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Approximating Fixed Point of (α, β, γ)-Nonexpansive Mapping Using Picard-Thakur Iterative Scheme

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Abstract. In this paper, we explore the fixed point approximation of (α, β, γ)–nonexpansive mappings using the Picard-Thakur iterative scheme. We identify the efficiency of Picard-Thakur iterative scheme through a numerical example by comparing it with other iterative schemes. We also proved strong and weak fixed point convergence theorems for (α, β, γ)–nonexpansive mappings by using Picard-Thakur iterative scheme.

1. Introduction

Let $K \neq \emptyset$ be a subset of a norm linear space $\mathcal{B}$. A mapping $\mathcal{H} : K \to K$ on K is called a contraction mapping if $\forall\, x, y \in K$, we have

$$\|\mathcal{H}x - \mathcal{H}y\| \leq \alpha\|x - y\|, \text{ for some fixed } \alpha \in [0, 1). \tag{1.1}$$

Some time, finding the solution of fixed point problem analytically is very difficult, that is why we need numerical approximate solutions. The numerical approximation of solutions for nonlinear operators is indeed an attractive area for researchers across various fields.

The Banach Contraction Principle (BCP) [5] which uses Picard [18] iterative scheme have indeed made meaningful contributions to fixed point theory but Picard iterative scheme fails to work for nonexpansive mappings (put $\alpha = 1$ in equation (1.1)). As time goes on, in 1965 (cf. Browder [6], Gohde [8] and others) proved fixed point theorems for nonexpansive mappings. The philosophy of Browder and Gohde were used by Kirk [14] in reflexive Banach spaces.

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In 2008 Suzuki [22] provided a new type of generalization of nonexpansive mappings and proved some related fixed point results for this class of mappings in Banach spaces. A self-map $\mathcal{H} : K \to K$ is said to be mapping with (C) property (also known as Suzuki mapping) if $\forall\, x, y \in K$, we have:

$$\frac{1}{2}\left\|x-\mathcal{H}x\right\| \leq \left\|x-y\right\| \Rightarrow \left\|\mathcal{H}x-\mathcal{H}y\right\| \leq \left\|x-y\right\|.$$

Aoyama and Kahsoka [4] suggest α–nonexpansive map $\mathcal{H}$ defined on a subset K of Banch space as if a real number can be defined $\alpha \in [0,1]$ such that for each element $x, y \in K$, we have:

$$\|\mathcal{H}x-\mathcal{H}y\|^2 \leq \alpha\|x-\mathcal{H}y\|^2 + \alpha\|y-\mathcal{H}x\|^2 + (1-2\alpha)\|x-y\|^2.$$

A self map on a subset K of a Banach space is said to be generalized α–nonexpansive if one can find a real number $\alpha \in [0,1]$ such that for $x, y \in K$, we have:

$$\frac{1}{2}\|x-\mathcal{H}x\| \leq \|x-y\| \Rightarrow \|\mathcal{H}x-\mathcal{H}y\| \leq \alpha\|x-\mathcal{H}y\| + \alpha\|y-\mathcal{H}x\| + (1-2\alpha)\|x-y\|.$$

Generalized (α, β)–nonexpansive mapping on a subset K of Banach space has been introduced by Ullah *et al.* defined as:

$$\frac{1}{2}\|x-\mathcal{H}x\| \leq \|x-y\| \Rightarrow \|\mathcal{H}x-\mathcal{H}y\| \leq \alpha\|x-\mathcal{H}y\| + \alpha\|y-\mathcal{H}x\| + \beta\|x-\mathcal{H}x\|,$$

$\forall x, y \in K$ where $\alpha, \beta \in \mathbb{R}^+$ and $\alpha + \beta < 1$.

