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Novel Results on Nigh Lindelöfness in Topological Spaces

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Abstract. The main objective of this research paper is to introduce the concept of nigh Lindelöfness in topological spaces and nigh topological spaces. This has led us to establish several generalizations and properties of nigh Lindelöf space that are related to the nearly nigh Lindelöf space, the nigh compactness in topological spaces, and their relations with other spaces. Several examples are discussed, and many well-known theorems are generalized concerning the nigh Lindelöf spaces.

1. Introduction

The Greek word topology means "the study of place". Although this well-defined field of mathematics first appeared in the early 20th century, it has taken many decades before any distinct conclusions were discovered [1–4]. Leonhard Euler is credited with being the first to apply topology in practice when he published his first work on the Seven Bridges of Königsberg in 1736. Augustin-Louis Cauchy, Johann Benedict Listing, Enrico Betti, Ludwig Schlafly, and Bernhard Riemann all contributed extra. Listing first used the term "topology" in his 1847 German work Vorstudien zur Topology. Listing's Funerary in the journal Nature in 1883 distinguished between "qualitative geometry and ordinary geometry" using the English term "topology."

Open cover is crucial in establishing a new class of sets in compact spaces, as well as certain key topological properties of these new ideas [5, 6]. Tong defined and explored a new compact space known as the nigh-compact space in 1982 by introducing the idea of the nigh sets employing open sets. Later, in 1997, Caldas defined the notions of s-compact space using semi-open sets. We obtain the implications of these new s-compact spaces with each other and with the standard s-compact space. In 2001, Jafari defined p-compact space using p-open sets and provided an introduction to

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the ideas involved. It is obtained the relation between various kinds of tiny spaces. Using the concept of α-open sets, the topologists Caldas et al. defined α-compact space in 2003.

This study aims to present the notion of nigh Lindelöfness in nigh topological spaces and topological spaces. This has allowed us to develop a number of features and generalizations of nigh Lindelöf space that have connections to nigh compactness in topological spaces, almost nigh Lindelöf space, and their relationships with other spaces. In relation to the near Lindelöf spaces, several well-known theorems are generalized and several examples are explored. The organization of this paper is given as follows: In the second section, we study the compact space as well as the locally compact space, studying their main theorems and examples of them. In section three, we study nigh Lindelöf space, discuss theorems and examples on it, and study the relationship between it and Lindelöf topological space. In section four, we include some theorems and several examples of nigh-nearly Lindelöf space and study the relation between it and nigh-Lindelöf space. Finally, we provide the conclusion of this work in the last section.

2. Preliminaries

We list the most crucial concepts and foundational theorems required for our investigation in this section. We also extract some additional useful results to finish our investigation.

Definition 2.1. *[7] Let $\xi = (\xi, \mu)$ be a topological space and $V = \{v_\mu : w_\mu \subset \xi\}$ be a family of subset of ξ. Then ξ is called a cover of ξ if $\xi = \bigcup \mu \in \xi v_\mu$. Also, we have:*

- *If v_μ is an open set in ξ for all $\mu \in \xi$ and $\xi = \bigcup \mu v_\xi$, then v is called open cover of ξ.*
- *If $\mu \in v$ is a cover of ξ, then μ is called subcover of ξ.*

Definition 2.2. *A topological space $\xi = (\xi, \mu)$ is called compact if and only if every open cover of ξ has a finite cover. Also, if $v = \{v_\mu : \mu \in T, w_\mu \subset \xi\}$ be an open cover of ξ, then ξ is compact.*

Theorem 2.1. *[8] Any closed bounded subset of $\mathbb{R}$ is compact.*

Definition 2.3. *[8] Given a topological space $\xi = (\xi, \mu)$ and a family of subsets A of k, we say that A has a finite intersection property (f.i.p) if and only if the intersection of the set A with a finite number of members is not empty.*

Theorem 2.2. *[8] Let $\xi = (\xi, \mu)$ be a topological space. Then ξ is compact if and only if every family of closed subset of ξ with (f.i.p) has non empty intersection.*

Remark 2.1. *Let (W, μ_w) be a subspace of $\xi = (\xi, \mu)$. We say (W, μ_w) is compact if and only if every open cover of w has a finite sub cover of w with respect to T_w. Observe that (W, T_w) is compact if and only if every open cover of W has finite subcover in μ.*

Theorem 2.3. *[9] Let W be a compact subset of T_2-space ξ. Then for all $x \notin W$, we can separate x and W in two disjoint open sets.*

Theorem 2.4. *[9] Let A and B be two disjoint compact subsets of a T_2-space ξ. Then we can separate A and B in two disjoint open sets in ξ.*

Theorem 2.5. *[10] Every compact T_2-space is T_4-space.*

Theorem 2.6. *[10] Every compact subset of T_2-space is closed.*

Theorem 2.7. *[9] Every subset in compact T_2-space is compact if and only if it is closed.*

Theorem 2.8. *[9] The compactness property is preserved under onto continuous function.*

Corollary 2.1. *[9] Compactness is topological property.*

Theorem 2.9. *[11] Let $f : \xi \to \gamma$ be a bijective continuous function and ξ be compact space and γ is a T_2-space. Then f is a homeomorphism function.*

Definition 2.4. *[11] The function $f : \xi \to \gamma$ is called perfect function if f is closed continuous function and for all $y \in \gamma$, $f^{-1}(y)$ is compact in ξ.*

