.7), ω_{n+1} does not lie between ω_i and ω_j.

$$\begin{cases} d_s^+(\epsilon_i, \omega_{n+1}) + d_s^+(\omega_{n+1}, \epsilon_j) = 2(p + q) \\ d_s^-(\epsilon_i, \omega_{n+1}) + d_s^-(\omega_{n+1}, \epsilon_j) = -2(p + q) \end{cases} \tag{3.8}$$

$$\text{But,} \quad d_s^+(\epsilon_i, \epsilon_j) < 2(p+q), \quad d_s^-(\epsilon_i, \epsilon_j) > -2(p+q) \tag{3.9}$$

From (3.8), (3.9), ω_{n+1} does not lie between ϵ_i and ϵ_j. So, ω_{n+1} is not an I-vertex. Hence, ω_{n+1} is a null vertex.

Case (2) n is even, $n \geq 4$.
The vertices ϵ_i, for all i are B-vertices of ω_{n+1}. Let $\frac{n}{2} = t, \quad \frac{n}{2} + 1 = s$.
When $i \leq t, \quad \omega_i$ are B-vertices of ϵ_{i+t}.
When $i > t, \quad \omega_i$ are B-vertices of ϵ_{i-t}.
Consider ω_{n+1}. The neighbours of ω_{n+1} are $\omega_i, 1 \leq i \leq n$.
$d_s(\omega_i, \omega_{n+1}) = (p, -p)$,
When $i \leq t$, $d_s(\omega_i, \omega_{i+t}) = (tq, -tq)$
When $i > t$, $d_s(\omega_i, \omega_{i-t}) = (tq, -tq)$
Given, $\frac{2p}{n} < q$. i.e., $p < tq, \quad -p > -tq$
Consider the neighbours $\omega_{i+t}, \omega_{i-t}$ of ω_{n+1}.

$$\begin{cases} d_s^+(\omega_i, \omega_{n+1}) < d_s^+(\omega_i, \omega_{i+t}) \\ d_s^-(\omega_i, \omega_{n+1}) > d_s^-(\omega_i, \omega_{i+t}), \end{cases} \quad i \leq t \tag{3.10}$$

$$\begin{cases} d_s^+(\omega_i, \omega_{n+1}) < d_s^+(\omega_i, \omega_{i-t}) \\ d_s^-(\omega_i, \omega_{n+1}) > d_s^-(\omega_i, \omega_{i-t}), \end{cases} \quad i > t \tag{3.11}$$

From (3.10) and (3.11), ω_{n+1} is not a B-vertex of ω_i.

$d_s(\epsilon_i, \omega_{n+1}) = ((p+q), -(p+q))$

For $i \leq t, \quad d_s(\epsilon_i, \omega_{i+t}) = (sq, -sq)$
For $i > t, \quad d_s(\epsilon_i, \omega_{i-t}) = (sq, -sq)$
Given, $p < \frac{n}{2}q, \quad -p > -\frac{n}{2}q$. i.e., $p + q < sq, \quad -p - q > -sq$,

$$\begin{cases} d_s^+(\epsilon_i, \omega_{n+1}) < d_s^+(\epsilon_i, \omega_{i+t}) \\ d_s^-(\epsilon_i, \omega_{n+1}) > d_s^-(\epsilon_i, \omega_{i+t}), \end{cases} \quad i \leq t \tag{3.12}$$

$$\begin{cases} d_s^+(\epsilon_i, \omega_{n+1}) < d_s^+(\epsilon_i, \omega_{i-t}) \\ d_s^-(\epsilon_i, \omega_{n+1}) > d_s^-(\epsilon_i, \omega_{i-t}), \end{cases} \quad i > t \tag{3.13}$$

From (3.12) and (3.13), ω_{n+1} is not a B-vertex of ϵ_i. For $i \neq j$,

$$\begin{cases} d_s^+(\epsilon_i, \omega_{n+1}) + d_s^+(\omega_{n+1}, \omega_j) = 2p + q \\ d_s^-(\epsilon_i, \omega_{n+1}) + d_s^-(\omega_{n+1}, \omega_j) = -(2p + q) \end{cases} \tag{3.14}$$

$$\text{But,} \quad d_s^+(\epsilon_i, \omega_j) < 2p + q, \quad d_s^-(\epsilon_i, \omega_j) > -(2p + q) \tag{3.15}$$

