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A Novel Approach for Time-Local Fractional Solutions of Certain Nonlinear Partial Differential Equations in Fractal Dimension

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Abstract. Time-local fractional approaches for nonlinear partial differential equations in fractal dimensions are essential for capturing the complex, irregular behaviors found in fractal systems. In this paper, a new modification of the local fractional Laplace variational iteration method (MLFLVIM) for obtaining analytical approximate solutions to the fractional gas dynamics equation, fractional Stefan equation, and fractional Newell-Whitehead-Segel equation within the context of fractal time space is presented. The proposed method (MLFLVIM) elegantly combines the local fractional Laplace transform (LFLT) with modified variational iteration method. Specifically, we first apply the (LFLT) to the given local fractional PDEs, yielding a transformed system of equations. We then apply modified variational iteration to this system. Finally, we use the inverse of (LFLT) to obtain the desired solution. To demonstrate the effectiveness of this approach, we implement it on three numerical physical problems. The results show that the (MLFLVIM) can successfully handle these nonlinear LFPDEs and provide accurate analytical approximation solutions.

1. Introduction

Local fractional derivatives are a generalization of classical derivatives to non-integer, or fractional, orders, providing a powerful tool for analyzing complex systems that exhibit non-standard behaviors such as fractality and heterogeneity. Unlike traditional derivatives, which are defined over integer orders, local fractional derivatives extend the concept of differentiation to fractional

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dimensions, offering a more nuanced description of changes in functions that vary irregularly or discontinuously. This approach is particularly useful in fields such as physics, engineering, and finance, where it aids in modeling phenomena like anomalous diffusion, viscoelastic materials, and other processes that cannot be accurately captured using classical calculus. Moreover, Local fractional derivatives have emerging applications in computer science and AI, particularly in fields such as signal processing, image analysis, and machine learning. By extending the traditional concept of differentiation to fractional orders, they allow for more accurate modeling of complex, non-linear systems and irregular data patterns, see [1, 2].

In the 19th century, influential mathematicians like Liouville and Riemann established the foundations for major advancements in the field of fractional calculus. This area of study has seen significant progress and transformation since those early developments. Local fractional calculus, a branch of this field, has found widespread applications across diverse domains such as anomalous diffusion processes, signal processing techniques, biomechanical modeling, and financial analyses, as highlighted in reference [3, 4]. One of the key properties of local fractional derivatives is their non-locality. Unlike classical derivatives which are local operators (depending only on the values of the function and its derivatives at a point), fractional derivatives incorporate a more global view of the function, taking into account its behavior over a range of values. This property makes them particularly well-suited for analyzing systems with memory or hereditary properties, see [5] This innovative calculus has garnered the interest of mathematicians, prompting them to explore and expand upon existing concepts while developing new results tailored to this calculus [6–14]. Over the recent years, a diverse range of analytical and semi-analytical techniques have been proposed and developed by researchers for solving many systems of fractional partial differential equations (FPDEs) [15–17].

Generally, mathematicians have struggled to solve FPDEs, using both numerical and analytical techniques. These types of equations are notoriously difficult to solve. This challenge motivated the development of new methods to address FPDEs.

The primary goal of this work is to introduce and expand the application of the proposed modified Laplace variational iteration method (MLVIM) [18], within the framework of fractal derivatives, to obtain local fractional solutions for the following three nonlinear PDEs:

Throughout the paper, we use $u = u(x,t)$

I. Gas dynamics equation (GDE):

$$\frac{\partial^\alpha u}{\partial t^\alpha} - u + u^2 + \frac{1}{2}\frac{\partial u^2}{\partial x} = f(x,t), \quad 0 \le x \le 1, \ t \ge 0, \tag{1.1}$$

with initial condition:

$$u(x,0) = g(x). \tag{1.2}$$

The gas dynamics equations are indeed mathematical formulations derived from the fundamental conservation laws in physics, such as the conservation of mass, momentum, and energy. These nonlinear equations, specifically for ideal gases, are used to model three types of nonlinear waves: shock fronts, contact discontinuities, and rare factions. These equations play a crucial role in understanding and analyzing the behavior of gases and the propagation of waves within them. [19]. Different analytical and numerical methods have been utilized to solve various types of gas dynamics equations in physics. For example, in [20] the investigators carried out the several plans to solve nonlinear system of equations of gas dynamic problems for a group of discontinuous functions. In the work [21], the authors employed novel homotopy perturbation techniques and the Laplace transform to derive an analytical solution for the gas dynamics equation.

II. Stefan equation (SE):

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2} \;\; 0 \le x \le t(s), \;\; s \ge 0, \tag{1.3}$$

with initial conditions:

$$\begin{aligned} u(x,0) &= -1, \;\; s \ge 0, \\ u(t(s),s) &= 0, \quad\; s \ge 0, \\ \frac{\partial u(x,t)}{\partial x} &= \frac{\partial t(s)}{\partial s}. \end{aligned} \tag{1.4}$$

The nonlinear Stefan problems, such as diffusion processes, melting, and freezing, constitute a broad domain with extensive engineering and industrial applications. These complex Stefan models represent heat-related phenomena involving state changes. They are defined by heat spreading through materials and have been extensively studied [23]. Huntul and Lesnic [26] investigated an inverse problem focused on determining the time-dependent thermal conductivity and the transient temperature that satisfy the heat equation with given boundary conditions. Many researchers have shown interest in studying the numerical and analytical solutions of the Stefan problem [30,31].

III. Newell-Whitehead-Segel equation (NWSE):

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = k\frac{\partial^2 u}{\partial x^2} + au - bu^q, \tag{1.5}$$

with initial condition:

$$u(x,0) = g(x), \tag{1.6}$$

where a, $b \in \mathbb{R}$, and $k, q \in \mathbb{N}$.

The NWSE is a nonlinear partial differential equation applied for modelling physical, chemical and biological systems including material movement through an environment. The NWSE has been widely used in diverse situations, such as astrophysics [24], plasma physics [25], and Bernard-Rayleigh convection of a fluid mixture around a bifurcation points [22]. Several scholars have focused their efforts on developing analytical and numerical solutions to this particular equation. As an illustration, in the year 2013, Ezzati and Shakibi derived solutions for two nonlinear forms of the NWS equations by employing a methodology that combined the ADM Method with the reduced differential transform technique [29].