Theorem 2.10. *[11] If the function $f : \xi \to \gamma$ is a perfect function and γ is compact, then ξ is compact (i.e. the compactness property is an inverse invert under perfect functions).*

Definition 2.5. *For all $y \in \gamma$, the set $\{f^{-1}(y) : y \in \gamma\}$ is called fibers of f. Hence, a function $f : \Gamma \to \delta$ is called perfect if and only if f is closed and continuous with compact fibers.*

Theorem 2.11. *Let ξ be a compact space. Then the projection $p : \xi \times \gamma \to \gamma$ is closed.*

Theorem 2.12. *[12] Let ξ and γ be arbitrary spaces and $f : \xi \to \gamma$ be a function such that f is closed subset of $\xi \times \gamma$. If $B \subseteq \gamma$ is compact, then $f^{-1}(B)$ is closed in ξ.*

Theorem 2.13. *[7] Let ξ be an arbitrary space and γ be a compact space. If $f : \xi \to \gamma$ is closed subset of $\xi \times \gamma$, then f is continuous.*

Theorem 2.14. *[7] Let ξ and γ be two compact spaces, then $\xi \times \gamma$ is compact.*

Definition 2.6. *[13] The locally compact space ξ is defined as for all $\gamma \in \xi$, there exists an open set u_γ in Γ containing γ such that $\bar{u}$ is compact.*

Theorem 2.15. *Every locally compact T_2-space is Tychonoff space.*

Theorem 2.16. *Let ξ be locally a compact space and A be a compact subset of ξ. If $A \subseteq V$ for which V open in ξ, then there exists an open set u in ξ such that $A \subseteq u \subseteq \bar{u} \subseteq V$ and $\bar{u}$ compact in ξ.*

Theorem 2.17. *[14] Let ξ be a locally compact space. Then any subspace of the form $f \cap V$ of ξ is locally compact in Γ, where f is closed and V open in ξ.*

Theorem 2.18. *Every locally compact subspace M of a T_2-space ξ is open in $\bar{M}$.*

3. Nigh Lindelöf Spaces

In this section, we aim to introduce the concept of nigh Lindelöf spaces, talk about other concepts that relate to the concept of nigh Lindelöf spaces, and discuss some theories related to this concept.

Definition 3.1. *A topological space $\xi = (\xi, \mu)$ is called a 2^{nd}-countable space if ξ has a countable base.*

Theorem 3.1. *If a topological space $\xi = (\xi, \mu)$ is 2^{nd}-countable space, then it is nigh Lindelöf space.*

Proof. Let ξ be a nigh 2^{nd}-countable space. Then ξ has a countable base $\underset{\sim}{B}$, say $\underset{\sim}{B} = \{B_i\}_{i=1}^{\infty}$. Let $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ be an open cover of ξ. Then u_α=union of same members of $\underset{\sim}{B}$. Now, since $\underset{\sim}{B}$ is a base of ξ, then u_α itself is a union of members of $\underset{\sim}{B}$. This union of $\underset{\sim}{B}$ is countable subcover of $\underset{\sim}{U}$, which covers ξ. Therefore, ξ is nigh Lindelöf. □

Remark 3.1. *Every nigh compact space is a nigh Lindelöf space, but the converse need not necessarily be true.*

Proof. Let ξ be a nigh compact space. Then every open cover of ξ has a finite subcover of ξ. Thus, each open cover of ξ has countable subcover of ξ. So, ξ is a nigh Lindelöf. On the other hand, to prove that the converse need not be true, we may note that $(\mathbb{R}, \tau_u)$ is a nigh Lindelöf space, which is not nigh compact. Now, since $\underset{\sim}{B} = \{(a, b) : a < b \ in \ \mathbb{Q}\}$ is a countable base of $\mathbb{R}$, then $\mathbb{R}$ is a second countable. Hence, due to each second countable space is Lindelöf, then $(\mathbb{R}, \tau_u)$ is nigh Lindelöf. □

Theorem 3.2. *A nigh Lindelöfness is preserved under onto nigh continuous function.*

Proof. Let $f : (\xi, \mu) \to (Y, \zeta)$ be a surjective continuous function, and ξ be a Lindelöf space. To show that Y is Lindelöf space, we assume that $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is an open cover of Y. Then, u_α is open for each $\alpha \in \Lambda$. Now, since f is continuous, then $f^{-1}(u_\alpha)$ is open in ξ, for each $\alpha \in \Lambda$. As a result, since f is surjective, then $f^{-1}(u_\alpha) = \{f^{-1}(u_\alpha)\} : \alpha \in \Lambda$ is an open cover of ξ. But, ξ is Lindelöf space, and so we can reduce $f^{-1}(u_\alpha)$ to a countable subcover, say $\{f^{-1}(u_\alpha) : \alpha \in \Lambda\}$, where $\Gamma \subset \Lambda$ and $|\Gamma| \leq w_0 = |\mathbb{N}|$. Hence, we have $\xi \subseteq \underset{\alpha \in \Gamma}{\cup} f^{-1}(; u_\alpha)$. Consequently, since f is onto, then $Y = f(\xi) \subseteq f(\underset{\alpha \in \Gamma}{\cup} f^{-1}(u_\alpha)) \subseteq \underset{\alpha \in \Gamma}{\cup} u_\alpha$. Therefore, $\underset{\sim}{\cup}$ has a countable subcover of Y, which implies that Y is Lindelöf space. □