From (3.14), (3.15)

$$\begin{cases} d_s^+(\epsilon_i, \omega_j) \neq d_s^+(\epsilon_i, \omega_{n+1}) + d_s^+(\omega_{n+1}, \omega_j) \\ d_s^-(\epsilon_i, \omega_j) \neq d_s^-(\epsilon_i, \omega_{n+1}) + d_s^-(\omega_{n+1}, \omega_j) \end{cases}$$

i.e., ω_{n+1} does not lie between ϵ_i and ω_j.

$$\begin{cases} d_s^+(\omega_i, \omega_{n+1}) + d_s^+(\omega_{n+1}, \omega_j) = 2p \\ d_s^-(\omega_i, \omega_{n+1}) + d_s^-(\omega_{n+1}, \omega_j) = -2p \end{cases} \quad (3.16)$$

Given, $q < \frac{4p}{n}$. So, $(\frac{n}{2})q < 2p, -(\frac{n}{2})q > -2p$. i.e., $tq < 2p, -tq > -2p$.

$$d_s^+(\omega_i, \omega_j) \leq tq < 2p, \quad d_s^-(\omega_i, \omega_j) > -2p. \quad (3.17)$$

From (3.16), (3.17), ω_{n+1} does not lie between ω_i and ω_j.

$$\begin{cases} d_s^+(\epsilon_i, \omega_{n+1}) + d_s^+(\omega_{n+1}, \epsilon_j) = 2(p+q) \\ d_s^-(\epsilon_i, \omega_{n+1}) + d_s^-(\omega_{n+1}, \epsilon_j) = -2(p+q) \end{cases} \quad (3.18)$$

$$\text{But,} \quad d_s^+(\epsilon_i, \epsilon_j) < 2(p+q), \quad d_s^-(\epsilon_i, \epsilon_j) > -2(p+q) \quad (3.19)$$

From (3.18), (3.19), ω_{n+1} does not lie between ϵ_i and ϵ_j. So, ω_{n+1} is not an I-vertex. Thus, ω_{n+1} is a null vertex. □

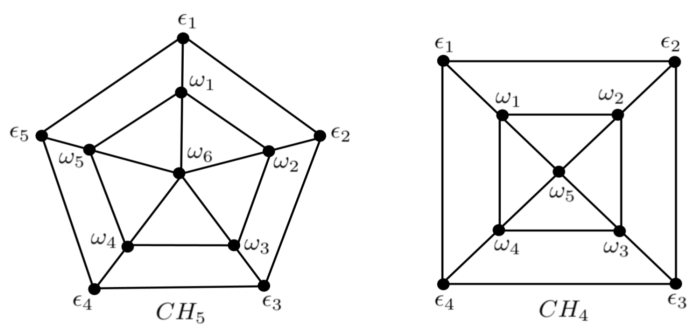

Figure 2. BF Closed helm graphs CH_5 and CH_4

Example 3.1. In the BF closed helm graph CH_5 (n is odd) in Figure 2 with vertices $\epsilon_i, \omega_i, 1 \leq i \leq 5$ and apex vertex ω_6, let $\mu(\omega_i, \omega_6) = (0.8, -0.8)$, $\mu(\epsilon_i, \epsilon_j) = \mu(\omega_i, \omega_j) = \mu(\epsilon_i, \omega_i) = (0.6, -0.6)$, $1 \leq i, j \leq 5$. Then,

$d_s(\omega_i, \omega_6) = (0.8, -0.8)$.

$d_s(\omega_1, \omega_3) = d_s(\omega_1, \omega_4) = (1.2, -1.2)$

The vertices $\epsilon_i, \omega_i, 1 \leq i \leq 5$ are B-vertices by definition.

