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A Novel Distribution in the Family of Lifetime Distributions for Enhancing Predictive Modeling for Medical and Engineering Data

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Abstract. In nearly all scientific disciplines, the statistical inference about the population rely on handling the sampled data. In the time to event analysis, there are many lifetime distributions to model variation of the lifetime observations based on the shape of hazard rate of the data. In the literature, it has been observed that for non-monotonic hazard the existing distributions do not provide good fits. Practically it is not possible for a distribution to fit any kind of data. Therefore, in this study, a new lifetime distribution is suggested called Flexible Exponentiated Weibull distribution (FEW) to model monotonic and non-monotonic hazard rate data. Maximum likelihood estimation approach is used to estimate the model parameters. In addition to these some prominent statistical properties like, reliability function, moments, hazard function, order statistics, quantile function and entropy measure are obtained. Two real data sets were taken to compare the proposed distribution with existing distributions, and the results showed that the proposed distribution is more flexible than other existing lifetime distributions. Furthermore, simulation study is carried out to check the consistency of model parameters that showed that the parameters are consistent when the sample size increases. These results establish a foundational rationale for selecting the suggested distribution as a model for such a data type. It shows that this distribution is more flexible and suitable for the data studied, making a strong case for choosing it over other options. These findings not only boost trust in the chosen model but also help in deciding how to model similar data in the future.

1. Introduction

Statistics is said to be the science of numerics, which deals with extracting form the real world and provides the tools to convert those numbers into useful information. In addition to this statistics is used to generalize the information from the sample statistic(s) to the population parameter(s). This is one of the main reasons that statistics is considered one of the main pillars of decision

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sciences. With the advancement of computers and the availability of statistical softwares, it has become an essential tool for research and decision-making in every field of life.

The techniques related to modeling lifetimes can be traced back centuries, from basic observations to complex statistical analyses. But a few decades ago, specifically during World War II motivated the interest to quantify the reliability of military equipments. Presently, with advancements in data science, we utilize intricate models to predict and comprehend lifespan dynamics with unprecedented accuracy [1].

Lifetime distributions are statistical techniques used to model the data of "time until an event of interest occurs". This event may be defined as failure of a mechanical system, death of a living organism and so on. The prominent goal of lifetime distribution is to estimate the probability of the defined event occurring at a given time and to understand the factors that may influence the time of the event. The lifetime distribution model assumes that the time to the event follows a certain probability distribution, such as the exponential, Rayleigh, or Lomax distribution, but one of the most widely used lifetime distribution is the Weibull distribution. These distributions play a vital role in predicting the lifetime data but every distribution has some limitation that compelled the researchers to modify the existing distribution so that to predict the event more precisely [2].

The exponential distribution is mostly used to model lifetimes for the constant failure rate. Similarly, the Lomax distribution is very sensitive to the outlier, but it is unable to capture the complex data. The Weibull distribution is applicable to predict the event in more complex data, but still it lacks to be used for the non-monotonic hazard rate functions. In daily life phenomenon, there are many situations that follow non-monotonic hazard rate function. The most common situation in reliability analysis can be found in the failure of automotive parts. Automotive parts such as engine components, transmissions, and braking systems are subjected to various forms of stress and wear that can cause them to fail over time. During its initial phase, a part may exhibit a heightened hazard rate attributed to manufacturing imperfections, these flaws influence the part to premature failure, resulting in a high initial failure rate. Subsequently, under normal operational conditions, the hazard rate may stabilize throughout its useful life. However, with a large lifetime, the part may undergo degradation from factors like vibration, corrosion, or extreme temperatures, inducing a gradual rise in the hazard rate during the wear-out phase, thus adhering to a non-monotonic failure rate function [3].

Similarly in Reliability analysis , during the covid-19 pandemic, it was observed that the hazard rate of corona patients was at its peak for the first two weeks and then started to decline thereafter. In the same way, the situations fluctuated country-wise where the covid-19 trend was increasing and decreasing; hence, the exponential and Weibull distribution were unable to model the data with such patterns. The researchers proposed numerous lifetime distributions to predict the pattern of uncertain future events. For example, Hadeel S. Klakattawi [4] proposed a modified form of the Weibull distribution called a new extended Weibull distribution for the Reliability analysis of cancer patients. Yoosefi, et.al, [5] produced an Exponentiated Weibull Distribution for the Reliability analysis of Colorectal Cancer Patients. The covid-19 pandemic created anxiety among the people and hence compelled the researchers to precisely forecast the situation so that government agencies and health practitioners may timely cope with the situation. For this purpose, Farooq et.al, [6] proposed a new lifetime distribution for modeling covid-19 death data, called Flexible Exponential Weibull distribution. In a similar study to model the HIV+ data, Eliva et.al, [7] proposed a single parameter lifetime distribution known as Odd Lindley Half Logistic (OLiHL) which showed that the proposed model performed better than the other existing distributions. Nasiru et.al, [8] modeled the mortality rate and recovery rate of the UK, Canada, and Spain by a new lifetime distribution known as Bounded Truncated Cauchy Power Exponential Distribution (BTCPE) which also revealed that the suggested BTCPE model produced a better fit as compared with other competing models available in the literature. To explore additional studies with a similar focus, we refer to [9–12].

The lifetime distributions also play a prominent role in reliability analysis, as these models test the failure functioning of the electronic equipment. In this connection, Sindhu and Atangana [13] suggested a new versatile distribution known as Modified Generalized Inverse Weibull Distribution (MGIWD) to examine the efficiency and lifetime of electronic devices which fitted best as compared to other similar distributions. The Weibull distribution is frequently applied by researchers to investigate the failure behavior of electronic products but still, the said distribution is unable to accurately encounter the failure pattern of these products. For this purpose, González et.al [14] proposed beta-Weibull distribution that provided a better fit for the lifetime of electronic devices. Researchers mostly examine the tensile strength data for reliability analysis which is mostly fitted by the Weibull distribution but the poor estimation methods often produce inefficient results. For this purpose, Wu et.al [15] proposed a new parameter estimation method that showed that the proposed method performs better than other estimation methods. For other studies in reliability analysis, we refer to [16–18].

The motivation behind this study is to address the limitations of the traditional Weibull distribution when dealing with non-monotonic data. The standard Weibull distribution is primarily designed to model monotonic data, where the hazard rate either continuously decreases or increases. However, real-world phenomena often exhibit complex and non-monotonic behavior, where the hazard rate may initially increase, reach a peak, and then decrease again. By modifying the Weibull distribution to handle non-monotonic data, we aim to provide a more flexible and accurate modeling approach for a wide range of practical scenarios. The standard Weibull distribution is limited in its ability to accommodate non-monotonic behavior. Introducing a new parameter allows the distribution to be more flexible and adaptable to a wider range of data patterns. This flexibility can significantly improve the model's goodness-of-fit and predictive performance when dealing with non-monotonic data.

Since the Wiebull distribution is one of the significant distributions of lifeime modeling. Therefore, the main potential benefits of modifying the Weibull distribution lie in its enhanced versatility and improved applicability to a broader range of real-world data scenarios. Such modifications can lead to more reliable predictions, better decision-making, and a deeper understanding of the underlying processes in systems characterized by non-monotonic behavior. Therefore, this study is driven by the desire to develop a more robust statistical tool that can effectively handle diverse datasets and contribute to advancements in various fields of research and industry.

2. Materials and Methods

Lifetime is considered as one of the most important random variable in daily life applications, and such type of random variables are usually modeled through probability distribution functions. A random variable that is characterized as a Flexible Exponentiated Weibull (FEW) distribution if it adheres to the cumulative distribution function (CDF) and the probability density function (PDF) specified in equation (2.1) and equation (2.2), respectively.

