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Approximating Fixed Point of (α,β,γ)–Nonexpansive Mappings Using JK Iterative Scheme

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Abstract. In this manuscript, we analyze the fixed point approximation of (α,β,γ)-nonexpansive mappings using the JK iteration scheme. To evaluate its efficiency, we perform a comparative analysis with other iterative schemes for (α,β,γ)-nonexpansive mappings. Additionally, we establish the convergence results for sequence generated by the JK iterative scheme. Our work generalizes several results from the existing literature.

1. Introduction and preliminaries

Let $\mathcal{V}$ be a normed linear space and E a non-empty subset of $\mathcal{V}$. A mapping $\mathcal{T}: E \to E$ is called a contraction mapping on E if $\forall x, y \in E$, we have

$$\|\mathcal{T}x - \mathcal{T}y\| \le \alpha\|x - y\|, \text{ for some fixed } \alpha \in [0,1). \tag{1.1}$$

Numerical approximation of solutions for nonlinear operators is an interesting field of mathematics that captures the interest of many researchers. One of the methods used for numerical approximation of solutions for nonlinear mappings is the fixed point approximation method, which states that the fixed point of $g(x) = x$ is the solution of $f(x) = x - g(x)$.

Initially, the Banach Contraction Principle (BCP) [1], supported by the Picard iterative scheme [2], made substantial contributions to fixed point theory. With the passage of time, several researchers (Browder [3], Gohde [4] and others) have introduced more advanced concepts by considering the

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closed, convex and bounded structures of the subsets in Uniformly Convex Banach (UCB) spaces to prove the fixed point existence theorems for nonexpansive mappings (put $\alpha = 1$ in equation (1.1)).

In 2008, Suzuki [5] introduced an extension of nonexpansive mappings, known as mappings satisfying condition (C), in Banach spaces and proved results on the existence of fixed points for this type of mappings.

Aoyama and Kohsaka [6] suggested a new generalized type of nonexpansive mappings known as α–nonexpansive mappings on Banach spaces defined as:

A mapping $\mathcal{T} : E \to E$ is called α- nonexpansive mapping if

$$\|\mathcal{T}x - \mathcal{T}y\|^2 \leq \alpha\|x - \mathcal{T}y\|^2 + \alpha\|y - \mathcal{T}x\|^2 + (1 - 2\alpha)\|x - y\|^2,$$

for any real number $\alpha \in [0, 1]$ and $x, y \in E$.

A self map on a subset E of a Banach space is called generalized α–nonexpansive if there exists a real number $\alpha \in [0, 1]$ such that for every pair of elements $x, y \in E$,

$$\frac{1}{2}\|x - \mathcal{T}x\| \leq \|x - y\| \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| \leq \alpha\|x - \mathcal{T}y\| + \alpha\|y - \mathcal{T}x\| + (1 - 2\alpha)\|x - y\|.$$

Ullah et al. [7] introduced the concept of generalized (α, β)–nonexpansive mappings on a subset E of Banach space given as

$$\frac{1}{2}\|x - \mathcal{T}x\| \leq \|x - y\| \Rightarrow \|\mathcal{T}x - \mathcal{T}y\| \leq \alpha\|x - \mathcal{T}y\| + \alpha\|y - \mathcal{T}x\| + \beta\|x - \mathcal{T}x\|.$$

$\forall x, y \in E$ where $\alpha, \beta \in \mathbb{R}^+$ and $\alpha + \beta < 1$.

Ullah et al. [7] introduced another generalization of nonexpansive mappings known as (α, β, γ)-nonexpansive mappings, defined as below:

Definition 1.1. *A mapping $\mathcal{T}$ on a subset E of a Banach space is called (α, β, γ)–nonexpansive if*

$$\|\mathcal{T}x - \mathcal{T}y\| \leq \alpha\|x - y\| + \beta\|x - \mathcal{T}x\| + \gamma\|x - \mathcal{T}y\| \ , \forall\, x, y \in E,$$

where $\alpha, \beta, \gamma \in \mathbb{R}^+$ are fixed scalars such that $\gamma \in [0, 1)$ and $\alpha + \gamma \leq 1$.

