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Legendre Polynomials and Techniques for Collocation in the Computation of Variable-Order Fractional Advection-Dispersion Equations

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Abstract. The paper discusses a numerical approach to solving complicated partial differential equations, with a particular emphasis on fractional advection-dispersion equations of space-time variable order. With the use of fractional derivative matrices, Legendre polynomials, and numerical examples and comparisons, it surpasses current methods by utilizing spectral collocation techniques. It resolves equations involving spatial and time variables that are variable-order fractional advection–dispersion (VOFADE). Legendre polynomials serve as basis functions in this method, whereas Legendre operational matrices are employed for fractional derivatives. The technique is more computationally efficient since it reduces fractional advection–dispersion equations to systems of algebraic equations. Numerical examples and a comparison with current approaches illustrate the method's superior performance in solving complicated partial differential equations, especially in the context of transport processes.

1. Introduction

A number of engineering procedures, physical characteristics, physiological models, and financial applications (e.g., [1] and the references therein) are found to be more realistically modeled using fractional derivative operators. Anomaly diffusion, non-exponential relaxation patterns, and viscoelastic materials are some of these modeling applications. Because the fractional derivative operators are non-local, it is challenging to solve fractional differential equations (FDEs) properly, despite their significance. Thus, in order to comprehend the physical behavior of these equations,

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numerical methods are crucial tools.Over the past ten years, a significant amount of work has been done on alternative numerical approaches for the solution of partial differential equations with fractions of constant order in space or time [2]. Samko et al. [4] originally suggested variable-order operators in 1993. With their work on fractional integration and differentiation where the order is a function rather than a constant of changeable order, they offer an extension of the traditional fractional calculus [5]. Hartley [7]Due to the variability of memory incorporation in different time periods and locations, a new cohort of mathematicians and physicists is intrigued by investigating physical issues that necessitate the use of variable-order derivatives [6]. The variable order operator is a function that fluctuates based on the independent variables of differentiation or other unrelated factors. Lorenzo and Hartley [7] present distributed order fractional operators. Given that the majority of these kinds of problems lack precise solutions, numerical methods have been demonstrated to be more effective for solving variable-order fractional differential equations [8]. Abdelkawy et al. [10] suggested a new spectral technique that may be used to properly solve the fractional order mobile-immobile advection-dispersion model with a time variable. By combining the operational matrix for variable-order fractional derivatives, known as the shifted Jacobi operational matrix with shifted Jacobi collocation, Bhrawy and Zaky [9] were able to get a numerical solution to the variable-order nonlinear cable issue. At now, FDEs are generally acknowledged as a plausible explanation for anomalous diffusion and relaxation phenomena that have been seen in several scientific and technological fields [12]. Among these domains are fluid transport in porous media, two-dimensional rotational flow, plasma diffusion, growth on surfaces, and diffusion on liquid surfaces. Nevertheless, some recent studies [14] have shown that certain intricate diffusion systems, which exhibit diffusion patterns that are influenced by changes in time or space, cannot be accurately described by fractional diffusion equations. The authors have introduced variable-order fractional diffusion equations to address these difficulties. These equations can contain a time- or space-dependent variable-order temporal fractional operator.

2. Elementary Principles

Definition 2.1. *The Riemann-Liouville integral (RLI) order is defined for values of ω that satisfy $0 < \omega < 1$. The function $v(\tau)$ determines the RLI order. [27]:*

$$D^{\omega} v\left(t\right)=\frac{1}{\Gamma(n-\omega)}\int_{0}^{t}\left(t-\tau\right)^{n-\omega-1}v^{n}\left(\tau\right)d\tau\ =I^{n-\omega}v^{n}\left(t\right),\ \ t>0. \tag{2.1}$$

Definition 2.2. *The Riemann-Liouville fractional integral of a certain order $\omega > 0$, given by [24]:*

$$I_{a+}^{\omega} f(t)=\frac{1}{\Gamma(\omega)}\int_{a}^{t}(t-s)^{\omega-1}f(s)ds,\ \ t>a. \tag{2.2}$$