Defining $G_y(x)$, the CDF of the selected baseline distribution, the pdf $f(x)$ can be obtained as follows:

$$F\left(x\right) = G_y\left(x\right)\left(e^{G_y(x)-1}\right)^{\frac{1}{a}} \tag{2.1}$$

and

$$f\left(x\right) = g_y\left(x\right)\left(e^{G_y(x)-1}\right)^{\frac{1}{a}}\left(1 + \frac{G_y\left(x\right)}{a}\right) \tag{2.2}$$

will provide the required distributions.

2.1. **Suggested Distribution.** In this section a new distribution of the New Flexible Family (NFF) is obtained using the CDF of Weibull distribution, referred to as FEW. The CDF of the Weibull distribution, as defined in reference [16], is expressed as follows:

$$G_y\left(x\right) = 1 - e^{-by^c} \tag{2.3}$$

Where b and c denote the scale and shape parameters, respectively. By substituting equation (2.3) into equations (2.1) and (2.2), the CDF and PDF of the suggested FEW distribution can be obtained as follows:

$$F\left(x\right) = \left(1 - e^{-bx^c}\right)\left(e^{1-e^{-ax^b}-1}\right)^{\frac{1}{a}} \quad a, b, c > 0 \tag{2.4}$$

$$f\left(x\right) = \frac{bcx^{c-1}\left(\left(a+1\right)e^{bx^c} - 1\right)e^{\frac{1-e^{-bx^C}}{a} - 2bx^c - \frac{1}{a}}}{a} \tag{2.5}$$

Scale parameter measures the scatter or spread of the distribution. It influences the variance or standard deviation of the distribution. When the scale parameter increases, the distribution spreads out. The shape parameter influences the shape of the distribution. It determines whether the distribution is skewed to the left or right, symmetric, or has multiple peaks. In the above functions a and b are the scale parameters, while c is shape parameter of the developed distribution. For more realistic results, these parameters need to be estimated precisely. Furthermore, this distribution will have better goodness of fit measurements.

Figure 1 below explains various functional forms of the CDF and PDF for the developed distribution based on different sets of parameters respectively.

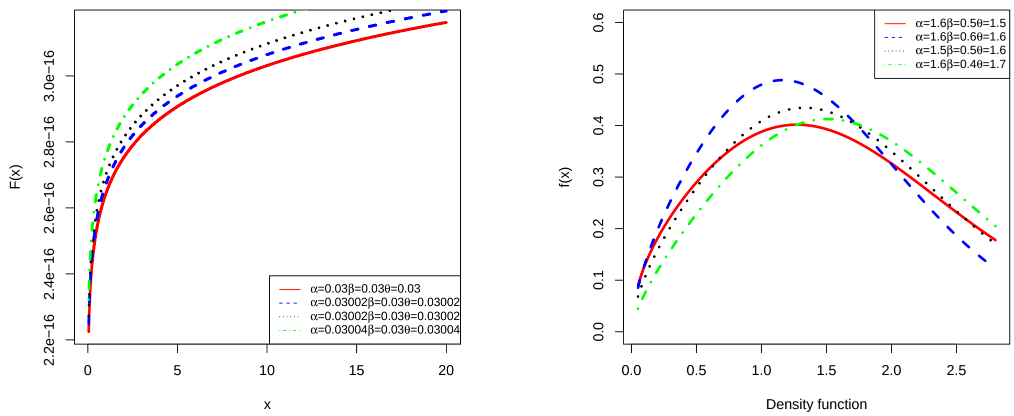

Figure 1. Graphical presentation of the CDF and PDF of FEW.

3. Statistical Properties

Derivation of the various statistical properties associated with the proposed lifetime model are presented in this section. The key properties are discussed and summarized below:

3.1. **Reliability function and Hazard rate function.** In lifetime modeling the estimation of reliability $R(x)$ and hazard rate $h(x)$ stands as fundamental components, these cab be defined as:

$$R(x) = 1 - F(x)$$

$$h(x) = \frac{f(x)}{1 - F(x)}$$

Using equation (2.4) and equation (2.5), these can be obtained through the following equations:

$$R(x) = 1 - \left(1 - e^{-bx^c}\right)\left(e^{1-e^{-ax^b}-1}\right)^{\frac{1}{a}} \quad a, b, c > 0$$

$$h(x) = \frac{bcx^{c-1}\left(\left(a+1\right)e^{bx^c} - 1\right)e^{\frac{1-e^{-bx^c}}{a} - 2bx^c - \frac{1}{a}}}{a\left(1 - \left(1 - e^{-bx^c}\right)\left(e^{1-e^{-ax^b-1}}\right)^{\frac{1}{a}}\right)}$$

Figure 2, shown below, explains the hazard rate function $h(x)$ for the different values of parameters.

The above graph distinctly illustrates that the hazard curve follows a non-monotonic hazard rate function.

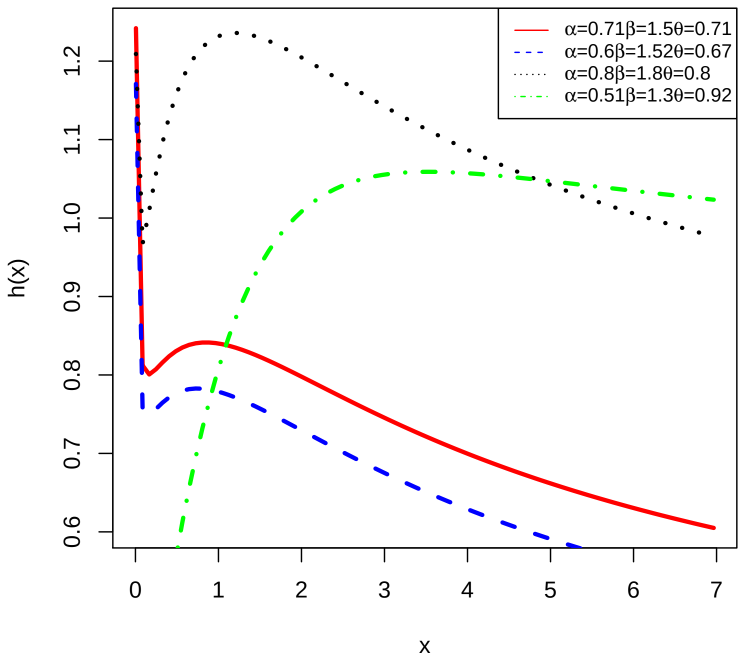

FIGURE 2. hazard rate function of FEW

3.2. **Quantile function.** Regarding the positional measures of the distribution, the quantile function can be delineated utilizing the subsequent relation:

$$p(X \leq x) = q \tag{3.1}$$

After substituting equation (2.4) into equation (3.1), resulting the following:

$$\left(1 - e^{-bx^{c}}\right)\left(e^{1-e^{-ax^{b}}-1}\right)^{\frac{1}{a}} = q$$

Solving the the above equation for obtaining the value of a random variable x for the required position at q, the final expression is obtained as:

$$x = \left(\frac{log\left(\frac{1}{1-aW\left(\frac{qe^{\frac{1}{a}}}{a}\right)}\right)}{b}\right)^{\frac{1}{c}}$$

The simplified expression includes the exponential term e, the quantile function q, and the Lambert function W.