Many researchers developed different iterative methods for approximating the fixed point of different generalizations of nonexpansive mappings [8–13] and many more. Let $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ be the sequences in $(0, 1]$. Following is the one step Mann [14] iteration process:

$$\left.\begin{array}{l} x_0 \in E \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\mathcal{T}x_n. \end{array}\right\} \tag{1.2}$$

Agarwal [8]suggested a two step iteration process which converges faster than the Mann iteration for contraction mappings in Banach spaces:

$$\left.\begin{array}{l} x_0 \in E, \\ y_n = (1 - \beta_n)x_n + \beta_n\mathcal{T}x_n, \\ x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\mathcal{T}y_n \\ . \end{array}\right\} \tag{1.3}$$

In 2000, Noor [12] introduced a three step iteration process as:

$$\left.\begin{aligned} &x_0 \in E, \\ &z_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T} x_n, \\ &y_n = (1-\beta_n)x_n + \beta_n \mathcal{T} z_n, \\ &x_{n+1} = (1-\alpha_n)x_n + \alpha_n \mathcal{T} y_n. \end{aligned}\right\} \tag{1.4}$$

Ahmad et al. [9] introduced a new iterative scheme named as JK iteration:

$$\left.\begin{aligned} &x_0 \in E, \\ &z_n = (1-\gamma_n)x_n + \gamma_n \mathcal{T} x_n, \\ &y_n = \mathcal{T} z_n, \\ &x_{n+1} = \mathcal{T}[(1-\alpha_n)\mathcal{T} z_n + \alpha_n \mathcal{T} y_n]. \end{aligned}\right\} \tag{1.5}$$

In this paper, we prove convergence results for the JK iteration process applied to (α,β,γ)-nonexpansive mappings.

Definition 1.2. *Let $\mathcal{V}$ denotes a Banach space and $\{x_n\} \subseteq \mathcal{V}$ be bounded. If $\emptyset \neq E \subseteq \mathcal{V}$ is convex and closed. Then the asymptotic radius of $\{x_n\}$ corresponding to E is defined as*

$$r(E,\{x_n\}) = \inf\{\limsup_{n\to\infty} \|x_n - s_0\| : s_0 \in E\}.$$

Similarly, the asymptotic center of the sequence $\{x_n\}$ corresponding to E is given as

$$\mathcal{A}(E,\{x_n\}) = \{s_0 \in E : \limsup_{n\to\infty} \|x_n - s_0\| = r(E,\{x_n\})\}.$$

Remark 1.1. *If $\mathcal{V}$ denotes a UCB space [15], then it is known that $\mathcal{A}(E,\{x_n\})$ contains a unique element. Also note that when E is both convex and weakly compact then $\mathcal{A}(E,\{x_n\})$ is convex(see e.g [16, 17] and others).*

Definition 1.3. *[18] A Banach space $\mathcal{V}$ is said to satisfy Opial's condition if for any subsequence $\{x_n\} \subseteq \mathcal{V}$ that converges weakly to $s_0 \in E$, then the following condition holds:*

$$\limsup_{n\to\infty} \|x_n - s_0\| < \limsup_{x\to\infty} \|x_n - e_0\| \; \forall e_0 \in \mathcal{V} \setminus \{s_0\}.$$

Every Hilbert space satisfies the Opial's condition.

Definition 1.4. *A mapping $\mathcal{T}$ defined on a subset E of a Banach space $\mathcal{V}$ is said to satisfy the condition (I) if there exists a function $q : [0,\infty) \to [0,\infty)$ such that $q(0) = 0$, $q(x) > 0$ for every $x \in (0,\infty)$ and $\|x - \mathcal{T}x\| \geq q(d(x, F_{\mathcal{T}}))$ whenever $x \in E$. Here, $d(x, F_{\mathcal{T}})$ is the distance of x to $F_{\mathcal{T}}$, where $F_{\mathcal{T}}$ represents collection of all fixed point of the mapping $\mathcal{T}$.*