Definition 2.3. *The function $y\left(\tau\right)$ is provided.The Caputo derivative, denoted by $\frac{d^{\omega}}{dt^{\omega}}$, is defined for $0 < \omega < 1$ as follows [5]:*

$$I^{\alpha} y\left(t\right)\ =\frac{1}{\Gamma\left(\omega\right)}\int_{0}^{t}\left(t-\tau\right)^{\omega-1}y\left(\tau\right)d\tau\ ,\ \ t>0. \tag{2.3}$$

Definition 2.4. *The Mittag-Leffler function can be represented as follows. [15]:*

$$E_{\omega}\left(t\right)=\sum_{k=0}^{\infty}\frac{t^{k}}{\Gamma(\omega k+1)}. \tag{2.4}$$

Definition 2.5. *The ABC operator, $y\left(t\right)$ in the RLI is given by [24]:*

$${}_{0}^{ABC}D_{t}^{\omega}y\left(t\right)=\frac{B\left(\omega\right)}{1-\omega}\frac{d}{dt}\int_{0}^{t}y\left(\tau\right)E_{\omega}\left(\frac{\omega}{1-\omega}\left(t-\tau\right)^{\omega}\right)d\tau,\ \ 0<\omega<1. \tag{2.5}$$

Where $B\left(\omega\right)$ meets the criteria $B\left(1\right)=B\left(0\right)=1$.

In order to introduce the space-time variable-order fractional advection, we will refer to [15] and [16]. the formula for dispersion

$${}_{0}^{C}D_{t}^{\zeta(t)}w(x,t)=\mu_{1}\,{}_{0}^{RL}D_{x}^{\rho(x,t)}w(x,t)-\lambda\,{}_{0}^{RL}D_{x}^{\zeta(x,t)}w(x,t)$$

$$(x,t)\in\Omega=\left([O,L]\times[O,T]\right), \tag{2.6}$$

With respect to the original boundary conditions

$$w(x,0)=\phi(x),\ x\in[0,L]$$

$$w(0,t)=f_{1}(t),w(L,t)=f_{2}(t)\ ,\quad t\in[0,T] \tag{2.7}$$

Where $\mu_{1},\lambda_{1}>0,0<\underline{\zeta}\leq\zeta(x)\leq\overline{\zeta}\leq1,1<\underline{\rho}\leq\rho(x,t)\leq\overline{\rho}\leq2,0<\underline{\zeta}\leq\zeta(x,t)\leq\overline{\zeta}\leq1.$

Here ${}_{0}^{RL}D_{x}^{\rho(x,t)}w\left(x,t\right)$ is the variable order Riemann- Liouville fractional derivative defined as

$${}_{0}^{RL}D_{x}^{\rho(x,t)}h\left(x\right)=\left[\frac{1}{\zeta(n-\rho(x,t))}\,\frac{d^{n}}{d{\xi_{1}}^{n}}\,\int_{0}^{\xi_{1}}\left(\xi_{1}-\vartheta\right)^{n-\rho(x,t)-1}h\left(\vartheta\right)d\vartheta\right]_{\xi_{1}=x} \tag{2.8}$$

And ${}_{0}^{C}D_{x}^{\zeta(x,t)}w(x,t)$ is the variable -order Caputo derivative defined as

$${}_{0}^{C}D_{x}^{\zeta(x,t)}h(x)=\frac{1}{\zeta(n-\zeta(x))}\,\frac{d^{n}}{d{\xi_{1}}^{n}}\,\int_{0}^{x}\frac{h^{(n)}\left(\vartheta\right)d\vartheta}{\left(x-\vartheta\right)^{\zeta(x)-n+1}}d\vartheta \tag{2.9}$$

where $n\ -\ 1\ <\ \zeta\ \leq\ n\ \in\ N\,n-1<\zeta\leq n\in N$ in this case. The objective of this work is to develop a numerical method that can enhance the accuracy of the numerical solutions for the ST-VO-FADE [2]. The technique utilizes the operational matrix of variable order differentiation and the [10] shifting Legendre-Gauss collocation approach to turn the ST VO-FADE into an algebraic system of equations [13]. Consequently, the computations need less exertion [25]. The subsequent sections of the paper are structured in this fashion. The number is. Section 2 covers essential concepts in fractional calculus [27], along with the characteristics of the shifted Legendre polynomials. Section 3 contains [**?**] operational matrices that display the variable-order fractional derivatives of the shifted 73 Legendre polynomials.