Remark 3.2. *Every compact subset in a nigh Hausdorff space is closed, but a Lindelöf subset in a nigh Hausdorff space need not be closed. For example, $(\mathbb{R}, \tau_u)$ is a nigh Hausdorff space and $(0, 1)$ is Lindelöf subset of $\mathbb{R}$. However, $(0, 1)$ is not nigh closed in $\mathbb{R}$.*

Definition 3.2. *A space (ξ, μ) is called nigh space if and only if a countable intersection of open sets is open.*

Theorem 3.3. *Every Lindelöf subset of a T_2-space is nigh closed.*

Proof. Let A be a Lindelöf subset of nigh T_2-space ξ. To show that A is a nigh closed set, it is enough to show that $\xi\backslash A$ is a nigh open set. For this purpose, we let $x \in \xi\backslash A$. Then, $x \notin \xi$, and so for each $a \in A$, we have $x \neq a$. Now, since ξ is T_2-space, then there exist two open sets u_a and v_a in ξ such that $x \in u_a$ and $a \in v_a$ with $u_a \cap v_a = \phi$. Hence, $\underset{\sim}{V} = \{v_a : a \in A\}$ forms an open cover of A, and since A is nigh Lindelöf subset of ξ, then $\underset{\sim}{V}$ can be reduced to a countable subcover, say $\{v_{a_\alpha}\}_{\alpha\in\Lambda_0}$. Hence, $A \subseteq \underset{\alpha\in\Lambda_0}{\cup} v_{a_\alpha}$. So, for each v_{a_α} and $\alpha \in \Lambda_0$, there is a corresponding u_{a_α} such that $x \in u_{a_\alpha}$ and $a \in v_{a_\alpha}$ with $u_{a_\alpha} \cap v_{a_\alpha} = \phi$. Now, let $v^* = \underset{\alpha\in\Lambda_0}{\cup} v_{a_\alpha}$ and $u^* = \underset{\alpha\in\Lambda_0}{\cap} u_{a_\alpha}$. Then, we have $A \subseteq v^*$ and $x \in u^*$ with $u^* \cap v^* \subseteq u_{a_\alpha} \cap v_{a_\alpha} = \phi$ for all $\alpha \in \Lambda_0$. Hence, $u^* \cap v^* = \phi$, which implies $u^* \cap \underset{\alpha\in\Lambda_0}{\cup} v_{a_\alpha} = \phi$. Therefore, $u^* \cap v^* = \phi$. Now, since $A \subseteq v^*$, then $u^* \cap A \subseteq u^* \cap v^* = \phi$. So, we have $u^* \cap v^* = \phi$. Consequently, we conclude that $x \in u^* \subset x - A$, where $u^* = \underset{\alpha\in\Lambda_0}{\cap} u_{a_\alpha}$ is open, since ξ is nigh space. Therefore, $x - A$ is nigh open, and hence A is nigh closed. □

Theorem 3.4. *If A is a Lindelöf subset of a nigh Hausdorff space ξ. Then, for each $x \notin A$, we can separated x and A in two disjoint open sets in ξ.*

Proof. For each $a \in A$, we have $a \neq x$, since $x \notin A$. Due to ξ is a Hausdorff space, then there exist nigh open sets $u_a(x)$ and $v(a)$ in ξ such that $x \in u_a(x)$ and $a \in v(a)$ with $u_a(x) \cap v(a) = \phi$. Hence, $\underset{\sim}{V} = \{v(a) :\ a \in A\}$ forms an open cover of A, and since A is a nigh Lindelöf subset of ξ, then $\underset{\sim}{V}$ can be reduced to a countable subcover of A, say $\underset{\sim}{V} = \{v(a_\alpha) : \alpha \in \Lambda_0\}$ $A \subseteq \underset{\alpha\in\Lambda}{\cup} v(a_\alpha) = V$. For all $v(a_\alpha)$ and $\alpha \in \Lambda_0$, there is corresponding nigh open sets $u_{a_\alpha}(x)$ containing ξ such that $u_{a_\alpha}(x) \cap v(a_\alpha) = \phi$. Now, let $u = \cap u_{a_\alpha}(x)$. Then, u is open set because ξ is nigh space. Consequently, $u \subseteq u_{a_\alpha}(x)$, for all $\alpha \in \Lambda_0$. So, we have $u \cap v(a_\alpha) \subseteq u_{a_\alpha} \cap v(a_\alpha) = \phi$. Hence, we obtain $u \cap v(a_\alpha) = \phi$. Thus, $u \cap \underset{\alpha\in\Lambda_0}{\cup} v(a_\alpha) = \phi$, which implies that $u \cap v = \phi$, so we have $x \in u$. Since $x \in u_a(\alpha)$, then $\alpha \in \Lambda_0$ and $A \subseteq v$ with $u \cap v = \phi$. So, we can separate x and A in two disjoint nigh open set in ξ. □

Theorem 3.5. *Every disjoint Lindelöf subset in nigh Hausdorff space can be separated by disjoint open sets in ξ.*