Lemma 1.1. *[7] Suppose $\mathcal{T}$ is (α,β,γ)–nonexpansive mapping on a subset E of a Banach space with a fixed point, s_0. Then $\|\mathcal{T}x - \mathcal{T}s_0\| \leq \|x - s_0\|$ holds for all $x \in E$.*

Lemma 1.2. *[7] Suppose $\mathcal{T}$ is (α,β,γ)–nonexpansive mapping on a subset E of a Banach space $\mathcal{V}$. Then the set $F_{\mathcal{T}}$ is closed. Moreover, $F_{\mathcal{T}}$ is convex provided that E is convex and the space $\mathcal{V}$ is strictly convex.*

Lemma 1.3. *[7] Suppose $\mathcal{T}$ is (α, β, γ)–nonexpansive mappings on a subset E of a Banach space. Then for all $x, y \in E$, we have*

$$\|x - \mathcal{T}y\| \leq \frac{(1+\beta)}{(1-\gamma)}\|x - \mathcal{T}x\| + \frac{\alpha}{(1-\gamma)}\|x - y\|.$$

Lemma 1.4. *[7] If $\mathcal{T}$ is (α, β, γ)–nonexpansive mapping,$\{x_n\}$ is weakly convergent to s_0 and $\lim_{n\to\infty} \|\mathcal{T}x_n - x_n\| = 0$, then $s_0 \in F_{\mathcal{T}}$ provided that $\mathcal{V}$ satisfies the Opial's condition.*

2. Main Results

This section presents convergence results for the JK iteration process (1.5) applied to (α, β, γ)-nonexpansive mappings.

Lemma 2.1. *Let $\mathcal{V}$ be a UCB space and $\emptyset \neq E \subseteq \mathcal{V}$ be closed and convex. If $\mathcal{T} : E \to E$ is (α, β, γ)–nonexpansive mapping satisfying $F_{\mathcal{T}} \neq \emptyset$ and $\{x_n\}$ a sequence of JK iterates (1.5). Then for each $s_0 \in F_{\mathcal{T}}$, it follows that, $\lim_{n\to\infty} \|x_n - s_0\|$ exists.*

Proof. If $s_0 \in F_{\mathcal{T}}$ is any element, then applying Lemma 1.1 on (1.5), we have

$$\begin{aligned} \|z_n - s_0\| &= \|(1-\gamma_n)x_n + \gamma_n\mathcal{T}x_n - s_0\| \\ &= \|x_n - x_n\gamma_n + \gamma_n\mathcal{T}x_n - s_0\| \\ &\leq (1-\gamma_n)\|x_n - s_0\| + \gamma_n\|x_n - s_0\| \\ &\leq \|x_n - s_0\|. \end{aligned} \tag{2.1}$$

Using (2.1) and Lemma1.1, we have

$$\begin{aligned} \|y_n - s_0\| &= \|\mathcal{T}z_n - s_0\| \\ &\leq \|z_n - s_0\|. \end{aligned} \tag{2.2}$$

From (2.1) and (2.2), we have

$$\begin{aligned} \|x_{n+1} - s_0\| &= \|T(1-\alpha_n)Tz_n + \alpha_n\mathcal{T}y_n) - s_0\| \\ &\leq \|(1-\alpha_n)Tz_n + \alpha_n\mathcal{T}y_n) - s_0\| \\ &= \|(1-\alpha_n)Tz_n + \alpha_n s_0 - \alpha_n s_0 + \alpha_n\mathcal{T}y_n - s_0\| \\ &= \|(1-\alpha_n)Tz_n + \alpha_n\mathcal{T}y_n - \alpha_n s_0 - s_0 + \alpha_n s_0\| \\ &= \|((1-\alpha_n)Tz_n + \alpha_n(\mathcal{T}y_n - s_0) - s_0(1-\alpha_n)\| \\ &\leq (1-\alpha_n)\|Tz_n - s_0\| + \alpha_n\|(\mathcal{T}y_n - s_0\| \\ &\leq (1-\alpha_n)\|z_n - s_0\| + \alpha_n\|y_n - s_0\| \\ &\leq (1-\alpha_n)\|x_n - s_0\| + \alpha_n\|x_n - s_0\| \\ &\leq \|x_n - s_0\|. \end{aligned} \tag{2.3}$$

It can be observed from (2.1), (2.2) and (2.3) that $\|x_{n+1} - s_0\| \leq \|x_n - s_0\|$ i.e $\{\|x_n - s_0\|\}$ is essentially bounded and also non-increasing. This means that $\lim_{n\to\infty} \|x_n - s_0\|$ exists for each element s_0 of $F_{\mathcal{T}}$. □

For the existence of a fixed point, this theorem elaborates the necessary and sufficient conditions.