3. Differentiated Legendre Polynomials

The three term recurrence relation may be used to create shifted Legendre polynomials, which are the subject of this text's discussion of their key characteristics [16] [27]. $[-1,1]$. $\mathcal{L}_0(z)=1$,

$$\mathcal{L}_1(x)=x,\ \mathcal{L}_{j_1+1}(z)=\frac{2j_1+1}{j_1+1}z\mathcal{L}_j(z)-\frac{j_1}{j_1+1}\mathcal{L}_{j_1-1}(z),\ j_1\geq 2$$

The shifted Legendre polynomials have been examined. $\mathcal{L}_{j_1}\ (2x/h_1\ -\ 1)$ be represented by $\mathcal{L}^{h_1}_{j_1}(x)$. Then $\mathcal{L}^{h_1}_{j_1}(x)$The recurrence formula provided can be utilized to generate.

$$\mathcal{L}^{h_1}_0(x)=1,\quad \mathcal{L}^{h_1}_1(x)=\frac{2x}{h_1}-1,$$

$$\mathcal{L}^{h_1}_{j_1+1}(z)=\frac{(2j_1+1)\,(2x-h_1)}{(j_1+1)\,h_1}\mathcal{L}^{h}_{j_1}(x)$$

$$-\frac{j_1}{j_1+1}\mathcal{L}^{h_1}_{j_1-1}(x),\ j_1\geq 2$$

Among the shifted Legendre polynomials' orthogonality properties, one is

$$\int_0^{h_1}\mathcal{L}^{h_1}_{i_1}(x)\ \mathcal{L}^{h_1}_{j_1}(x)\ dx=\rho_{j_1} \tag{3.1}$$

Where

$$\rho_{j_1}=\frac{\delta_{i_1j_1}h}{2j_1+1}$$

It is explicitly provided by [16] that $\mathcal{L}^{h_1}_{j_1}(x)$ of degree j has an analytical representation.

$$\mathcal{L}^{h_1}_{j_1}(x)=\sum_{k_1=0}^{j_1}\epsilon^{h_1}_{j_1,k_1}x^{k_1} \tag{3.2}$$

Where

$$\epsilon^{h_1}_{j_1,k_1}=\frac{(-1)^{j_1+k_1}\ (j_1+k_1)!}{(j_1-k_1)!(k_1)!^2{h_1}^2} \tag{3.3}$$

The expression can be reformulated using matrix notation.

$$\Delta_{h_1,M_1}(x)=E_{h_1}x_{M_1}(x), \tag{3.4}$$

where $\epsilon^{h_1}_{j_1,k_1}\ for\ j_1,k_1=0,1,\ldots M_1$ are the entries of matrix E_{h_1},

$$\Delta_{h_1,M_1}(x)=\left[\mathcal{L}^{h_1}_0(x),\mathcal{L}^{h_1}_1(x),\ldots,\mathcal{L}^{h_1}_{M_1}(x)\right]^T \tag{3.5}$$

$$X_{M_1}(x)=\left[1,x.x^2,\ldots,x^{M_1}\right]^T.$$

The determinability of the matrix E_{h_1} and the vector $X_{M_1}(x)$ is assessed. In addition, the orthogonality property of the Legendre polynomials with shifts (3.1) can be expressed in terms of $\Delta_{h_1,M_1}(x)$.

$$X_{M_1}(x) = E_h^{-1}\Delta_{h_1,M_1}(x) \tag{3.6}$$

When the Legendre polynomials with shifts reach their terminal, their values are determined by

$$\mathcal{L}_{j_1}^{h_1}(0) = (-1)^j, \mathcal{L}_{j_1}^{h_1}(h) = 1 \tag{3.7}$$

They will be demonstrated later and have great significance.

