Int. J. Anal. Appl. (2024), 22:146

International Journal of Analysis and Applications

# Nigh-Locally Compactness in Topological Spaces

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**Abstract.** The focus of this paper is the introduction of the notion of a nigh-locally compact topological space. To do this, concepts of "nigh-topological space" and "nigh-compact space" would be defined, and various conclusions and theorems would be derived. This would lead to a well-defined notion of a nigh-locally compact topological space, from which we would obtain a number of theorems and instances concerning this innovative idea.

## 1. Introduction

Assume $(\kappa, \eta)$ is a topological space. Typically, $\kappa$ is referred to as nigh-locally compact whenever every node $x$ within $\kappa$ has a nigh-compact neighborhood. In other words, if there are a nigh-compact set $K$ and an open set $U$ for which $x \in U \subseteq K$. There are alternative widely accepted definitions, all of which hold if $\kappa$ is a pre-regular space (or Hausdorff space). Yet generally speaking, they are not comparable. In what follows, we list some of these definitions for completeness:

- Each point in $\kappa$ has a neighborhood that is almost compact.
- There is a closed compact neighborhood for each point in $\kappa$.
- Each point in $\kappa$ has a neighborhood that is comparatively close together.
- There is a nigh-local basis of pretty compact neighborhoods at each location in $\kappa$.
- There is a nigh-local base of compact neighborhoods at each location in $\kappa$.
- There is a nigh-local base of closed compact neighborhoods at each location in $\kappa$.
- Given that $\kappa$ is Hausdorff, it satisfies all or any of the preceding Conditions.

It is important to remember that a Hausdorff space is a near Tychonoff space if it is nigh-locally compact. It is for this reason that the article on near Tychonoff spaces has instances of nigh-Hausdorff spaces that are not nigh-locally compact, but there are also examples of Tychonoff

Received: Jul. 16, 2024.

2020 *Mathematics Subject Classification.* 54E45.

*Key words and phrases.* compact space; nigh-compact space; locally compact space; nigh-locally compact space.

https://doi.org/10.28924/2291-8639-22-2024-146
ISSN: 2291-8639

spaces that are not nigh-locally compact. However, for further description of more concepts and notions, the reader may refer to the references [1–6].

In this study, the notion of a nigh-locally compact topological space is introduced. This would involve defining the terms "nigh-topological space" and "nigh-compact space," as well as deriving a number of conclusions and theorems. Eventually, we would have a well-defined concept of a nigh-locally compact topological space, from which we might derive several theorems and examples pertaining to this novel concept. The remaining portions of this article are structured as follows: Section 2 recalls important notions and definitions needed later. Section 3 is divided into three subsections; the first one is about the nigh-topological space, the second one is about the nigh-compact space, and the third one is about the nigh-locally compact space. Finally, Section 4 contains the conclusion of this work.

## 2. Preliminaries

We list the most crucial concepts and foundational theorems required for our investigation in this section. We also extract some additional useful results to finish our investigation.

**Definition 2.1.** *[7] Let $\xi = (\xi, \mu)$ be a topological space and $V = \{v_\mu : w_\mu \subset \xi\}$ be a family of subset of $\xi$. Then $\xi$ is called a cover of $\xi$ if $\xi = \bigcup \mu \in \xi v_\mu$. Also, we have:*

- *If $v_\mu$ is an open set in $\xi$ for all $\mu \in \xi$ and $\xi = \bigcup \mu v_\xi$, then $v$ is called open cover of $\xi$.*
- *If $\mu \in v$ is a cover of $\xi$, then $\mu$ is called subcover of $\xi$.*

**Definition 2.2.** *A topological space $\xi = (\xi, \mu)$ is said to be compact if and only if each open cover of $\xi$ contains a finite cover. Also, $\xi$ is compact if $v = \{v_\mu : \mu \in T, w_\mu \subset \xi\}$ is an open cover of $\xi$.*

**Theorem 2.1.** *[8] A compact subset of $\mathbb{R}$ is any closed bounded subset.*

**Definition 2.3.** *[8] We say that a family of subsets A of k has a finite intersection property $(f.i.p.)$ given a topological space $\xi = (\xi, \mu)$ if and only if the intersection of the set A with a finite number of members is not empty.*

**Theorem 2.2.** *[8] Suppose we have a topological space $\xi = (\xi, \mu)$. Then, $\xi$ is considered compact if and only if each family of closed subsets of $\xi$ that has $(f.i.p)$ has an intersection that is not empty.*

**Remark 2.1.** *Consider a subspace of $\xi = (\xi, \mu)$ to be $(W, \mu_w)$. If each open cover of $w$ has a finite sub cover of $w$ with respect to $T_w$, then we say that $(W, \mu_w)$ is compact. Observe that every open cover of W has a finite subcover in $\mu$ if and only if $(W, T_w)$ is compact.*

**Theorem 2.3.** *[9] Consider W is a compact subset of $T_2$-space $\xi$. For all $x \notin W$, we can separate $x$ and W in two disjoint open sets.*

**Theorem 2.4.** *[9] Assume that $\xi$ is a $T_2$-space and that A and B are two disjoint compact subsets. Then, A and B can be divided into two disjoint open sets in $\xi$.*

**Theorem 2.5.** *[10] Every compact $T_2$-space is $T_4$-space.*

**Theorem 2.6.** *[10] All of $T_2$-space's compact subsets are closed.*

**Theorem 2.7.** *[9] In compact $T_2$-space, a subset is considered compact if and only if it is closed.*

**Theorem 2.8.** *[9] The compactness property is preserved under onto continuous function.*

**Corollary 2.1.** *[9] Compactness is topological property.*

**Theorem 2.9.** *[11] The function $f$ is a homeomorphism function if $\xi$ is a compact space, $\gamma$ a $T_2$-space, and $f : \xi \to \gamma$ is a bijective continuous function.*

**Definition 2.4.** *[11] If $f$ is a closed continuous function and $f^{-1}(y)$ is compact in $\xi$ for all $y \in \gamma$, then the function $f : \xi \to \gamma$ is referred to as a perfect function.*

**Theorem 2.10.** *[11] If $\gamma$ is compact and $f : \xi \to \gamma$ is a perfect function, then $\xi$ is compact. In other words, the compactness property is an inverse invert under perfect functions.*

**Definition 2.5.** *The set $\{f^{-1}(y) : y \in \gamma\}$ is said to be fibers of $f$ for all $y \in \gamma$. Therefore, a function $f : \Gamma \to \delta$ is considered perfect if and only if $f$ has compact fibers for which it is closed and continuous.*

**Theorem 2.11.** *The projection $p : \xi \times \gamma \to \gamma$ is closed provided that $\xi$ is compact space.*

**Theorem 2.12.** *[12] Consider two arbitrary spaces, $\xi$ and $\gamma$, and a function $f : \xi \to \gamma$, for which $f$ is a closed subset of $\xi \times \gamma$. Then, $f^{-1}(B)$ is closed in $\xi$ if $B \subseteq \gamma$ is compact.*

**Theorem 2.13.** *[7] Let $\gamma$ be a compact space and $\xi$ an arbitrary space. A function $f : \xi \to \gamma$ is continuous if and only if it is a closed subset of $\xi \times \gamma$.*

**Theorem 2.14.** *[7] The space $\xi \times \gamma$ is compact if $\xi$ and $\gamma$ are compact spaces.*

**Definition 2.6.** *[13] For any $\gamma \in \xi$, there is an open set $u_\gamma$ in $\Gamma$ containing $\gamma$ for which $\bar{u}$ is compact; this defines the locally compact space $\xi$.*

**Theorem 2.15.** *Tychonoff space is a locally compact $T_2$-space.*

**Theorem 2.16.** *Assume that $A$ is a compact subset of a locally compact space $\xi$. There exists an open set $u$ in $\xi$ for which $A \subseteq u \subseteq \bar{u} \subseteq V$ for which $\bar{u}$ is compact in $\xi$ if $A \subseteq V$ for which $V$ open in $\xi$.*

**Theorem 2.17.** *[14] Every subspace of the form $f \cap V$ of a locally compact space $\xi$ is locally compact, where $f$ is closed and $V$ open in $\xi$.*

**Theorem 2.18.** *Every locally compact subspace $M$ of a $T_2$-space $\xi$ is open in $\bar{M}$.*

## 3. Nigh-locally compactness in topological spaces

The notion of a nigh-locally compact topological space is presented in this section. This would entail defining the terms "nigh-compact space" and "nigh-topological space", as well as drawing certain inferences and theorems.

3.1. **Nigh-topological space.** In this subsection, we describe a recent type of topological space known as a nigh-topological space, investigate its prosperities, and provide some new operations and results on a nigh-topological space [11].

**Definition 3.1.** *If a pair $(\kappa, \eta)$, that comprises a set $\kappa$ and a family $\eta$ of subsets of $\kappa$ is satisfied with the following states:*

(1) $\phi \in \eta$ *and* $\kappa \in \eta$,
(2) *A union of any number of members in $\eta$ is a member of $\eta$,*
(3) *Any two members of $\eta$ that intersect also belong to $\eta$,*

*then it is referred to as a topological space.*

**Definition 3.2.** *Assume that $(\kappa, \eta)$ is a topological space for which $B \subseteq \kappa$, then $B$ is called:*

(1) *a regular open set in $\kappa$ if $B = \overline{B}^{o}$.*
(2) *a regular closed set in $\kappa$ if $B = \overline{B^{o}}$.*
(3) *a semi open set in $\kappa$ if $\exists$ an open set $W$ for which $W \subseteq B \subseteq \bar{W}$.*

**Definition 3.3.** *[5] Let $\delta$ be a subset of $\kappa$ and $(\kappa, \eta)$ be a topological space. A set $\delta$ is said to be nigh-open if there are two open sets, $\nu$ and $\xi$, for which $\xi \subseteq \delta \subseteq Ext(\nu)$ and $\nu \cap \delta = \phi$. A nigh-closed set is the complement of a nigh-open set.*

**Remark 3.1.** *Based on the previous definition, we have:*

(1) $\nu \cap \xi = \phi$.
(2) *The first open set is denoted by $\xi$, and the second by $\nu$.*

**Theorem 3.1.** *In any topological space, every open set is also a nigh-open set.*

*Proof.* Consider $\delta$ is an open set in topological space $(\kappa, \eta)$. Then, we have $\delta \subseteq \delta \subseteq Ext(\phi)$ and $\delta \cap \phi = \phi$. Thus, $\delta$ is a nigh-open set. □

Herein, it is important to note that the opposite of the aforementioned theorem need not hold. For instance, if we consider the usual topology $\mathbb{R}$, and take the set $(-1, 2]$, which is a nigh-open as there are two open sets, $((0, 1)$ and $(4, 5)$ for which $(0, 1) \subseteq (-1, 2] \subseteq Ext(4, 5)$, however, given the standard topology on $\mathbb{R}$, the set $(-1, 2]$ is not an open set.