Fixed point theory plays an important role in different fields of mathematics. Some time finding the solution of fixed point problem analytically is very difficult, and hence there is a need of approximate solutions. Several iterative schemes have been developed by many researchers for solving fixed point problems for different operators over different spaces. Some of well known iterations are Mann [15] , Ishikawa [9], Noor [16], Agarwal [3], Abbas [1], Nawaz et al. [28] etc. In 2022 Jia et al. [10] proposed Picard-Thakur itrative scheme which is defined as:

$$\left.\begin{aligned} &a_1 \in K, \\ &a_{n+1} = \mathcal{H}b_n \\ &b_n = (1-\alpha_n)\mathcal{H}d_n + \alpha_n\mathcal{H}c_n, \\ &c_n = (1-\beta_n)d_n + \beta_n\mathcal{H}d_n, \\ &d_n = (1-\gamma_n)a_n + \gamma_n\mathcal{H}a_n. \end{aligned}\right\} \tag{1.2}$$

It has been proved that the new iterative scheme converges faster than the leading iterative schemes like Thakur, Picard-Ishikawa, Picard-S, Picard- S* iterative schemes. By using Picard-Thakur iterative scheme we prove some fixed point results for (α, β, γ)-nonexpansive mappings. We also discuss a numerical example to show the efficiency of Picard-Thakur iterative scheme by comparing it with other well-known iterative schemes.

Definition 1.1. *[25]Let $K \neq \emptyset$ be a subset of a norm linear space $\mathcal{B}$. A mapping $\mathcal{H}$ on K is called (α, β, γ)–nonexpansive if*

$$\|\mathcal{H}x-\mathcal{H}y\| \leq \alpha\|x-y\| + \beta\|x-\mathcal{H}x\| + \gamma\|x-\mathcal{H}y\| \;\forall\, x, y \in K$$

where $\alpha, \beta, \gamma \in \mathbb{R}^+$ are fixed scalers such that $\gamma \in [0,1)$ and $\alpha + \gamma \leq 1$.

Definition 1.2. *Let $\mathcal{B}$ be a Banach space and $\mathcal{B} \supseteq \{p_n\}$ be bounded. If $\emptyset \neq K \subseteq \mathcal{B}$ is convex and closed. Then asymptotic radius of $\{p_n\}$ corresponding to K is defined as:*

$$r(K, \{p_n\}) = \inf\{\limsup_{n\to\infty} \|p_n - p^*\| : p^* \in K\}$$

Likewise, the asymptotic center of the sequence $\{p_n\}$ corresponding to K is given as,

$$\mathcal{A}(K, \{p_n\}) = \{s \in K : \limsup_{n\to\infty} \|p_n - p^*\| = r(K, p_n)\}$$

Remark 1.1. *$\mathcal{B}$ is represent to be a UCB space [7] , then it is established that $\mathcal{A}(K, \{p_n\})$ contains a unique element. Moreover it is noted that when K is both convex and weakly compact, then $\mathcal{A}(K, \{p_n\})$ is convex . (see e.g., [2, 23] and others).*

Definition 1.3. *[17] A Banach space $\mathcal{B}$ is said to be satisfy the Opial's condition if for $\{p_n\} \subseteq \mathcal{B}$ converging weakly to $p^* \in K$, then the following condition holds:*

$$\limsup_{n\to\infty} \|p_n - p^*\| < \limsup_{n\to\infty} \|p_n - e_0\| \ \forall e_0 \in \mathcal{B}\backslash\{p^*\}.$$

Noted that every Hilbert space satisfies Opial's condition.

Definition 1.4. *[20] A mapping $\mathcal{H}$ that is prescribed on a subset K of Banach space $\mathcal{B}$ is said to have the condition (I) iff there is a function $q : [0,\infty) \to [0,\infty)$ such that $q(0) = 0$, $q(x) > 0$ for every $x \in [0,\infty) - \{0\}$ and $\|x - \mathcal{H}x\| \geq q(d(x, F_{\mathcal{H}}))$ when $x \in K$. Here $d(x, F_{\mathcal{H}})$ is the distance of x to $F_{\mathcal{H}}$.*

Lemma 1.1. *[25] Let $\mathcal{H}$ is (α, β, γ)–nonexpansive mapping on a subset K of a Banach space with a fixed point, particularly, p^*. Then $\|\mathcal{H}x - \mathcal{H}p^*\| \leq \|x - p^*\|$ containing $x \in K$ and $p^* \in F_{\mathcal{H}}$.*