Proof. Assume that A and B are two disjoint Lindelöf subsets of nigh Hausdorff. For each $a \in A$, we have $a \notin B$ because $A \cap B = \phi$. Thus, by Theorem (3.1.4), there exist two open sets u_a and v_a in ξ such that $a \in u_a$ and $B \subseteq v_a$ with $u_a \cap v_a = \phi$. Hence, $\underset{\sim}{U} = \{u_a : a \in A\}$ forms an open cover of A, and since A is Lindelöf subset of ξ, then $\underset{\sim}{U}$ can be reduced to a countable subcover, say $\underset{\sim}{U} = \{u_{a_\alpha} : \alpha \in \Lambda_0\}$, Λ_0, which is countable. Thus, $A \subseteq \underset{\alpha\in\Lambda_0}{\cup} u_{a_\alpha} = u$, and so u is open set. So, for each u_{a_α} and $\alpha \in \Lambda_0$, there are two corresponding open sets v_{a_α} and $\alpha \in \Lambda_0$ such that $B \subset v, v_{a_\alpha} \cap u_{a_\alpha} = \phi$. Now, let $v = \underset{\alpha\in\Lambda_0}{\cap} v_{a_\alpha}$. Then, $B \subseteq v$ and v is open in ξ. Since ξ is a nigh space, then $A \subseteq u$ and $B \subseteq v$ for which u and v are two open sets in ξ. Consequently, due to $v \subset v_{a_\alpha}$ for all $\alpha \in \Lambda_0$, then $v \cap u_{a_\alpha} \subseteq u_{a_\alpha} \cap v_{a_\alpha} = \phi$. So, we have $\phi = v \cap u_{a_\alpha} = v \cap (\underset{\alpha\in\Lambda_0}{\cup} u_{a_\alpha}) = v \cap u$. Hence, A and B can be separated in two disjoint open sets in ξ, as required. □

Theorem 3.6. *Let ξ be a Lindelöf space and Y be a nigh space. Then the projection $P : \xi \times \xi \to Y$ is closed.*

Proof. Let $y \in Y$ and G be an open set in $\xi \times \xi$ such that $P^{-1} \in G$. To show that there is an nigh open set v containing y in a space Y such that $P^{-1}(v) \subseteq G$, we should first notice that G is open in $\xi \times \xi$. Then for every $(x, y) \in G$, there exist two P-open basic sets u_x and v_x in ξ and Y respectively such that $x \in u_x$ and $y \in v_x$ with $(x, y) \in u_x \times v_x \subseteq G$, where (x, y) in $\xi \times Y$. Hence, $\underset{\sim}{U} = \{u_x : x \in \xi\}$ forms a λ_i-open cover of ξ. Now, since ξ is a Lindelöf space, then $\underset{\sim}{U}$ can be reduced to a λ_j-countable subcover, say $\{u_{x_\alpha} : \alpha \in \Lambda_0\}, |\Lambda_0| \leq w_0 = |\mathbb{N}|$. So, we have $\xi \subseteq \underset{\alpha \in \Lambda_0}{\cup} = u$, and for all $\{u_{x_\alpha} : \alpha \in \Lambda_0\}$ corresponding to $\{v_{x_\alpha} : \alpha \in \Lambda_0\}$ such that $y \in v_{x_\alpha}$, we let $v = \underset{\alpha \in \Lambda_0}{\cap} v_{x_\alpha}$. Thus, due to Λ_0 is countable and Y is a nigh space, then v is open and $P^{-1}(v) \subseteq \xi \times v \subseteq u \times v \subseteq G$, i.e. $P^{-1}(v) \subseteq G$. Therefore, P is closed, and this completes the proof. □

Theorem 3.7. *Let $f : \xi \to Y$ be a closed continuous surjective function and the fibers $f^{-1}(y)$ be a Lindelöf space, for all $y \in Y$. If Y is Lindelöf, then ξ is so.*

Proof. Let $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ be an open cover of ξ. Then for all $y \in Y$, $f^{-1}(y) \subseteq \xi$. Thus, $\underset{\sim}{U}$ is an open cover of $f^{-1}(y)$. Since $f^{-1}(y)$ is a Lindelöf space, then $\underset{\sim}{U}$ can be reduced to a countable subcover, say $\{u_{\alpha y}\}$. Hence, we have $f^{-1}(y) \subseteq \underset{\alpha y \in \Lambda_0}{\cup} u_{\alpha y}$, which implies $f^{-1}(y) \cap (\xi - \underset{\alpha y \in \Lambda_0}{\cup} u_{\alpha y}) = \phi$. Consequently, we obtain $y \cap f(\xi - \underset{\alpha y \in \Lambda_0}{\cup} u_{\alpha y}) = \phi$. Hence, we get $y \in O_y = y - f(\xi - \underset{\alpha y \in \Lambda_0}{\cup} u_{\alpha y})$. Now, due to $u_{\alpha y}$ open in ξ, then $\underset{\alpha y \in \Lambda_0}{\cup} u_{\alpha y}$ is open in ξ. Thus, $\xi - \underset{\alpha y \in \Lambda_0}{\cup} u_{\alpha y}$ is closed in ξ, which implies that $f(\xi - \underset{\alpha y \in \Lambda_0}{\cup} u_{\alpha y})$ is closed in Y because f is closed. So, $O_y = Y - f(\xi - \underset{\alpha y \in \Lambda_0}{\cup} u_{\alpha y})$ is an open set in Y. Therefore, for each $y \in Y$, we have $y \in O_y$. Thus, $\underset{\sim}{O} = \{O_y : y \in Y\}$ is an open cover of Y, and since Y is Lindelöf, then $\underset{\sim}{O}$ can be reduced to a countable subcover, say $\{O_{y_r}\}_{r \in \Gamma_0}$, such that Γ_0 is countable. Hence, we have $Y \subseteq \underset{r \in \Gamma_0}{\cup} O_{y_r}$, and thus $\xi = f^{-1}(\xi) \subseteq \underset{r \in \Gamma_0}{\cup} f^{-1}\left(O_{y_r}\right) = \underset{r \in \Gamma_0}{\cup} x - (x - \underset{r \in \Gamma_0}{\cup} u_{\alpha y}) = \underset{\substack{\alpha y \in \Lambda_0 \\ \in \Gamma_0}}{\cup} u_{\alpha_{y_r}}$. Therefore, $\underset{\sim}{U}$ can be reduced to a countable subcover of ξ, and so ξ is Lindelöf space. □