**Theorem 3.2.** *Given a topological space $(\kappa, \eta)$ where $\delta$ is a nigh-open set. Then, $Int(\overline{\nu}) \subset Ext(\delta) \subseteq Ext(\xi)$, where $\xi$ and $\nu$ are the first and second open sets, respectively.*

*Proof.* Suppose there is a nigh-open set $\delta$ in $\kappa$. Then, there are open sets, $\xi$ and $\nu$, for which $\xi \subseteq \delta \subseteq Ext(\nu)$ and $\nu \cap \delta = \phi$. As a consequence, we get

$$\overline{\delta} \subseteq \overline{Ext(\nu)} = \bigcap_{\substack{F \text{ closed} \\ Ext(\nu) \subseteq F}} F.$$

As a result, we have

$$\bigcup_{\substack{F^c \text{ open set} \\ F^c \subset (Ext(\nu))^c}} F^c = \overline{\nu} \subset \overline{\delta}^c = Ext(\delta).$$

Now, putting $w = F^c$ yields that $w$ is an open set. As a result, we obtain

$$\bigcup_{\substack{w \text{ open} \\ w \subset \overline{\nu}}} w \subset Ext(\delta).$$

But, we have

$$\bigcup_{\substack{w \text{ open} \\ w \subset \overline{\nu}}} w = Int(\delta).$$

Now, since $\xi \subseteq \delta$, then $Ext(\delta) \subseteq Ext(\delta)$, which implies $Int(\overline{\nu}) \subseteq Ext(\delta) \subseteq Ext(\xi)$. □

**Definition 3.4.** *Let $\eta \subseteq p(\kappa)$ for which $\kappa$ be a non-empty set. If the following are satisfied:*

(1) $\phi, \kappa \in \eta$.
(2) *Any two nigh-open sets that intersect are also nigh-open sets.*
(3) *A nigh-open set is the union of any family of nigh-open sets.*

*then $\eta$ is called a nigh-topology on $\kappa$.*

**Theorem 3.3.** *A nigh-topological space is a topological space.*

*Proof.* Let us consider a topological space $(\kappa, \eta)$. In order to demonstrate the nigh-topological space of $(\kappa, \eta)$, we examine the following cases:

(1) By the definition of the topological space, it is clear that $\kappa, \phi \in \eta$.
(2) Let $\delta$ and $\rho$ be two nigh-open sets. Then, $\exists$ open sets $\xi_1, \xi_2, \nu_1$ and $\nu_2$ for which

$$\xi_1 \subseteq \delta \subseteq Ext(\nu_1) \text{ and } \xi_2 \subseteq \rho \subseteq Ext(\nu_2).$$

This consequently implies

$$\xi_1 \cap \xi_2 \subseteq \delta \cap \rho \subseteq Ext(\nu_1) \cap Ext(\nu_2) \subseteq Ext(\nu_1 \cap \nu_2).$$

Now, it is a time to observe that $\xi_1 \cap \xi_2 = \xi$, which is an open set. Besides, $\nu_1 \cap \nu_2 = \nu$, which is an open set as well (i.e., $\xi \subseteq \delta \cap \rho \subseteq Ext(\nu)$ and $\delta \cap \rho$ is a nigh-open set).

(3) Let $\delta = \{\delta_\alpha : \alpha \in \Gamma\}$ be a family of nigh-open sets. We observe that $\delta_\alpha$ is a nigh-open set for every $\alpha \in \Gamma$. So, $\exists$ two open sets $\xi_\alpha$ and $\nu_\alpha$ for which $\xi_\alpha \subseteq \delta_\alpha \subseteq Ext(\nu_\alpha)$, for every $\alpha \in \Gamma$. Consequently, we have

$$\bigcup_{\alpha \in \Gamma} \xi_\alpha \subseteq \bigcup_{\alpha \in \Gamma} \delta_\alpha \subseteq \bigcup_{\alpha \in \Gamma} Ext(\nu_\alpha) \subseteq Ext(\phi).$$

Therefore, $\bigcup_{\alpha \in \Gamma} \delta_\alpha$ is a nigh-open set.

□

**Definition 3.5.** *Assume that $\delta$ is a subset of $\kappa$ and that $(\kappa, \eta_{(n)})$ is a non-topological space. Then:*

(1) *A nigh-limit point of a set* $\delta$ *is* $F \in \Gamma$ *if, for any nigh-open set* $\nu$ *that contains* $v$*, we have:*

$$\begin{cases} \nu \cap \delta_v \neq \phi, \text{ if } F \notin \delta \\ \nu \cap \delta_v \backslash_{\{F\}} \neq \phi, \text{ if } F \in \delta \end{cases}.$$

(2) *The definition of the nigh-derived set of* $\delta$*, represented by* $\delta'_v$*, is*

$$\delta'_v = \{\nu \in \Gamma : F \text{ is nigh-limite point of } \delta\}.$$

(3) *The definition of the nigh-closure set of* $\delta$*, represented by* $CL_{(v)}(\delta)$*, is* $CL_{(v)}(\delta) = \delta \cup \delta'_{(v)}$.

(4) *The definition of the nigh-interior set of* $\delta$ *is* $Int_{(v)}(\delta) = \left(CL_{(v)}(\delta^c)\right)^c$.

(5) *The definition of the nigh-exterior set of* $\delta$ *is* $Ext_{(v)}(\delta)$ *and* $Ext_{(v)}(\delta) = \left(CL_{(v)}(\delta)\right)^c$.

(6) *The definition of the nigh-boundary set of* $\delta$ *is* $Bd_{(v)}(\delta)$ *and* $Bd_{(v)} = CL_{(v)}(\delta) \cap CL_{(v)}(\delta^c)$.

**Theorem 3.4.** *Given a nigh-topological space* $(\kappa, \eta_{(v)})$*, we have:*

(1) $\phi'_{(v)} = \phi$.
(2) $CL_{(v)}(\phi) = \phi$ *and* $CL_{(v)}(\kappa) = \kappa$.
(3) $Int_{(v)}(\phi) = \phi$ *and* $Int_{(v)}(\kappa) = \kappa$.
(4) $Ext_{(v)}(\phi) = \kappa, Ext_{(v)}(\kappa) = \phi$.

*Proof.* (1) Suppose not! So, there is $F \in \kappa$ for which $F \in \phi'_{(v)}$. This confirms that $\kappa$ is a limit point of $\phi_{(v)}$. Then, for all nigh-open sets containing $\kappa$, we have $\nu \cap \phi \neq \phi$, which is a contradiction. Thus, the result is hold.

(2) Observe that we can have $CL_{(v)}(\phi) = \phi \cup \phi'_{(v)} = \phi \cup \phi = \phi$. This immediately gives $CL_{(v)}(\kappa) = \kappa \cup \kappa'_{(v)} = \kappa$.

(3) Here, we have $Int_{(v)}(\phi) = \left(CL_{(v)}(\phi^c)\right)^c = \left(CL_{(v)}(\kappa)\right)^c = \kappa^c = \phi$, which implies $Int_{(v)}(\kappa) = \left(CL_{(v)}(\kappa^c)\right)^c = \left(CL_{(v)}(\phi)\right)^c = \phi^c = \kappa$.

(4) We can obtain $Ext_{(v)}(\phi) = \left(CL_{(v)}(\phi)\right)^c = \phi^c = \kappa$. Accordingly, we get $Ext_{(v)}(\kappa) = \left(CL_{(v)}(\kappa)\right)^c = \kappa^c = \phi$.

□

**Theorem 3.5.** *Given a non-topological space* $(\kappa, \eta)$*, let* $\delta$ *and* $\rho$ *be two subsets of* $\kappa$*. Then:*

(1) *If* $\delta \subseteq \rho$*, then* $\delta'_{(v)} \subseteq \rho'_{(v)}$.
(2) *If* $\delta \subseteq \rho$*, then* $CL_{(v)}(\delta) \subseteq CL_{(v)}(\rho)$.
(3) *If* $\delta \subseteq \rho$*, then* $Int_{(v)}(\delta) \subseteq Int_{(v)}(\rho)$.
(4) *If* $\delta \subseteq \rho$*, then* $Ext_{(v)}(\rho) \subseteq Ext_{(v)}(\delta)$.

*Proof.* (1) Let $F \in \delta'_{(v)}$. So, $\Gamma$ is a nigh-limit point of $\delta$. Consequently, for all nigh-open set $\mu$ containing $v$, we obtain:

$$\begin{cases} \nu \cap \delta_v \neq \phi, \text{ if } F \notin \delta \\ \nu \cap \delta_{v)} \backslash_{\{F\}} \neq \phi, \text{ if } F \in \delta. \end{cases}.$$

Now, since $\delta \subseteq \rho$, then we have:

$$\begin{cases} \nu \cap \rho_v \neq \phi, \; if\, F \notin \rho \\ \nu \cap \rho_{v)} \backslash_{\{F\}} \neq \phi, \; if\, F \in \rho. \end{cases}.$$

Thus, $\kappa$ is a nigh-limit point of $\rho$, which implies $F \in \rho'_{(v)}$, and hence the result hold.

(2) Due to $\delta \subseteq \rho$, then we have $\delta'_{(v)} \subseteq \rho'_{(v)}$ and $\delta \cup \delta'_{(v)} \subseteq \rho \cup \rho'_{(v)}$, (i.e., $CL_{(v)}(\delta) \subseteq CL_{(v)}(\rho)$).

(3) Because of $\delta \subseteq \rho$, then $\rho^c \subseteq \delta^c$. So, by using Theorem 2, we can have

$$CL_{(v)}(\rho^c) \subseteq CL_{(v)}(\delta^c),$$

and

$$\left(CL_{(v)}(\delta^c)\right)^c \subseteq \left(CL_{(v)}(\rho^c)\right)^c.$$

As a result, we have $Int_{(v)}(\delta) \subseteq Int_{(v)}(\rho)$.

(4) Since $\delta \subseteq \rho$, then we have

$$CL_{(v)}(\delta) \subseteq CL_{(v)}(\rho),$$

and

$$\left(CL_{(v)}(\rho^c)\right)^c \subseteq \left(CL_{(v)}(\delta^c)\right)^c.$$

This means that $Ext_{(v)}(\rho) \subseteq Ext_{(v)}(\delta)$.

