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# A Note on Exact Frames in Banach Spaces

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**Abstract.** In this article, we have defined $\mu$-exact and finitely exact Banach frames and discuss their existence and relationship. A necessary condition for the existence of a $\mu$-exact Banach frame is given. Also, we discuss quasi-complementary subspaces and prove a result using exact Banach frames. Finally, as an application, we discuss boundedness of an isometry using exact retro Banach frame sequences.

## 1. Introduction

Duffin and Schaeffer [1] presented frames explicitly within the framework of nonharmonic Fourier analysis. Daubechies, Grossmann, and Meyer [2] brought frames back to limelight. Thanks to their many good qualities, frames are widely used in wireless communications, sigma-delta quantization, filter bank theory, image processing, and signal processing. One of the inherent characteristics of a frame is its ability to extract function properties and recreate it just using the frame coefficients, which are a series of scalars.

A sequence $\Psi = \{g_n\}_{n=1}^{\infty}$ in a separable Hilbert space $\mathcal{H}$ is called a frame for $\mathcal{H}$, if one can find scalars $\mathcal{A}, \mathcal{B} > 0$ satisfying the inequality

$$\mathcal{A}\|h\|^2 \leq \sum_{n=1}^{\infty} |\langle h, g_n\rangle|^2 \leq \mathcal{B}\|h\|^2, \text{ for all } h \in \mathcal{H}. \tag{1.1}$$

The scalars denoted as $\mathcal{A}$ and $\mathcal{B}$ in 1.1 are identified as the lower and upper frame bounds, respectively. The frame bounds need not be unique. In case $\mathcal{A} = \mathcal{B}$, then $\Psi$ is termed as an $\mathcal{A}$-tight

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frame and in the case when $\mathcal{A} = \mathcal{B} = 1$, $\Psi$ is termed as a Parseval frame. The inequality (1.1) is named as the frame inequality. The operator $T : \ell^2(\mathbb{N}) \to \mathcal{H}$ given as

$$T(\{c_k\}) = \sum_{k=1}^{\infty} c_k g_k, \ \{c_k\} \in \ell^2(\mathbb{N})$$

is called the pre-frame operator and the adjoint operator $T^* : \mathcal{H} \to \ell^2(\mathbb{N})$ of it is named as the analysis operator which is given by $T^*(x) = \{\langle x, g_k \rangle\}, x \in \mathcal{H}$. Composing $T$ and $T^*$ we obtain the frame operator $S = TT^* : \mathcal{H} \to \mathcal{H}$ given by

$$S(h) = \sum_{k=1}^{\infty} \langle h \, , \ g_k \rangle g_k, \ \ h \in \mathcal{H}.$$

Note that on $\mathcal{H}$, the frame operator $S$ is an invertible, positive, bounded, and self-adjoint operator. This provides the reconstruction formula that follows:

$$h = SS^{-1}h = \sum_{k=1}^{\infty} \langle S^{-1}h \, , \ g_k \rangle g_k = \sum_{k=1}^{\infty} \langle h \, , \ S^{-1} g_k \rangle g_k, \text{ for all } \ h \in \mathcal{H}. \tag{1.2}$$

In the remaining part of this paper, throughout $\mathbb{B}$ indicate a Banach space over the scalar field $\mathcal{K}(\mathcal{R}$ or $\mathcal{C})$, and $\mathcal{H}$ is a separable Hilbert space. The related Banach space of scalar valued sequences indexed by $\mathbb{N}$ is indicated by $\mathbb{B}_d$.