Theorem 2.1. *Let $\mathcal{V}$ represents a UCB space and $\emptyset \neq E \subseteq \mathcal{V}$ be closed and convex. If $\mathcal{T} : E \to E$ is (α, β, γ)–nonexpansive mapping and $\{x_n\}$ is a sequence of JK iterates (1.5). Then, $F_{\mathcal{T}} \neq \emptyset \iff \{x_n\}$ is bounded and satisfies* $\lim_{n\to\infty} \|x_n - \mathcal{T}x_n\| = 0$.

Proof. To prove this, we first assume that $F_{\mathcal{T}} \neq \emptyset$. Therefore, for any $s_0 \in F_{\mathcal{T}}$, Lemma 2.1 suggests that $\{x_n\}$ is bounded and $\lim_{n\to\infty} \|x_n - s_0\|$ exists for $s_0 \in F_{\mathcal{T}}$. Consider

$$\lim_{n\to\infty} \|x_n - s_0\| = e. \tag{2.4}$$

Now from (2.1)

$$\begin{aligned} \|z_n - s_0\| &\leq \|x_n - s_0\| \\ \Rightarrow \limsup_{n\to\infty} \|z_n - s_0\| &\leq \limsup_{n\to\infty} \|x_n - s_0\| = e. \end{aligned} \tag{2.5}$$

Since $s_0 \in F_{\mathcal{T}}$, we can apply Lemma 1.1 to get

$$\begin{aligned} \|\mathcal{T}x_n - s_0\| &\leq \|x_n - s_0\| \\ \Rightarrow \limsup_{n\to\infty} \|\mathcal{T}x_n - s_0\| &\leq \limsup_{n\to\infty} \|x_n - s_0\|. \end{aligned} \tag{2.6}$$

Now from (2.3), we have

$$\|x_{n+1} - s_0\| \leq \|z_n - s_0\|.$$

Using this together with (2.4), we obtain

$$e \leq \liminf_{n\to\infty} \|z_n - s_0\|. \tag{2.7}$$

From (2.5) and (2.7), we obtain

$$\lim_{n\to\infty} \|z_n - s_0\| = e \tag{2.8}$$

Since $\|z_n - s_0\| = \|(1-\gamma_n)(x_n - s_0) + \gamma_n(\mathcal{T}x_n - s_0)\|$, so using this together with (2.8), we get

$$e = \lim_{n\to\infty} \|(1-\gamma_n)(x_n - s_0) + \gamma_n(\mathcal{T}x_n - s_0)\|. \tag{2.9}$$

Considering (2.4), (2.6)and (2.9) along with the Lemma 1.1, we get

$$\lim_{n\to\infty} \|x_n - \mathcal{T}x_n\| = 0.$$

Conversely, we shall assume that $\{x_n\}$ is essentially bounded with the property $\lim_{n\to\infty}\|x_n - \mathcal{T}x_n\| = 0$. Need to prove that $F_{\mathcal{T}} \neq \emptyset$. To do this, we consider any $s_0 \in \mathcal{A}(E, \{x_n\})$. By Lemma 1.3, we have

$$\begin{aligned} r(\mathcal{T}s_0, \{x_n\}) &= \limsup_{n\to\infty}\|x_n - \mathcal{T}s_0\| \\ &\leq \limsup_{n\to\infty}\frac{(1+\beta)}{(1-\gamma)}\|x_n - \mathcal{T}x_n\| + \limsup_{n\to\infty}\frac{\alpha}{(1-\gamma)}\|x_n - s_0\| \\ &= \limsup_{n\to\infty}\|x_n - s_0\| \\ &= r(s_0, \{x_n\}). \end{aligned}$$

Thus $\mathcal{T}s_0 \in \mathcal{A}(E, \{x_n\}$. But the set $\mathcal{A}(E, \{x_n\})$ contains only one point, therefore $\mathcal{T}s_0 = s_0$. It implies $s_0 \in F_{\mathcal{T}}$ i.e $F_{\mathcal{T}} \neq \emptyset$. □

Now we will prove weak convergence theorem.