Lemma 1.2. *[25] Let $\mathcal{H}$ is (α, β, γ)–nonexpansive mapping on a subset K of a Banach space $\mathcal{B}$. Then the set $F_{\mathcal{H}}$ is closed. Also, the set $F_{\mathcal{H}}$ is convex providing that K is convex and the space $\mathcal{B}$ is strictly convex.*

Lemma 1.3. *[25] Suppose $\mathcal{H}$ is (α, β, γ)–nonexpansive mappings on a subset K of a Banach space. Then for all $x, y \in K$, we have*

$$\|x - \mathcal{H}y\| \leq \tfrac{(1+\beta)}{(1-\gamma)}\|x - \mathcal{H}x\| + \tfrac{\alpha}{(1-\gamma)}\|x - y\|.$$

Lemma 1.4. *[25] If $\mathcal{H}$ is (α, β, γ)–nonexpansive mapping condition, $\{p_n\}$ is weakly convergent to p^* and $\lim_{n\to\infty} \|\mathcal{H}p_n - p_n\| = 0$, then $p^* \in F_{\mathcal{H}}$ provided that $\mathcal{B}$ is equipped with the Opial's condition.*

2. Main Results

Now we study the convergence behavior of fixed points for (α, β, γ)–nonxpansive mapping through Picard-Thakur iterative scheme (1.2).

Lemma 2.1. *Let $K \neq \emptyset$ be a closed and convex subset of a norm linear space $\mathcal{B}$. If $\mathcal{H} : K \to K$ is (α, β, γ)–nonxpansive mapping satisfying $F_{\mathcal{H}} \neq \emptyset$ and $\{p_n\}$ a sequence of Picard-Thakur iterates (1.2). Then for each $p^* \in F_{\mathcal{H}}$, it follows that, $\lim_{n\to\infty} \|p_n - p^*\|$ exists.*

Proof. If $p^* \in F_{\mathcal{H}}$, then applying Lemma 1.1 with (1.2), we have

$$\begin{aligned} \|d_n - p^*\| &= \|(1-\gamma_n)p_n + \gamma_n \mathcal{H}p_n - p^*\| \\ &= \|(1-\gamma_n)p_n + \gamma_n p^* - \gamma_n p^* + \gamma_n \mathcal{H}p_n - p^*\| \\ &\leq (1-\gamma_n)\|p_n - p^*\| + \gamma_n \|p_n - p^*\| \\ &\leq \|p_n - p^*\|. \end{aligned} \tag{2.1}$$

Using (2.1) and Lemma1.1, we have

$$\begin{aligned} \|c_n - p^*\| &= \|(1-\beta_n)d_n + \beta_n \mathcal{H}d_n - p^*\| \\ &= \|(1-\beta_n)d_n + \beta_n p^* - \beta_n p^* + \beta_n \mathcal{H}d_n - p^*\| \\ &\leq (1-\beta_n)\|d_n - p^*\| + \beta_n \|d_n - p^*\| \\ &\leq \|d_n - p^*\|. \end{aligned} \tag{2.2}$$

From (2.1) and (2.2) we have

$$\begin{aligned} \|b_n - p^*\| &= \|(1-\alpha_n)\mathcal{H}d_n + \alpha_n \mathcal{H}c_n - p^*\| \\ &= \|(1-\alpha_n)\mathcal{H}d_n + \alpha_n p^* - \alpha_n p^* + \alpha_n \mathcal{H}c_n - p^*\| \\ &\leq (1-\alpha_n)\|\mathcal{H}d_n - p^*\| + \alpha_n \|c_n - p^*\| \\ &\leq \|p_n - p^*\|, \end{aligned} \tag{2.3}$$

and

$$\|p_{n+1} - p^*\| = \|\mathcal{H}p_n - p^*\| \leq \|p_n - p^*\|. \tag{2.4}$$

It can be observed from (2.4) that $\|p_{n+1} - p^*\| \leq \|p_n - p^*\|$ i.e $\{\|p_n - p^*\|\}$ is essentially bounded and also non-increasing. This means that $\lim_{n\to\infty} \|p_n - p^*\|$ exists for each element p^* of $F_{\mathcal{H}}$. □

This theorem explains the necessary and sufficient assumptions required for the existence of fixed point.