Assuming that w(x) is a square integrable function in [0, L], it may be expressed in terms of shifted Legendre polynomials as

$$w_1(x) = \sum_{j_1=0}^{\infty} c_{j_1} L_{j_1}^{h_1}(x), \tag{3.8}$$

where the coefficients c_{j_1} are given by

$$c_{j_1} = \frac{1}{\rho_{j_1}} \int_0^{h_1} w_1(x) L_{j_1}^{h_1} dx, j_1 = 0, 1, ... \tag{3.9}$$

When we take into account that the first(M + 1)-terms approximate $w_1(x)$, we may write

$$w_{M_1(x)} = \sum_{j_1=0}^{\infty} c_{j_1} L_{j_1}^{h_1}(x) = C^T \Delta_{h_1,M_1(x)}, \tag{3.10}$$

in which $C^T = [c_0, c_1, ..., c_M] representstheshiftedLegendrecoefficientvectorC.A$

4. Distinction Matrices

The shifted Legendre vector's fractional derivative of variable order is presented in this part in the Capu to definition meaning. The syntax of the first-order derivative expression is as follows. [18].

$$\frac{d}{dx}\Delta_{h_1,M_1}(x) = \mathbf{D}_{h_1}^{(1)} \Delta_{h_1,M_1}(x) \tag{4.1}$$

where the first derivative's operational matrix of $\Delta_{h_1,M_1}(x)$ is $\mathbf{D}_{h_1}^{(1)}$. with $(M_1 + 1) \times (M_1 + 1)$ as its dimension

$$\frac{d}{dx}\Delta_{h_1,M_1(x)} = E_{h_1} \frac{d}{dx} X_{M_1}(x) = E_{h_1} \Xi_{M_1} X_{M_1}(x), \tag{4.2}$$

first derivative of $X_{M_1}(x)$ has a dimension of $(M_1 + 1) \times (M_1 + 1)$, and its operational matrix is denoted by Ξ_{M_1}. This is the outcome of

$$\Xi_{M_1} = \begin{cases} j_1 + 1, & \text{for} \quad i_1 = j_1 + 1, j_1 = 0, 1, ..., M_1, \\ 0, & \quad otherwise. \end{cases} \tag{4.3}$$

It is now simple to write by using (16) and (10).

$$\frac{d}{dx}\Delta_{h_1,M_1}(x) = E_{h_1} \Xi_{M_1} E_{h_1}^{-1} \Delta_{h_1,M_1}(x) = \mathbf{D}_{h_1}^{(1)} \Delta_{h_1,M_1}(x) \tag{4.4}$$

Thus, it follows that

$$\mathbf{D}_{h_1}^{(1)} = E_{h_1} \Xi_{M_1} E_{h_1}^{-1} \tag{4.5}$$

By continuously using (18), the relation is given.

$$\frac{d^P}{dx^P}\Delta_{h_1,M_1}(x) = (\mathbf{D}_{h_1}^{(1)})^P\Delta_{h_1,M_1}(x) = \mathbf{D}_{h_1}^P\Delta_{h_1,M_1}(x) \quad ,p = 1, 2, ..., \tag{4.6}$$

where $p \in N$.

Theorem [17] suggests that the matrix of operations for It is possible to extend The derivative of shifted Legendre polynomials for fractional derivatives with variable order.

$${}_0^C\mathbf{D}_x^{\zeta(x,t)}\Delta_{h_1,M_1}(x) = \mathbf{D}_{h_1,\zeta(x,t)}\Delta_{h_1,M_1}(x) \tag{4.7}$$

where $n-1 < \zeta_{min} < \zeta(x,t) < \zeta_{max} < n$ and $\mathbf{D}_{h_1,\zeta(x,t)}$ is an $(M_1+1)\times(M_1+1)$ matrix of the following form:

$$\mathbf{D}_{h_1,\zeta(x,t)} = X^{-\zeta(x,t)}E_{h_1}\mathbf{B}E_{h_1}^{-1},$$

where E_h is specified in equation (8) and $\mathbf{B}$ is a square matrix with dimensions $(M_1+1)\times(M_1+1)$ and its elements, $b_{ij}; 0 \le i_1, j_1 \le M_1$ are given as follows:

$$b_{i_1 j_1} = \begin{cases} \frac{\zeta(i_1+1)}{\zeta(i_1+1-\zeta(x,t))}, & \text{for} \quad i_1 = j_1, j_1 = n, n+1, ..., M_1, \\ 0, & \textit{otherwise}. \end{cases}$$