The remaining parts of this work are structured as follows: In Section 2, The fundamental information on local fractional calculus is provided. In Section 3, the analysis of the (MLFLVIM) is given. In Section 4, we demonstrate the method on three special numerical examples of well-known physical problems and provide graphs of the proposed solutions to illustrate the accuracy and effectiveness of this approach. To conclude, we summarize our findings in Section 5.

2. Local Fractional Calculus

In this section, the essential concepts of local fractional calculus that are necessary to understand the original contributions that will be discussed later are presented. We direct the readers to the references [27,28] which provide rigorous definitions and detailed mathematical proofs related to the topics being discussed. Throughout this paper, let $\alpha \in (0,1]$.

Definition 2.1. A map $g : (a,b) \to \mathbb{R}$ is said to be continuous in the sense of Local fractional derivative at a certain t_0 if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $t \in (a,b)$ and $|t - t_0| < \delta$ the following condition holds:

$$\Big|g(t) - g(t_0)\Big| < \varepsilon^{\alpha}. \tag{2.1}$$

Definition 2.2. ***local Fractional Partial Derivative(LFPD)***
Let $h(x,t)$ be local fractional continuous in t. The LFPD of $h(x,t)$ of order α at a certain point (x,t_0) in terms of t is defined as

$$\frac{\partial^{\alpha}}{\partial t^{\alpha}}\, h(x,t_0) = \lim_{t\to t_0} \frac{\Delta^{\alpha}\Big[h(x,t) - h(x,t_0)\Big]}{(t-t_0)^{\alpha}},$$

where

$$\Delta^{\alpha}\Big[h(x,t) - h(x,t_0)\Big] \cong \Gamma(1+\alpha)\,\Big[h(x,t) - h(x,t_0)\Big].$$

Definition 2.3. The Local fractional Mittage-Leffler function is defined as: for every $\tau \in \mathbb{R}$,

$$E^{L}_{\alpha}(\tau) = \sum_{i=0}^{\infty} \frac{\tau^{i\alpha}}{\Gamma(1+i\alpha)}. \tag{2.2}$$

Lemma 2.1. Suppose g,h are partially local fractional differentiable in variable t of order α, then:

i. $\frac{\partial^{\alpha}}{\partial t^{\alpha}}\, g(x,t) = 0, \quad$ if g is a constant function in t.;
ii. $\frac{\partial^{\alpha}}{\partial t^{\alpha}}(ag+bh) = a(\frac{\partial^{\alpha}}{\partial t^{\alpha}}g) + b(\frac{\partial^{\alpha}}{\partial t^{\alpha}}h) \ \ for \ \ a,b \in \mathbb{R};$
iii. $\frac{\partial^{\alpha}}{\partial t^{\alpha}}(gh) = g(\frac{\partial^{\alpha}}{\partial t^{\alpha}}h) + h(\frac{\partial^{\alpha}}{\partial t^{\alpha}}g);$
iv. $\frac{\partial^{\alpha}}{\partial t^{\alpha}}\Big(\frac{g}{h}\Big) = \dfrac{h\frac{\partial^{\alpha}}{\partial t^{\alpha}}g - g\frac{\partial^{\alpha}}{\partial t^{\alpha}}h}{h^2}$, given that $h \neq 0$;
v. $\frac{\partial^{\alpha}}{\partial t^{\alpha}}\left(\dfrac{t^{p\alpha}}{\Gamma(1+q\alpha)}\right) = \dfrac{t^{(q-1)\alpha}}{\Gamma(1+(q-1)\alpha)} \ \ \forall q \in \mathbb{N};$
vi. $\frac{\partial^{\alpha}}{\partial t^{\alpha}}\ (E^{L}_{\alpha}(t)) = E^{L}_{\alpha}(t);$

vii. $\frac{\partial^\alpha}{\partial t^\alpha}\left(E^L_\alpha(-t)\right) = -E^L_\alpha(-t)$;

Definition 2.4. ***Local Fractional Integral(LFI)***

Let ϕ be local fractional continuous on $[a,b]$. The LFI of $\phi(t)$ of order α is given as:

$$LI^{(\alpha)}(\phi) = \frac{1}{\Gamma(1+\alpha)} \int_a^b \phi(s)(ds)^\alpha \tag{2.3}$$

$$= \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta s_k \to 0} \sum_{k=1}^{M} \phi(s_k)(\Delta s_k)^\alpha, \tag{2.4}$$

where, $M \in \mathbb{N}$ $\Delta s_k = s_k - s_{k-1}$, and $[s_k, s_{k-1}]$ for $k = 1, 2, ..., M$, $s_1 = a < s_1 < ... < s_{M-1} < s_M = b$ form a partition of $[a,b]$.

Definition 2.5. Let ϕ be local fractional continuous on $[a,b]$. The LF-Laplace transform of ϕ of order α is given as

$$\mathscr{L}_{L,\alpha}\{\phi\}(s) = \frac{1}{\Gamma(1+\alpha)} \int_0^{+\infty} E^L_\alpha(-s^\alpha t^\alpha)\,\phi(t)(dt)^\alpha, \tag{2.5}$$

Denote the LFLT of ϕ by $\Phi^{L,\alpha}(s)$.

The LFLT of ϕ exists if the following inequality holds true:

$$\frac{1}{\Gamma(1+\alpha)} \int_0^\infty |\phi(t)|(dt)^\alpha < \infty.$$

Definition 2.6. The inverse of LFLT is defined as:

$$\phi(t) = \mathscr{L}^{-1}_{L,\alpha}\{\,\Phi^{L,\alpha}(s)\,\} \tag{2.6}$$

Lemma 2.2. ***LF Laplace Transform's Properties***

i. $\mathscr{L}_{L,\alpha}\{\,c_1\,\theta(t) \pm c_2\,\delta(t)\,\} = c_1\mathscr{L}_{L,\alpha}\{\theta(t)\} \pm c_2\mathscr{L}_{L,\alpha}\{\delta(t)\}$;

ii. $\mathscr{L}_{L,\alpha}\{1\} = \frac{1}{s^\alpha}$;

iii. $\mathscr{L}_{L,\alpha}\left\{\frac{t^{m\alpha}}{\Gamma(1+m\alpha)}\right\} = \frac{1}{s^{\alpha(m+1)}}, m \in \mathbb{N}$;

iv. $\mathscr{L}_{L,\alpha}\left\{E^L_\alpha(a^\alpha t)\right\} = \frac{1}{s^\alpha - a^\alpha}$, provided that $s^\alpha > a^\alpha$;

Theorem 2.1. Suppose that $\mathscr{L}_{L,\alpha}[\psi(x,t)] = \Psi^{L,\alpha}(x,s)$ exists and $\psi(x,t)$ approaches 0 as $t \to \infty$, then

$$\mathscr{L}_{L,\alpha}\Big[\frac{\partial^\alpha \psi(x,t)}{\partial t^\alpha}\Big] = s^\alpha \Psi^{L,\alpha}(x,s) - \psi(x,0) \tag{2.7}$$

3. Analysis of MLFLVIM

In this section, we describe the procedure of the introduced technique known as (MLFLVIM), for the following general form of a nonlinear Local Fractional PDE:

$$\frac{\partial^\alpha u}{\partial t^\alpha} + N(x,u,t) = 0, \tag{3.1}$$

with I.C.

$$u(x,0) = f(x), \tag{3.2}$$

where N represents a nonlinear operator that depends on the function u and its derivatives.