3.3. **Moments.** For the shape and more properties like skewness and kourtosis of the distribution, moment function is required. According to the definition, the r^{th} moments can be obtained using the expression provided as follows:

$$E\left(x^{r}\right)=\int_{0}^{\infty} x^{r} f\left(x\right) dx \tag{3.2}$$

After substituting equation (2.5) into equation (3.2), we obtain the following result:

$$E\left(x^{r}\right)=\int_{0}^{\infty} x^{r}\left(\frac{bcx^{c-1}\left(1-e^{-bx^{c}}\right)e^{-\left(\frac{e^{-bx^{c}}}{a}\right)-bx^{c}}}{a}+bcx^{c-1}e^{-\left(\frac{e^{-bx^{c}}}{a}\right)-bx^{c}}\right)dx$$

$$E\left(x^{r}\right)=\int_{0}^{\infty} x^{r}\left(\frac{bcx^{c-1}e^{-\left(\frac{e^{-bx^{c}}}{a}\right)-bx^{c}}-bcx^{c-1}e^{-\left(\frac{e^{-bx^{c}}}{a}\right)-2bx^{c}}}{a}+bcx^{c-1}e^{-\left(\frac{e^{-bx^{c}}}{a}\right)-bx^{c}}\right)dx \tag{3.3}$$

We can break down the above equation into parts and solve each part separately, as shown below:

$$E(x^{r})=I+II+III$$

$$I=\frac{bc}{a}\int_{0}^{\infty} x^{r+c-1}e^{-\left(\frac{e^{-bx^{c}}}{a}\right)-bx^{c}}dx$$

$$I=\frac{bc}{a}\int_{0}^{\infty} x^{r+c-1}e^{-bx^{c}}\sum_{k=0}^{\infty}\left(\left(\frac{-e^{-\frac{bx^{c}}{a}}}{k!}\right)^{k}\right)dx$$

Following simplification, we arrive at the following expression:

$$I=-\sum_{k=0}^{\infty}\frac{bc}{ak!}\int_{0}^{\infty} x^{r+c-1}e^{-bx^{c}-\frac{k}{a}bx^{c}}dx$$

Upon applying the gamma function and further simplifying the expression, we obtain:

$$I=\sum_{k=0}^{\infty}\frac{b}{ak!}\left(\frac{\Gamma(\frac{r+c}{c},b\left(1+\frac{k}{a}\right)x^{c})}{\left(b+\frac{kb}{a}\right)^{\frac{r+c}{c}}}\right) \tag{3.4}$$

Taking into account Part II,

$$II=-\frac{bc}{a}\int_{0}^{\infty} x^{r+c-1}e^{-\left(\frac{e^{-bx^{c}}}{a}\right)-2bx^{c}}dx$$

$$II=-\frac{bc}{a}\int_{0}^{\infty} x^{r+c-1}e^{-2bx^{c}}\sum_{k=0}^{\infty}\left(\frac{-e^{-\frac{bx^{c}}{a}}}{k!}\right)^{k}dx$$

$$II=\sum_{k=0}^{\infty}\frac{bc}{ak!}\int_{0}^{\infty} x^{r+c-1}e^{-2bx^{c}}e^{-\left(\frac{kbx}{a}+2b\right)x^{c}}dx$$

Upon applying the gamma function and performing further simplifications, we arrive at:

$$II=-\sum_{k=0}^{\infty}\frac{b}{ak!}\left(\frac{\Gamma\left(\frac{r+c}{c},b\left(2b+\frac{kb}{a}\right)x^{c}\right)}{\left(2b+\frac{kb}{a}\right)^{\frac{r+c}{c}}}\right) \tag{3.5}$$

Now, focusing on Part III,

$$III = bc\int_0^\infty x^{r+c-1}e^{-\left(\frac{e^{-bx^c}}{a}\right)-bx^c}dx$$

$$III = bc\int_0^\infty x^{r+c-1}e^{-bx^c}\sum_{k=0}^{\infty}\left(\frac{-e^{-\frac{bx^c}{a}}}{k!}\right)^k dx$$

$$III = -\sum_{k=0}^{\infty}\frac{bc}{k!}\int_0^\infty x^{r+c-1}e^{-\left(\frac{kb}{a}+b\right)x^c}dx$$

Upon applying the gamma function and simplifying the expression for Part III, we get:

$$III = \sum_{k=0}^{\infty}\frac{b}{k!}\left(\frac{\Gamma\left(\frac{r+c}{c}, b\left(1+\frac{k}{a}\right)x^c\right)}{\left(b+\frac{kb}{a}\right)^{\frac{r+c}{c}}}\right) \tag{3.6}$$

After substituting equations (3.4), equation (3.5), and equation (3.6) into equation (3.3), we obtain the following result:

$$E\left(x^r\right) = \sum_{k=0}^{\infty}\left[\left(\frac{\Gamma\left(\frac{r+c}{c}, b\left(a+\frac{k}{a}\right)x^c\right)}{a\left(b+\frac{kb}{a}\right)^{\frac{r+c}{c}}}\right) - \left(\frac{\Gamma\left(\frac{r+c}{c}, b\left(2b+\frac{kb}{a}\right)x^c\right)}{a\left(2b+\frac{kb}{a}\right)^{\frac{r+c}{c}}}\right) + \left(\frac{\Gamma\left(\frac{r+c}{c}, b\left(1+\frac{k}{a}\right)x^c\right)}{\left(b+\frac{kb}{a}\right)^{\frac{r+c}{c}}}\right)\right]$$

3.4. **Renyi Entropy.** Renyi entropy is calculated for a probability distribution to quantify the uncertainty or randomness inherent in that distribution. Renyi entropy allows for the comparison of different distributions and the assessment of changes in uncertainty under various conditions or transformations, aiding in decision-making and inference tasks.

The Renyi entropy of the FEW distribution with the random variable X is defined as:

$$R_H\left(x\right) = \frac{1}{1-p}log\int_0^\infty \left(f\left(x\right)\right)^p dx \tag{3.7}$$

By plugging the function from equation (2.4) into equation (3.7), we arrive at the following outcome:

$$R_H\left(x\right) = \frac{1}{1-p}log\int_0^\infty\left(\frac{bcx^{c-1}e^{-\left(\frac{e^{-bx^c}}{a}\right)-bx^c} - bcx^{c-1}e^{-\left(\frac{e^{-bx^c}}{a}\right)-2bx^c}}{a} + bcx^{c-1}e^{-\left(\frac{e^{-bx^c}}{a}\right)-bx^c}\right)^p dx \tag{3.8}$$

The given equation can be divided into segments, and we can address and solve each part separately, as follows:

$$I = \frac{(bc)^p}{a^p\left(1-p\right)}log\int_0^\infty x^{p(c-1)}e^{-\left(\frac{e^{-bx^c}}{a}\right)-bx^c}dx$$

Simplifying the above expression and applying the gamma function, we obtained the following result:

$$I = \frac{(bc)^p}{(1-p)}\sum_{k=0}^{\infty}\frac{p^k}{a^{k+p}k!}log\left[\frac{\Gamma\left(\left(\frac{(c-1)(p+1)}{c}\right), b\left(p+k\right)x^c x^{(c-1)(p+1)}\right)}{c\left(b\left(p+k\right)x^c\right)^{\frac{(c-1(p+1))}{c}}}\right] \tag{3.9}$$

Taking into account Part II,

$$II=\frac{1}{1-p}log\int_0^{\infty}\left(\frac{bcx^{c-1}e^{-\left(\frac{e^{-bx^c}}{a}\right)-2bx^c}}{a}\right)^p dx$$

Simplifying the above expression and applying the gamma function, we obtained the following result:

$$II=\frac{(bc)^p}{(1-p)}\sum_{k=0}^{\infty}\frac{p^k}{a^{k+p}k!}log\int_0^{\infty}x^{(p(c-1))}e^{-(kb+2pb)x^c}dx \tag{3.10}$$