Theorem 2.1. *Let $\mathcal{B}$ represents a UCB space and $K \neq \emptyset$ be a subset of $\mathcal{B}$, where K is closed and convex. Let $\mathcal{H} : K \to K$ is (α, β, γ)–nonxpansive mapping satisfying $F_{\mathcal{H}} \neq \emptyset$ and $\{p_n\}$ is a sequence generated by Picard-Thakur iterates (1.2). Then, $F_{\mathcal{H}} \neq \emptyset \iff \{p_n\}$ is bounded and satisfies $\lim_{n\to\infty} \|p_n - \mathcal{H}p_n\| = 0$.*

Proof. We first suppose that $F_{\mathcal{H}} \neq \emptyset$. Therefore, for any $p^* \in F_{\mathcal{H}}$, Lemma 2.1 indicate that $\{p_n\}$ is bounded and $\lim_{n\to\infty} \|p_n - p^*\|$ exists. Consider

$$\lim_{n\to\infty} \|p_n - p^*\| = q. \tag{2.5}$$

Need to prove that $\lim_{n\to\infty} \|p_n - \mathcal{H}p_n\| = 0$. From (2.1) we have

$$\begin{aligned} \|d_n - p^*\| &\leq \|p_n - p^*\| \\ \Rightarrow \limsup_{n\to\infty} \|d_n - p^*\| &\leq \limsup_{n\to\infty} \|p_n - p^*\| = q. \end{aligned} \tag{2.6}$$

Since $p^* \in F_{\mathcal{H}}$, we can apply Lemma 1.1 to get

$$\begin{aligned} \|\mathcal{H}p_n - p^*\| &\leq \|p_n - p^*\| \\ \Rightarrow \limsup_{n\to\infty} \|\mathcal{H}p_n - p^*\| &\leq \limsup_{n\to\infty} \|p_n - p^*\|. \end{aligned} \tag{2.7}$$

Now from (2.4), we have

$$\|p_{n+1} - p^*\| \leq \|d_n - p^*\|.$$

Using this together with (2.5), we obtain

$$q \leq \liminf_{n\to\infty} \|d_n - p^*\|. \tag{2.8}$$

From (2.6) and (2.8), we obtain

$$\lim_{n\to\infty} \|d_n - p^*\| = q. \tag{2.9}$$

Since $\|d_n - p^*\| = \lim_{n\to\infty} \|(1-\gamma_n)p_n + \gamma_n\mathcal{H}p_n - p^*\|$, so using this together with (2.9), we get

$$q = \lim_{n\to\infty} (1-\gamma_n)\|(p_n - p^*)\| + \gamma_n\|(\mathcal{H}p_n - p^*)\|. \tag{2.10}$$

Considering (2.5), (2.7) and (2.10) along with the Lemma 1.1, one gets

$$\lim_{n\to\infty} \|p_n - \mathcal{H}p_n\| = 0.$$

Conversely, we shall assume that $\{p_n\}$ is essentially bounded with the property $\lim_{n\to\infty} \|p_n - \mathcal{H}p_n\| = 0$. Need to prove that $F_{\mathcal{H}} \neq \emptyset$. To do this, we consider any $p^* \in \mathbb{A}(E, \{p_n\})$. By Lemma 1.3, we have

$$\begin{aligned} r(\mathcal{H}p^*, \{p_n\}) &= \limsup_{n\to\infty} \|p_n - \mathcal{H}p^*\| \\ &\leq \frac{(1+\beta)}{(1-\gamma)}\|p_n - \mathcal{H}p_n\| + \frac{\alpha}{(1-\gamma)}\|p_n - p^*\| \\ &= \limsup_{n\to\infty} \|p_n - p^*\| \\ &= r(p^*, \{p_n\}). \end{aligned}$$

Thus $\mathcal{H}p^* \in \mathcal{A}(E, \{p_n\}$. Since the set $\mathcal{A}(E, \{p_n\})$ contains only one point, therefore $\mathcal{H}p^* = p^*$. It implies that $p^* \in F_{\mathcal{H}}$ i.e $F_{\mathcal{H}} \neq \emptyset$. □

Now we will prove weak convergence theorem.