Apply LFLT on Eq. (3.1), we get

$$\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u}{\partial t^\alpha} + N(x,u,t)\right] = 0, \tag{3.3}$$

We multiply the equation Eq. (3.3) by a Local fractional Lagrange multiplier $\lambda_\alpha(s)$, we acquire:

$$\lambda_\alpha(s)\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u}{\partial t^\alpha} + N(x,u,t)\right] = 0. \tag{3.4}$$

Rewrite Eq. (3.4) as follows:

$$U^{L,\alpha}(x,s) = U^{L,\alpha}(x,s) + \lambda_\alpha(s)\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u}{\partial t^\alpha} + N(x,u,t)\right]. \tag{3.5}$$

Inspired by He's Variational Iteration Method (VIM), we can formulate the following iterative equation:

$$U_{n+1}^{L,\alpha}(x,s) = U_n^{L,\alpha}(x,s) + \lambda_\alpha(s)\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_n}{\partial t^\alpha} + N(x,\tilde{u}_n,t)\right]. \tag{3.6}$$

To determine the expression for $\lambda(s)$, we take the variation of Eq. (3.6). Then, by applying the restricted variations, $\delta\tilde{u}_n = 0$ and $\delta\tilde{U}_n(x,0) = 0$, we obtain the following result:

$$\delta\, U_{n+1}^{L,\alpha}(x,s) = s^\alpha\lambda_\alpha(s)\delta\, U_n^{L,\alpha}(x,s) + \delta\, U_n^{L,\alpha}(x,s) = 0 \tag{3.7}$$

This process leads to the stationary conditions:

$$1 + s^\alpha\lambda_\alpha(s) = 0, \text{or } \breve{}(\mathrm{s}) = -\frac{1}{\mathrm{s}^{\mathrm{ff}}} \tag{3.8}$$

By substituting the derived expression for $\lambda(s)$ into Eq. (3.6), we obtain the iteration equation

$$U_{n+1}^{L,\alpha}(x,s) = U_n^{L,\alpha}(x,s) - \frac{1}{s^\alpha}\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_n}{\partial t^\alpha} + N(x,u_n,t)\right].. \tag{3.9}$$

Now, by applying the inverse of the LFLT to the expression given in Eq. (3.9), we obtain the following result:

$$u_{n+1}(x,t) = u_n - \mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_n}{\partial t^\alpha} + N(x,u_n,t)\right]\right]. \tag{3.10}$$

We begin by considering the zeroth approximation$u_0(x,t)$, and by substituting this value into the expression given in Eq. (3.10), we can iteratively compute u_1, u_2, $\cdots$, $u_n, \cdots$, and so forth. As a result, the proposed local fractional solution to the given Equations Eqs. (3.1) and (3.2) can be obtained as follows:

$$u(x,t) = \lim_{n\to\infty} u_n(x,t). \tag{3.11}$$

The convergence behavior of this iterative process is contingent upon the specific form of the nonlinear term present in the equations. For more comprehensive insights and details regarding the convergence analysis, one can refer to the reference [32]

4. Applications

In this section, we present three well-known partial differential equations with local fractional derivatives in fractal dimension. These example problems were chosen because they possess closed-form analytical solutions in sense of fractal derivative. Their closed-form analytical solutions will facilitate analyzing and studying the accuracy assessment of numerical results.

Example 4.1. Consider the following NWSE in sense of local fractional derivative with $k = 1$, $a = -2$, and $b = 0$:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = -2u + \frac{\partial^2 u}{\partial x^2}, \tag{4.1}$$

subject to I.C.

$$u(x,0) = e^x. \tag{4.2}$$

By employing the proposed methodology, we can derive a recurrence relationship as follows:

$$U_{n+1}^{L,\alpha}(x,s) = \lambda_\alpha(s)\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_n}{\partial t^\alpha} - \frac{\partial^2 u_n}{\partial x^2} + 2u_n\right] + U_n^{L,\alpha}. \tag{4.3}$$

Performing the variational operation on the equation Eq. (4.3), we obtain the subsequent expression:

$$\delta\, U_{n+1}^{L,\alpha}(x,s) = \lambda_\alpha(s)\delta\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_n}{\partial t^\alpha} + 2\tilde{u}_n - \frac{\partial^2 \tilde{u}_n}{\partial x^2}\right] + \delta\, U_n^{L,\alpha}. \tag{4.4}$$

By applying the constrained variations $\delta\tilde{u}_n = 0$, we reach at the following result:

$$\delta\, U_{n+1}^{L,\alpha}(x,s) = s^\alpha\lambda_\alpha(s)\delta\, U_n^{L,\alpha} + \delta\, U_n^{L,\alpha} = 0. \tag{4.5}$$

This process yields the stationary condition $1 + s^\alpha\lambda(s) = 0$, which consequently leads to $\lambda(s) = -\frac{1}{s^\alpha}$. Substituting this expression for $\lambda(s)$ into Eq. (4.3), we obtain the following result:

$$U_{n+1}^{L,\alpha}(x,s) = U_n^{L,\alpha} - \frac{1}{s^\alpha}\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_n}{\partial t^\alpha} + 2u_n - \frac{\partial^2 u_n}{\partial x^2}\right]. \tag{4.6}$$

Performing the inverse Laplace transform operation on Eq. (4.6), we obtain:

$$u_{n+1}(x,s) = u_n - \mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_n}{\partial t^\alpha} - \frac{\partial^2 u_n}{\partial x^2} + 2u_n\right]\right]. \tag{4.7}$$