The notion of frame was generalized or extended to Banach spaces in many ways by several authors namely, Feichtinger and Gröchenig [3], Gröchenig [4], Casazza et al. [5] and Terekhin [6]. One of the extensions like atomic decomposition is a concept that is generalized from the notion of frame in Hilbert spaces to Banach spaces. The concept of atomic decomposition for specific Function spaces was initially introduced by Coifman and Weiss [7]. A further development saw the idea of atomic decomposition extended to specific Banach spaces by Feichtinger and Gröchenig [3]. Another extension is the concept of Banach frame for a Banach Space. The idea of Banach frame was first given by Gröchenig [4]. He defined it as follows:

Let $\mathbb{B}$ be a Banach space and $\mathbb{B}_d$ be a related Banach space. Let $\{y_n\} \subset \mathbb{B}^*$ and $S : \mathbb{B}_d \to \mathbb{B}$ be given. Then the pair $(\{y_n\}, \mathcal{S})$ is termed as a Banach frame for $\mathbb{B}$ w. r. t $\mathbb{B}_d$, if

(1) $\{y_n(x)\} \in \mathbb{B}_d$, for each $x \in \mathbb{B}$.

(2) $\exists$ scalars $A_1$ and $A_2$ with $0 < A_1 \leq A_2 < \infty$ satisfying the inequality

$$A_1 \|x\|_{\mathbb{B}} \leq \|\{y_n(x)\}\|_{\mathbb{B}_d} \leq A_2 \|x\|_{\mathbb{B}}, \ \ x \in \mathbb{B}. \tag{1.3}$$

(3) $S(\{y_n(x)\}) = x, \ \ x \in \mathbb{B}$ where the operator $S$ is bounded and linear.

Scalars $A_1$ and $A_2$ are stated as a lower and an upper frame bound of $(\{y_n\}, S)$ and $S : \mathbb{B}_d \to \mathbb{B}$ is stated as the reconstruction operator. The expression in (1.3) is called the frame inequalty. In case $A_1 = A_2$, then $(\{y_n\}, S)$ is termed as a tight frame for $\mathbb{B}$ and if $A_1 = A_2 = 1$, then $(\{y_n\}, S)$ is said to be a normalized tight Banach frame. The Banach frame $(\{y_n\}, S)$ is called exact if $\exists$ no reconstruction operator $S_0$ such that $(\{y_n\}_{n \neq i}, S_0)$ $(i \in \mathbb{N})$ is a Banach frame for $\mathbb{B}$.

Next, we give the definition of retro Banach frame introduced by Jain et al. [8].

The pair $(\{g_n\}, \mathcal{T})$ $(\{g_n\} \subset \mathbb{B}, \mathcal{T} : \mathbb{B}^*_d \to \mathbb{B}^*)$ is called a retro Banach frame for $\mathbb{B}^*$ with respect to $\mathbb{B}^*_d$, if

(1) $\{y(g_n)\} \in \mathbb{B}^*_d$, for each $y \in \mathbb{B}^*$.
(2) $\exists$ scalars $A_1$ and $A_2$ with $0 < A_1 \le A_2 < \infty$ such that

$$A_1\|y\|_{\mathbb{B}^*} \le \|\{y(g_n)\}\|_{\mathbb{B}^*_d} \le A_2\|y\|_{\mathbb{B}^*}, \quad x \in \mathbb{B}. \tag{1.4}$$

(3) $\mathcal{T}$ is bounded, linear and satisfies $\mathcal{T}(\{y(g_n)\}) = y, \quad y \in \mathbb{B}^*$.

The Scalars $A_1$ and $A_2$, are stated as a lower and an upper retro Banach frame bound of $(\{g_n\}, \mathcal{T})$. The operator $\mathcal{T} : \mathbb{B}^*_d \to \mathbb{B}^*$ is termed as the reconstruction operator. The expression in (1.4) is stated as the retro frame inequality. The other type of retro Banach frames are defined as in case of Banach frames. A sequence $\{g_n\} \subset \mathbb{B}$ is called a retro Banach frame sequence if it is a retro Banach frame for $\overline{span}\{g_n\}$.

For further details concerning frames in Banach spaces and allied topics, one may refer to [9–18].

## 2. Main Results

In this section, we shall define two type of exactness of Banach frames and investigate relationship between them. We begin with the following definition.