□

**Theorem 3.6.** *If $\delta \subseteq \kappa$ in which $(\kappa, \eta)$ is a nigh-topological space, then:*

(1) *$CL_{v)}(\delta)$ is a nigh-closed set.*
(2) *$Int_{(v)}(\delta)$ is a nigh-open set.*
(3) *$Ext_{(v)}(\delta)$ is a nigh-open set.*
(4) *$Bd_{(v)}(\delta)$ is a nigh-closed set.*

*Proof.* (1) In case $F \in \left(CL_{(v)}(\delta)\right)^c$, then $F \notin \delta \cup \delta'_{(v)}$. Thus, we get $F \notin \delta$ and $F \notin \delta'_{(v)}$. Thus, there is a nigh-open set $\nu$ for which $\nu \cap \delta_{(v)} = \phi$ (say *). Then, we obtain

$$\begin{aligned} \nu \cap CL_{(v)(\delta)} &= \nu \cap (\delta \cap \delta'_{(v)}) \\ &= (\nu \cap \delta) \cup (\nu \cap \delta'_{(v)} \\ &= \phi \cup (\nu \cap \delta'_{(v)}). \end{aligned}$$

This means that $\nu \cap CL_{(v)(\delta)} = \nu \cap \delta'_{(v)}$. Now, if $F \in (\nu \cap \delta'_{(v)})$, then $F \in \nu$ and $\nu \cap \delta_{(v)} \neq \phi$, which is a contradiction with (*). So, we get $(\nu \cap \delta'_{(v)}) = \phi$, and hence $\nu \cap CL_{(v)}(\delta) = \phi$. Thus, we obtain

$$F \in \nu_F \subseteq \left(CL_{(v)}(\delta)\right)^c$$

, which consequently implies

$$\left(CL_{(v)}(\delta)\right)^c = \bigcup_{\substack{F \in \nu_F \\ \nu_F \text{is nigh-open set}}} \nu_F.$$

Thus, it follows that $\left(CL_{(v)}(\delta)\right)^c$ is a nigh-open set, and so $CL_{(v)}(\delta)$ is a nigh-closed set.

(2) With the use if Theorem 1, we can notice that $CL_{(v)}(\delta^c)$ is nigh-closed set. Hence, $\left(CL_{(v)}(\delta)\right)^c$ is a nigh-open set (i.e., $Int_{(v)}(\delta)$ is a nigh-open set).

(3) Due to $Ext_{(v)}(\delta) = \left(CL_{(v)}(\delta)\right)^c$. Then, we can assert that $Ext_{(v)}(\delta)$ is a nigh-open set.

(4) Due to $Bd_{(v)}(\delta) = CL_{(v)}(\delta) \cap CL_{(v)}(\delta^c)$. Then, we can assert that $Bd_{(v)}(\delta)$ is a nigh-closed set.

□

**Theorem 3.7.** *We have $CL_{(v)}(\delta) = \delta$ if and only if $\delta$ is a nigh-closed set.*

*Proof.* ⇒) Trivial.

⇐) Consider $\delta$ is a nigh-closed set. It is clear that $\delta \subseteq CL_{(n)}$ (say *). Now, we want to prove that $CL_{(n)} \subseteq \delta$. To do so, we let $F \in \delta'$. Now, to show $F \in \delta$, we assume not, i.e. $F \notin \delta$. Then, we have $F \in \delta^c$, which is a nigh-open set. In this regard, since $F \in \delta'$, then $\delta^c \cap \delta \neq \phi$, which is contradiction. So, we obtain $F \in \delta$, i.e. $\delta' \subseteq \delta$ and $\delta \subseteq \delta$. Therefore, we get $\delta' \cup \delta \subseteq \delta$, i.e. $CL_{(n)}(\delta) \subset \delta$, and hence the result hold. □

**Theorem 3.8.** *Consider $(\kappa, \eta)$ is a nigh-topological space for which $\delta \subseteq \kappa$, then:*

(1) $CL_{(n)}(\delta) = \bigcap\{\varsigma; \varsigma$ *is nigh-closed set and* $\delta \subseteq \varsigma\}$, *i.e.* $CL_{(n)}(\delta)$ *is the smallest nigh-closed set containing* $\delta$.

(2) $Int_{(n)}(\delta) = \cup\{T : T$ *is nigh-open set and* $T \subseteq \delta\}$.

(3) $Ext_{(n)}(\delta) = \cup\{W : W$ *is nigh-open set and* $W \subseteq \delta^c\}$.

*Proof.* (1) Observe that $\delta \subseteq \varsigma$. By previous theorem, we can have $CL_{(v)}(\delta) \subseteq CL_{(v)}(\varsigma) = \varsigma$ for which $CL_{(v)}(\delta)$ is closed set. Therefore, $CL_{(v)}(\delta)$ is one member of $\varsigma'^s$, which implies

$$\bigcap_{\substack{\varsigma \text{ is nigh-closed set} \\ \delta\subseteq\varsigma}} \{\varsigma\} \subseteq CL_{(v)}(\delta).$$

On the other hand, as $\delta \subseteq \varsigma$, we can obtain

$$CL_{(v)}(\delta) \subseteq CL_{(v)}(\varsigma),$$

which gives

$$CL_{(v)}(\delta) \subseteq \bigcap_{\substack{\varsigma \text{ is nigh-closed set} \\ \delta\subseteq\varsigma}} \{CL_{(v)}(\varsigma)\} = \bigcap_{\substack{\varsigma \text{ is nigh-closed set} \\ \delta\subseteq\varsigma}} \{\varsigma\}.$$

(2) Herein, we have $Int_n(\delta) = \left(CL_{(n)}(\delta^c)\right)^c = \cap\{\varsigma : \varsigma$ is nigh-closed set and $\delta^c \subseteq \varsigma\}$, which implies $\left(CL_{(n)}(\delta^c)\right)^c = \cup\{\varsigma^c : \varsigma^c$ *is nigh-closed set and* $\varsigma^c \subseteq \delta\}$. Now, by letting $\varkappa^c = T$, we get $Int_{(n)}(\delta) = \cup\{T : T$ *is nigh-open set and* $T \subseteq \delta\}$.

(3) By the previous part, one might have $Ext_{(n)}(\delta) = Int_{(n)}(\delta^c) = \cup\{W : W$ *is nigh-open set and* $W \subseteq \delta^c\}$.

□

3.2. **Nigh-compact space.** The so-called nigh-compact space is described in this subsection, and its properties are examined through the presentation of certain novel findings and theorems.

**Definition 3.6.** *[15] Consider $(\kappa, \eta)$ is a topological space and $V = \{v_\zeta : \zeta \in \omega\}$ is a family of subsets of $\kappa$. Then $V$ is said to be a cover of $\kappa$ if $\kappa = \bigcup \zeta \in \omega v_\zeta$. Furthermore, we have the following states:*

(1) *If $v_\zeta$ is a nigh-open set in $\kappa$ for all $\zeta \in \omega$ and $\kappa = \bigcup \zeta \in \omega v_\zeta$, then $V$ is called a nigh-open cover of $\kappa$.*
(2) *If $B \subseteq V$ is a cover of $\kappa$, then $B$ is called a subcover of $V$ for $\kappa$.*

**Definition 3.7.** *If there is a finite subcover for each nigh-open cover of $\kappa$, then $(\kappa, \eta)$ is a nigh-compact topological space.*

**Example 3.1.** *[11] It can be noticed that $(\mathbb{R}, \eta_u)$ is not a nigh-compact space.*

*Proof.* It should be noted that there is a nigh-open cover $\varphi_n = \{(-n, n) : n \in \mathbb{N}\}$ of $\mathbb{R}$ that has no finite subcover because if this is not true, i.e. $\varphi_n$ has a finite subcover of $\mathbb{R}$, say $T$, then $T$ can be expressed in the from $T = \{(-n_1, n_1), (-n_2, n_2), \ldots, (-n_k, n_k)\}$. Thus, we have $\mathbb{R} = \bigcup_{i=1}^{n}, (-n_i, n_i)$. Now, if we let $M = \max\{n_1, n_2, n_3, \ldots, n_k\}$, then $n_i \leq M$ and $-n \geq -M$m for all $i = 1, 2, \ldots, k$. Therefore, we obtain $\mathbb{R} = \bigcup^{n_{i=1}} (-n, n) \subseteq \bigcup_{i=1}^{n} (-M, M) = (-M, M)$, and so we have $\mathbb{R} \subseteq (-M, M)$, which is a contradiction. Thus, $\exists$ a nigh-open cover of $\mathbb{R}$ that has no finite subcover of $\mathbb{R}$. Hence, $(\mathbb{R}, \eta_u)$ is not a nigh-compact space. □

**Example 3.2.** *Every $[c, d]$ is a nigh-compact space in $(\mathbb{R}, \eta_u)$.*

*Proof.* Suppose not! That is $[c, d]$ is not nigh-compact. Then, there is a nigh-open cover of $[c, d]$, say $U$, for which it has no finite subcover of $[c, d]$. Due to $[c, d] = [c, \frac{c+d}{2}] \cup [\frac{c+d}{2}, d]$, then either $[c, \frac{c+d}{2}]$ or $[\frac{c+d}{2}, d]$ can not be covered by finite members of $U$. Now, as $[c_1, d_1] = [c_1, \frac{c+d}{2}] \cup [\frac{c+d}{2}, d_1]$, then either $[c_1, \frac{c+d}{2}]$ or $[\frac{c+d}{2}, d_1]$ can not be covered by finite members of $U$. Suppose that $[c_1, d_1]$ is one of these so that it can not be covered by finitely numbers of $U$. Then, we have $b_2 - a_2 = \frac{1}{2^2}(c, d)$. If we continue by the argument, we get $\{[c_n, d_n]\}_{n=1^\infty}$, which is a sequence of intervals for which $[c_n, d_n]$ can not be covered by finitely number members of $U$, for all $n = 1, 2, \ldots$. Consequently, we have $d_n - c_n = \frac{c-d}{2^n} \to 0$ as $n \to \infty$. Therefore, by the cantor nested intervals theorem, we have $\Gamma p \in \bigcap_{i=1}^{\infty} [c_n, d_n]$, and so $p \in [c_n, d_n]$, for all $n = 1, 2, \ldots$. Since $p \in [a, b]$, then $O$ is a nigh-open interval for which $p \in O \subseteq [c, d]$, where $O \in U$ and $O = (p - t, p + t)$ for which $t > 0$. Hence, we obtain $p \in (p - t, p + t)$, and by taking $n$ as large as enough for which $\frac{1}{2^n} < t$, we get $p \in [c_n, d_n]$ by one of member of $U$, which is a contradiction! Frorm this discussion, the desired result hold. □