Theorem 2.2. *Let $\mathcal{B}$ represents a UCB space and $K \neq \emptyset$ be a subset of $\mathcal{B}$, where $\mathcal{B}$ be a closed and convex. Let $\mathcal{H} : K \to K$ be a (α, β, γ)–nonxpansive mapping satisfying $F_{\mathcal{H}} \neq \emptyset$ and $\{p_n\}$ be a sequence of Picard-Thakur iterates (1.2). Then $\{p_n\}$ converges weakly to a point of $F_{\mathcal{H}}$ provided that $\mathcal{B}$ is satisfying Opial's condition.*

Proof. As given $\mathcal{B}$ is a UCB space and according to the Theorem 2.1, $\{p_n\}$ is bounded. It follows that there is a point, namely, $p_1 \in K$ such that a subsequence, namely, $\{o_{n_m}\}$ of $\{p_n\}$ weakly converges to it. From Theorem 2.1, it is clear that $\lim_{m\to\infty} \|p_{n_m} - \mathcal{H}t_{n_m}\| = 0$. Using Lemma 1.2, we have $p_1 \in F_{\mathcal{H}}$. We want to prove that the point p_1 is the only weak limit of $\{p_n\}$. Contrary suppose that p_1 cannot become a weak limit for $\{p_n\}$ i.e there exists another subsequence, namely, $\{p_{n_s}\}$ of $\{p_n\}$ with a weak limit, namely, $p_2 \neq p_1$. From Theorem 2.1, it is annotated that $\lim_{s\to\infty} \|p_{n_s} - \mathcal{H}p_{n_s}\| = 0$. Applying Lemma 1.2, we get $p_2 \in F_{\mathcal{H}}$. Using Opial's condition of $\mathcal{B}$ along with the Theorem 2.1, we get

$$\begin{aligned}\lim_{n\to\infty} \|p_n - p_1\| &= \lim_{p\to\infty} \|p_{n_m} - p_1\| < \lim_{m\to\infty} \|p_{n_m} - p_2\| \\ &= \lim_{r\to\infty} \|p_n - p_2\| = \lim_{s\to\infty} \|p_{n_s} - p_2\| \\ &< \lim_{s\to\infty} \|p_{n_s} - p_1\| = \lim_{n\to\infty} \|p_n - p_1\|.\end{aligned}$$

Thus, we get $\lim_{n\to\infty} \|p_n - p_1\| < \lim_{n\to\infty} \|p_n - p_1\|$, which is a contradiction. Hence proved. □

Now we prove strong convergence theorem for Picard-Thakur iterative scheme.

Theorem 2.3. *Suppose K be a convex and compact subset of a UCB space $\mathcal{B}$ and let $\mathcal{H} : K \to K$ be a (α, β, γ)–nonxpansive mapping satisfying $F_{\mathcal{H}} \neq \emptyset$ and let $\{p_n\}$ be a sequence of Picard-Thakur iterates (1.2). Then the sequence $\{p_n\}$ converges strongly to some fixed point of $F_{\mathcal{H}}$.*

Proof. Given that K is convex and compact, so sequence $\{p_n\} \subseteq K$ has a convergent subsequence. We represent this subsequence by $\{p_{n_m}\}$ with a strong limit $s^* \in K$ i.e $\lim_{n_m\to\infty} \|p_{n_m} - s^*\| = 0$. Then applying Lemma 1.3 for $x = p_{n_m}$ and $y = s^*$, we have

$$\|p_{n_m} - \mathcal{H}s^*\| \leq \frac{(1+\beta)}{(1-\gamma)}\|p_{n_m} - \mathcal{H}p_{n_m}\| + \frac{\alpha}{(1-\gamma)}\|p_{n_m} - s^*\|. \tag{2.11}$$

By Theorem 2.1, $\lim_{n_q\to\infty} \|p_{n_q} - \mathcal{H}p_{n_q}\| = 0$ and also $\lim_{n_m\to\infty} \|p_{n_m} - \mathcal{H}t_{n_m}\| = 0$. Accordingly (2.11) provide $\lim_{n_m\to\infty} \mathcal{H}s^* = \mathcal{H}s^* \Rightarrow \mathcal{H}s^* = s^*$.