**Definition 2.1.** *A Banach frame $\{x_n\} \subset \mathbb{B}^*$ is called a $\mu$-exact Banach frame for $\mathbb{B}$ if for a sequence $\{\mu_n\}$ of positive real numbers*

$$\left|\lambda_k^{(n)}\right| \le \mu_k, \lim_{n\to\infty}\sum_{k=1}^{\infty}\lambda_k^{(n)}x_k = 0, \quad k, n \in \mathbb{N} \implies \lim_{n\to\infty}\lambda_k^{(n)} = 0, \quad k \in \mathbb{N}. \tag{2.1}$$

One may observe that an exact Banach frame for a Banach space $\mathbb{B}$ is always $\mu$-exact. Indeed, if $(\{x_n\}, \mathcal{T})\left(\{x_n\} \in \mathbb{B}^*, \mathcal{T} : \mathbb{B}_d \to \mathbb{B}\right)$ is an exact Banach frame, then one can find a sequence $\{y_n\} \in \mathbb{B}$ such that $x_i\left(y_j\right) = \delta_{ij}$, for all $i, j \in \mathbb{N}$. Suppose that for a sequence $\{\mu_n\}$ of positive real numbers

$$\left|\lambda_k^{(n)}\right| \le \mu_k \text{ and } \lim_{n\to\infty}\sum_{k=1}^{\infty}\mu_k^{(n)}x_k = 0, \quad \forall\, k, n \in \mathbb{N}.$$

Then

$$\lim_{n\to\infty}\mu_k^{(n)} = \lim_{n\to\infty}\left(\sum_{j=1}^{\infty}\mu_j^{(n)}x_j\right)(y_k) = 0.$$

However, an exact Banach frame need not be $\mu$-exact.

**Example 2.1.** *Let $(\{x_n\}, \mathcal{S})$ be an exact retro Banach frame for $\mathbb{B}$ which is not a Schauder basis for $\mathbb{B}$. Let $g \in \mathbb{B}$ be such that it admit no representation of the type $g = \sum_{i=1}^{\infty} a_i x_i$. Then, one can always express $g = \lim_{n\to\infty}\sum_{i=1}^{m_n} a_i^{(n)} x_i$, where $\sup_{1\le n<\infty} |a_i^{(n)}| < \infty (i \in \mathbb{N})$.*

*Define a sequence $\{z_n\}$ in $\mathbb{B}$ by $z_1 = g, z_n = x_{n-1}, n = 1, 2, \ldots.$ Then $\{z_n\}$ is exact but not $\mu$-exact for $\mu_1 \geq 1$ and $\mu_k \geq \sup_{1\leq n<\infty} |a^{(n)}_{k-1}|, \forall\, k = 2, 3, \ldots.$ Indeed, since $(\{x_n\}, \mathcal{S})$ is exact, $\{z_n\}$ is exact. However if we write*

$$c^{(n)}_1 = 1,\ c^{(n)}_j = \begin{cases} -a^{(n)}_{j-1}, & j = 2, 3, \ldots, m_n + 1 \\ 0\ , & j = m_n + 2, m_n + 3, \ldots, n \in \mathbb{N}, \end{cases}$$

*then $|c^{(n)}_j| \leq \mu_j, \forall z, n \in \mathbb{N}$ and $\lim_{n\to\infty} \sum_{j=1}^{\infty} c^{(n)}_j z_j = \lim_{n\to\infty} \left[g - \sum_{z=1}^{m_n} a^{(n)}_j x_j\right] = 0$, but $\lim_{n\to\infty} c^{(n)}_1 = 1$.*

In the following result, we shall provide a necessary condition for a normalize $\mu$ -exact Banach frame for $\mathbb{B}$.

**Theorem 2.1.** *If $\{\mu_n\}$ is a sequence of positive real numbers with $\inf_{1\leq k\leq\infty} \mu_k > 0$, then for a $\mu$-exact Banach frame $\{x_n\}$ with $\|x_n\| = 1, n \in \mathbb{B}$, we have*

$$\sum_{n=1}^{\infty} \lambda_n x_n = 0 \implies \lambda_n = 0,\ \ \forall\ \ n \in \mathbb{N}, \tag{2.2}$$