As expressions I and III are equivalent, we can streamline the analysis as follows:

$$III=\frac{(bc)^p}{(1-p)}\sum_{k=0}^{\infty}\frac{p^k}{a^k k!}log\left[\frac{\Gamma\left(\frac{(c-1)(p+1)}{c},b\,(p+k)\,x^c x^{(c-1)(p+1)}\right)}{c\,(b\,(p+k)\,x^c)^{\frac{(c-1)(p+1)}{c}}}\right] \tag{3.11}$$

After substituting equations (3.9), equation (3.10), and equation (3.11) into equation (3.8), we obtain the following result:

$$R_H(x)=\frac{(bc)^p}{1-p}\sum_{k=0}^{\infty}\frac{p^k}{a^k k!}\left[\left(\frac{\Gamma\left(\frac{(c-1)(p+1)}{c},b(p+k)x^c x^{(c-1)(p+1)}\right)}{a^p c(b(p+k)x^c)^{\frac{(c-1)(p+1)}{c}}}\right)-\left(\frac{\Gamma\left(\frac{(c-1)(p+1)}{c},b(2p-k)x^c x^{(c-1)(p+1)}\right)}{c(b(2p+k)x^c)^{\frac{(c-1)(p+1)}{c}}}\right)+\left(\frac{\Gamma\left(\frac{(c-1)(p+1)}{c},b(p+k)x^c x^{(c-1)(p+1)}\right)}{c(b(p+k)x^c)^{\frac{(c-1)(p+1)}{c}}}\right)\right] \tag{3.12}$$

3.5. **Order Statistics.** In the lifetime modeling order statistics are very important, in most systems for the reliability, one need smallest and largest observations. For the properties of these observations their corresponding distributions can be obtained in the following way.

Let $X_i, (i \leq n)$ be the i^{th} ordered statistics from FEW, then its PDF can be computed from the expression given by:

$$f_{(i,n)}(x)=\frac{n!}{(i-1)!\,(n-i)!}f(x)\,F(x)^{i-1}\left(1-F(x)^{n-i}\right)$$

Using equation (2.4) and equation (2.5), the smallest and largest order statistics of FEW can be obtained respectively by using $i=1$ and $i=n$ as:

$$f_{(1,n)}(x)=\frac{nbcx^{(c-1)}\left((a+1)\,e^{bx^c}-1\right)e^{\frac{1-e^{-bx^c}}{a}-2bx^c-\frac{1}{a}}\left(1-\left(1-e^{-bx^c}\right)\left(e^{1-e^{-ax^b}}-1\right)^{\frac{1}{a}}\right)^{n-1}}{a}$$

$$f_{(n,n)}(x)=\frac{nbcx^{(c-1)}\left((a+1)\,e^{bx^c}-1\right)e^{\frac{-e^{-bx^c}}{a}-2bx^c}\left(\left(1-e^{-bx^c}\right)\left(e^{\frac{-e^{-ax^b}}{a}}\right)\right)^{n-1}}{a}$$

3.6. **Skewness and Kurtosis.** In order to examine the shape of the suggested FEW distribution, skewness and kurtosis are obtained through the following quantile function $Q(\cdot)$. The formulas for Bowley's skewness and Moore's kurtosis are presented below:

$$S=\frac{Q\left(\frac{6}{8}\right)+Q\left(\frac{2}{8}\right)-2Q\left(\frac{4}{8}\right)}{Q\left(\frac{6}{8}\right)-Q\left(\frac{2}{8}\right)}$$

$$k = \frac{Q\left(\frac{7}{8}\right) + Q\left(\frac{3}{8}\right) - Q\left(\frac{5}{8}\right) - Q\left(\frac{1}{8}\right)}{Q\left(\frac{6}{8}\right) - Q\left(\frac{2}{8}\right)}$$

Here S is for skewness and k is for kurtosis.

In the following Table 1, the Skewness and Kurtosis of FEW distribution for various values of the parameters.

Table 1. Skewness and Kurtosis for different values of parameters.

a	**b**	**c**	**Skewness**	**Kurtosis**
1	2	2	0.02419435	1.228909
1	2	5	-0.06720608	1.266964
1	5	2	0.02419435	1.228909
10	5	2	0.06746432	1.202879
10	10	10	-0.08464995	1.251273
0.1	0.001	0.1	0.8795218	6.55319
0.1	0.1	0.01	1	196104660
10	0.1	0.01	1	1.267869e+17
0.1	10	20	-0.01307878	1.250528
20	0.01	20	-0.1014318	1.263692

The shapes of the suggested distribution are checked for various values of the parameters. Like other lifetime distribution, the suggested distribution is positively skewed and platykurtic.

4. Estimation of FEW Distribution's Parameters

For the model fitting of the suggested distribution, the parameter estimates are required. In this subsequent section the maximum likelihood estimators of the suggested FEW distribution parameters are obtained.

4.1. **Maximum Likelihood Estimation (MLE).** To obtain the corresponding estimators for the suggested distribution, let a random sample $X = (x_1, x_2, \cdots, x_n)$ drawn from the FEW distribution, then its PDF is expressed as:

$$f(x) = bcx^{c-1}\left(1 - e^{-bx^c}\right)e^{\frac{e^{-bx^c}}{a} - bx^c} + bcx^{c-1}e^{\frac{e^{-bx^c}}{a} - bx^c}$$

$$f(x) = bcx^{c-1}e^{\frac{e^{-bx^c}}{a} - bx^c} - bcx^{c-1}e^{\frac{e^{-bx^c}}{a} - 2bx^c} + bcx^{c-1}e^{\frac{e^{-bx^c}}{a} - bx^c}$$

The likelihood function for the FEW distribution is formulated as:

$$L = b^n c^n \sum_{i=1}^{n} x^{c-1} e^{\frac{\sum_{i=1}^{n} e^{-bx^c}}{a} - b\sum_{i=1}^{n} x^c} - b^n c^n \sum_{i=1}^{n} x^{c-1} e^{\frac{\sum_{i=1}^{n} e^{-bx^c}}{a} - 2b\sum_{i=1}^{n} x^c} + b^n c^n \sum_{i=1}^{n} x^{c-1} e^{\frac{\sum_{i=1}^{n} e^{-bx^c}}{a} - b\sum_{i=1}^{n} x^c} \tag{4.1}$$

The corresponding log-likelihood function for the FEW distribution is:

$$log\left(L\right)=\left(nlog\left(b\right)+nlog\left(c\right)+\left(c-1\right)\sum_{i=1}^{n}log\left(x_i\right)-\frac{\sum_{i=1}^{n}e^{-bx^c}}{a}-b\sum_{i=1}^{n}x^c\right)-$$
$$\left(nlog\left(b\right)+nlog\left(c\right)+\left(c-1\right)\sum_{i=1}^{n}log\left(x_i\right)-\frac{\sum_{i=1}^{n}e^{-bx^c}}{a}-2b\sum_{i=1}^{n}x^c\right)+$$
$$\left(nlog\left(b\right)+nlog\left(c\right)+\left(c-1\right)\sum_{i=1}^{n}log\left(x\right)-\frac{\sum_{i=1}^{n}e^{-bx^c}}{a}-b\sum_{i=1}^{n}x^c\right)$$

Upon simplification, the expression becomes:

$$log\left(L\right)=\frac{\sum_{i=1}^{n}e^{-bx^c}}{a}+2b\sum_{i=1}^{n}x^c+nlog\left(b\right)+nlog\left(c\right)+\left(c-1\right)\sum_{i=1}^{n}log\left(x_i\right)$$