Theorem 3.8. *The product of two Lindelöf spaces one of them is $P - T_2$-space is Lindelöf space.*

Proof. Let ξ and Y be two Lindelöf spaces and Y be a $P - T_2$-space. Then, by previous theorem, the projection function $P : \xi \times Y \to Y$ is P-closed. Also, for all $y \in Y$, we have $P^{-1}(y) = \xi \times Y = \xi$. Since ξ is Lindelöf space, then $P^{-1}(y)$ is Lindelöf space and P is continuous and onto. Thus, P is a perfect function, and since Y is a Lindelöf space, then by the previous theorem, $\xi \times Y$ is so. □

Example 3.1. *Let $\xi = \mathbb{R}$ be a set of real numbers, then $\left(\mathbb{R}^2, \tau_u\right)$ is a nigh Lindelöf topological space, where τ_u is the usual topology defined on $\mathbb{R}$ as*

$$\tau_u = \{A \subset \mathbb{R} : \forall x \in A, \exists (a_x, b_x) \text{ such that } x \in (a_x, b_x) \subset A\}.$$

Example 3.2. *Let $\xi = \mathbb{R}$ be a set of real numbers, then $\left(\mathbb{R}^2, \tau_s \times \tau_s\right)$ is a nigh Lindelöf topological space, where τ_s is the Sorgenfrey topology defined as*

$$\tau_s = \{[a, b) : a < b \text{ in } \mathbb{R}\}.$$

This topology has a clopen set of the form $[a, b)$, and it is clear that this represents an example of Lindelöfness topological space.

4. On Nigh Nearly Lindelöf Space

We plan to present the notion of the near-neighbor Lindelöf space in this section. In light of this goal, we present a few more ideas and theories that are connected to the notion of the near-Lindelöf space as well as some instances for additional clarification.

Definition 4.1. *Suppose that $\xi = (\xi, \mu)$ is a topological space. We say that $A \subset \xi$ is a nearly open set if $A = (\bar{A})^\circ$.*

Definition 4.2. *Suppose that $\xi = (\xi, \mu)$ is a topological space. Then $A \subset \xi$ is called a nigh nearly open set if $A = (\bar{A})^\circ$ in λ_1 and $A = (\bar{A})^\circ$ in λ_2.*

Definition 4.3. *A family $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is called nearly open cover if:*

(1) *u_α is a nigh nearly open set for all $\alpha \in \Lambda$.*
(2) *$\bigcup_{\alpha \in \Lambda} u_\alpha = \xi$.*

Definition 4.4. *Suppose that $\xi = (\xi, \mu)$ is a topological space and $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is a nigh nearly open cover of ξ. Then $\underset{\lambda}{S} = \left\{u_{\alpha_\lambda} : \lambda \in \Gamma, \Gamma \subset \Lambda\right\}$ is called nearly subcover of ξ if $\bigcup_{\alpha \in \Gamma} u_{\alpha_\lambda} = \xi$.*

Definition 4.5. *Suppose that $\xi = (\xi, \mu)$ is a topological space. Then, we say that ξ is nearly compact space if every nearly open cover of ξ has a finite nearly subcover.*

Definition 4.6. *Suppose that $\xi = (\xi, \mu)$ is a topological space. Then ξ is called nigh nearly compact space if every nigh nearly open cover has a finite nearly subcover.*

Definition 4.7. *Suppose that $\xi = (\xi, \mu)$ is a topological space. Then ξ is called nearly Lindelöf space if every nearly open cover of ξ has a countably nearly subcover.*

Definition 4.8. *Suppose that $\xi = (\xi, \mu)$ is a topological space. Then ξ is called nigh nearly Lindelöf space if every nigh nearly open cover has a countably nearly subcover.*

Definition 4.9. *A topological space $\xi = (\xi, \mu)$ is called S-nearly Lindelöf if it is nearly Lindelöf and nigh nearly Lindelöf spaces.*

Remark 4.1. *Every nigh nearly compact space is a nigh nearly Lindelöf space, but the converse need not necessarily true.*

Proof. Let ξ be a nigh nearly compact space. Then every nearly open cover of ξ has a finite subcover of ξ. Thus, each nearly open cover of ξ has a countable subcover of ξ. So, ξ is a nigh nearly Lindelöf space. Now, to show that the converse need not be true, we need to provide an example of a space that is a nigh nearly Lindelöf space, but it is not nigh nearly compact space. This space is $(\mathbb{R}, \tau_u)$ for which τ_u is a usual topology. Now, because of $B = \{(a, b) : a < b \in \mathbb{Q}\}$ is a countable base of

$(\mathbb{R}, \tau_u)$, then $(\mathbb{R}, \tau_u)$ is a second countable space. Since each 2^{nd}-countable space is Lindelöf space, then $(\mathbb{R}, \tau_u)$ is a Lindelöf space, and so $(\mathbb{R}, \tau_u)$ is a nigh Lindelöf space. Besides, it is also nigh nearly Lindelöf space. Now, due to $(\mathbb{R}, \tau_u)$ is not compact space because $\underset{\sim}{U} = \{(-n, n) : n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$ that has no finite subcover, then $(\mathbb{R}, \tau_u)$ is not a nigh compact space, and therefore it is not a nigh nearly compact space. □