5. Legendre Technique for the Spectral Displacement

The shifted Legendre collocation technique, which uses The double-shifted Legendre polynomials in series form are employed to approximate the solution of w(x, t) and solve the ST-VO-FADE (1)-(2) problem.

$$w(x,t) \simeq w_{N,M}(x,t) = \sum_{i=0}^{N}\sum_{j=0}^{M} a_{i,j}\mathcal{L}_i^\tau(t)\mathcal{L}_j^L(x) = \Delta_{\tau,N}^T(t)\mathbf{A}\Delta_{L,M}(x) \tag{5.1}$$

The matrix A contains unknown entries with an order of $(N_1+1)\times(M_1+1)$. It can now be expressed as follows by applying Equations (20), (21) and (22).

$$\begin{aligned} {}_0^{RL}D_x^{\zeta(x,t)}w_1(x,t) &= {}_0^C D_x^{\zeta(x,t)}w_1(x,t) + \frac{f_1(t)x^{-\zeta(x,t)}}{\Gamma(1-\zeta(x,t))} \\ &\simeq \Delta_{\tau_1,N_1}^T(t)\mathbf{A}\mathbf{D}_{L,\zeta(x,t)}\Delta_{L_1,M_1}(x) + \frac{f_1(t)x^{-\zeta(x,t)}}{\Gamma(1-\zeta(x,t))} \end{aligned} \tag{5.2}$$

$$\begin{aligned} {}_0^{RL}D_x^{\varrho(x,t)}w_1(x,t) &= {}_0^C D_x^{\varrho(x,t)}w(x,t) + \frac{f_2(t)x^{-\varrho(x,t)}}{\Gamma(1-\varrho(x,t))} \\ &\quad + \frac{\partial_x W_1(0,t)x^{1-\varrho(x,t)}}{\Gamma(2-\varrho(x,t))} \\ &\simeq \Delta_{\tau_1,N_1}^T(t)\mathbf{A}\mathbf{D}_{L_1,\varrho(x,t)}\Delta_{L_1,M_1}(x) + \frac{f_2(t)x^{-\varrho(x,t)}}{\Gamma(1-\beta_1(x,t))} \\ &\quad + \Delta_{\tau_1,N_1}^T(t)\mathbf{A}\mathbf{D}_{L_1,\varrho(x,t)}^{(1)}\Delta_{L_1,M_1}(0) + \frac{x^{1-\beta_1(x,t)}}{\Gamma(2-\varrho(x,t))}, \end{aligned} \tag{5.3}$$

$${}_0^C D_x^{\zeta(x)}w(x,t) = \Delta_{\tau_1,N_1}^T(t)\mathbf{D}_{\tau_1,zeta(x)}\mathbf{A}\Delta_{L_1,M_1}(x)$$

$$w_{N_1,M_1}(x,0) = \Delta^T_{\tau_1,N_1}(0)\mathbf{A}\Delta_{L_1,M_1}(x)$$

$$w_{N_1,M_1}(0,t) = \Delta^T_{\tau_1,N_1}(t)\mathbf{A}\Delta_{L_1,M_1}(0) \tag{5.4}$$

$$w_{N_1,M_1}(L,t) = \Delta^T_{\tau_1,N_1}(T)\mathbf{A}\Delta_{L_1,M_1}(L)$$

Equations (1)–(2) in [11] Equations (22)–(26) when used produces

$$O_{\tau_1,N(t)}{}^T D^T_{\tau_1,\zeta(x)} A O_{L,M(x)} \tag{5.5}$$

$$= \mu_1 O^T_{\tau_1,N(t)} A D_{L,\zeta(x,t)} O_{L,M(x)} + \frac{f_1(t)x^{-\zeta(x,t)}}{\Gamma_1(1-\zeta(x,t))} \tag{5.6}$$