By selecting the initial condition $u_0(x,t) = e^x$, and substituting it in Eq. (4.7), we can derive the following sequence of successive approximations:

For $n = 0$:

$$
\begin{aligned}
u_1(x,t) &= u_0 - \mathscr{L}^{-1}_{L,\alpha}\left[\frac{1}{s^\alpha}\mathscr{L}_{L,\alpha}\left[\frac{\partial u_0}{\partial t} - \frac{\partial^2 u_0}{\partial x^2} + 2u_0\right]\right],\\
&= e^x - \mathscr{L}^{-1}_{L,\alpha}\left[\frac{1}{s^\alpha}\mathscr{L}_{L,\alpha}\left[-\frac{\partial^2 u_0}{\partial x^2} + 2u_0\right]\right],\\
&= e^x - \mathscr{L}^{-1}_{L,\alpha}\left[\frac{e^x}{s^{2\alpha}}\right],\\
&= e^x\Big(1 - \frac{t^\alpha}{\Gamma(1+\alpha)}\Big).
\end{aligned} \tag{4.8}
$$

For $n = 1$:

$$
\begin{aligned}
u_2(x,t) &= u_1 - \mathscr{L}^{-1}_{L,\alpha}\left[\frac{1}{s^\alpha}\mathscr{L}_{L,\alpha}\left[\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} + 2u_1\right]\right],\\
&= e^x\Big(1 - \frac{t^\alpha}{\Gamma(1+\alpha)}\Big) - \mathscr{L}^{-1}_{L,\alpha}\left[\left[\frac{e^x}{s^{3\alpha}}\right]\right],\\
&= e^x\Big(1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\Big).
\end{aligned} \tag{4.9}
$$

For $n = 2$:

$$
\begin{aligned}
u_3(x,t) &= u_2 - \mathscr{L}^{-1}_{L,\alpha}\left[\frac{1}{s^\alpha}\mathscr{L}_{L,\alpha}\left[\frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial x^2} + 2u_2\right]\right],\\
&= e^x\Big(1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\Big) - \mathscr{L}^{-1}_{L,\alpha}\left[\frac{e^x}{s^{4\alpha}}\right],\\
&= e^x\Big(1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\Big).
\end{aligned} \tag{4.10}
$$

Continuing with this iterative process, we obtain the general formulation for u_n as follows:

$$
u_n(x,t) = e^x\Big(1 - \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots + \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\Big), \ \forall\ n \geq 0. \tag{4.11}
$$

Hence, the approximate local fractional analytical solution is:

$$
\begin{aligned}
u(x,t) &= \lim_{n\to\infty} u_n(x,t)\\
&= e^x \lim_{n\to\infty} \sum_{j=0}^{n} \frac{(-1)^j t^{j\alpha}}{\Gamma(1+j\alpha)}\\
&= e^x E^L_\alpha(-t).
\end{aligned} \tag{4.12}
$$

This solution aligns perfectly with the exact solution of the given problem in the sense of local fractional derivative.

It is worth to mention that, in case of $\alpha = 1$, the current solution becomes $u(x,t) = e^{x-t}$ which is the exact solution of the ordinary problem with classical derivatives. The plot of this solution is depicted in Figure 1, generated using Python for four distinct values of α.

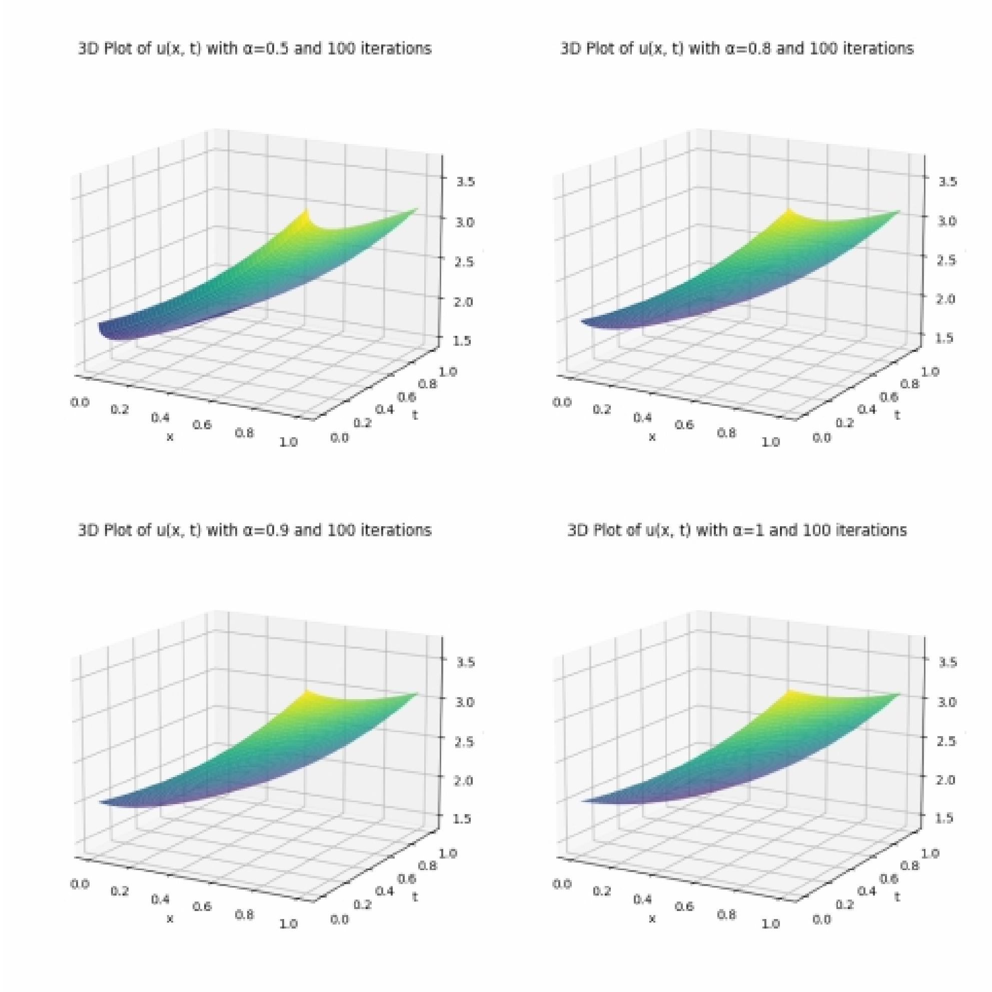

Figure 1. The graph represents the local fractional solution of NWSE Eq. (4.1) with four distinct values of α