Theorem 2.2. *Let $\mathcal{V}$ represents a UCB space and $\emptyset \neq E \subseteq \mathcal{V}$ be closed and convex. If $\mathcal{T} : E \to E$ is (α, β, γ)–nonexpansive mapping satisfying $F_{\mathcal{T}} \neq \emptyset$ and $\{x_n\}$ a sequence of JK iterates (1.5) then $\{x_n\}$ converges weakly to a point of $F_{\mathcal{T}}$ provided that $\mathcal{V}$ is proclaiming opial's condition.*

Proof. As given $\mathcal{V}$ is a UCB space and according to the Theorem 2.1, $\{x_n\}$ is bounded. It follows that there is a point, namely, $x_0 \in E$ such that a subsequence, namely, $\{x_{n_m}\}$ of $\{x_n\}$ weakly converges to it. From Theorem 2.1, it is clear that $\lim_{m\to\infty}\|x_{n_m} - \mathcal{T}x_{n_m}\| = 0$. Using Lemma 1.2, $x_0 \in F_{\mathcal{T}}$. We want to prove that the point x_0 is the only weak limit of $\{x_n\}$, contrary we suppose that x_0 cannot become a weak limit for $\{x_n\}$ i.e there exists another subsequence, namely, $\{x_{n_s}\}$ of $\{x_n\}$ with a weak limit, namely, $x_0' \neq x_0$. From Theorem 2.1, it is annotated that $\lim_{s\to\infty}\|x_{n_s} - \mathcal{T}x_{n_s}\| = 0$. Applying Lemma 1.2 $x_0' \in F_{\mathcal{T}}$. Using Opial's condition of $\mathcal{V}$ along with the Theorem 2.1, we get

$$\begin{aligned} \lim_{n\to\infty}\|x_n - x_0\| &= \lim_{n\to\infty}\|x_{n_m} - x_0\| < \lim_{m\to\infty}\|x_{n_m} - x_0'\| \\ &= \lim_{n\to\infty}\|x_n - x_0'\| = \lim_{s\to\infty}\|x_{n_s} - x_0'\|. \\ &< \lim_{s\to\infty}\|x_{n_s} - x_0\| = \lim_{n\to\infty}\|x_n - x_0\|. \end{aligned}$$

Thus, we get $\lim_{n\to\infty}\|x_n - x_0\| < \lim_{n\to\infty}\|x_n - x_0\|$, which is a contradiction. Hence proved. □

Now proving strong convergence of *JK* iterative scheme.

Theorem 2.3. *Let E be a convex and compact subset of a UCB space $\mathcal{V}$ and $\mathcal{T} : E \to E$ be a (α, β, γ)–nonexpansive mapping satisfying $F_{\mathcal{T}} \neq \emptyset$ and $\{x_n\}$ a sequence of JK iterates (1.5). Then sequence $\{x_n\}$ converges strongly to some fixed point of $F_{\mathcal{T}}$.*

Proof. Since E is convex and compact, therefore sequence $\{x_n\} \subseteq E$ has a convergent subsequence. We denote this sequence by $\{x_{n_m}\}$ with a strong limit $s_0 \in E$ i.e $\lim_{n_m\to\infty}\|x_{n_m} - s_0\| = 0$. By applying Lemma 1.3, we have

$$\|x_{n_m} - \mathcal{T}s_0\| \leq \frac{(1+\beta)}{(1-\gamma)}\|x_{n_m} - \mathcal{T}x_{n_m}\| + \frac{\alpha}{(1-\gamma)}\|x_{n_m} - s_0\|. \tag{2.10}$$

If we let $m \to \infty$, then $\mathcal{T}s_0 = s_0$ which means $s_0 \in Fix(\mathcal{T})$. Since by Lemma 2.1 $\lim_{n\to\infty} \|x_n - s_0\|$ exists for every $s_0 \in Fix(\mathcal{T})$, so $\{x_n\}$ converges strongly to s_0. □

Now we prove strong convergence theorem without compactness condition on E.