ω_6 is not a B-vertex of ω_1, since,

$$d_s^+(\omega_1, \omega_6) < d_s^+(\omega_1, \omega_j),$$

$$d_s^-(\omega_1, \omega_6) > d_s^-(\omega_1, \omega_j), \text{for the neighbours } \omega_j, j = 3, 4 \text{ of } \omega_6.$$

Similarly, ω_6 is not a B-vertex of the other vertices $\omega_i, 2 \leq i \leq 5$.
$d_s(\epsilon_i, \omega_6) = (1.4, -1.4)$.
$d_s(\epsilon_1, \omega_3) = d_s(\epsilon_1, \omega_4) = (1.8, -1.8)$.
ω_6 is not a B-vertex of ϵ_1 since,

$$d_s^+(\epsilon_1, \omega_6) < d_s^+(\epsilon_1, \omega_j),$$
$$d_s^-(\epsilon_1, \omega_6) > d_s^-(\epsilon_1, \omega_j), \text{for the neighbours } \omega_j, j = 3, 4 \text{ of } \omega_6.$$

Similarly, ω_6 is not a a B-vertex of $\epsilon_i, 2 \leq i \leq 5$.
$d_s^+(\omega_i, \omega_j) \leq 1.2, \quad d_s^+(\omega_i, \omega_6) + d_s^+(\omega_6, \omega_j) = 1.6,$
$d_s^-(\omega_i, \omega_j) \geq -1.2, \quad d_s^-(\omega_i, \omega_6) + d_s^-(\omega_6, \omega_j) = -1.6$
$d_s^+(\omega_i, \omega_j) \neq d_s^+(\omega_i, \omega_6) + d_s^+(\omega_6, \omega_j)$.
$d_s^-(\omega_i, \omega_j) \neq d_s^-(\omega_i, \omega_6) + d_s^-(\omega_6, \omega_j)$.
$\Rightarrow \omega_6$ does not lie between ω_i and ω_j.
$d_s^+(\epsilon_i, \omega_j) \leq 1.8, \quad d_s^+(\epsilon_i, \omega_6) + d_s^+(\omega_6, \omega_j) = 2.2,$
$d_s^-(\epsilon_i, \omega_j) \geq -1.8, \quad d_s^-(\epsilon_i, \omega_6) + d_s^-(\omega_6, \omega_j) = -2.2,$
$d_s^+(\epsilon_i, \omega_j) \neq d_s^+(\epsilon_i, \omega_6) + d_s^+(\omega_6, \omega_j)$.
$d_s^-(\epsilon_i, \omega_j) \neq d_s^-(\epsilon_i, \omega_6) + d_s^-(\omega_6, \omega_j)$.
$\Rightarrow \omega_6$ does not lie between ϵ_i and ω_j.
$d_s^+(\epsilon_i, \epsilon_j) \leq 1.2, \quad d_s^+(\epsilon_i, \omega_6) + d_s^+(\omega_6, \epsilon_j) = 2.8,$
$d_s^-(\epsilon_i, \epsilon_j) \geq -1.2, \quad d_s^-(\epsilon_i, \omega_6) + d_s^-(\omega_6, \epsilon_j) = -2.8,$
$d_s^+(\epsilon_i, \epsilon_j) \neq d_s^+(\epsilon_i, \omega_6) + d_s^+(\omega_6, \epsilon_j)$.
$d_s^-(\epsilon_i, \epsilon_j) \neq d_s^-(\epsilon_i, \omega_6) + d_s^-(\omega_6, \epsilon_j)$.
$\Rightarrow \omega_6$ does not lie between ϵ_i and ϵ_j. So, ω_6 is not an I-vertex.
Hence, ω_6 is a null vertex.