To obtain the Maximum Likelihood Estimates (MLE) of the unknown parameters, simplify the provided equation by taking its derivative with respect to the parameters a, b, and c.

$$\frac{d}{da}log\left(L\right)=\frac{\sum_{i=1}^{n}e^{-bx^c}}{a^2}$$

$$\frac{d}{db}log\left(L\right)=2\sum_{i=1}^{n}x^c+\frac{n}{b}+\frac{\sum_{i=1}^{n}e^{-bx^c}x^c}{a}$$

$$\frac{d}{dc}log\left(L\right)=2b\sum_{i=1}^{n}x^c log\left(x\right)+\frac{n}{c}+\sum_{i=1}^{n}log\left(x\right)+\frac{b\sum_{i=1}^{n}e^{-bx^c}x^c log\left(x\right)}{a}$$

$$\frac{d^2}{da^2}log\left(L\right)=\frac{-2\sum_{i=1}^{n}e^{-bx^c}}{a^3}$$

$$\frac{d^2}{db^2}log\left(L\right)=\frac{-n}{b^2}-\frac{b\sum_{i=1}^{n}e^{-bx^c}x^{2c}}{a^3} \tag{4.2}$$

$$\frac{d^2}{dc^2}log\left(L\right)=2b\sum_{i=1}^{n}log\left(x^2\right)x^c-\frac{n}{c^2}+\frac{b\sum_{i=1}^{n}e^{-bx^c}x^c log\left(x^2\right)}{a}-\frac{b^2\sum_{i=1}^{n}e^{-bx^c}x^{2c}log\left(x^2\right)}{a}$$

$$\frac{d^2}{dadb}log\left(L\right)=-\frac{b\sum_{i=1}^{n}e^{-bx^c}}{a^2}$$

$$\frac{d^2}{dadc}log\left(L\right)=-\frac{b\sum_{i=1}^{n}e^{-bx^c}x^c log\left(x\right)}{a^2}$$

$$\frac{d^2}{dbdc}log\left(L\right)=\frac{\sum_{i=1}^{n}e^{-bx^c}x^c log\left(x\right)\left(2ae^{bx^c}-bx^c+1\right)}{a}$$

The asymptotic confidence interval for the unknown parameters a, b, and c can be derived under the assumption MLEs are approximately normally distributed with mean (a, b, c) and an inverse Fisher information observed covariance matrix denoted as FI^{-1}. This matrix is defined as:

$$FI^{-1} = \begin{bmatrix} \frac{d^{2log(L)}}{da^2} & \frac{d^{2log(L)}}{dadb} & \frac{d^{2log(L)}}{dadc} \\ \frac{d^{2log(L)}}{dadb} & \frac{d^{2log(L)}}{db^2} & \frac{d^{2log(L)}}{dbdc} \\ \frac{d^{2log(L)}}{dadc} & \frac{d^{2log(L)}}{dbdc} & \frac{d^{2log(L)}}{dc^2} \end{bmatrix}$$

The variance-covariance matrix related to these parameter estimates is provided by:

$$FI = \begin{bmatrix} var\left(\hat{a}\right) & cov\left(\hat{a},\hat{b}\right) & cov\left(\hat{a},\hat{c}\right) \\ cov\left(\hat{a},\hat{b}\right) & var\left(\hat{b}\right) & cov\left(\hat{b},\hat{c}\right) \\ cov\left(\hat{a},\hat{c}\right) & cov\left(\hat{b},\hat{c}\right) & var\left(\hat{c}\right) \end{bmatrix}$$

Therefore, the asymptotic $(1-\alpha)100\%$ confidence intervals for the parameters can be computed as:

$$\hat{a} \pm Z_{\frac{\alpha}{2}} \sqrt{var\left(\hat{a}\right)}$$

$$\hat{b} \pm Z_{\frac{\alpha}{2}} \sqrt{var\left(\hat{b}\right)} \tag{4.3}$$

$$\hat{c} \pm Z_{\frac{\alpha}{2}} \sqrt{var\left(\hat{c}\right)}$$

5. Applications

This section will evaluate the performance of the suggested model by examining various goodness-of-fit measures, like AIC, CAIC, BIC, HQIC. These measures provide insights into how well the model fits the data. Through this evaluation, we measure the model's performance and suitability for our objectives. It's important to emphasize that the model with a lower value of these criteria is regarded as the best model among the alternatives.

Case Study 1: data set for COVID-19 deaths in Pakistan (in Millions)

The dataset presented below is sourced from the Coronavirus Pandemic (COVID-19) statistics and research repository, which can be found at https://github.com/owid/covid-19-data. It contains records of daily deaths per million in Pakistan and covers the time period from 02/05/2020 to 04/07/2021.

0.009, 0.014, 0.014, 0.023, 0.027, 0.032, 0.036, 0.041, 0.05, 0.054, 0.063, 0.095, 0.118, 0.122, 0.154, 0.181, 0.186, 0.213, 0.24, 0.258, 0.276, 0.294, 0.299, 0.389, 0.412, 0.421, 0.435, 0.503, 0.579, 0.611, 0.647, 0.761, 0.797, 0.91, 0.96, 1.073, 1.145, 1.218, 1.272, 1.322, 1.412, 1.553, 1.743, 1.888, 1.992, 2.069, 2.155, 2.327, 2.553, 2.648, 2.712, 2.879, 2.983, 3.196, 3.336, 3.445, 3.486, 3.776, 3.776, 3.952, 4.088, 4.251, 4.459, 4.604, 4.83, 4.984, 5.129, 5.283, 5.419, 5.546, 5.704, 5.962, 6.315, 6.714, 6.985, 7.338, 7.642, 8.013, 8.321, 8.76, 9.063, 9.358, 9.833, 10.209, 10.666, 11.15, 11.15, 11.549, 12.354, 12.852, 13.468, 14.002, 14.618, 15.311, 15.849, 16.252, 16.728, 16.999, 17.669, 17.936, 18.267, 18.643, 18.864, 19.485, 19.897, 20.25, 20.603, 20.603, 20.911, 21.558, 21.907, 22.282, 22.559, 22.898, 23.192, 23.527, 23.84, 24.084, 24.383, 24.564, 24.564, 24.999, 25.207, 25.347, 25.528, 25.7, 25.845, 26.09, 26.198, 26.357, 26.357, 26.447, 26.551, 26.674, 26.818, 26.941, 26.941, 27.054, 27.158, 27.158, 27.226, 27.321, 27.398, 27.47, 27.534, 27.602, 27.67, 27.747, 27.792, 27.855, 27.896, 27.955, 27.955, 28.023, 28.073, 28.109, 28.154, 28.208, 28.267, 28.267, 28.317, 28.371, 28.403, 28.444, 28.448, 28.466, 28.494, 28.512, 28.647, 28.679, 28.702, 28.702, 28.724, 28.747, 28.788, 28.815, 28.838, 28.851, 28.878, 28.896, 28.924, 28.942, 28.969, 29.01, 29.041, 29.046, 29.064, 29.082, 29.118, 29.141, 29.173, 29.204, 29.231, 29.272, 29.308, 29.331, 29.354, 29.422, 29.458, 29.485, 29.485, 29.53, 29.585, 29.625, 29.662 ,29.689, 29.743, 29.788, 29.824, 29.883, 29.942, 29.974, 30.051, 30.123, 30.146, 30.209, 30.295, 30.341, 30.399, 30.454, 30.494, 30.508, 30.535, 30.599, 30.671, 30.762, 30.811, 30.888, 30.943, 31.006,31.088, 31.205, 31.341, 31.432, 31.545, 31.586, 31.69, 31.785, 31.939, 32.106, 32.183, 32.328, 32.414, 32.563, 32.731, 32.812, 34.229, 34.419, 34.687, 34.841, 35.058, 35.325, 35.506, 35.75, 35.954, 36.149, 36.33, 36.629, 36.968, 37.145, 37.394, 37.588, 37.851, 38.019, 38.421, 38.693, 38.947, 39.173, 39.494, 39.82, 39.983, 40.314, 40.789, 41.106, 41.486, 41.876, 42.238, 42.518, 42.89, 43.265, 43.768, 44.153, 44.438, 44.701, 44.95, 45.235, 45.484, 45.746, 46.068, 46.439, 46.679, 46.855, 47.123, 47.358.