$$-\lambda_1 O^T_{\tau_1,N(t)} A D_{L,\varrho(x,t)} O_{L,M(x)} + \frac{f_2(t)x^{-\varrho(x,t)}}{\Gamma_1(1-\varrho(x,t))} \tag{5.7}$$

$$+O^T_{\tau_1,N(t)} A D^1_L O_{L,M(0)} + \frac{x^{1-\varrho(x,t)}}{\Gamma_1(2-\varrho(x,t))} + g(x,t) \tag{5.8}$$

$$O_{\tau_1,N(0)}{}^T A O_{L,M(x)} = \Phi_1(x) \tag{5.9}$$

$$O_{\tau_1,N(t)}{}^T A O_{L,M(0)} = g_1(t) \tag{5.10}$$

$$O_{\tau_1,N(t)}{}^T A O_{L,M(l)} = g_2(t) \tag{5.11}$$

We now solve (27)–(28) using the collocation strategy [27] immediately. using $L_M l(x)$ and its shifted Legendre–Gauss–Lobatto roots, denoted by the nodes x_i $(0 \le i \le M)$

$t_j (0 \le j \le N-1)$ is the shifting Legendre root of $L_N \tau_1(x)$ as a result. Due to the substitution of these nodes in equations, the collaborative approach may be written as (27)–(28).

$$O_{\tau_1,N(t_i)}{}^T D^T_{\tau_1,\zeta(x_j)} A O_{L,M(x_j)} \tag{5.12}$$

$$= \mu_1 O^T_{\tau_1,N(t_i)} A D_{L,\zeta(x,t_i)} O_{L,M(x_j)} + \frac{f_1(t_i)x^{-\zeta(x_j,t_i)}}{\Gamma_1(1-\zeta(x,t_i))} \tag{5.13}$$

$$\lambda_1 O^T_{\tau_1,N(t_i)} A D_{L,\beta_1(x_j,t_i)} O_{L,M(x_j)} + \frac{f_2(t_i)x^{-\beta_1(x,t_i)}}{\Gamma_1(1-\beta_1(x_j,t_i))} + O^T_{\tau_1,N(t_i)} A D^1_L O_{L,M(0)} \tag{5.14}$$

$$+\frac{x^{1-\varrho(x,t_i)}}{\Gamma_1(2-\varrho(x,t))} + g(x,t_i) \tag{5.15}$$

$$O_{\tau_1,N(0)}{}^T A O_{L,M(x_j)} = \Phi_1(x_j) \tag{5.16}$$

$$O_{\tau_1,N(t_i)}{}^T A O_{L,M(0)} = g_1(t_i) \tag{5.17}$$

$$O_{\tau_1,N(t_i)}{}^T AO_{L,M(l)} = g_2(t_i) \tag{5.18}$$

$$(0 \le i \le M), t_j(0 \le j \le N-1) \tag{5.19}$$

6. Application

6.1. Fractional-Time Models.

The fractional mobile-immobile equation (FMIM) and the time-fractional FADE are two frequently employed time-fractional partial differential equations (PDEs)., with duality explaining the link between space and time FADE. [23] [25].

6.2. Time-fractional-fade.

The equation for the time-fractional advection dispersion, as presented by Liu et al. [12] or Zaslavsky [30], is

$$\left(\frac{\partial}{\partial t}\right)^{\zeta} = -v_1 \frac{\partial C_1}{\partial x_1} + D\frac{\partial^2 C_1}{\partial x^2} \tag{6.1}$$

where the Caputo derivative of order $0 < \zeta < 1$ corresponds to the initial term on the positive half-axis of (5.19). The value reduces to the conventional ADE when the damping ratio (ζ) equals 1. The velocity parameter v is distinct from the spatial dispersion coefficient, which is measured in units of L^2/T^{ζ}. This formula, which incorporates time-fractional properties, provides the scaling limit of a random path with continuous time (CTRW).To simplify matters, let's assume that v_1 is equal to zero.As stated in the section, let's assume that $S_n = X_1 + \ldots + X_n$ represents a random walk of particle jumps. Now envision a situation where there is an unpredictable interval of waiting, denoted as W_n, before the occurrence of the nth jump. The number of leaps is also obtained when the nth leap is determined by a walk at random $T_n = W_1 + \ldots + W_n$. $N(t) = \max\{n \ge 0 : T_n \le t\}$