By Lemma 2.1 $\lim_{n\to\infty} \|p_n - s^*\|$ exist. Hence we have proved that $s^* \in F_{\mathcal{H}}$ and $p_n \to p_1$. □

Now we removed compactness condition on K and proved strong convergence theorem as follows.

Theorem 2.4. *Let K be a closed and convex subset of UCB space $\mathcal{B}$. Let $\mathcal{H} : K \to K$ be a (α, β, γ)–nonxpansive mapping satisfying $F_{\mathcal{H}} \neq \emptyset$ and $\{p_n\}$ be a sequence of Picard-Thakur iterates (1.2). Then $\{p_n\}$ converges strongly to a point $F_{\mathcal{H}}$ whenever $\liminf_{n\to\infty} d(p_n, F_{\mathcal{H}}) = 0$*

Proof. From Lemma 2.1, $\lim_{n\to\infty} \|p_n - s^*\|$ exists for any $s^* \in F_{\mathcal{H}}$. It follows that $\liminf_{n\to\infty} d(p_n, F_{\mathcal{H}})$ also exists. Accordingly $\liminf_{n\to\infty} d(p_n, F_{\mathcal{H}}) = 0$. Hence two subsequences of p_n namely $\{p_{n_m}\}$ and $\{p_m\}$ exists in $F_{\mathcal{H}}$ with property $\|p_{n_m} - p_m\| \leq \frac{1}{2^m}$. We need to prove that $\{p_m\}$ is Cauchy in $F_{\mathcal{H}}$. To do this, using Lemma 2.1, we can write that $\{p_n\}$ is nonincreasing. Thus, we have

$$\|p_{m+1} - p_m\| \leq \|p_{m+1} - p_{n_{m+1}}\| + \|p_{n_{m+1}} - p_m\| \leq \frac{1}{2^{m+1}} + \frac{1}{2^m}.$$

It follows that $\lim_{m\to\infty} \|p_{m+1} - p_m\| = 0$. Hence it is proved that $\{p_m\}$ is cauchy in $F_{\mathcal{H}}$. According to the Lemma 1.2 we get that $F_{\mathcal{H}}$ is closed. Hence $\{p_m\}$ converges to some $q_0 \in F_{\mathcal{H}}$. By Lemma 2.1, $\lim_{n\to\infty} \|p_n - s^*\|$ exists and hence s^* is the strong limit of $\{p_n\}$. □

Following is the strong convergence results using condition (I) [21] .

Theorem 2.5. *Let K be a closed convex subset of a UCBS $\mathcal{B}$ and $\mathcal{H} : K \to K$ be any (α, β, γ)–nonexpansive mappings with $F_{\mathcal{H}} \neq \emptyset$. Assume that $\{p_n\}$ be a sequence of a Picard-Thakur iterates (1.2). If $\mathcal{H}$ possess condition (I), then the sequence $\{p_n\}$ converges strongly to some fixed point of $\mathcal{H}$.*

Proof. We verify this result using the Theorem 2.4. For this, from the Theorem 2.1, we have $\liminf_{n\to\infty} \|\mathcal{H}p_n - p_n\| = 0$. By applying condition (I) of $\mathcal{H}$, we have $\liminf_{n\to\infty} d(p_n, F_{\mathcal{H}}) = 0$. It follows from Theorem 2.4 that $\{p_n\}$ has a strong limit in $F_{\mathcal{H}}$. This completes the proof. □

3. Numerical Example

Picard-Thakur iterative scheme indubitably exhibit faster convergence rate as compare to other iterative scheme using in connection with (α, β, γ)–nonxpansive mapping. Observation are given below with the help of numerical example.