Example 4.2. Consider the local fractional SE in fractal dimension as follows:

$$\frac{\partial^{\alpha} u}{\partial t^{\alpha}} = \frac{\partial^2 u}{\partial x^2}, \tag{4.13}$$

subject to I.C.

$$u(x,0) = \cosh(x) - x. \tag{4.14}$$

By employing the proposed methodology, we can derive a recurrence relationship as follows:

$$U_{n+1}^{L,\alpha}(x,s) = \lambda_{\alpha}(s)\mathscr{L}_{L,\alpha}\left[\frac{\partial^{\alpha} u_n}{\partial t^{\alpha}} - \frac{\partial^2 u_n}{\partial x^2}\right] + U_n^{L,\alpha}. \tag{4.15}$$

Performing the variational operation on the equation Eq. (4.15), we obtain the subsequent expression:

$$\delta\, U_{n+1}^{L,\alpha}(x,s) = \lambda_{\alpha}(s)\delta\mathscr{L}_{L,\alpha}\left[\frac{\partial^{\alpha} u_n}{\partial t^{\alpha}} - \frac{\partial^2 \tilde{u}_n}{\partial x^2}\right] + \delta\, U_n^{L,\alpha}. \tag{4.16}$$

By applying the constrained variations $\delta\tilde{u}_n = 0$, we reach at the following result:

$$\delta\, U_{n+1}^{L,\alpha}(x,s) = s^{\alpha}\lambda_{\alpha}(s)\delta\, U_n^{L,\alpha} + \delta\, U_n^{L,\alpha} = 0. \tag{4.17}$$

This process yields the stationary condition $1 + s^{\alpha}\lambda(s) = 0$, which consequently leads to $\lambda(s) = -\frac{1}{s^{\alpha}}$. Substituting this expression for $\lambda(s)$ into Eq. (4.15), we obtain the following result:

$$U_{n+1}^{L,\alpha}(x,s) = U_n^{L,\alpha} - \frac{1}{s^{\alpha}}\mathscr{L}_{L,\alpha}\left[\frac{\partial^{\alpha}u_n}{\partial t^{\alpha}} - \frac{\partial^2 u_n}{\partial x^2}\right]. \tag{4.18}$$

Performing the inverse Laplace transform operation on Eq. (4.18), we obtain:

$$u_{n+1}(x,s) = u_n - \mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}_{L,\alpha}\right]\left[\frac{\partial^{\alpha}u_n}{\partial t^{\alpha}} - \frac{\partial^2 u_n}{\partial x^2}\right]. \tag{4.19}$$

By selecting the initial condition $u_0(x,t) = u(x,0) = \cosh(x) - x$, and substituting it in Eq. (4.19), we can derive the following sequence of successive approximations:

For $n = 0$:

$$\begin{aligned}
u_1(x,t) &= u_0 - \mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}_{L,\alpha}\right]\left[\frac{\partial^{\alpha}u_0}{\partial t^{\alpha}} - \frac{\partial^2 u_0}{\partial x^2}\right].\\
&= -x + \cosh(x) - \mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}_{L,\alpha}\right]\Big[\cosh(x)\Big].\\
&= -x + \cosh(x) - \cosh(x)\mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^{2\alpha}}\right].\\
&= -x + \cosh(x)\left[1 - \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right].
\end{aligned} \tag{4.20}$$

For $n = 1$:

$$\begin{aligned}
u_2(x,t) &= u_1 - \mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}_{L,\alpha}\right]\left[\frac{\partial^{\alpha}u_1}{\partial t^{\alpha}} - \frac{\partial^2 u_1}{\partial x^2}\right].\\
&= -x + \cosh(x)\left[1 - \frac{t^{\alpha}}{\Gamma(1+\alpha)}\right] - \mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}_{L,\alpha}\right]\left[-2\cosh(x) + \cosh(x)\frac{t^{\alpha}}{\Gamma(1+\alpha)}\right].\\
&= -x + \cosh(x)\left[1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right].
\end{aligned} \tag{4.21}$$

For $n = 2$:

$$\begin{aligned}
u_3(x,t) &= u_2 - \mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}_{L,\alpha}\right]\left[\frac{\partial^{\alpha}u_2}{\partial t^{\alpha}} - \frac{\partial^2 u_2}{\partial x^2}\right].\\
&= -x + \cosh(x)\left[1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} - \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right] - \cosh(x)\mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^{\alpha}}\mathscr{L}_{L,\alpha}\right]\left[-2\frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right]\\
&= -x + \cosh(x)\left[1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)}\right].
\end{aligned} \tag{4.22}$$

Continuing with this iterative process, we obtain the general formulation for u_n as follows:

$$u_n(x,t) = -x + \cosh(x)\left[1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} - \frac{t^{3\alpha}}{\Gamma(1+3\alpha)} + \cdots + \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right],\ \forall\ n \geq 0. \tag{4.23}$$

Hence, the approximate local fractional analytical solution is:

$$\begin{aligned} u(x,t) &= \lim_{n\to\infty} u_n(x,t). \\ &= -x + \cosh(x) \lim_{n\to\infty} \sum_{j=0}^{n} \frac{t^{j\alpha}}{\Gamma(1+j\alpha)}. \\ &= -x + \cosh(x) E_\alpha^L(t). \end{aligned} \tag{4.24}$$

This solution aligns perfectly with the exact solution of the given problem in the sense of local fractional derivative.

It is worth to mention that, in case of $\alpha = 1$, the current solution becomes $u(x,t) = -x + \cosh(x)e^t$ which is the exact solution of the ordinary problem with classical derivatives. The plot of this solution is depicted in Figure 2, generated using Python for four distinct values of α.