*where $\{\lambda_n\}$ is any sequence of scalars in $\mathbb{K}$.*

*Proof.* Suppose on the contrary that (2.2) is not true. Then, one can find a sequence of scalars $\{r_n\} \subseteq \mathbb{K}$ with $\sup_{1\leq n\leq\infty} |r_n| \neq 0$ such that $\sum_{p=1}^{\infty} r_p x_p = 0$. This yields that

$$\sup_{1\leq p\leq\infty} \left|r_p\right| = \sup_{1\leq p\leq\infty} \|r_p x_p\| < \infty.$$

Write

$$a = \frac{\inf_{1\leq p\leq\infty} \mu_p}{\sup_{1\leq p\leq\infty} |r_p|} \quad \text{and} \quad r^{(n)}_p = a r_p,\ p, n \in \mathbb{N}.$$

Then, we compute

$$\left|r^{(n)}_p\right| = \left|a r_p\right| \leq \inf_{1\leq p\leq\infty} \mu_p \leq \mu_p, \quad \forall\ \ p, n \in \mathbb{N}.$$

This gives

$$\lim_{n\to\infty} \sum_{p=1}^{n} r^{(n)}_p x_p = \lim_{n\to\infty} \sum_{p=1}^{n} a r_p x_p = a \lim_{n\to\infty} \sum_{p=1}^{n} r_p x_p = 0.$$

Since $\{x_n\}$ is $\mu$-exact, we get

$$a r_p = \lim_{n\to\infty} r^{(n)}_p = 0,\ \forall n \in \mathbb{N}.$$

Also, since $a \neq 0$, we conclude that $r_p = 0,\ p \in \mathbb{N}$. This contradicts the assumption that $\sup_{1\leq p<\infty} |r_p| \neq 0$. □

Next, we give another notion called finitely exact Banach frames.

**Definition 2.2.** *A Banach frame $(\{x_n\}, \mathcal{T})$ is called finitely exact if for every finite sub-sequence $\left\{x_{n_p}\right\}_{p=1}^{k}$ of $\{x_n\}$*

$$\sum_{p=1}^{k} \lambda_p x_{n_p} = 0 \implies \lambda_p = 0, \quad p = 1, 2 \ldots, k.$$

In the following result we prove that a $\mu$-exact Banach frame is always finitely exact.

**Theorem 2.2.** *If a Banach frame $(\{x_n\}, \mathcal{T})$ is $\mu$-exact, then it is finitely exact.*

*Proof.* Suppose on the contrary that the Banach frame $(\{x_n\}, \mathcal{T})$ is $\mu$-exact but not finitely exact. Then, one can always find scalars $\lambda_1, \lambda_2, \ldots, \lambda_p$ with $\sup_{1\le k\le p} |\lambda_k| \neq 0$ and such that $\sum_{k=1}^{p} \lambda_k x_k = 0$. Since $(\{x_n\}, \mathcal{T})$ is $\mu$-exact there exist a sequence $\{\mu_n\}$ of positive real numbers satisfying (2.1). Write

$$a = \frac{\inf_{1\le k\le p} \mu_k}{\sup_{1\le k\le p} |r_k|}$$

and

$$\lambda_i^{(n)} = \begin{cases} a\lambda_i, \text{if } 1 \le i \le p \\ 0, \text{if } i \ge p+1. \end{cases}$$

Clearly $a \neq 0$. Also, as in Theorem 2.3, $\lambda_i = 0, 1 \le i \le p$. This contradicts the assumption that $\sup_{1\le k\le p} |\lambda_k| \neq 0$. □

The converse statement of Theorem 2.2 is not true.