Theorem 4.1. *Let $\xi = (\xi, \mu)$ be a topological space. Then we have a hereditary nearly Lindelöf space, and so ξ is an S-nearly Lindelöf space.*

Proof. Assume that $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\} \cup \{v_\beta : \beta \in \Gamma\}$ is a nearly λ_1, λ_2-open cover of (ξ, μ) such that $u_\alpha \in \lambda_1$ and $v_\beta \in \lambda_2$, for every $\alpha \in \Lambda$ and $\beta \in \Gamma$. Because of $U = \cup\{u_\alpha : \alpha \in \Lambda\}$ is a λ_1-nearly Lindelöf, then there is a countable set $\Lambda_1 \subset \Lambda$ for which $U = \cup\{u_\alpha : \alpha \in \Lambda_1\}$. In the same way, since $V = \cup\{v_\beta : \beta \in \Gamma\}$ is a λ_2-nearly Lindelöf, then there is a countable set $\Gamma_1 \subset \Gamma$ for which $V = \cup\{v_\beta : \beta \in \Gamma_1\}$. Hence, it clear that $\{u_\alpha : \alpha \in \Lambda_1\} \cup \{v_\beta : \beta \in \Gamma_1\}$ is a countable subcover of $\underset{\sim}{U}$ for ξ. □

Theorem 4.2. *A nigh nearly Lindelöf space is preserved under onto nigh continuous function.*

Proof. Let $f : (\xi, \mu) \to (Y, \zeta)$ be a surjective continuous function and ξ be a nearly Lindelöf space. To show that Y is a nearly Lindelöf space, we assume that $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ is a nearly open cover of Y. Then u_α is an open set, for all $\alpha \in \Lambda$. Now, due to f is continuous function, then $f^{-1}(u_\alpha)$ is an open set in ξ, for each $\alpha \in \Lambda$. Also, since f is surjective, then $f^{-1}(u_\alpha) = \{f^{-1}(u_\alpha) : \alpha \in \Lambda\}$ is an open cover of ξ. But ξ is nearly Lindelöf space, which allows us to reduce $f^{-1}(u_\alpha)$ to a countable subcover, say $\{f^{-1}(u_\alpha) : \alpha \in \Lambda\}$, where $\Gamma \subset \Lambda$ and $|\Gamma| \leq w_0 = |\mathbb{N}|$. Hence, we have $\xi \subseteq \underset{\alpha\in\Gamma}{\cup} f^{-1}(u_\alpha)$. Now, since f is onto, then $Y = f(x) \subseteq F(\underset{\alpha\in\Gamma}{\cup} f^{-1}(u_\alpha)) \subseteq \underset{\alpha\in\Gamma}{\cup} u_\alpha$, which implies that $\underset{\sim}{\cup}$ has a countable subcover of Y. Therefore, Y is a nearly Lindelöf space. □

Remark 4.2. *Every compact subset in a nigh nearly T_2-space is closed. However, the nearly Lindelöf subset in a nigh nearly T_2-space need not be closed. For example, we can assert that $(\mathbb{R}, \tau_u)$ is a nigh nearly T_2-space such that $(0, 1)$ is a nearly Lindelöf subset of $\mathbb{R}$, however $(0, 1)$ is not a nigh nearly closed in $\mathbb{R}$.*

Definition 4.10. *A space (ξ, μ) is called a nigh nearly nigh space if the countable intersection of nearly open sets is open.*

Theorem 4.3. *Every nearly Lindelöf subset of nearly T_2-nigh space is closed.*

Proof. Let A be a nearly Lindelöf subset of a nearly $P - T_2$-space ξ. To show that A is a nigh nearly closed set, it is enough to show that $\xi \backslash A$ is nigh nearly open. For this purpose, we let $x \in \xi \backslash A$. Then $x \notin \xi$, and so we have $x \neq a$, for each $a \in A$. Now,since ξ is a T_2-space, then there exist two nearly open sets u_a and v_a in ξ such that $x \in u_a$ and $a \in v_a$ with $u_a \cap v_a = \phi$. Hence, $\underset{\sim}{V} = \{v_a : a \in A\}$ forms a nearly open cover of A, and since A is a P-nearly Lindelöf subset of ξ, then $\underset{\sim}{V}$ can be reduced to a countable subcover, say $\{v_{a_\alpha}\}_{\alpha\in\Lambda_0}$. Consequently, we have $A \subseteq \underset{\alpha\in\Lambda_0}{\cup} v_{a_\alpha}$. So, for each

v_{a_α}, $\alpha \in \Lambda_0$, there is a corresponding u_{a_α} such that $x \in u_{a_\alpha}$ and $a \in v_{a_\alpha}$ with $u_{a_\alpha} \cap v_{a_\alpha} = \phi$. In this regard, we let $V^* = \underset{\alpha\in\Lambda_0}{\cup} v_{a_\alpha}$ and $U^* = \underset{\alpha\in\Lambda_0}{\cap} u_{a_\alpha}$. Then, we obtain $A \subseteq V^*$ and $x \in U^*$ such that $U^* \cap V^* \subseteq u_{a_\alpha} \cap v_{a_\alpha} = \phi$, for all $\alpha \in \Lambda_0$. Hence, we have $U^* \cap V^* = \phi$, and therefore $U^* \cap \underset{\alpha\in\Lambda_0}{\cup} v_{a_\alpha} = \phi$. As a result, we get $U^* \cap V^* = \phi$. Now, due to $A \subseteq V^*$, then we have $U^* \cap A \subseteq U^* \cap V^* = \phi$. So, we can obtain $U^* \cap V^* = \phi$. Thus, we get $x \in U^* \subset x - A$, where $U^* = \underset{\alpha\in\Lambda_0}{\cap} u_{a_\alpha}$ is open as ξ is a nigh nearly space. Therefore, $\xi - A$ is a nigh nearly open set, which implies that A is a nigh nearly closed set. □