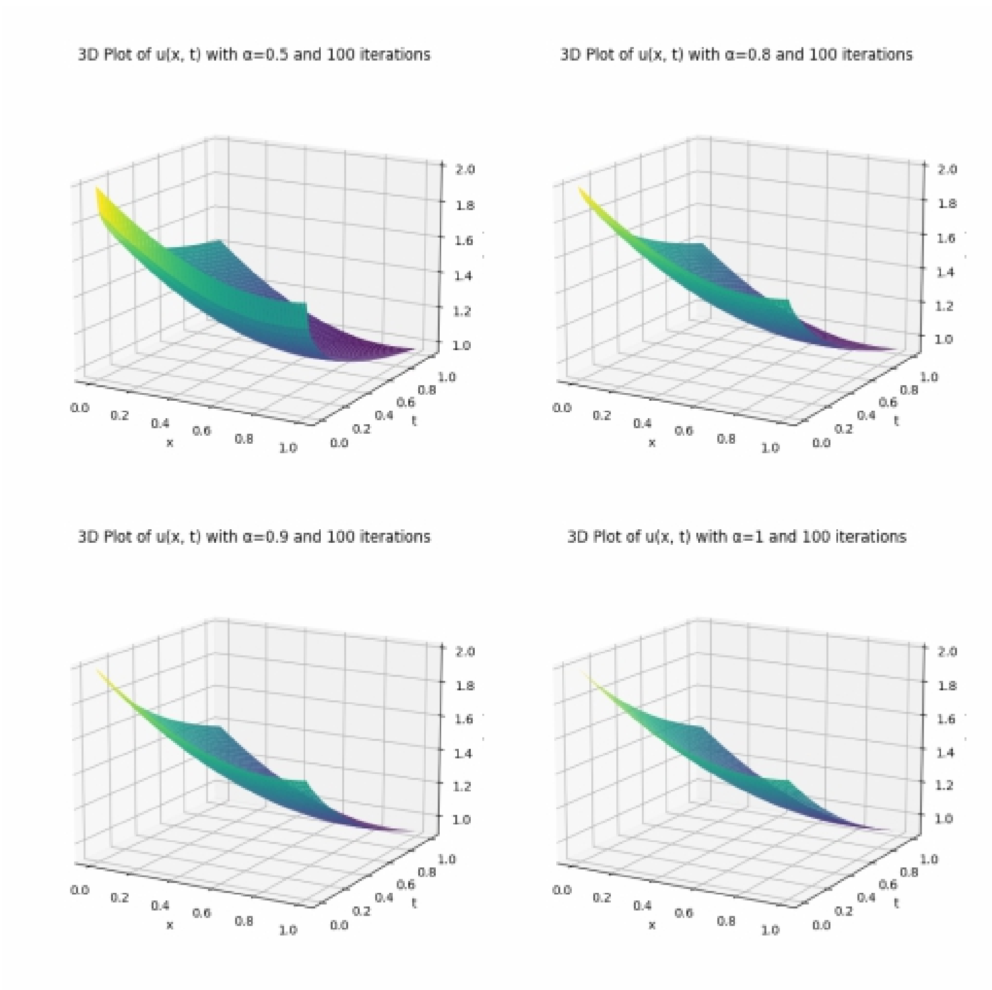

Figure 2. The graph represents the local fractional solution of SE Eq. (4.13) with four distinct values of α.

Example 4.3. Consider the local fractional NGDE in fractal dimension as follows:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = u - \frac{1}{2}\frac{\partial u^2}{\partial x} - u^2, \quad 0 \leq x \leq 1, \;\; t \geq 0, \tag{4.25}$$

subject to I.C.

$$u(x,0) = e^{-x}. \tag{4.26}$$

By employing the proposed methodology, we can derive a recurrence relationship as follows:

$$U_{n+1}^{L,\alpha}(x,s) = \lambda_\alpha(s)\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_n}{\partial t^\alpha} - u_n + u_n^2 + \frac{1}{2}\frac{\partial u_n^2}{\partial x}\right] + U_n^{L,\alpha}. \tag{4.27}$$

Performing the variational operation on the equation Eq. (4.27), we obtain the subsequent expression:

$$\delta\, U_{n+1}^{L,\alpha}(x,s) = \lambda_\alpha(s)\delta\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_n}{\partial t^\alpha} - \tilde{u}_n(x,t) + \frac{1}{2}\frac{\partial \tilde{u}_n^2}{\partial x} + \tilde{u}_n^2\right] + \delta\, U_n^{L,\alpha} \tag{4.28}$$

By applying the constrained variations $\delta\tilde{u}_n = 0$, we reach at the following result:

$$\delta\, U_{n+1}^{L,\alpha}(x,s) = s^\alpha\lambda_\alpha(s)\delta\, U_n^{L,\alpha} + \delta\, U_n^{L,\alpha} = 0. \tag{4.29}$$

This process yields the stationary condition $1 + s^\alpha\lambda(s) = 0$, which consequently leads to $\lambda(s) = -\frac{1}{s^\alpha}$. Substituting this expression for $\lambda(s)$ into Eq. (4.27), we obtain the following result:

$$U_{n+1}^{L,\alpha}(x,s) = U_n^{L,\alpha} - \frac{1}{s^\alpha}\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_n}{\partial t^\alpha} - u_n + u_n^2 + \frac{1}{2}\frac{\partial u_n^2}{\partial x}\right]. \tag{4.30}$$

Performing the inverse Laplace transform operation on Eq. (4.30), we obtain:

$$u_{n+1}(x,s) = u_n - \mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_n}{\partial t^\alpha} - u_n + u_n^2 + \frac{1}{2}\frac{\partial u_n^2}{\partial x}\right]\right]. \tag{4.31}$$

By selecting the initial condition $u_0(x,t) = u(x,0) = e^{-x}$, and substituting this value into Eq. (4.31), we can derive the following sequence of successive approximations:
For $n = 0$:

$$\begin{aligned} u_1(x,t) &= u_0 - \mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_0}{\partial t^\alpha} - u_0 + u_0^2 + \frac{1}{2}\frac{\partial u_0^2}{\partial x}\right]\right]. \\ &= e^{-x} + e^{-x}\mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^{2\alpha}}\right]. \\ &= e^{-x}[1 + \frac{t^\alpha}{\Gamma(1+\alpha)}]. \end{aligned} \tag{4.32}$$

For $n = 1$:

$$u_2(x,t) = u_1 - \mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^\alpha}\mathscr{L}_{L,\alpha}\left[\frac{\partial^\alpha u_1}{\partial t^\alpha} - u_1 + u_1^2 + \frac{1}{2}\frac{\partial u_1^2}{\partial x}\right]\right]. \tag{4.33}$$

$$= e^{-x}[1 + \frac{t^\alpha}{\Gamma(1+\alpha)}] + e^{-x}\mathscr{L}_{L,\alpha}^{-1}\left[\frac{1}{s^{3\alpha}}\right]. \tag{4.34}$$

$$= e^{-x}[1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)}]. \tag{4.35}$$

Continuing with this iterative process, we obtain the general formulation for u_n as follows:

$$u_n(x,t) = e^{-x}\left[1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots + \frac{t^{n\alpha}}{\Gamma(1+n\alpha)}\right], \quad \forall\, n \geq 0. \tag{4.36}$$

Therefore, the approximate local fractional analytical solution is:

$$\begin{aligned} u(x,t) &= e^{-x} \lim_{n\to\infty} \sum_{j=0}^{n} \frac{t^{j\alpha}}{\Gamma(1+j\alpha)} \\ &= e^{-x} E_{\alpha}^{L}(t). \end{aligned} \tag{4.37}$$

This solution aligns perfectly with the exact solution of the given problem in the sense of local fractional derivative.