**Corollary 3.1.** *Every finite space is nigh-compact.*

*Proof.* Let $(\kappa, \eta)$ be a finite topological space and let $W = \{w_\xi\}_{\xi \in \delta}$ be a nigh-open cover of $\kappa$. Then, for all $x$ in $\kappa$, $\exists\, \Gamma \xi \in \delta$ for which $x \in w_\xi$. Define $O = \{w_{\xi_t} : t = 1, 2, \ldots, n, \xi \in \delta\}$. Then, $O$ is finite and $\kappa = \bigcup_{t=1}^{n} W_{\xi_t}$. Hence, $O$ is a finite subcover of $\kappa$, and so $\kappa$ is nigh-compact. □

**Example 3.3.** *$(\kappa, \eta_{cof})$ is a nigh-compact space.*

*Proof.* Let $U = \{u_\xi\}_{\xi\in\delta}$ be a nigh-open cover of $\kappa$. Then, $u_\xi$ is a nigh-open in $\kappa$ for every $\xi \in \delta$. Consequently, $u_\xi$ is a nigh-open set in $\kappa$. Thus, $\kappa - u_\xi$ is a finite set, and hence, by the previous corollary, it is nigh-compact. So, every nigh-open cover of $\kappa - u_\xi$ has a finite subcover of $\kappa - u_\xi$. But $U$ is a nigh-open cover of $\kappa$, and so $U$ is a nigh-open cover of $\kappa - u_\xi$, say $\{u_i\}_{i=1}^m$. Hence, $\kappa - u_\xi$ as $\kappa - u_\xi \subseteq \kappa$. Therefore, $U$ has a finite subcover that covers $\kappa - u_\xi$, say $\{u_i\}_{i=1}^m$. Therefore, we have $\kappa - u_\xi = \bigcup_{i=1}^m u_i$, which implies $\kappa = (\bigcup_{i=1}^m u_i) \cup u_\xi$. Therefore, $\kappa$ has a subcover $B = \{u_1, \ldots, u_m, u_\xi\}$, and so $\kappa$ is nigh-compact. □

**Corollary 3.2.** *Every subspace of* $(\kappa, \eta_{cof})$ *is nigh-compact.*

*Proof.* Since every subspace of $(\kappa, \eta_{cof})$ is a co-finite topological space, then by previous theorem, we can infer that every subspace of a co-finite topological space is compact. □

**Example 3.4.** $(\mathbb{R}, \eta_{dis})$ *is not nigh-compact.*

*Proof.* Let $V = \{\{x\} : x \in \mathbb{R}\}$. Due to $\mathbb{R}$ is defined by $\eta_{dis}$, then $\{x\}$ is nigh-open in $\mathbb{R}$. Therefore, $V$ is a nigh-open cover of $\mathbb{R}$. Now, let $\acute{V} \subset V$, i.e. $\acute{V}$ is proper in $V$, then $\Gamma y \in \mathbb{R}$ and $y \notin \acute{V}$. Therefore, $\acute{V}$ is not a nigh-cover of $\mathbb{R}$, and so $\acute{V}$ is not a subcover of $V$. Thus, $V$ has no finite subcover which contradicts 1! Therefore, $(\mathbb{R}, \eta_{dis})$ is not nigh-compact. □

**Theorem 3.9.** *The set E of real numbers is nigh-closed and bounded if and only if the set E is nigh-compact.*

*Proof.* Consider $E \subseteq \mathbb{R}$ is nigh-compact, then, for all $e \in E$, we get $e \in (e-1, e+1) = u_e : u_e$ is nigh-open in $\mathbb{R}$. Then, $\{u_e : e \in E)$ is a nigh-open cover of $E$, and because of $E$ is nigh-compact, then $E \subseteq \bigcup_{i=1}^n u_{e_i}$. As a result, $\Gamma e_1, e_2, \ldots, e_n \in E$ for which $e_i \in u_{e_i}$, for all $i = 1, 2, \ldots, n$. Now, let $M = max\{e_1, e_2, \ldots, e_n\}$ and $m = min\{e_1, e_2, \ldots, e_n\}$. Then, we have $E \subseteq \bigcup_{i=1}^n u_{e_i} \subseteq [m-1, m+1]$, and so $E$ is bounded. In this regard, since $E$ is nigh-compact in $T_2$-space, then $E$ is nigh-closed. Conversely, suppose $E$ is nigh-closed and bounded in $\mathbb{R}$ for which $E$ is bounded, then $E \subseteq [a, b]$ for some $a \prec b$ in $\mathbb{R}$. Due to $E$ is closed in nigh-compact subset $[a, b]$, then $E$ is nigh-compact. □

**Definition 3.8.** *[8] Let A be a family of subsets of* $\kappa$ *and* $(\kappa, \eta)$ *be a topological space. If the intersection of the finite number of members set A is not empty, we say that A has a finite intersection property* $(f.i.p.)$.

**Theorem 3.10.** *[1] If* $(\kappa, \eta)$ *is a topological space, then* $\kappa$ *is a nigh-compact space if and only if* $(f.i.p)$ *has a non-empty intersection for every family of a nigh-closed subset of* $\kappa$.

*Proof.* $\Rightarrow$) Consider $\kappa$ is a nigh-compact space. If we assume that $\exists$ a family of closed subsets of $\kappa$, say $F = \{F_\alpha : \alpha \in \beta\}$, with $f.i.p$ for which $\bigcap_{\alpha\in\beta} F_\alpha = \phi$, then we have

$$\bigcup_{\alpha\in\beta}(\kappa - F_\alpha) = \kappa - \bigcap_{\alpha\in\beta}(F_\alpha) = \kappa.$$

Due to $F_\alpha$ is a nigh-closed set in $\kappa$ for all $\alpha \in \beta$, then $\kappa - F_\alpha$ is a nigh-open set in $\kappa$ for all $\alpha \in \beta$. Thus, $U = \{\kappa - F_\alpha : \alpha \in \beta\}$ is a nigh-open cover of $\kappa$, and hence by compactness of $\kappa$, $U$ has a finite subcover of $\kappa$. As a result, we have $\kappa = \bigcup_{i=1}^n(\kappa - F_i) = \kappa - \bigcap_{i=1}^n F_i$, and so $\phi = \bigcap_{i=1}^n F_i$, which is a contradiction with $F$ has $f.i.p$. Therefore, we can assert that every family of closed subsets of $\kappa$ with $f.i.p$ has non empty intersection.

$\Leftarrow$) Suppose that every family of closed subsets of $\kappa$ with $f.i.p$ has non empty intersection. Now, if we assume $\kappa$ is not a nigh-compact space, then $\exists$ a nigh-open cover of $\kappa$, say $U = \{u_\alpha : \alpha \in \beta\}$, that can not be reduced to a finite subcover of $\kappa$. Therefore, we obtain

$$\phi = \kappa - \bigcup_{\alpha\in\beta} u_\alpha = \bigcap_{\alpha\in\beta}(\kappa - u_\alpha).$$

Consequently, due to $u_\alpha$ is open for all $\alpha \in \beta$, then $\{\kappa - u_\alpha : \alpha \in \beta\}$ is a family of nigh-closed subsets of $\kappa$. Consequently, we have the following claim.

Claim: $f$ has $f.i.p$.

To demonstrate the above statement, we assume it is not true. Then, $\exists\ u_1, u_2, \ldots, u_n$ for which $\bigcap_{i=1}^{n}(\kappa - u_i) = \phi$. Thus, we have $\kappa = \bigcup_{i=1}^{n} u_i$, and so $U$ has a finite subcover of $\kappa$, which is a contradiction. Hence, $F = \{\kappa - u_\alpha\}$ has $f.i.p$, and thus by the assumption $\bigcap_\alpha \in \beta u_\alpha \neq \phi$, we can obtain

$$\kappa \neq \kappa - \bigcap_{\alpha\in\beta}(\kappa - u_\alpha) = \bigcup_{\alpha\in\beta}(\kappa - (\kappa - u_\alpha)) = \bigcup_{\alpha\in\beta} u_\alpha.$$

This means that we have $\kappa \neq \bigcup_{\alpha\in\beta} u_\alpha$, which is contradiction. Therefore, $\kappa$ is a nigh-compact space. □

**Remark 3.2.** *Consider a subspace of $(\kappa, \eta)$ to be $(W, \eta_w)$. If there is a finite subcover of $W$ with respect to $T_w$ for every nigh-open cover of $W$, then $(W, \eta_w)$ is a nigh-compact space. Furthermore, $(W, T_w)$ is a nigh-compact space if and only if each nigh-open cover of $W$ in $\eta$ has a finite subcover.*

**Theorem 3.11.** *Every nigh-closed subset of a nigh-compact space is nigh-compact.*

*Proof.* Suppose that $W$ is a nigh-closed subset in a nigh-compact space $\kappa$. Let $U = \{u_\alpha : \alpha \in \beta\}$ ba a nigh-open cover of $W$. Then $\kappa = W \cup (\kappa - W) = \bigcup_{\alpha\in\beta} u_\alpha \cup (\kappa - W)$ is a nigh-open cover of $\kappa$. Due to $\kappa$ is a nigh-compact space, then $W \cup (\kappa - W)$ can be reduced to a finite subcover, say $\kappa = (\bigcup_{i=1}^{n} u_{\alpha_i}) \cup (\kappa - W)$. Therefore, we have $W = \bigcup_{i=1}^{n} u_{\alpha_i}$. Consequently, $\{u_{\alpha_i} : i = 1, \ldots, n\}$ forms a finite subcover of $W$. Therefore, $W$ is nigh-compact. □

**Theorem 3.12.** *Let $W$ be a nigh-compact subset in $T_2$-space $\kappa$. Then for all $x \notin W$, we can separate $x$ and $W$ into two disjoint open sets.*

*Proof.* For all $w \in W$, we have $w \neq x$ with $x \notin W$. Since $\kappa$ is a $T_2$-space, then $\exists$ two open subset $u_w(x)$ and $v(w)$ in $\kappa$ for which $x \in u_w(x)$ and $w \in v(w)$ for which $u_w(x) \cap v(w) = \phi$. Hence, $v = \{v(w) : w \in W\}$ forms a nigh-open cover of $W$. But $W$ is nigh-compact. So, $V$ can be reduced to a finite subcover of $W$, say $\{v(w_1), v(w_2), \ldots, v(w_n)\}$. Hence, we obtain $W \subseteq \bigcup_{i=1}^{n} V(w_\Gamma)$ (say (1)). Therefore, for all $V(w_k), \Gamma = 1, \ldots, n$, there is corresponding open sets $u_{w_k}(\Gamma)$ condoling $\Gamma$ for which $u_{w_k}(\Gamma) \cap v(w_k) = \phi$ (since $\kappa$ is a $T_2 - space$). Now, let $u = \bigcap_{k=1}^{n} u_{w_k}$, then $\Gamma \in u$ for which $u$ is an open set in $\kappa$ and $u \cap V(w_\Gamma) = \phi$, for all $k = 1, \ldots, n$. Also, for all $\Gamma = 1, \ldots, n$, we obtain