Theorem 4.4. *Let A be a nearly Lindelöf subset of a nigh nearly T_2-space ξ. Then for each $x \notin A$, we can separate x and A into two disjoint nearly open sets in ξ.*

Proof. For each $a \in A$, we have $a \neq x$. Due to $x \notin A$ and ξ is a T_2-space, then there exist two nigh nearly open sets $u_a(x)$ and $v(a)$ in ξ such that $x \in u_a(x)$ and $a \in v(a)$ with $u_a(x) \cap v(a) = \phi$. Hence, $\underset{\sim}{V} = \{v(a) : a \in A\}$ forms an open cover of A. Since A is a nigh nearly Lindelöf subset of ξ, then $\underset{\sim}{V}$ can be reduced to a countable subcover of A, say $\underset{\sim}{V} = \{v(a_\alpha) : \alpha \in \Lambda_0\}$. Thus, we have $A \subseteq \underset{\alpha\in\Lambda}{\cup} v(a_\alpha) = V$. In this regard, for all $v(a_\alpha)$ and $\alpha \in \Lambda_0$, there is a corresponding nigh nearly open set $u_{a_\alpha}(x)$ containing ξ such that $u_{a_\alpha}(x) \cap v(a_\alpha) = \phi$. Now, we let $U = \cap u_{a_\alpha}(x)$. Then, it is clear that u is an open set because of ξ is a nigh nearly space. Thus, we have $u \subseteq u_{a_\alpha}(x)$, for all $\alpha \in \Lambda_0$. So, we obtain $u \cap v(a_\alpha) \subseteq u_{a_\alpha} \cap v(a_\alpha) = \phi$, and hence $u \cap v(a_\alpha) = \phi$. In light of this discussion, we have $u \cap \underset{\alpha\in\Lambda_0}{\cup} v(a_\alpha) = \phi$. This implies that $u \cap v = \phi$, and so $x \in u$. Now, since $x \in u_a(\alpha)$, then we have $\alpha \in \Lambda_0$ and $A \subseteq v$ with $u \cap v = \phi$. So, we can separate ξ and A into two disjoint P-nearly open sets in ξ. □

Theorem 4.5. *Every disjoint nearly Lindelöf subset od a nigh nearly Hausdorff space can be separated by disjoint open sets in ξ.*

Proof. Assume A and B are two disjoint nearly Lindelöf subsets of a nigh nearly Hausdorff space. For each $a \in A$, we have $a \notin B$ as $A \cap B = \phi$. Thus, by the previous theorem, there exist two nearly open sets u_a and v_a in ξ such that $a \in u_a$ and $B \subseteq v_a$ with $u_a \cap v_a = \phi$. Hence, $\underset{\sim}{U} = \{u_a : a \in A\}$ forms an open cover of A. Due to A is a nearly Lindelöf subset of ξ, then $\underset{\sim}{U}$ can be reduced to a countable subcover, say $\underset{\sim}{U} = \{u_{a_\alpha} : \alpha \in \Lambda_0\}$ for which Λ_0 is countable. Therefore, $A \subseteq \underset{\alpha\in\Lambda_0}{\cup} u_{a_\alpha} = U$, and hence U is open. So for each u_{a_α}, we have $\alpha \in \Lambda_0$. Then, there is a corresponding open set v_{a_α} for which $\alpha \in \Lambda_0$ such that $B \subset v$ and $v_{a_\alpha} \cap u_{a_\alpha} = \phi$. Now, let $v = \underset{\alpha\in\Lambda_0}{\cap} v_{a_\alpha}$. Then, we have $B \subseteq v$ for which v is open in ξ. Since ξ is a nigh nearly space, then $A \subseteq u$ and $B \subseteq v$ such that u and v are open sets in ξ. Now, due to $v \subset v_{a_\alpha}$, for all $\alpha \in \Lambda_0$, then $v \cap u_{a_\alpha} \subseteq u_{a_\alpha} \cap v_{a_\alpha} = \phi$. As a result, we have $\phi = v \cap u_{a_\alpha} = v \cap (\underset{\alpha\in\Lambda_0}{\cup} u_{a_\alpha}) = v \cap u$. Hence, A and B can be separated into two disjoint nearly open sets in ξ. □

Theorem 4.6. *Let ξ be a nearly Lindelöf space and Y be a nigh nearly space. Then the projection $P : \xi \times \xi \to Y$ is closed.*