With $[\ W_n > t] = A_1 t^{-\zeta}$ for some $A_1 > 0$ and $0 < \zeta < 1$, then

$$n_1{}^{-\frac{1}{\zeta}} \sum_{j=1}^{[n_1 t]} W_j = n_1{}^{-\frac{1}{\aleph}} T_{[nt]} \Rightarrow D_t \tag{6.2}$$

A stable subordinator is a different type of stable Lévy motion, obtained through counting and the CTRW limit. It simulates moments between particle motions, with waiting times illustrating the hydrological perspective.

Subordination creates closed-form solutions in ADE, with an inverse stable subordinator randomized time variable. Equation (21) represents subordination integral with pulse starting condition.

$$C_1\left(x_1, t\right) = \int_0^{\infty} h_{\zeta}\left(u_1,\ t\right) \frac{1}{\sqrt{4\pi D_1 u_1}} \exp\left(-\frac{\left(x_1 - v_1 u_1\right)^2}{4D_1 u_1}\right) du_1, \tag{6.3}$$

The PDF of the inverse (ζ)-stable subordinator $E(t)$ is denoted by $h_{zeta}(u_1,\ t)$. The density $h_\zeta(u_1,\ t)$ can be expressed using a stable density [16] and can be evaluated using easily accessible techniques. [11] [13]].

6.3. **Fractional-mobile-immobile-equation.**

The conventional mobile-immobile model is generalized by the fractional mobile-immobile (FMIM) model, which divides concentration into immobile and mobile phases. The FMIM model uses power law memory to describe the transition from immobile to mobile phase. [27]. [25]. [10]. function $f(t) = t^{-\zeta}/\Gamma(1-\zeta)$ with $0 < \zeta < 1$. Benson and Meerschaert developed a CTRW model for FMIM, separating particle types and using a power law to describe immobile particle waiting periods, similar to studies on river transit. [26]. [20] [27]

We take into account the next time-the domain's fractional mobile-immobile solute transport [27] equation $\omega \equiv \omega_x \times \Omega_y = (0,1)\times(0,T]$:

$$\begin{matrix} ABC \\ 0 \end{matrix} D_t^\alpha u + \beta(x,y)u_y - \nu(x,y)u_z z + \mu(x,y)u_z + \gamma(x,y)u = g(x,y), (x,y) \in \omega, \tag{6.4}$$

with the initial condition

$$u(x,0) = \psi(x), \quad on \quad H_z = \omega_z \times 0, \tag{6.5}$$

and the boundary conditions

$$u(0,y) = \phi_0(y), \quad on \quad H_0 = 0 \times \bar{\omega}_y, \tag{6.6}$$

$$u(1,y) = \phi_1(y), \quad on \quad H_1 = 1 \times \bar{\omega}_y, \tag{6.7}$$

where the domain boundary is represented by $\partial\omega = H_x \cup H_0 \cup H_1$ and $0 < \alpha < 1$. The following functions are regarded as sufficiently smooth: $\beta(x,y), \nu(x,y), \mu(x,y), and \gamma(x,y), \Psi(x), \phi_0(y), \psi_1(y), and g(x,y)$. Moreover, the ABC time fractional derivative is defined as $\begin{matrix} ABC \\ 0 \end{matrix} D_t^\alpha u$.

$$\begin{matrix} ABC \\ 0 \end{matrix} D_t^\alpha u(x,y) = \frac{\mathfrak{N}_f(\alpha)}{1-\alpha} \tag{6.8}$$

where $\mathfrak{N}_f(\alpha)$ is termed a normalization function satisfying $\mathfrak{N}_f(0) = \mathfrak{N}_f(1) = 1$, and the term $E_{\alpha,\beta}(T)$ is defined as

$$E_{\alpha,\beta}(T) = \sum_{r=0}^{\infty} \frac{T^r}{(\Gamma\alpha r + \beta)}$$

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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