$u \subseteq u_{w_\Gamma}(\Gamma)$. Thus, we have $u \cap U_v \subseteq u_{w_k}(\Gamma) \cap v(w_\Gamma) = \phi$, which implies $u \cap V(w_k) = \phi$, for all $k = 1, \ldots, n$. Hence, we get $(u \cap U_v(w_1)) \cup (u \cap U_v(w_2)) \cup \ldots \cup (u \cap U_v(w_n)) = \phi$, which gives $u \bigcup_{k=1}^{n} V(a_k) = \phi$. Now, we let $V = \bigcup_{n=1}^{k} v(w_k)$. Then, $V$ is a nigh-open set as it is a union of finite nigh-open sets in $\Gamma$. Now, with the use of (1), we get $A \subseteq V$, and so $\Gamma$ has two nigh-open sets $u$ and $v$ in $\kappa$ for which $\Gamma \in u$ and $W \subseteq v$ with $u \cap v = \phi$. □

**Theorem 3.13.** *Given a $T_2$-space $\kappa$, let $A$ and $B$ be two disjoint nigh-compact subsets. After that, $A$ and $B$ can be divided into two disjoint near-open sets in $\kappa$.*

*Proof.* For all $x \in A$, we have $x \notin B$. Due to $A \cap B = \phi$, then by the previous theorem, $\exists$ two nigh-open sets $u_\Gamma$ and $v_\Gamma$ in $\kappa$ for which $x \in u_\Gamma$ and $B \subseteq v_\Gamma$ with $u_\Gamma \cap v_\Gamma = \phi$. Hence, $U = \{u_\Gamma : \Gamma \in A\}$ forms a nigh-open cover of $A$. But $A$ is compact, then $U$ can be reduced to a finite subcover, say $\{u_{\Gamma_1}, u_{\Gamma_2}, \ldots, u_{\Gamma_n}\}$. Thus, we obtain $A \subseteq \bigcup_{i=1}^{n} u_{\Gamma_i}$. Now, let $u = \bigcup_{i=1}^{n} u_{\Gamma_i}$. Then, $u$ is a nigh-open set in $\kappa$ with $A \subseteq u$ (say (1). In this regard, there is a corresponding set $V_{\Gamma_i}$ for which $B \subseteq V_{\Gamma_i}$, for all $u_{\Gamma_i}$ with $i = 1, 2, \ldots, n$. Thus, we have $B \subseteq \bigcap_{i=1}^{n} V_{\Gamma_i}$, which implies that $V$ is a nigh-open set and $B \subseteq V$ (say (2)). Now, due to $u_{\Gamma_i} \cap v_{\Gamma_i} = \phi$, for all $i = 1, 2, \ldots, n$, then $\bigcup_{i=1}^{n} u_{\Gamma_i} \cap V_{\Gamma_i} = \phi$, for all $i = 1, 2, \ldots, n$. Thus, we get $u \cap v_{\Gamma_i} = \phi$, for all $i = 1, 2, \ldots, n$. But $V = \bigcap_{i=1}^{n} V_{\Gamma_i} \subseteq V_{\Gamma_i}$, for all $i = 1, 2, \ldots, n$. This consequently implies that $u \cap v \subseteq V_{\Gamma_i} \cap u = \phi$. Hence, $v \cap u = \phi$ (say (3)). Therefore, by (1), (2) and (3), we can infer that $\Gamma$ is a nigh-open set for which $u$ and $v$ are in $\Gamma$ for which $A \subseteq u$ and $B \subseteq v$ with $u \cap v = \phi$. □

**Theorem 3.14.** *Every nigh-compact $T_2$-space is a $T_4$-space.*

*Proof.* Consider $\kappa$ is a nigh-compact $T_2$-space. Then, $\kappa$ is a $T_1$-space (say (1)). Assume that $A$ and $B$ are two nigh-closed disjoint subsets of $\kappa$. Due to $\kappa$ is a nigh-compact space, then $A$ and $B$ are nigh-compact. This implies that $A$ and $B$ are two nigh-compact subsets of $T_2$-space. Hence, by the previous theorem, we can separate $A$ and $B$ into two disjoint nigh-open sets. So, $\kappa$ is normal. Consequently, by (1) and (2), we can confirm that $\kappa$ is a $T_4$-space. □

**Theorem 3.15.** *Every nigh-compact subset of a $T_2$-space is nigh-closed.*

*Proof.* Suppose that $A$ is a nigh-compact subset of a $T_2$-space $\kappa$. Let $x \notin A$, then by previous theorem, $\exists$ two nigh-open sets $U$ and $V$ in $\kappa$ for which $x \in U$ and $A \subseteq V$ with $U \cap V = \phi$. So, we have $U \subseteq \kappa - V$, and since $A \subseteq V$, we can obtain $\kappa - V \subseteq \kappa - A$. Thus, we have $x \in U \subseteq \kappa - V \subseteq \kappa - A$, which means that $x \in U \subseteq k - A$. Consequently, due to $U$ is a nigh-open set, then $\kappa - A$ is a nigh-open set, and so $A$ is nigh-closed. □

**Theorem 3.16.** *[9] Every subset of a nigh-compact $T_2$-space is nigh-compact if and only if it is nigh-closed.*

*Proof.* $\Rightarrow$) Suppose that $A$ is a nigh-compact set in a nigh-compact $T_2$-space $\kappa$. Then, $A$ is a nigh-compact in a $T_2$-space $\kappa$. So by previous theorem, $A$ is nigh-closed.
$\Leftarrow$) Suppose that $A$ is nigh-closed in a nigh-compact $T_2$-space. Then, $A$ is nigh-closed in a nigh-compact space $\kappa$. Thus by previous theorem, $A$ is nigh-compact. □

**Theorem 3.17.** *Let $(\kappa, \eta)$ be a $T_3$-space $\kappa$. If $A$ is a nigh-compact subset of $\kappa$ for which $A \subseteq u$, for some nigh-open set $u$, then $\exists$ a nigh-open set $V \in \kappa$ for which $A \subseteq V \subseteq \bar{V} \subseteq u$.*

*Proof.* For all $x \in A$, we have $x \in u$. Since $\kappa$ is a regular space, then by the previous theorem, there is a nigh-open set $V_x$ for which $A \subseteq V \subseteq \bar{V} \subseteq u$ (say (1)). Hence, $V = \{V_X : X \in A\}$ is a nigh-open cover of $A$, Consequently, due to $A$ is nigh-compact, then we can reduce $V$ to a finite subcover of $A$, say $\{u_{\Gamma_1}, u_{\Gamma_2}, \ldots, u_{\Gamma_n}$. Therefore, we obtain $A \subseteq \bigcup_{i=1}^{n} u = u$. Now, let $V = \bigcup_{i=1}^{n} V_{\Gamma_i}$, then we have $A \subseteq V \subseteq \bar{V} \subseteq u$. Also, if we let $V = \bigcup_{i=1}^{n} V_{\Gamma_i}$, then we get $A \subseteq V \subseteq \bar{V} \subseteq u$ as required. □

**Theorem 3.18.** *[9] The nigh-compactness property is preserved under onto continuous function.*

*Proof.* Consider $\kappa$ is a nigh-compact space and $f : \kappa \to \gamma$ is onto continuous function. To prove that $\gamma$ is nigh-compact, we first assume $U = \{u_\alpha : \alpha \in \beta\}$ is a nigh-open cover of $\gamma$. Then, $u_\alpha$ is a nigh-open set in $\gamma$, for all $\alpha \in \beta$. Due to $f$ is continuous, then $f^{-1}(U) = \{f^{-1}(u_\alpha) : \alpha \in \beta\}$ is a nigh-open cover of $\kappa$. Since $\kappa$ is nigh-compact, then $f^{-1}(U)$ can be reduced to a finite subcover of $\kappa$. Therefore, we have

$$\gamma = f(\kappa) = f(\bigcup_{i=1}^{n} f^{-1}(u_{\alpha_i})) = \bigcup_{i=1}^{n} u_{\alpha_i}.$$

Hence, $\{u_{\alpha_1}, \ldots, u_{\alpha_n}\}$ is a finite subcover of $U$, which consequently implies that $\gamma$ is nigh-compact. □

**Corollary 3.3.** *The nigh-compactness is a topological property.*

*Proof.* Consider $f : \kappa \to \gamma$ is a homomorphism function and $\kappa$ is a nigh-compact space. Then, $f$ is continuous and onto function. Thus by the above theorem, we can assert that $\gamma$ is nigh-compact, and this proves that the compactness is a topological property. □

**Example 3.5.** *The set $(0, 1)$ is not nigh-compact in $(\mathbb{R}, \eta_u)$.*

*Proof.* Since $(0, 1) \cong (-1, 1)$ by a function $f : (0, 1) \to (-1, 1)$ for which $f(\Gamma) = 2\Gamma - 1$, ($f$ is bijection homeomorphism $f : (-1, 1) \to \mathbb{R}$ for which $f(\Gamma) = \tan(\frac{\pi}{2}\Gamma)$), then by the transitive of the relation $\cong$, we conclude that $(0, 1) \cong \mathbb{R}$. Now, due to $(\mathbb{R}, \eta_u)$ is not nigh-compact, then $(0, 1)$ with the usual topology is not a nigh-compact set. □

**Theorem 3.19.** *[2] Let $f : \kappa \to \gamma$ be a bijective continuous function and $\kappa$ is a nigh-compact $\gamma$ $T_2$-space, then $f$ is a homeomorphism function.*

*Proof.* Consider $F$ is a nigh-closed subset of $\kappa$. Because of $\kappa$ is a nigh-compact space, then by the previous theorem, $F$ is nigh-compact in $\kappa$. Due to $f$ is continuous and onto, then $f$ preserves the compactness property. Hence, $f(F)$ is nigh-compact in $\gamma$, and since $\gamma$ is a $T_2$-space, then $f(F)$ is nigh-closed in $\gamma$. As a result, $f$ is nigh-closed. Consequently, due to $f$ is continuous, then $f$ is a homeomorphism function. □