Given below Figure 3 presents both the histogram and theoretical density plots, along with empirical and theoretical cumulative distribution functions (CDFs). These plots are generated for the COVID-19 death data for Pakistan. The proposed distribution is compared to other existing distributions such as the exponential Weibull, Weibull, alpha power inverted exponential, exponential, and exponentiated exponential distributions. This comparison demonstrates that the proposed model fits the data quite accurately when compared to the mentioned lifetime distributions.

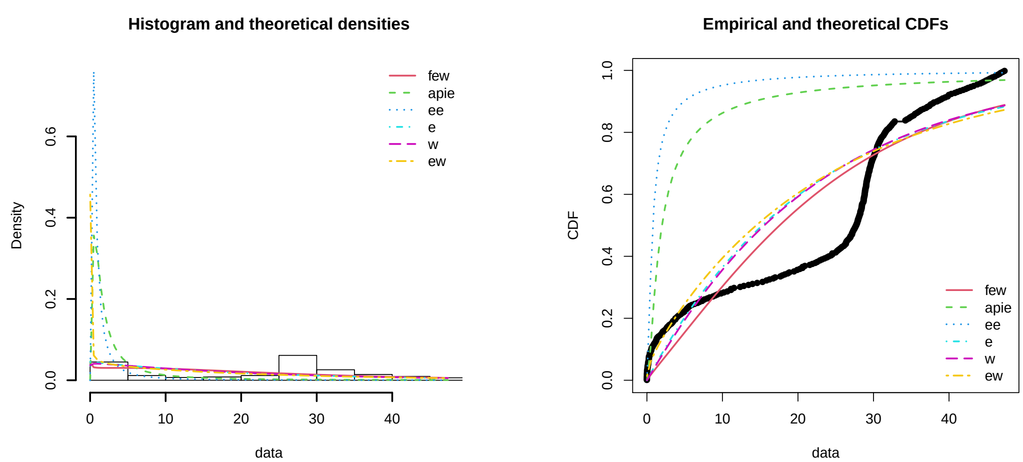

Figure 3. Theoretical and empirical PDF and CDF of FEW

Given below Figure 4 exhibits the Q-Q and P-P plots for the COVID-19 deaths data per millions for Pakistan. The close theoretical and applied curves shows the best fit of the model.

In both the plots the theoretical and empirical curves are close enough to suggest that the proposed model offers a fairly satisfactory fit to the data, aligning with theoretical and empirical

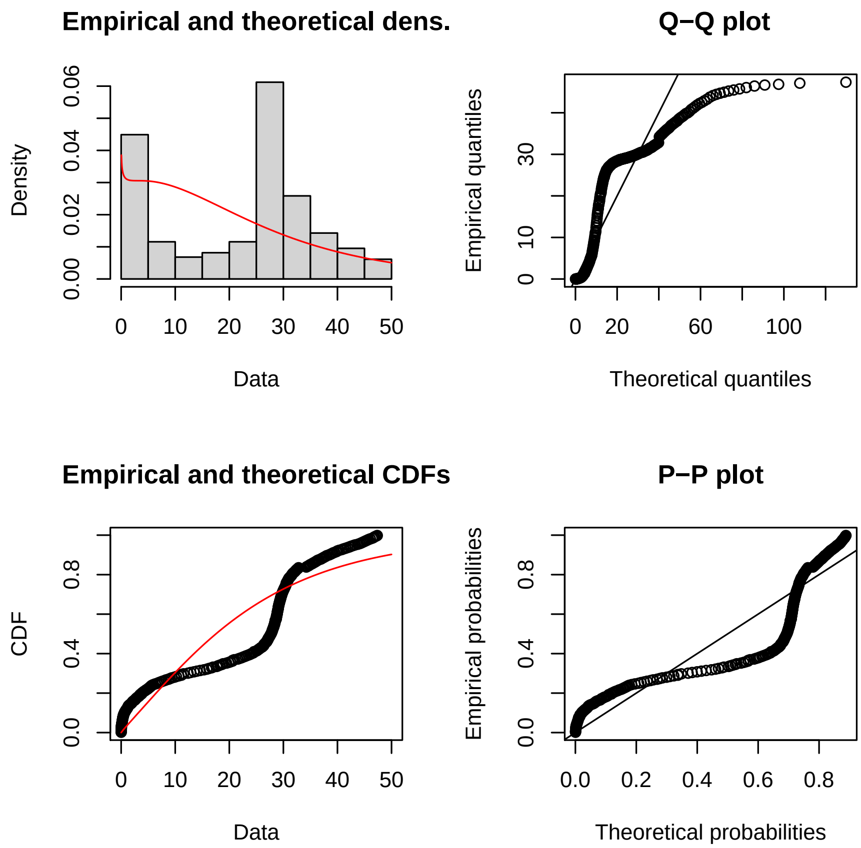

Figure 4. Q-Q (Quantile-Quantile) and P-P (Probability-Probability) plot for FEW

densities. Furthermore, the effectiveness of the proposed model is strengthened by referencing Table 2 and Table 3 below. These tables provide a comparative analysis of the proposed model against other distributions, including exponential Weibull, Weibull, alpha power inverted exponential, exponential, and exponentiated exponential distributions. The comparison shows that the proposed model performs better across various evaluation criteria.

The above table shows the MLEs and other properties of the estimates of the suggested distribution and other distributions for comparison. It is evident in the above table that -Log(L) and D are the minimum of the suggested distribution as compared to other popular distributions usually used for fitting such types of data.

In the above table, the measurements for the goodness of fit are given for the suggested distribution and other popular distributions for comparison. For the best fit, it is required to get small values for these measurements. All the values are much smaller for the suggested distribution as compare to other familiar lifetime distribution usually used for such kind of data. These are strong evidences to use the suggested distribution for non-monotonic hazard rate data.

Table 2. MLE and standard errors for Covid-19 deaths data of Pakistan.

Model	W	A	MLE	SD	-log(L)	D
FEW	4.321297	22.46489	0.98508077 0.08092659 0.92046609	0.19153846 0.01557337 0.05009739	1190.461	0.22868
E	4.704847	24.56227	0.04532665	0.002642217	1203.454	0.26749
W	4.679397	24.4247	0.04243439 1.02011264	0.00814312 0.05432640	1203.33	0.26687
APIE	NaN	NaN	8.1120718 0.4462725	0.90595823 0.02718059	1729.361	0.62095
EW	4.711372	24.59633	3.834743 0.999751 3.796775	NaN NaN NaN	1204.855	0.23054
AIFW	2.756311	13.82258	0.01984159 0.05013188	0.002082701 0.002646834	1229.493	0.34411

Table 3. Goodness of fit measures for Covid-19 data of Pakistan.