Example 3.1. *Let $K = [0, 1] \subset \mathcal{B}$ and norm on K be defined as $\|.\| = |.|$, $\alpha = \frac{1}{2}$, $\beta = \frac{2}{3}$, $\gamma = \frac{1}{2}$. Defined function $\mathcal{H} : K \to K$ as*

$$\mathcal{H}x = \begin{cases} \frac{x}{4} & \text{if } x \in [0, 0.5) \\ \frac{x}{5} & \text{if } x \in [0.5, 1]. \end{cases}$$

then $\mathcal{H}$ is $\left(\frac{1}{2}, \frac{2}{3}, \frac{1}{2}\right)$–nonexpansive mapping.But not nonexpansive mapping.

Proof. First we show that given mapping is not nonexpensive. For this we take $x = 0.4$ and $y = 0.2$, then it is not nonexpensive mapping. Now we prove that the given mapping is $\left(\frac{1}{2}, \frac{2}{3}, \frac{1}{2}\right)$–nonexpansive. We proceed as follows

Case(I): If $x, y \in [0, 0.5)$, then

$$\begin{aligned} \alpha\|x-y\| + \beta\|x - \mathcal{H}x\| + \gamma\|x - \mathcal{H}y\| &= \frac{1}{2}\|x-y\| + \frac{2}{3}\|x - \frac{x}{4}\| + \frac{1}{2}\|x - \frac{y}{4}\| \\ &= \frac{1}{2}|x-y| + \frac{2}{3}|x - \frac{x}{4}| + \frac{1}{2}|x - \frac{y}{4}| \\ &\geq \frac{1}{2}|x-y| + \frac{2}{3}|x - \frac{x}{4}| \\ &\geq \frac{1}{2}|x-y| \\ &\geq \frac{1}{4}|x-y| \\ &\geq \|\mathcal{H}x - \mathcal{H}y\|. \end{aligned}$$

Case(II): If $x, y \in [0.5, 1]$, then

$$\begin{aligned} \alpha\|x-y\| + \beta\|x - \mathcal{H}x\| + \gamma\|x - \mathcal{H}y\| &= \frac{1}{2}|x-y| + \frac{2}{3}\|x - \frac{x}{5}\| + \frac{1}{2}\|x - \frac{y}{5}\| \\ &\geq \frac{1}{2}|x-y| + \frac{2}{3}|x - \frac{x}{5}| \\ &\geq \frac{1}{2}|x-y| \\ &\geq \frac{1}{5}|x-y| \end{aligned}$$

$$\geq \|\mathcal{H}x - \mathcal{H}y\|.$$

Case(III): If $x \in [0, 0.5)$ and $y \in [0.5, 1]$, then

$$\begin{aligned}
\alpha\|x-y\| + \beta\|x-\mathcal{H}x\| + \gamma\|x-\mathcal{H}y\| &= \frac{1}{2}\|x-y\| + \frac{2}{3}\|x-\frac{x}{4}\| + \frac{1}{2}\|x-\frac{y}{5}\| \\
&\geq \frac{1}{2}|(x-y)-(x-\frac{y}{5})| + \frac{2}{3}|x-\frac{x}{4}|| \\
&= \frac{1}{5}|2y| + \frac{1}{3}|x| \\
&\geq \frac{1}{4}|x| + \frac{1}{5}|y| \\
&\geq |\frac{1}{4}x - \frac{1}{5}y| \\
&\geq \|\mathcal{H}x - \mathcal{H}y\|.
\end{aligned}$$

Case(IV): If $x \in [0.5, 1]$ and $y \in [0, 0.5)$ then

$$\begin{aligned}
\alpha\|x-y\| + \beta\|x-\mathcal{H}x\| + \gamma\|x-\mathcal{H}y\| &= \frac{1}{2}\|x-y\| + \frac{2}{3}\|x-\frac{x}{5}\| + \frac{1}{2}\|x-\frac{y}{4}\| \\
&\geq \frac{1}{2}|(x-y)-(x-\frac{y}{4})| + \frac{2}{3}|x-\frac{x}{5}|| \\
&= \frac{1}{8}|3y| + \frac{1}{15}|8x| \\
&\geq \frac{1}{4}|y| + \frac{1}{5}|x| \\
&\geq |\frac{1}{4}y - \frac{1}{5}x| \\
&\geq \|\mathcal{H}x - \mathcal{H}y\|.
\end{aligned}$$