Example 3.2. In the BF closed helm graph CH_4 (n is even) in Figure 2 with vertices $\epsilon_i, \omega_i, 1 \leq i \leq 4$ and apex vertex ω_5, let $\mu(\omega_i, \omega_5) = (0.8, -0.8)$, $\mu(\epsilon_i, \epsilon_j) = \mu(\omega_i, \omega_j) = \mu(\epsilon_i, \omega_i) = (0.5, -0.5), \quad 1 \leq i, j \leq 4$.
Then, $d_s(\omega_i, \omega_5) = (0.8, -0.8)$
$d_s(\omega_1, \omega_3) = (1, -1)$.
The vertices $\epsilon_i, \omega_i, 1 \leq i \leq 4$ are B-vertices by definition.
ω_5 is not a B-vertex of ω_1, since,

$$d_s^+(\omega_1, \omega_5) < d_s^+(\omega_1, \omega_3),$$
$$d_s^-(\omega_1, \omega_5) > d_s^-(\omega_1, \omega_3), \text{ for the neighbour } \omega_3 \text{ of } \omega_5.$$

Similarly, ω_5 is not a B-vertex of $\omega_i, 2 \leq i \leq 4$.
$d_s(\epsilon_i, \omega_5) = (1.3, -1.3)$

$d_s(\epsilon_1, \omega_3) = (1.5, -1.5)$

ω_5 is not a B-vertex of ϵ_1 since,

$$d_s^+(\epsilon_1, \omega_5) < d_s^+(\epsilon_1, \omega_3),$$
$$d_s^-(\epsilon_1, \omega_5) > d_s^-(\epsilon_1, \omega_3), \text{ for the neighbour } \omega_3 \text{ of } \omega_5.$$

Similarly, ω_5 in not a B-vertex of $\epsilon_i, 2 \leq i \leq 4$.

$d_s^+(\omega_i, \omega_j) \leq 1, \quad d_s^+(\omega_i, \omega_5) + d_s^+(\omega_5, \omega_j) = 1.6$

$d_s^-(\omega_i, \omega_j) \geq -1, \quad d_s^-(\omega_i, \omega_5) + d_s^-(\omega_5, \omega_j) = -1.6$

$d_s^+(\omega_i, \omega_j) \neq d_s^+(\omega_i, \omega_5) + d_s^+(\omega_5, \omega_j)$.

$d_s^-(\omega_i, \omega_j) \neq d_s^-(\omega_i, \omega_5) + d_s^-(\omega_5, \omega_j)$.

$\Rightarrow \omega_5$ does not lie between ω_i and ω_j.

$d_s^+(\epsilon_i, \omega_j) \leq 1.5, \quad d_s^+(\epsilon_i, \omega_5) + d_s^+(\omega_5, \omega_j) = 2.1,$

$d_s^-(\epsilon_i, \omega_j) \geq -1.5, \quad d_s^-(\epsilon_i, \omega_5) + d_s^-(\omega_5, \omega_j) = -2.1$

$d_s^+(\epsilon_i, \omega_j) \neq d_s^+(\epsilon_i, \omega_5) + d_s^+(\omega_5, \omega_j)$.

$d_s^-(\epsilon_i, \omega_j) \neq d_s^-(\epsilon_i, \omega_5) + d_s^-(\omega_5, \omega_j)$.

$\Rightarrow \omega_5$ does not lie between ϵ_i and ω_j.

$d_s^+(\epsilon_i, \epsilon_j) \leq 1, \quad d_s^+(\epsilon_i, \omega_5) + d_s^+(\omega_5, \epsilon_j) = 2.6,$

$d_s^-(\epsilon_i, \epsilon_j) \geq -1, \quad d_s^-(\epsilon_i, \omega_5) + d_s^-(\omega_5, \epsilon_j) = -2.6.$

$d_s^+(\epsilon_i, \epsilon_j) \neq d_s^+(\epsilon_i, \omega_5) + d_s^+(\omega_5, \epsilon_j)$.

$d_s^-(\epsilon_i, \epsilon_j) \neq d_s^-(\epsilon_i, \omega_5) + d_s^-(\omega_5, \epsilon_j)$.

$\Rightarrow \omega_5$ does not lie between ϵ_i and ϵ_j. Thus, ω_5 is not an I-vertex.

Hence, ω_5 is a null vertex.

4. Conclusion

We introduced the concept of null vertex in BFGs and investigated the presence of null vertex in BF closed helm graphs. BFGs find applications in various fields, including decision-making, image processing, pattern recognition and modeling systems where positive and negative relationships need to be considered.

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