Models	AIC	CAIC	BIC	HQIC
FEW	2386.921	2387.004	2397.972	2391.347
E	2408.907	2408.921	2412.591	2410.382
W	2410.659	2410.701	2418.027	2413.61
APIE	3463.063	3463.104	3470.43	3466.013
EW	2415.711	2415.794	2426.762	2420.136
AIFW	2462.986	2463.028	2470.354	2465.937

In summary, Figure 4, in conjunction with the theoretical and empirical densities presented in Figure 3, and the comparison results in Table 2 and Table 3 collectively highlight the model's ability to effectively fit the data when contrasted with the aforementioned distribution alternatives.

Figure 5 below illustrates the hazard rate function for the COVID-19 deaths data in Pakistan. The plot demonstrates that the curve intersects the diagonal line, signifying that the data follows a non-monotonic failure rate function.

Case Study 2: Life of Fatigue fracture of a material, such as Kevlar

The data provided below pertains to an application in the field of reliability analysis. More specifically, it corresponds to the lifespan of the fatigue fracture of Kevlar 373/epoxy material subjected to constant sustained pressure at a 90% stress level. This specific case has been studied by Barlow et al. [20], Yolanda et al. [21], and Andrews and Herzberg [22].

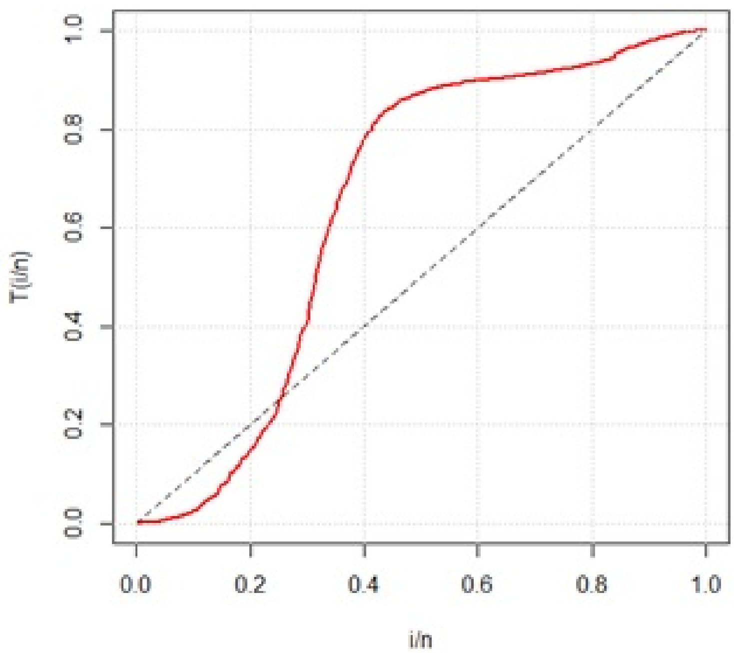

Figure 5. TTT plot of the Covid-19 data

0.0251, 0.0886, 0.0891, 0.2501, 0.3113, 0.3451, 0.4763, 0.5650, 0.5671, 0.6566, 0.6748, 0.6751, 0.6753, 0.7696, 0.8375, 0.8391, 0.8425, 0.8645, 0.8851, 0.9113, 0.9120, 0.9836, 1.0483, 1.0596, 1.0773 , 1.1733, 1.2570, 1.2766, 1.2985, 1.3211, 1.3503, 1.3551, 1.4595, 1.4880, 1.5728, 1.5733, 1.7083, 1.7263, 1.7460, 1.7630, 1.774, 1.8275, 1.8375, 1.8503, 1.8808, 1.8878, 1.8881, 1.9316, 1.9558, 2.0048, 2.0408, 2.0903, 2.1093, 2.1330, 2.210, 2.2460, 2.287, 2.3203, 2.3470, 2.3513, 2.4951, 2.5260, 2.9911, 3.0256, 3.2678, 3.404, 3.4846, 3.7433, 3.7455, 3.9143, 4.8073, 5.4005, 5.4435, 5.5295 ,6.5541 ,9.0960

Figure 6 below displays the histogram and theoretical densities, as well as the empirical and theoretical cumulative distribution functions (CDFs). These plots represent the dataset related to the fatigue fracture of Kevlar 373/epoxy under constant sustained pressure at a stress level of 90%. In order to assess the fitting performance, the proposed distribution is compared with other existing distributions such as exponential Weibull, Weibull, alpha power inverted exponential, exponential, and exponentiated exponential distributions. The comparison reveals that the proposed model provides a precise fit to the data when compared to these lifetime distributions.

The adequacy of these plots can also be substantiated by referencing the Tables 4 and Table 5 of model selection criteria presented below. These tables provide additional evidence of the effectiveness and superiority of the proposed model.

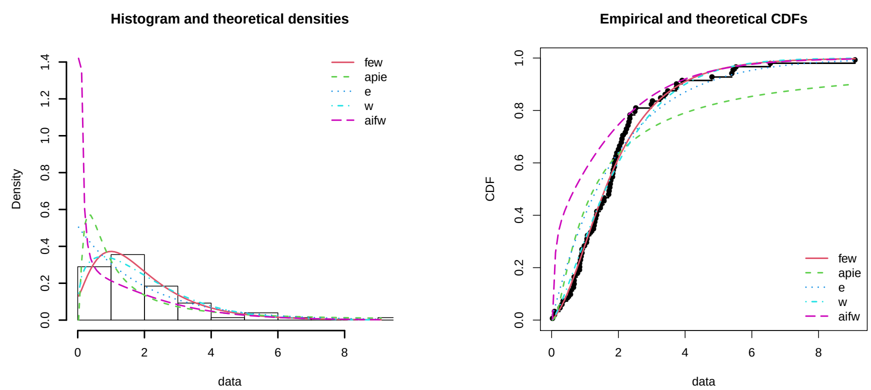

Figure 6. Theoretical and empirical PDF and CDF of FEW

Table 4. MLE and standard errors for Life of Fatigue fracture of Kevlar.

Model	W	A	MLE	SE	-log(L)	D
FEW	0.07769439	0.4510368	0.3135046 1.2429705 0.8044758	0.1471746 0.3409119 0.1458529	120.837	0.089848
E	0.1192814	0.707398	0.5104098	0.05854779	127.1143	0.16634
W	0.1301932	0.7650084	5.0517542 1.3197378	0.06212357 0.11347659	122.526	0.10969
APIE	0.8830294	5.15633	8.1120718 0.4182497	1.78994039 0.06242814	151.0588	0.22857
EW	4.711372	24.59633	3.834743 0.999751 3.796775	NaN NaN NaN	1204.855	0.23054
AIFW	0.2736417	1.670403	0.04337202 0.61650142	0.009503889 0.065074660	146.5769	0.37454

To compare the distribution, for the second data set it is clear that -Log(L) and D are minimum of the suggested distribution as compare to other popular distribution usually used for fitting such types of data.

The comparison between the proposed model and other prominent models, including exponential Weibull, Weibull, alpha power inverted exponential, exponential, and exponentiated exponential distributions, is carried out. The comparison analysis clearly indicates that the proposed model stands out from the rest, showcasing superior performance evidenced by consistently attaining the lowest values across the model selection criteria. These findings emphasize the effectiveness and reliability of the proposed model in addressing the given task or problem.

Table 5. Goodness of fit measures for Life of Fatigue fracture of Kevlar.