**Definition 3.9.** *[3] A function $f : \kappa \to \gamma$ is called a perfect function if $f$ is a nigh-closed continuous function and $f^{-1}(y)$ is nigh-compact in $\kappa$, for all $y \in \gamma$.*

**Theorem 3.20.** *[11] If $f : \kappa \to \gamma$ is a perfect function and $\gamma$ is nigh-compact, then $\kappa$ is nigh-compact (i.e., the compactness property is an inverse invert under a perfect function).*

*Proof.* Consider $U = \{u_\alpha : \alpha \in \beta\}$ is a nigh-open set of $\kappa$. Since we have $f^{-1}(y) \subseteq \kappa$, then $U$ is a nigh-open cover of $f^{-1}(y), forally \in \gamma$. But $f^{-1}(y)$ is nigh-compact. Thus, $U$ can be reduced to a finite subcover of $f^{-1}(y)$, say $\{u_\alpha\}_\alpha$, for which $\beta_y \subseteq \beta$ and $\beta_y$ is finite. Hence, we obtain $f^{-1}(y) \subseteq \bigcup_{\alpha\in\beta} u_\alpha$, and so $f^{-1}(Y) \cap (K - \bigcup_{\alpha\in\beta} u_\alpha) = \phi$ is nigh-open in $Y$. Due to $\bigcup_{\alpha\in\beta} u_\alpha$ is a union of finite members of $U$, then $U$ a nigh-open cover of $\kappa$. Thus, $F(\Gamma - \bigcup_{\alpha\in\beta} u_\alpha)$ is nigh-closed. Now, we have for all $y \in \gamma$, $y \in O_y$ for which $O_y$ can be reduced to a finite subcover of $\gamma$, say $\{O_{y_1}, O_{y_2}, \ldots, O_{y_n}\}$. Hence, we obtain $\gamma \subseteq \bigcup_{i=1}^n O_{y_i}$, which implies that

$$k \subseteq f^{-1}(\gamma) = \bigcup_{i=1}^{n} f^{-1}(O_{y_i} = \bigcup_{i=1}^{n} f^{-1}(\gamma - f(k - \bigcup_{i=1}^{n} u_{\alpha_i} = \bigcup_{i=1}^{n} u_{\alpha_i}$$

for which $1 \le i \le n$. Thus, $\{u_{\Gamma_i}\}$ is a subcover of a nigh-open cover $\Gamma$ that covers $\Gamma$, and hence $\Gamma$ is nigh-compact. □

**Definition 3.10.** *For all $y \in \gamma$, the set $\{f^{-1}(y) : y \in \gamma\}$ is called filter of $f$. Hence, the function $f : \Gamma \to \delta$ is called perfect if and only if $f$ is nigh-closed and continuous with nigh-compact filter.*

**Theorem 3.21.** *Let $\kappa$ be a nigh-compact space. Then, the projection $p : \kappa \times \gamma \to \gamma$ is nigh-closed.*

*Proof.* Let $y \in \gamma$ and $O$ be a nigh-open set in $\kappa \times \gamma$ for which $p^{-1}(y) = \kappa \times \{\gamma\} \subseteq O$. By the previous theorem, there is a nigh-open set $v$ in $\gamma$ that contains $y$ for which $f^{-1}(V) \subseteq O$. Due to $O$ is a nigh-open set of $\kappa \times \gamma$, then for all $(x, y) \in \kappa \times \{\gamma\}$, there are two nigh-open basic sets $u_{\kappa\Gamma}$ and $v_{y_\kappa}$ in $\kappa$ and $\gamma$ respectively for which $(x, y) \in u_\kappa \times V_{y_\kappa} \subseteq O$. Hence, $U = \{u_x : x \in \kappa\}$ forms a nigh-open cover of $\kappa$. But $\kappa$ is nigh-compact, so $U$ can be reduced to a finite subcover, say $\{u_{x_1}, u_{x_2}, \ldots, u_{x_n}\}$. Hence, for all $u_{x_i}$, $i = 1, 2, \ldots, n$, there is a corresponding $V_{y_x}$ for which $y \in V$. Therefore, we have $p^{-1}(V) = \kappa \times \{\gamma\} \subseteq u \times V \subseteq O$. As a result, we have $p^{-1}(V) \subseteq O$, which implies that $p$ is nigh-closed. □

**Theorem 3.22.** *Let $\kappa$ and $\gamma$ be two arbitrary spaces and $f : \kappa \to \gamma$ be a function for which $f$ is a nigh-closed subset of $\kappa \times \gamma$. If $B \subseteq \gamma$ is nigh-compact, then $f^{-1}(B)$ is nigh-closed in $\kappa$.*

*Proof.* To show that $f^{-1}(B)$ is nigh-closed in $\kappa$, it is enough to show that $\kappa - f^{-1}(B)$ is open in $\kappa$. To this end, we let $x \in (\kappa - f^{-1}(B))$. Then, we have $f(x) \in (\gamma - B)$, and so we obtain $f(x) \notin B$. This implies $x \notin f^{-1}(B)$ (say (1)). Now, for all $b \in B$, we have $(x, b) \notin f$. Due to $f$ is a nigh-closed subset of $\kappa \times \gamma$, then $(x, b) \in (\kappa \times \gamma) - f$ is a nigh-open set of $\kappa \times \gamma$. Also, since $f$ is a nigh-closed subset of $\kappa \times \gamma$, then there are two nigh-open basic sets $u_b(x)$ and $v(b)$ in $\kappa$ and $\gamma$ respectively for which $(x, b) \in u_b(x) \times v(b) \subseteq (\kappa \times \gamma) - f$. Consequently, for all $z \in \kappa$, we have $(z, f(z)) \notin u_b(x)$, because if this is not hold, then $(z, f(z)) \in (\kappa \times \gamma) - f$. Therefore, we get $(z, f(z)) \notin f$, which is a contradiction. Now, $V = \{v(b) : b \in B\}$ is a nigh-open cover of $B$, and since $B$ is nigh-compact, then it can be reduced to a finite subcover, say $\{v(b_1), v(b_2), \ldots, v(b_n)\}$. Also, for all $v(b_i)$, $i = 1, 2, \ldots, n$, there is a corresponding $u_{b_i}(x)$ for which $x \in U_{b_i}(x)$. In this regard, we let $U(x) = \bigcup_{i=1}^{n} u_{b_i}$, and so $U(x)$ is a nigh-open set in $\kappa$ with $x \in U(x)$. Therefore, we have $U(x) \cap f^{-1}(B) = \phi$, and consequently $x \in u(x) \subseteq \kappa - f^{-1}(B)$. Hence, due to $u(x)$ is nigh-open in $\kappa$, then $\kappa - f^{-1}(B)$ is nigh-open. Thus, $f^{-1}(B)$ is nigh-closed in $\kappa$. □

**Theorem 3.23.** *[7] Let $\kappa$ be an arbitrary space and $\gamma$ be a nigh-compact space. If $f : \kappa \to \gamma$ is a nigh-closed subset of $\kappa \times \gamma$, then $f$ is continuous.*

*Proof.* Let $B$ be a nigh-closed subset in $\gamma$. Due to $\gamma$ is nigh-compact, then by the previous theorem, $B$ is nigh-compact in $\gamma$ (ad any nigh-closed subset of a nigh-compact space is nigh-compact). Therefore, by the previous theorem, $f^{-1}(B)$ is nigh-closed, and hence $f$ is continuous. □

**Theorem 3.24.** *[7] Let $\kappa$ and $\gamma$ be two nigh-compact spaces. Then, $\kappa \times \gamma$ is a nigh-compact space.*

*Proof.* Define a projection function $p_y : \kappa \times \gamma \to \gamma$ as $p_y(x, y) = y$. Then, it is clear that $p_y$ is continuous and surjective function. Since $\kappa$ is nigh-compact, then $p_y$ is a nigh-closed function. Consequently, due to for all $y \in \gamma$, we have $p^{-1}(y) = \kappa \times \{y\} \cong \kappa$ and $\kappa$ is compact, then $p_y$ is a nigh-closed continuous function with nigh-compact fibers. Therefore, $f$ is perfect function, and since $\gamma$ is nigh-compact, then by the previous theorem, $\kappa \times \gamma$ is nigh-compact too (as the compactness property is an inverse inerrant under perfect function). □

3.3. **Nigh-locally compact space.** This subsection describes the so-called nigh-locally compact space and examines its features by presenting some new results and theorems.

**Definition 3.11.** *A topological space $(\kappa, \eta)$ is said to be a nigh-locally compact space if for all $\gamma \in \kappa$, $\exists$ a nigh-open set $u_\gamma$ in $\kappa$ containing $\gamma$ for which $\bar{u_\gamma}$ is nigh-compact.*

**Example 3.6.** *The topological space $(\mathbb{R}, \eta_u)$ is a nigh-locally compact space, but it is not nigh-compact.*

*Proof.* To prove this example, we should consider the following states:

- Observe that $\mathbb{R}$ is a nigh-locally compact space as for all $\Gamma \in \mathbb{R}$, we have open intervals $u = (\Gamma - 1, \Gamma + 1)$ for which $\bar{u} = [\Gamma - 1, \Gamma + 1]$ is nigh-compact.
- On the other hand, one may notice that $\mathbb{R}$ is not nigh-compact, as there is a nigh-open cover $\varphi_n = \{(-n, n) : n \in \mathbb{N}\}$ of $\mathbb{R}$ that has no finite subcover, because if this is not true (i.e., $\varphi_n$ has a finite subcover of $\mathbb{R}$, say $\varrho$), then $\varrho$ forms $\sigma = \{(-n_1, n_1), (-n_2, n_2), \ldots, (-n_k, n_k)\}$, and thus $\mathbb{R} = \bigcup_{i=1}^{n}(-n, n)$.