It is worth to mention that, in case of $\alpha = 1$, the current solution becomes $u(x,t) = e^{-x+t}$ which is the exact solution of the ordinary problem with classical derivatives. The plot of this solution is depicted in Figure 3, generated using Python for four distinct values of α.

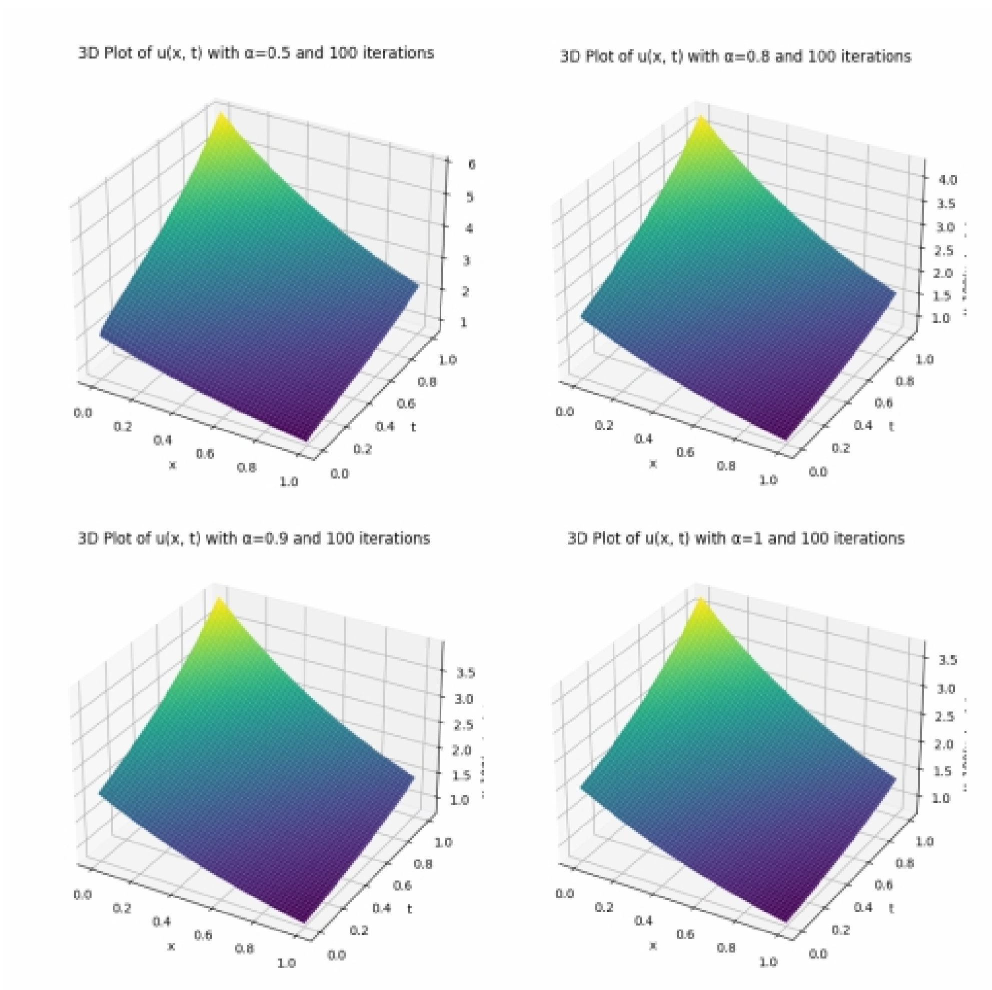

FIGURE 3. The graph represents the local fractional solution of NGDE Eq. (4.25) with four distinct values of α

5. Conclusion

In this paper, we effectively applied the proposed method to derive approximate local fractional analytical solutions for the Nonlinear Gas Dynamics Equations (NGDE), the Stefan Equation (SE), and Newell-Whitehead-Segel equation (NWSE) involving local fractional derivatives. The above applications demonstrate that the proposed technique is a straightforward, effective, and precise technique that rapidly approaches the closed-form solution in the neighborhood of the initial point, providing an accurate power series solution. Furthermore, it operates without the need for variable discretization, substantial machine memory, or the computational time required by other methods. This method is sufficiently qualified to reduce the computational size and time. Its advantages make it a more efficient and practical approach compared to traditional methods. As perspective of this work, we plan to explore the use of local fractional derivatives in image processing and machine learning to enhance performance. By applying this method, we aim to improve edge detection and texture analysis in images. In machine learning, it can offer a new approach to feature extraction, helping to model complex, non-linear patterns in data. This technique could lead to more accurate and efficient algorithms for various applications.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] F. Mainardi, Fractional Calculus and Waves in Linear Viscoelasticity: An Introduction to Mathematical Models, Imperial College Press, 2010. https://doi.org/10.1142/p614.

[2] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley, 1993. https://cir.nii.ac.jp/crid/1130282270123129216.

[3] I. Podlubny, Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications, Academic Press, San Diego, 1998.

[4] A. Carpinteri, P. Cornetti, A. Sapora, Nonlocal Elasticity: An Approach Based on Fractional Calculus, Meccanica 49 (2014), 2551–2569. https://doi.org/10.1007/s11012-014-0044-5.

[5] S. Seddighi Chaharborj, S. Seddighi Chaharborj, Z. Seddighi Chaharborj, P.S. Phang, Theoretical Fractional Formulation of a Three-Dimensional Radio Frequency Ion Trap (Paul-Trap) for Optimum Mass Separation, Eur. J. Mass Spectrometry 27 (2021), 73–83. https://doi.org/10.1177/14690667211026790.

[6] Y.M. Chu, M. Jneid, A. Chaouk, M. Inc, H. Rezazadeh, A. Houwe, Local Time Fractional Reduced Differential Transform Method for Solving Local Time Fractional Telegraph Equations, Fractals 32 (2024), 2340128. https://doi.org/10.1142/S0218348X2340128X.