Proof. Let $y \in Y$, and G be an open set in $\xi \times \xi$ such that $P^{-1} \in G$. To show that there is a nigh nearly open set v containing y in the space Y such that $P^{-1}(v) \subseteq G$, we assume that G is open in $\xi \times \xi$. So, for each $(x, y) \in G$, there exist two nigh nearly open basic sets u_x and v_x in ξ and Y respectively such that $x \in u_x$ and $y \in v_x$ with $(x, y) \in u_x \times v_x \subseteq G$, where (x, y) in $\xi \times Y$. Hence, $\underset{\sim}{U} = \{u_x : x \in \xi\}$ forms an open cover of ξ, Now, since ξ is a nearly Lindelöf space, then $\underset{\sim}{U}$ can be reduced to a countable subcover, say $\{u_{x_\alpha} : \alpha \in \Lambda_0\}$ such that $|\Lambda_0| \leq w_0 = |\mathbb{N}|$. So, we obtain $\xi \subseteq \underset{\alpha\in\Lambda_0}{\cup} = u$ for all $\{u_{x_\alpha} : \alpha \in \Lambda_0\}$ corresponding $\{v_{x_\alpha} : \alpha \in \Lambda_0\}$ such that $y \in v_{x_\alpha}$. In this regard, we let $v = \underset{\alpha\in\Lambda_0}{\cap} v_{x_\alpha}$. Due to Λ_0 is countable and Y is a nigh nearly space, then v is open and $P^{-1}(v) \subseteq \xi \times v \subseteq u \times v \subseteq G$, i.e. $P^{-1}(v) \subseteq G$. This consequently implies that P is closed. □

Theorem 4.7. *Let $f : \xi \to Y$ be a closed continuous surjective function and $f^{-1}(y)$ be nearly Lindelöf, for all $y \in Y$. If Y is a v nearly Lindelöf, then ξ is so.*

Proof. Let $\underset{\sim}{U} = \{u_\alpha : \alpha \in \Lambda\}$ be a nearly open cover of ξ. Then for all $y \in Y$, we have $f^{-1}(y) \subseteq \xi$. Thus, $\underset{\sim}{U}$ is an open cover of $f^{-1}(y)$. Due to $f^{-1}(y)$ is nearly Lindelöf, then $\underset{\sim}{U}$ can be reduced to a countable subcover, say $\{u_{\alpha y}\}$. Hence, we have $f^{-1}(y) \subseteq \underset{\alpha y\in\Lambda_0}{\cup} u_{\alpha y}$, which implies $f^{-1}(y) \cap (\xi - \underset{\alpha y\in\Lambda_0}{\cup} u_{\alpha y}) = \phi$. So, we get $y \cap f(\xi - \underset{\alpha y\in\Lambda_0}{\cup} u_{\alpha y}) = \phi$. As a result, we obtain $y \in O_y = y - f(\xi - \underset{\alpha y\in\Lambda_0}{\cup} u_{\alpha y})$. So, as $u_{\alpha y}$ is open in ξ, then $\underset{\alpha y\in\Lambda_0}{\cup} u_{\alpha y}$ is open in ξ. Thus, $\xi - \underset{\alpha y\in\Lambda_0}{\cup} u_{\alpha y}$ is closed in ξ. This implies that $f(\xi - \underset{\alpha y\in\Lambda_0}{\cup} u_{\alpha y})$ is closed in Y as f is closed. So, we obtain $O_y = Y - f(\xi - \underset{\alpha y\in\Lambda_0}{\cup}$ for which $u_{\alpha y})$ is open in Y. Therefore, for all $y \in Y$, we have $y \in O_y$. Thus, $\underset{\sim}{O} = \{O_y : y \in Y\}$ is a nearly open cover of Y. In this connection, due to Y is a Lindelöf space, then $\underset{\sim}{O}$ can be reduced to a countable subcover, say $\{O_{y_r}\}_{r\in\Gamma_0}$, such that Γ_0 is countable. Hence, we obtain $Y \subseteq \underset{r\in\Gamma_0}{\cup} O_{y_r}$. Thus, we have $\xi = f^{-1}(\xi) \subseteq \underset{r\in\Gamma_0}{\cup} f^{-1}\left(O_{y_r}\right) = \underset{r\in\Gamma_0}{\cup} x - (x - \underset{r\in\Gamma_0}{\cup} u_{\alpha y}) = \underset{\substack{\alpha y\in\Lambda_0\\ \in\Gamma_0}}{\cup} u_{\alpha_{y_r}}$. Therefore, $\underset{\sim}{U}$ is reduced to a countable subcover of ξ, which implies that ξ is Lindelöf space. □

Theorem 4.8. *The product of two nearly Lindelöf spaces one of them is nigh T_2-space is nearly Lindelöf.*

Proof. Let ξ and Y be two nearly Lindelöf spaces for which Y is a P-nearly T_2-space. Then, by previous theorem, the projection function $P : \xi \times Y \to Y$ is nigh closed, and for all $y \in Y$, we have $P^{-1}(y) = \xi \times Y = \xi$. Now, due to ξ is Lindelöf space, then $P^{-1}(y)$ is nearly Lindelöf for which P is continuous and onto. Thus, P is a perfect function, and because of Y is a nearly Lindelöf space, then by the previous theorem, $\xi \times Y$ is also nearly Lindelöf space. □

Example 4.1. *The two nigh Lindelöf topological spaces τ_u and τ_s defined, respectively, in Examples 3.1 and 3.2 are also examples on the nigh nearly Lindelöf topological space, as every nigh Lindelöf space is a nigh nearly Lindelöf space.*

5. Conclusion

This paper has presented the idea of nigh Lindelöfness in nigh topological spaces and topological spaces. As a result, we have established a number of generalizations and characteristics of nigh Lindelöf space that are connected to nigh compactness in topological spaces, nearly nigh Lindelöf space, and their relationships with other spaces. Numerous well-known theorems on the near Lindelöf spaces have been generalized, and several examples have been discussed as well.

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