□

**Theorem 3.25.** *Every nigh-locally compact $T_2$-space is Tychonoff space.*

**Theorem 3.26.** *Let $\kappa$ be a nigh-locally compact $T_2$-space and $A$ be a nigh-compact subset of $\kappa$. If $A \subseteq V$ for which $V$ is a nigh-open set in $\kappa$, then $\exists$ a nigh-open set $u$ in $\kappa$ for which $A \subseteq u \subseteq \bar{u} \subseteq V$ and $\bar{u}$ is nigh-compact in $\kappa$.*

*Proof.* Assume that $\kappa$ is a nigh-locally compact $T_2$-space. Then, $\kappa$ is a Tychonoff space, and so it is a $T_3$-space. Consequently, $\kappa$ is a regular space. Thus, for all $x \in A$, we have $x \in v$. Consequently, since $V$ is an open set, then $\exists$ a nigh-open set $V_X$ in $\kappa$ for which $x \in v_x \subseteq \bar{v_x} \subseteq V$. Now, due to $\kappa$ is a nigh-locally compact space, then there is a nigh-open set $w_x$ for which $x \in w_x$ for which $\bar{w_x}$ is nigh-compact. Thus, we have $x \in v_x \cap w_x = u_x$, which is a nigh-open set in $\kappa$. Hence, $\{u_x : x \in A\}$ forms a nigh-open cover of $A$. As a result, since $A$ is nigh-compact, then $A \subseteq \bigcup_{i=1}^{n}(u_{x_i}) = u$ for which $u$ is a nigh-open set in $\kappa$. Therefore, we obtain $u_{x_i} = v_{x_i} \cap w_{x_i}$, for all $i = 1, 2, \ldots, n$. Thus, we have $\bar{u_{x_i}} \subseteq \bar{w_{x_i}}$. Now, due to every nigh-closed set in a nigh-compact space is nigh-compact, then $u_{x_i}$ is nigh-compact as $\bar{w_{x_i}}$. Because of any finite union of a nigh-compact space is nigh-compact, then $\bigcup_{i=1}^{n}(\bar{u_{x_i}} = \bigcup_{i=1}^{n}(u_{x_i}) = \bar{u}$ is compact. Hence, we have $A \subseteq u \subseteq \bar{u}$. But $u_{x_i} = v_{x_i} \cap w_{x_i}$, and so $\bar{u_{x_i}} \subseteq \bar{v_{x_i}}$. Therefore, $u_{x_i} \bar{\subseteq} v_{x_i}$, for all $i = 1, 2, \ldots, n$. Thus, we get

$$\bar{u} = \overline{\bigcup_{i=1}^{n}(u_{x_i})} = \bigcup_{i=1}^{n}(\bar{u_{x_i}}) \subseteq \bigcup_{i=1}^{n}(\bar{v_{x_i}}) \subseteq V,$$

as $\bar{v_{x_i}} \subseteq V$. □

**Theorem 3.27.** *Let $\kappa$ be a nigh-locally compact space. Then, any subspace of the form $f \cap V$ of $\kappa$ is nigh-locally compact in $\kappa$, where $f$ is nigh-closed and $V$ is a nigh-open set in $\kappa$.*

*Proof.* To prove this result, we should consider the following states:

- To show that the nigh-locally compactness is hereditary with respect to a nigh-open set, we let $V$ be a nigh-open set in a nigh-locally compact space $\kappa$. Now, to show that $V$ is nigh-locally compact, we let $x \in v$. Then, we have $x \in \kappa$, and since $\kappa$ is a $T_3$-space, then it is regular. Also, since $V$ is a nigh-open set, then $\exists$ a nigh-open set $u$ in $\kappa$ for which $x \in u \subseteq \bar{u} \subseteq v$. Consequently, since $\kappa$ is a nigh-locally compact space, then $\exists$ a nigh-open set $W$ in $\kappa$ for which $x \in W$ for which $\bar{W}$ is nigh-compact. Thus, we have $x \in u \cap W = M$, where $M$ is a nigh-open set in $V$. Due to $M \subseteq u$ and $u \subseteq V$, then $M \subseteq V$ and $M \subseteq W$. Thus, we have $\bar{M} \subseteq \bar{W}$. Consequently, due to $\bar{W}$ is nigh-compact for which $\bar{M}$ is nigh-closed in $\bar{W}$, then $\bar{M}$ is nigh-compact. Thus, for all $x \in V$, $\exists$ an open set $M$ containing $x$ for which $\bar{M}$ is nigh-compact. Therefore, $V$ is nigh-locally compact.
- To show that the locally compactness is hereditary with respect to the closed set, we let $F$ be a nigh-closed set in a nigh-locally compact space $\kappa$. Also, we let $x \in F$, then $x \in \kappa$. Due to $\kappa$ is nigh-locally compact, then $F \cap W$ and $F \bar{\cap} W^F = F \bar{\cap} W \cap F = F \cap W \subseteq \bar{W}$. So, $\exists$ a nigh-open set $F \cap W$ for which $F \cap W^F$ is nigh-compact. Therefore, $F$ is nigh-locally compact.
- To show that every subspace $F \cap V$ of a nigh-locally compact space is nigh-locally compact, where $F$ is a nigh-closed set and $V$ is a nigh-open set in $\kappa$, we should note that due to $F$ is nigh-locally compact. Then, and by the previous part, we observe that the nigh-locally compactness is hereditary with respect to nigh open sets. Therefore, $F \cap V$ is nigh-locally compact in $F$, and so it is in $\kappa$.

□

**Theorem 3.28.** *Every nigh-locally compact dense subspace M of a$T_2$-space κ is nigh-open in $\bar{M}$, (i.e., every nigh-locally compact subspace M of a $T_2$-space κ can be represented as $M = F \cap V$, where F is nigh-closed and V is nigh-open in κ).*

*Proof.* To prove this result, it is enough to show that every nigh-locally compact dense subspace $M$ of $\kappa$ is nigh-open in $M = k$. To this end, we let $x \in M$. Now, due to $M$ is nigh-locally compact, then $\exists$ a nigh-open set $u$ in $M$ for which $x \in u$ for which $u \in M$ is nigh-compact in $M$, and so it is nigh-compact in $\kappa$. Now, since $\kappa$ is a $T_2$-space, then by the fact that asserts every nigh-compact subset of $T_2$-space is nigh-closed, we have $\bar{u} \in M$ is nigh-closed in $\kappa$ for which $u \subseteq \bar{u}$ for which $u \subseteq M$. Thus, we have $u \subseteq \bar{u} \cap M$ (say (1)). In the same regard, due to $u$ is a nigh-open set in $M$, then $\exists$ a nigh open $W$ in $\kappa$ for which $u = W \cap M$. But $x \in u$, and so $x \in W$. This implies that $x \in W \subseteq \bar{W} = W \bar{\cap} \kappa = W \bar{\cap} \bar{M} = W \bar{\cap} M$ (since $W$ is a nigh-open set in $\kappa$). Thus, by the previous theorem that says if $T$ nigh-open in $\kappa$, then $T \cap \bar{A} = T \bar{\cap} A$, for all $A \subseteq \kappa$), we have $x \in W \subseteq M$. Also, since $W$ is nigh-open in $\kappa$, then $M$ is nigh-open in $k = \bar{M}$, which completes the proof of this result. □

**Corollary 3.4.** *Every subspace M of a nigh-locally compact space κ is nigh-locally compact if and only if M can be written as $M = V \cap F$, where V is nigh-open and F is nigh-closed in κ.*

**Corollary 3.5.** *A space κ is nigh-locally compact if and only if it is homomorphic to a nigh-open subspace of a nigh-compact space.*

*Proof.* ⇒) Suppose $\kappa$ is a nigh-locally compact space. Then, $\kappa$ is Tychonoff. As every Tychonoff space is emendable in a nigh-compact space (i.e., every Tychonoff space is homomorphic to a nigh-open subspace in a nigh compact space), then $\kappa$ is homeomorphic to a nigh-open subspace in a nigh compact space.
⇐) Suppose that $\kappa$ is homeomorphic to a nigh-open subspace in a nigh compact space. Then, it is clear that $\kappa$ is a nigh-locally compact space. □

**Theorem 3.29.** *Let $f : \kappa \to \gamma$ be a continuous nigh-open function and γ be $T_2$-space. If κ is a nigh-locally compact space, then γ is nigh-locally compact.*

*Proof.* Let $y \in \gamma$ and $\Gamma \in f^{-1}(y)$. Then, we have $\Gamma \in \kappa$. Now, due to $\kappa$ is a nigh-locally compact space, then $\exists$ a nigh-open set $u$ in $\kappa$ for which $\Gamma \in u$ for which $\bar{u}$ is nigh-compact. Consequently, as $f : \kappa \to \gamma$ is nigh-open, then $f(U)$ is also nigh-open in $\gamma$ for which $y \in f(U)$ is continuous in $\gamma$. In the same regard, due to the compactness is preserved under continuity, then $f(\bar{u})$ in $\gamma$. Consequently, due to $\gamma$ is a $T_2$-space, then $f(\bar{u})$ is nigh-closed in $\gamma$. Thus, we obtain $f(\bar{u}) = f(\bar{\bar{u}})$. But $u \subseteq \bar{u}$, and so $f(u) \subseteq (\bar{u})$. Hence, $f(\bar{u}) \subseteq barf(\bar{u} = f(\bar{u}$ is nigh-compact in $\gamma$ for which $f(\bar{u})$ is nigh-compact. Therefore, for all $y \in \gamma$, we have $\Gamma$ is a nigh-open set for which $f(U)$ in $\gamma$ and $f(\bar{u})$ is nigh-compact. Hence, $\gamma$ is nigh-locally compact. □

**Definition 3.12.** *A space $\kappa$ is called $\kappa$-space if for all nigh-closed (open) subset in $\kappa$, we have $A \cap Z$ is a subset in every nigh compact subspace $Z$ of $\kappa$.*

**Theorem 3.30.** *Every nigh-locally compact space is a $\kappa$-space.*

*Proof.* Suppose that $\kappa$ is a nigh-locally compact space and $A \cap Z$ is an open set in every nigh-compact subset $Z$ of $\kappa$. To show that $\kappa$ is $\kappa$-space, it is enough to show that $A$ is a nigh-open set in $\kappa$. For this purpose, we let $a \in A$, then we have $a \in \kappa$. Now, since $\kappa$ is a nigh-locally compact space, then $\exists$ a nigh-open set $V$ in $\kappa$ for which $a \in V$ for which $\bar{V}$ is nigh-compact. Hence, we obtain $A \in \bar{V}$, and hence it is nigh-open in $\kappa$. But $A \cap V = A \cap (\bar{V} \cap V) = (A \cap V) \cap V$ is nigh-open in $\kappa$. Therefore, we have $a \in A, a \in V$, and so $a \in (A \cap V) \subseteq A$. This implies that $A$ is a nigh-open set, and hence $\kappa$ is $\kappa$-space. □

**Definition 3.13.**
- *A space $\kappa$ is called Frechet space if and only if for all $A \subseteq \kappa$ and for all $a \in \bar{A}$, $\exists$ $a_n \in A$ for which $a_n \to a$.*
- *A space $\kappa$ is called sequential space if and only if a subset $A$ of $\kappa$ is nigh-closed if and only if for all $(a_n) \in A$ for which $A$ contains its limit.*