**Example 2.2.** *Let $\{x_n\}$ be a Schauder basis in $\mathbb{B}$. Then there exist a sequence $\{y_n\} \subset \mathbb{B}^*$ such that $y_i(x_j) = \delta_{ij}, \forall\, i, j \in \mathbb{N}$. Then there exist a bounded linear operator $\mathcal{T} : \mathbb{B}_d \to \mathbb{B}$ such that $(\{y_n\}, \mathcal{T})$ is an exact Banach frame for $\mathbb{B}$. Write*

$$x = \sum_{j=1} \frac{1}{2^i \|x_i\|} x_i.$$

*Then $y_j(x) \neq 0, \ \forall\, j \in \mathbb{N}$. Define $\{z_n\} \subset \mathbb{B}$ by $z_1 = x, z_n = x_{n-1}, n = 2, 3, \ldots$. Then, it is easy to verify that $\{z_n\}$ is finitely exact. If we take $\{\mu_n\}$ as in Definition 2.1 such that $\inf_{1\le j<\infty} \mu_j > 0$, then $\{z_n\}$ is not $\mu$-exact.*

Next, we discuss quasi-complementary subspaces related to a given exact Banach frame.

Recall that two subspaces $X$ and $Y$ of a normed linear space $N$ are called quasi-complementary if $X \cap Y = \{0\}$ and $X + Y$ is dense in $N$. In this case $X$ and $Y$ are called quasi-complements of each other. Towards the existence of quasi-complementary subspaces in a normed linear space N, one may notice that if $\{e_n\}$ is a sequence of unit vectors in $N = \ell^p (1 < p < \infty)$, then $X = [e_{n_k}]$ and $Y = [e_j]_{j\in\mathbb{N}\setminus\{n_k\}}$, where $n_k$ is any increasing sequence in $\mathbb{N}$, are quasi-complementary subspace of $N$.

In view of the above discussion, we prove the following lemma.

**Lemma 2.1.** *If $(\{x_n\}, \mathcal{T})$ is an exact Banach frame for $\mathbb{B}$ and $\{n_k\}$ is an increasing sequence in $\mathbb{N}$, then there exists two subspaces $X$ and $Y$ of $\mathbb{B}$ such that $X \cap Y = 0$.*

*Proof.* Since the Banach frame $(\{x_n\}, \mathcal{T})$ is exact, there exist a sequence $y_n \subset \mathbb{B}$ such that $x_i(y_j) = \delta_{i,j}, \forall i, j \in \mathbb{N}$. Write $X = [y_{n_k}]$ and $Y = [y_i]_{i\in\mathbb{N}\setminus\{n_k\}}$. Let $x \in X \cap Y$ be any element. Then

$$x_{n_k}(x) = 0 = x_j(x), \forall j \in \mathbb{N}\setminus\{n_k\}.$$

Using lower frame inequality of the Banach frame $(\{x_n\}, \mathcal{T})$, $x = 0$. □

Note: From Lemma 2.1, one may notice that the existence of an exact Banach frame for $\mathbb{B}$ is not a sufficient condition for the existence of quasi-complementary subspaces in a Banach space $\mathbb{B}$. In order to have a sufficient condition for the existence of quasi-complementary subspaces in a Banach space $\mathbb{B}$, we define the following:

**Definition 2.3.** *Let $(\{x_n\}, \mathcal{T})$ be a Banach frame for $\mathbb{B}$ with respect to $\mathbb{B}_d$ and let $\{y_n\}$ be a sequence in $\mathbb{B}$ such that $x_i(y_j) = \delta_{i,j}, \forall i, j \in \mathbb{N}$. If there exist an associated Banach space $(\mathbb{B}_d)^*$ and a reconstruction operator $\mathcal{S} : (\mathbb{B}_d)^* \to \mathbb{B}$ such that $(\{y_n\}, \mathcal{S})$ is a retro Banach frame for $(\mathbb{B}_d)^*$, then the system of frames $\{(\{x_n\}, \mathcal{T}), (\{y_n\}, \mathcal{S})\}$ is called a frame system for $\mathbb{B}$.*

In the following result, we prove that if a Banach frame $\mathbb{B}$ has a frame system, then it has quasi-complementary subspaces.