[7] A. Chaouk, M. Jneid, Analytic Solution for Systems of Two-Dimensional Time Conformable Fractional PDEs by Using CFRDTM, Int. J. Math. Math. Sci. 2019 (2019), 7869516. https://doi.org/10.1155/2019/7869516.

[8] K. Abuasbeh, M. Awadalla, M. Jneid, Nonlinear Hadamard Fractional Boundary Value Problems with Different Orders, Rocky Mt. J. Math. 51 (2021), 17–29. https://doi.org/10.1216/rmj.2021.51.17.

[9] M. Jneid, A. Chaouk, The Conformable Reduced Differential Transform Method for Solving Newell-Whitehead-Segel Equation With Non-Integer Order, J. Anal. Appl. 18 (2020) 35–51.

[10] M. Jneid, New Conformable Fractional HPT for Solving Systems of Linear and Nonlinear Conformable Fractional PDEs, Italian J. Pure Appl. Math. 48 (2022), 1242–1253.

[11] M. Jneid, A.E. Chakik, Analytical Solution for Some Systems of Nonlinear Conformable Fractional Differential Equations, Far East J. Math. Sci. 109 (2018), 243–259. https://doi.org/10.17654/MS109020243.

[12] M. Jneid, M. Awadalla, On the Controllability of Conformable Fractional Deterministic Control Systems in Finite Dimensional Spaces, Int. J. Math. Math. Sci. 2020 (2020), 9026973. https://doi.org/10.1155/2020/9026973.

[13] M. Jneid, A. Chaouk, Recent Advancement on Analytical Solution for Linear and Nonlinear Systems of Partial Differential Equations Involving Time Conformable Fractional Derivatives, in: Theory and Applications of Mathematical Sciences, Book Publisher International, 2020.

[14] M. Jneid, Exact Controllability of Semilinear Control Systems Involving Conformable Fractional Derivatives, AIP Conf. Proc. 2159 (2019), 030017. https://doi.org/10.1063/1.5127482.

[15] L. Akinyemi, M. Şenol, S.N. Huseen, Modified Homotopy Methods for Generalized Fractional Perturbed Zakharov-Kuznetsov Equation in Dusty Plasma, Adv. Differ. Equ. 2021 (2021), 45. https://doi.org/10.1186/s13662-020-03208-5.

[16] M. Jneid, The Fractional Annihilator Technique for Solving Nonhomogeneous Fractional Linear Differential Equations, Int. J. Math. Anal. 12 (2018), 61–70. https://doi.org/10.12988/ijma.2018.712157.

[17] L. Akinyemi, O.S. Iyiola, Analytical Study of $(3+1)$-Dimensional Fractional-Reaction Diffusion Trimolecular Models, Int. J. Appl. Comput. Math. 7 (2021), 92. https://doi.org/10.1007/s40819-021-01039-w.

[18] M. Jneid, Solving Newell-Whitehead-Segel, Stefan and Nonlinear Gas Dynamics Equations via Modified Laplace Variational Iteration Technique, AIP Conf. Proc. 3051 (2024), 080003. https://doi.org/10.1063/5.0191598.

[19] A.D. Polyanin, V.F. Zaitsev, Handbook of Nonlinear Partial Differential Equations, Chapman & Hall/CRC, Boca Raton, 2004.

[20] M. Rasulov, T. Karaguler, Finite Difference Schemes for Solving System Equations of Gas Dynamic in a Class of Discontinuous Functions, Appl. Math. Comput. 143 (2003), 145–164. https://doi.org/10.1016/S0096-3003(02)00353-3.

[21] H. Aminikhah, A. Jamalian, Numerical Approximation for Nonlinear Gas Dynamic Equation, Int. J. Part. Differ. Equ. 2013 (2013), 846749. https://doi.org/10.1155/2013/846749.

[22] H.C. Rosu, O. Cornejo-Pérez, Supersymmetric Pairing of Kinks for Polynomial Nonlinearities, Phys. Rev. E. 71 (2005), 046607. https://doi.org/10.1103/PhysRevE.71.046607.

[23] D.A. Tarzia, Explicit and Approximated Solutions for Heat and Mass Transfer Problems With a Moving Interface, in: M. El-Amin (Ed.), Advanced Topics in Mass Transfer, InTech Open Access Publisher, Rijeka, 439-484, (2011).

[24] A. Sen, S. Tiwari, S. Mishra, P. Kaw, Nonlinear Wave Excitations by Orbiting Charged Space Debris Objects, Adv. Space Res. 56 (2015), 429–435. https://doi.org/10.1016/j.asr.2015.03.021.

[25] J. Tamang, K. Sarkar, A. Saha, Solitary Wave Solution and Dynamic Transition of Dust Ion Acoustic Waves in a Collisional Nonextensive Dusty Plasma with Ionization Effect, Physica A: Stat. Mech. Appl. 505 (2018), 18–34. https://doi.org/10.1016/j.physa.2018.02.213.

[26] M.J. Huntul, D. Lesnic, An Inverse Problem of Finding the Time-Dependent Thermal Conductivity from Boundary Data, Int. Commun. Heat Mass Transfer 85 (2017), 147–154. https://doi.org/10.1016/j.icheatmasstransfer.2017.05.009.

[27] X.J. Yang, D. Baleanu, H.M. Srivastava, Local Fractional Integral Transforms and their Applications, Academic Press, pp. 1–55, 2016.

[28] X.J. Yang, Advanced Local Fractional Calculus and Its Applications, World Science Publisher, 2012.

[29] R. Ezzati, K. Shakibi, Using Adomian's Decomposition and Multiquadric Quasi-Interpolation Methods for Solving Newell–Whitehead Equation, Procedia Computer Sci. 3 (2011), 1043–1048. https://doi.org/10.1016/j.procs.2010.12.171.

[30] N.G. Burago, A.I. Fedyushkin, Numerical Solution of the Stefan Problem, J. Phys.: Conf. Ser. 1809 (2021), 012002. https://doi.org/10.1088/1742-6596/1809/1/012002.

[31] J. Matlak, D. Slota, Solution of the Stefan Problem by Involving the Material Shrinkage, Techn. Trans. Mech. 2-M (2015), 157–164.

[32] M. Tatari, M. Dehghan, On the Convergence of He's Variational Iteration Method, J. Comput. Appl. Math. 207 (2007), 121–128. https://doi.org/10.1016/j.cam.2006.07.017.