Models	AIC	CAIC	BIC	HQIC
FEW	247.674	248.0073	254.6662	250.4684
E	256.2287	256.2827	258.5594	257.1601
W	249.0521	249.2165	253.7136	250.915
APIE	306.2199	306.3843	310.8814	308.0829
AIFW	297.1537	297.3181	301.8152	299.0167

Similarly, for the second data set the measurements for the goodness of fit are given. Again all the required values are much smaller for the suggested distribution as compare to other familiar lifetime distribution usually used for such kind of data. These are strong evidences to use the suggested distribution for non-monotonic hazard rate data. The Figure 7 below presents the Q-Q and P-P plots for the data on the fatigue fracture of Kevlar 373/epoxy under constant sustained pressure at 90% stress level. These plots demonstrate that the proposed model provides a more reasonable fit to the data.

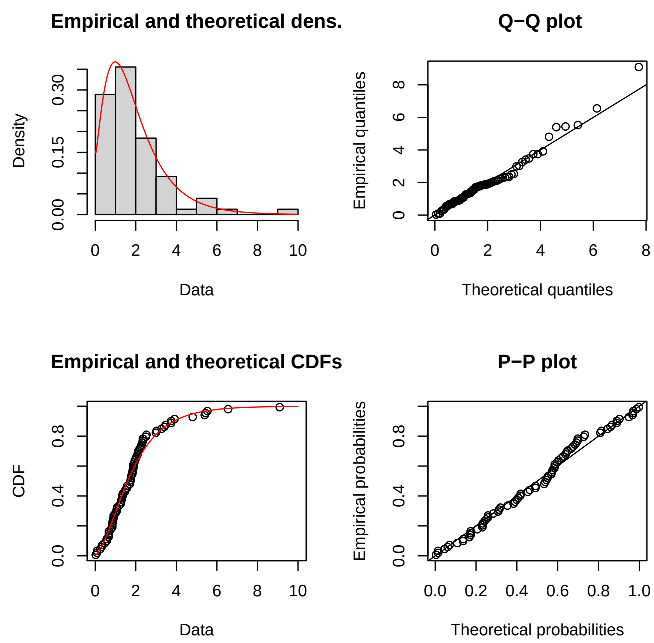

Figure 7. Theoretical and empirical PDF and CDF with Q-Q plot and P-P plot for FEW

In summary, the graphs depicted in Figure 6 and Figure 7, along with the comparison to alternative distributions and the model selection criteria detailed in Table 4 and Table 5, collectively demonstrate the precision of the proposed model in fitting and its adaptability.

Figure 8 below illustrates the TTT plot of the Life of Fatigue fracture of Kevlar. The graph clearly indicates that the line does not intersect the diagonal line, indicating that the data follows a monotonic failure function.

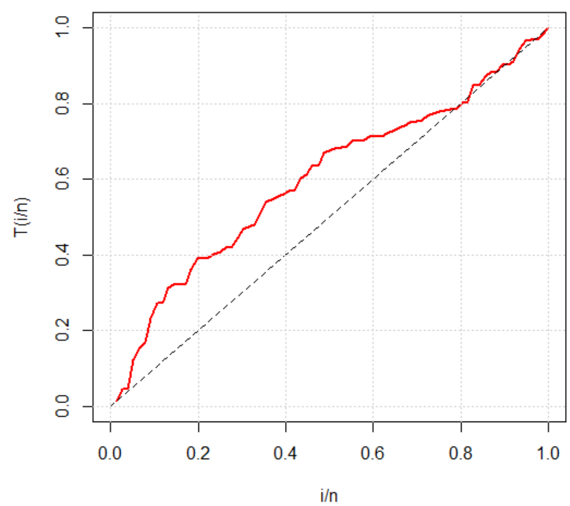

Figure 8. TTT plot of the Life of Fatigue fracture of Kevlar.

6. Monte Carlo (MC) Simulation of the FEW

In this section, we delve into the details of a Monte Carlo (MC) simulation conducted to evaluate the reliability of the parameters in the proposed distribution. The simulation involves testing two different sets of parameter values for this distribution: specifically, $a = 19, b = 8, c = 4$ and $a = 29, b = 15, c = 8$.

To evaluate consistency, we measured bias and mean squared errors (MSEs) across different sample sizes of $n = 100, 200, 400$, and 700 for each parameter set. The simulation was iterated 50 times for each sample size. The overall expressions for computing bias and mean squared error are as follows:

$$MSE = \frac{1}{W}\sum_{i=1}^{W} (\hat{a}_i - \alpha)^2 \tag{6.1}$$

$$MSE = \frac{1}{W}\sum_{i=1}^{W} (\hat{a}_i - \alpha) \tag{6.2}$$

The parameters of the proposed model are deemed consistent when both bias and MSE decrease as the sample size increases. Table 6 above provides the outcomes of the MC simulation, which was conducted to evaluate the performance of the proposed distribution. The primary goal of this simulation was to investigate how the parameters behave across different sample sizes.

Table 6. Average values of MSE and Biases

parameters	n	MSE0	MSE1	MSE2	BIAS0	BIAS1	BIAS2
A0=19,B0=8,C0=4,w=50	100	252.5022	1.679872	0.8599912	13.04612	0.3774243	0.5846472
	200	226.8206	0.8875751	0.7103874	12.6853	0.2604816	0.5679445
	400	208.7978	0.4218343	0.4733561	8.08156	0.1957855	0.3907876
	700	199.7342	0.1890359	0.3665986	6.893152	0.1387868	0.3247679
A0=29, B0=15, C0=8, w=50	100	508.4049	10.08599	1.167481	20.99003	0.483743	0.5070308
	200	446.0664	6.06047	0.8903861	18.97199	0.329115	0.4504953
	400	397.4329	2.741369	0.5654377	18.09046	0.1950704	0.3524617
	700	343.9165	0.8831448	0.2709185	12.34644	0.06653786	0.2674926

By measuring the MSE and biases, the MC simulation reveals that as the sample size increases, both the MSE and biases of the parameters tend to decrease and approach zero. These findings suggest that the consistency of the model parameters improves as the sample size increases.

7. Conclusion

Statistics provides sophisticated techniques for making statistical decisions about the population based on sample data. Lifetime distribution is a statistical technique used to model the time until an event of interest occurs. The methods for modeling lifetimes have roots tracing back centuries, but in the last few decades, that has sparked a keen interest in quantifying the reliability or survival lifetime and the factors related to these. Principally, it is not possible for any distribution to fit in all kinds of situations.

It has been observed that for non-monotonic hazard the existing distributions do not provide good fits. Therefore, in the present study, a novel lifetime distribution termed the FEW distribution has been introduced to cover the existing problem. Like other lifetime distribution, the suggested distribution is skewed positively and platykurtic. The corresponding statistical properties like, MLEs, an asymptotic confidence bound, order statistics, moments function, hazard function, quantile function, and entropy are obtained and discussed for the proposed distribution.

Furthermore, to evaluate the flexibility of the proposed distribution compared to existing lifetime distributions, two real datasets were analyzed. The results demonstrated that the proposed distribution exhibits more flexibility in the form to cope non-monotonic hazard function as compared to the other existing distributions. This is the prime application of the developed distribution is to model both non-monotonic and monotonic hazard rate functions. The assessment of the estimates' performance is also conducted, wherein the derived values for the goodness of fit substantiate the superior performance of the proposed distribution. In addition to these, hazard plot and TTT plots, using datasets justifies monotonicities of the data.

For the performance evaluation, a simulation study was conducted to assess the consistency and bias of the estimated parameters. The findings from the proposed distribution indicate that as the sample size increases, both the mean squared error (MSE) and bias of the parameters approach zero. The results show that the suggested distribution is the best choice for this type of data. The results proved that it works better than other options, giving strong reasons to use it. This selection is further reinforced by the certainty imparted in the model's performance through a simulation study.

These findings provides strong evidences that the proposed model has the potential to attract researchers to apply it in various fields of studies, such as business, agriculture, engineering etc.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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