Hence $\mathcal{H}$ is $\left(\frac{1}{2}, \frac{2}{3}, \frac{1}{2}\right)$–nonexpansive mapping. □

Table 1 and Table 2 shows the tabular comparison of Picard-Thakur [24] iteration scheme with M [26], Abbas [1], Agarwal [3] iteration schemes for initial values 0.4 and -0.2 respectively with $\alpha_n = 0.22$, $\beta_n = 0.66$, $\gamma_r = 0.25$ and the graphical comparison is given in Figure 1, which shows that Picard-Thakur iteration scheme converges faster as compare to the other iterative schemes.

Table 1. Numerical results produced by Picard-Thakur, M, Abbas and Agarwal iterative schemes for $\mathcal{H}$ of the Example 3.1.

n	**Picard-Thakur**	**M**	**Abbas**	**Agarwal**
1	0.4	0.4	0.4	0.4
2	0.01810046875	0.02087500000	0.03496187500	0.08911000000
3	0.00081906742	0.00108941406	0.00305583175	0.01985148025
4	0.00003706376	0.00005685379	0.00026709401	0.00442241351
5	0.00000167717	0.00000296705	0.00002334526	0.00098520317
6	0.00000007589	0.00000015484	0.00000204048	0.00021947863
7	0.00000000343	0.00000000808	0.00000017834	0.00004889435
8	0.00000000015	0.00000000042	0.00000001558	0.00001089243
9	0.00000000000	0.00000000002	0.00000000136	0.00000000013
10	0.0000000000	0.00000000000	0.00000000011	0.0000000012

Table 2. Numerical results produced by Picard-Thakur, M, Abbas and Agarwal approximation schemes for $\mathcal{H}$ of the Example 3.1.

n	**Picard-Thakur**	**M**	**Abbas**	**Agarwal**
1	-0.2	-0.2	-0.2	-0.2
2	-0.00565657600	-0.00659200000	-0.01245632000	-0.03535360000
3	-0.00015998420	-0.00021727232	-0.00077579953	-0.00062493851
4	-0.00000452480	-0.00000716129	-0.00004831803	-0.00110469131
5	-0.000000797400	-0.0000002360	-0.00000300932	-0.00019527407
6	-0.00000000360	-0.00000000777	-0.00000018742	-0.00003451820
7	0.000000000000	-0.00000000256	-0.00000001167	-0.00000610171
8	0.000000000000	0.000000000000	-0.00000000727	-0.00000107858
9	0.000000000000	0.000000000000	-0.00000000004	-0.00000019065
10	0.000000000000	0.000000000000	0.000000000000	-0.00000003370

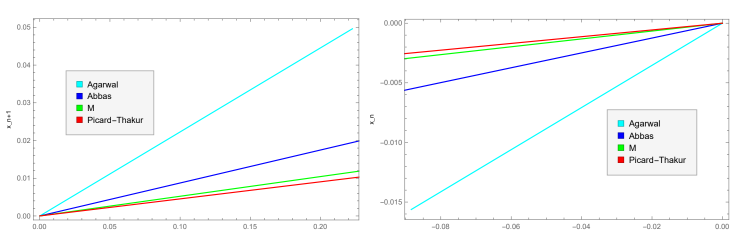

Figure 1. Graphical analysis of iteration schemes towards the fixed point of $\mathcal{H}$ in Example 3.1.

4. Conclusions

In this paper, convergence performance of Picard-Thakur iterative scheme is examined using numerical tabulation and graphs in relationship with (α,β,γ)–nonxpansive mappings. Strong and weak fixed point convergence results using Picard-Thakur iteration scheme for (α,β,γ)-nonxpansive mappings are proved.

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