**Proposition 2.1.** *If a Banach space $\mathbb{B}$ has a frame system, then there exist quasi-complementary subspaces in $\mathbb{B}$.*

*Proof.* Let $\{(\{x_n\}, \mathcal{T}), (\{y_n\}, \mathcal{S})\}$ be a frame system in $\mathbb{B}$. Let $\{x_k\}$ be any increasing sequence in $\mathbb{N}$. Write $X = [y_{n_k}]$ and $Y = [y_i]_{i \in \mathbb{N} \setminus \{n_k\}}$. Then, by Lemma 2.1, $X \cap Y = 0$. By hypothesis, $X \cup Y = [y_n] = \mathbb{B}$ and so $X + Y$ is dense in $\mathbb{B}$. □

**Theorem 2.3.** *Let $\{g_n\} \subset \mathbb{B}$ be an exact retro Banach frame sequence in $\mathbb{B}$ and let $\delta \in (0, 1)$. There exist a sequence $\{\varepsilon_n\}$ with $\varepsilon_n > 0, \forall n \in \mathbb{N}$ such that whenever $\{h_n\}$ is sequence in $\mathbb{B}$ satisfying*

$$\|g_n - h_n\| \leq \varepsilon_n, \ \forall n \in \mathbb{N} \tag{2.3}$$

*the linear map $w$ taking $g_i$ to $h_i$ $(i \in \mathbb{N})$ is a $\delta$-isometry of $[g_n]$ onto $[h_n]$.*

*Proof.* Since $\{g_n\}$ is an exact retro Banach frame sequence, there exist a sequence $\{y_n\}$ in $\mathbb{B}^*$ such that $y_i(g_j) = \delta_{ij}, \ \forall i, j \in \mathbb{N}$. Write

$$\varepsilon_n = \frac{\delta}{\|y_n\| 2^{n+1}}, \ n \in \mathbb{N}.$$

Let $a_i$ are any scalars and $g = \sum_{i=1}^{n} a_i g_j$. Then

$$\begin{aligned} \left\| \sum_{i=1}^{n} a_i \left[g_i - h_i\right] \right\| &= \left\| \sum_{i=1}^{n} y_i(g) \left[g_i - h_i\right] \right\| \\ &\leq \sum_{i=1}^{n} \left| y_i(g) \right| \varepsilon_n \\ &< \delta \left\| \sum_{i=1}^{n} a_i g_i \right\| \end{aligned}$$

This gives

$$(1 - \delta) \left\| g \right\| \leq \left\| w(g) \right\| \leq (1 - \delta) \left\| g \right\|. \tag{2.4}$$

Note that the set consisting of all the finite linear combinations of type $\sum_{i=1}^{n} a_i g_i$ is dense in the closed linear space of $\{g_n\}$. We conclude that the inequality (2.4) is true for all $g \in [g_n]$.

□

The condition of exactness of the retro Banach frame sequence in Theorem 2.3 is necessary for the boundedness of the isometry $w$. We prove the following result in this direction.

**Proposition 2.2.** *Let $\{g_n\} \subseteq \mathbb{B}$ be a retro Banach frame sequence which is not exact. Let $\delta \in (0,1)$ and $\{\varepsilon_n\}$ be a sequence of non-negative scalars. Then the linear map $w$ taking $g_i$ to $h_i$ $(i \in \mathbb{N})$ and satisfying (2.3) is unbounded.*

*Proof.* Let $g_j \in [g_i]_{i \neq j}$. Consider $h_i = g_i, i = 1, 2, \ldots, j-1, j+1, \ldots,$ $h_j \neq g_j$ and $\|g_i - h_j\| = \varepsilon_j$. Then (2.3) is satisfied. Also, since $g_j \in [g_i]_{i \neq j}$, for a given $p$ there exists

$$z^p = g_j - \sum_{i \neq j} a_i^p g_i$$

satisfying

$$\|z^p\| \leq \frac{\varepsilon_n}{p}.$$

Then, we obtain

$$\begin{aligned} \|w(z^p)\| &= \left\| h_j - \sum_{i \neq j} a_i^p g_i \right\| \\ &\geq \left(1 - \frac{1}{p}\right) \varepsilon_n. \end{aligned}$$

Therefore

$$\|w\| \geq \frac{\|w(z^p)\|}{\|z^p\|} \geq p - 1.$$

Hence, in view of the fact that $p$ is arbitrarily chosen number,$\|w\| \to \infty$. □

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