**Theorem 3.31.** *Every first countable space is a $\kappa$-space.*

*Proof.* Suppose that $\kappa$ is a first countable space. Let $A \subseteq Z$ be nigh-closed in every nigh-compact subset $Z$ of $\kappa$. To show that $\kappa$ is a $\kappa$-space, it is enough to show that $A$ is nigh-closed in $\kappa$. For this goal, we let $a \in \bar{A}$. Now, due to every first countable space is a Frechet space, then $\exists$ $(a_n) \in A$ for which $a_n \to a$. Hence, $a_n \cup a$ is a nigh-compact subset of $\kappa$. Thus, $A \cap ((a_n) \cup a)$ is closed in $(a_n) \cup a$, and so $a \in A$. This implies $\bar{A} \subseteq A$. But $A \subseteq \bar{A}$, and so $\bar{A} = A$. Therefore, $A$ is a nigh-closed set, which means that $\kappa$ is $\kappa$-space. □

**Theorem 3.32.** *Every Frechet space is a sequential space.*

*Proof.* Suppose that $\kappa$ is a Frechet space. To show that $\kappa$ is a sequential space, it is enough to show that $A$ is a nigh-closed set in $\kappa$ if and only if for all $(a_n) \in A$, we have $(a_n) \to a$, which implies $a \in A$. So we have the following states: $\to$) Suppose that $A$ is a nigh-closed set in $\kappa$ and $(a_n) \in A$ for which $a_n \to a$. Then, for all nigh-open set $u$ in $\kappa$ containing $a$, we have $\Gamma m \in \mathbb{N}$ for which $a_n \in u$, for all $n \geq m$. Thus, we have $a_n \in u$ for which $a_n \in A$, for all $n \geq m$. This implies that $A \cap u \neq \phi$, and since $a \in u$ for which $u$ is a nigh-open set in $\kappa$, then $a \in \bar{A} = A$ as $A$ is nigh-closed.
$\leftarrow$) Suppose that the condition here is hold. To show that $A$ is a nigh-closed, we let $a \in \bar{A}$. Now, since $\kappa$ is a Frechet space, then $\exists$ $a_n \in A$ for which $a_n \in A$ for which $a_n \to a$. Hence, by the condition assumed, we conclude that $a \in A$, and so $\bar{A} \subseteq A$. But $A \subseteq \bar{A}$, and so $\bar{A} = A$. Hence, $A$ is a nigh-closed set. □

**Theorem 3.33.** *Every $T_2$-sequential space is a $\kappa - space$.*

*Proof.* Suppose that $\kappa$ is a $T_2$-sequential space. To show that $\kappa$ is a $\gamma$-space, we suppose that $A \cap Z$ is a nigh-closed set in every nigh-compact subset $Z$. Also, to show that $\kappa$ is a $\gamma$-space, it is enough to show that $A$ is a nigh-closed set. For this purpose, we assume not! (i.e., we assume that $A$ is not a nigh-closed set). Now, since $\kappa$ is a sequential space, then $\exists\ (a_n) \in A$ for which $a_n \to a$ and $a \notin A$. Thus, if we assume that $z = \{a, a_1, a_2, \ldots, a_n\} = (a_n) \cup \{a\}$, then $Z$ is nigh-compact. Consequently, we have $A \cap ((a_n) \cup \{a\})$, which implies that $z$ is nigh-closed (say (1)). In the same regard, due to $\kappa$ is a $T_2$-space, then $Z \cong A(w_0)$. Therefore, $a$ is a unique cluster point of $(a_n)$. As a result, since $a \notin A$, we obtain $A \cap Z = A \cap (a_n) \cup \{a\}$, which is not a nigh-closed set, and this is a contradiction with (1). Hence, $A$ is a nigh-closed set, which implies that $\kappa$ is a $\gamma$-space. □

**Corollary 3.6.** *Let $\kappa$ be a sequential space. Then, $f : \kappa \to Y$ is continuous function if and only if $f(\lim x_i) \subseteq \lim f(x_i)$.*

*Proof.* Suppose that $f : \kappa \to Y$ is a continuous function and $x \in \lim f(x_i)$. To show that $f(\lim x_i) \subseteq \lim f(x_i)$, it is enough to show that $f(x) \in \lim f(x_i)$. For this goal, we let $V$ be a nigh-open set $\gamma$ for which $f(x) \in V$. Then, $x \in f^{-1}(V)$ is a nigh-open set in $\kappa$. Due to $f$ is continuous and $x \in \lim f(x_i)$, then $\exists\, m \in \mathbb{N}$ for which $x_i \in f^{-1}(V)$, for all $i \neq m$. Also, since $f(x) \in V$ for which $V$ is a nigh-open set in $\gamma$, then $f(x_i) \to f(x)$. Hence, we have $f(x) \in \lim f(x_i)$, and so $f(\lim x_i) \subseteq \lim f(x_i)$. Conversely, we suppose that $f(\lim x_i) \subseteq lim f(x_i)$. Here, we want to show that $f : \kappa \to Y$ is continuous. To this end, we let $A$ be a nigh-closed set in $\gamma$. So, it is enough to show that $f^{-1}(A)$ is nigh-closed in $\kappa$. Now, by using the definition of the sequential space that says "A space $\kappa$ is sequential if $T$ is nigh-closed in $\kappa$ if and only if for all $(t_n) \in T, t_n \to t$, we have $T \in t$", we let $x_i \subseteq f^{-1}(A) and x_i \to x$ (i.e., $x \in lim x_i$). Thus, we have the following claim:
claim: $x \in f^{-1}(A)$.
To prove this claim, we should note that $x \in \lim x_i$. Then, by the assumption $f(x_i) \to f(x)$, we have $f(x) \in f(\lim x_i) \subseteq \lim f(x_i)$. So, if we assume that $u$ is nigh-open in $\gamma$, we obtain $f(x) \in u$. Consequently, $\exists\ m \in \mathbb{N}$ for which $f(x_i) \in u$, for all $i \geq m$. But $x_i \subseteq f^{-1}(A)$, and so $f(x_i) \in A$, $\forall i = 1, 2, \ldots$. This implies $u \cap A \neq \phi$, and since $f(x) \in u$ for which $u$ is a nigh-open set in $\gamma$, then $f(x) \in \bar{A} = A$. Now, due to $A$ is nigh-closed, then we have $x \in f^{-1}(A)$. As a result, since $\kappa$ is a sequential space, then $f^{-1}(A)$ is nigh-closed in $\kappa$. Therefore, $f$ is continuous. □

**Corollary 3.7.** *If every sequence in $\kappa$ has at most one limit, then $\kappa$ is a $T_1$-space. Moreover, if $\kappa$ is a first countable space, then $\kappa$ is a $T_2$-space.*

*Proof.* Assume that $\kappa$ is not a $T_1$-space. Then, if we assume that $x \neq y$ implies that any open set $u$ can contain $x, y \in u$, we infer that $y_n = y$, for $n = 1, 2, \ldots$. Therefore, we have $(y_n) \to y$, and so for all nigh-open set $u$ containing $y$, we have $x \in u$. Therefore, we have $y_n \to x$. Consequently, due to every sequence in $\kappa$ has at most one limit, then we have $x = y$, which is a contradiction with $x \neq y$. Hence, $\kappa$ is $T_1$-space.

□

## 4. Conclusion

This study has introduced the concept of a topological space that is nigh-locally compact. The definitions of "nigh-compact space" and "nigh-topological space," as well as a number of other deductions and theorems, have been presented in order to accomplish that goal.

**Conflicts of Interest:** The author declares that there are no conflicts of interest regarding the publication of this paper.

## References

[1] R. Alharbi, J. Oudetallah, M. Shatnawi, I.M. Batiha, On c-Compactness in Topological and Bitopological Spaces, Mathematics 11 (2023), 4251. https://doi.org/10.3390/math11204251.

[2] J. Oudetallah, N. Abu-Alkishik, I.M. Batiha, Nigh-Open Sets in Topological Space, Int. J. Anal. Appl. 21 (2023), 83. https://doi.org/10.28924/2291-8639-21-2023-83.

[3] J. Oudetallah, R. Alharbi, I.M. Batiha, On r-Compactness in Topological and Bitopological Spaces, Axioms 12 (2023), 210. https://doi.org/10.3390/axioms12020210.

[4] J. Oudetallah, I.M. Batiha, On Almost Expandability in Bitopological Spaces, Int. J. Open Probl. Comp. Sci. Math. 14 (2021), 43–48.

[5] J. Oudetallah, M.M. Rousan, I.M. Batiha, On D-Metacompactness in Topological Spaces, J. Appl. Math. Inf. 39 (2021), 919–926. https://doi.org/10.14317/JAMI.2021.919.

[6] A.A. Hnaif, A.A. Tamimi, A.M. Abdalla, I. Jebril, A Fault-Handling Method for the Hamiltonian Cycle in the Hypercube Topology, Comp. Mater. Cont. 68 (2021), 505–519. https://doi.org/10.32604/cmc.2021.016123.

[7] J.C. Kelly, Bitopological Spaces, Proc. London Math. Soc. s3-13 (1963), 71–89. https://doi.org/10.1112/plms/s3-13.1.71.

[8] T.C.K. Raman, On Generalized *b*-Closed Sets In Bitopological Spaces, Int. J. Sci. Eng. Techn. 5 (2016), 124–127.

[9] N. Levine, Semi-Open Sets and Semi-Continuity in Topological Spaces, The Amer. Math. Mon. 70 (1963), 36–41. https://doi.org/10.1080/00029890.1963.11990039.

[10] M.C. Datta, Projective Bitopological Spaces, J. Aust. Math. Soc. 13 (1972), 327–334. https://doi.org/10.1017/s1446788700013744.

[11] J. Oudetallah, Nearly Expandability in Bitopological Spaces, Adv. Math., Sci. J. 10 (2020), 705–712. https://doi.org/10.37418/amsj.10.2.2.

[12] P. Fletcher, H.B. Hoyle, III, C.W. Patty, The Comparison of Topologies, Duke Math. J. 36 (1969), 325–331. https://doi.org/10.1215/s0012-7094-69-03641-2.

[13] Y.W. Kim, Pairwise Compactness, Publ. Math. Debrecen 15 (2022), 87–90. https://doi.org/10.5486/pmd.1968.15.1-4.12.

[14] I.L. Reilly, Bitopological Local Compactness, Indag. Math. (Proceedings) 75 (1972), 407–411. https://doi.org/10.1016/1385-7258(72)90037-6.

[15] J. Swart, Total Disconnectedness in Bitopological Spaces and Product Bitopological Spaces, Indag. Math. (Proceedings) 74 (1971), 135–145. https://doi.org/10.1016/s1385-7258(71)80020-3.