Theorem 2.4. *Suppose that E is closed and convex subset of UCB space $\mathcal{V}$. If $\mathcal{T} : E \to E$ be a (α, β, γ)–nonexpansive mapping satisfying $F_{\mathcal{T}} \neq \emptyset$ and $\{x_n\}$ a sequence of JK iterates (1.5). Then $\{x_n\}$ converges strongly to a point of $F_{\mathcal{T}}$ whenever* $\liminf_{n\to\infty} d(x_n, F_{\mathcal{T}}) = 0$

Proof. For any $s_0 \in E$, from Lemma 2.1 $\lim_{n\to\infty} \|x_n - s_0\|$ exists. It follows that $\liminf_{n\to\infty} d(x_n, F_{\mathcal{T}})$ also exist. Accordingly $\liminf_{n\to\infty} d(x_n, F_{\mathcal{T}}) = 0$. Hence two subsequences of x_n namely $\{x_{n_m}\}$ and $\{s_m\}$ exist in $F_{\mathcal{T}}$ with property $\|x_{n_m} - s_m\| \leq \frac{1}{2^m}$. We need to prove that $\{s_m\}$ is cauchy in $F_{\mathcal{T}}$. To do this, using Lemma 2.1 we can write that $\{x_n\}$ is nonincreasing. Thus, we have

$$\|s_{m+1} - s_m\| \leq \|s_{m+1} - x_{n_{m+1}}\| + \|x_{n_{m+1}} - s_m\| \leq \frac{1}{2^{m+1}} + \frac{1}{2^m}.$$

It follows that $\lim_{m\to\infty} \|s_{m+1} - s_m\| = 0$. Hence, this proves that $\{s_m\}$ is Cauchy in $F_{\mathcal{T}}$. According to Lemma 1.2 that $F_{\mathcal{T}}$ is closed, hence $\{s_m\}$ converges to some $q_0 \in F_{\mathcal{T}}$. By Lemma 2.1, $\lim_{n\to\infty} \|x_n - s_0\|$ exists and hence s_0 is the strong limit of $\{x_n\}$. □

Following is the strong convergence results using condition (I).

Theorem 2.5. *Suppose E be a closed convex subset of a UCB space $\mathcal{V}$ and $\mathcal{T} : E \to E$ be any (α, β, γ)-nonexpansive mappings with $F_{\mathcal{T}} \neq \emptyset$. Assume that $\{x_n\}$ is a JK iterative sequence(1.5). If $\mathcal{T}$ possess condition (I), then the sequence $\{x_n\}$ converges strongly to some fixed point of $\mathcal{T}$.*

Proof. We establish this result by applying Theorem 2.5. For this, from Theorem 2.1, we have $\liminf_{n\to\infty} \|\mathcal{T}x_n - x_n\| = 0$. By applying condition (I) of $\mathcal{T}$, we have $\liminf_{n\to\infty} d(x_n, F_{\mathcal{T}}) = 0$. It follows from Theorem 2.5 that $\{x_n\}$ has a strong limit in $F_{\mathcal{T}}$. This complete the proof. □

3. Numerical Example

The JK iteration scheme indubitably exhibit a faster convergence rate as compared to other iterative schemes using in connection with (α, β, γ)–nonexpansive mapping. Observations are provided below using a numerical example.

Example 3.1. *Let $E = [0, 3] \subset \mathcal{V}$ and norm on E be defined as $\|.\| = |.|$, Let $\alpha = \frac{1}{3}, \beta = \frac{1}{3}, \gamma = \frac{2}{3}$. Defined function $\mathcal{T} : E \to E$ as*

$$\mathcal{T}x = \begin{cases} \frac{x+1}{2} & \text{if } x \in [0, 2] \\ 1 & \text{if } x \in (2, 3]. \end{cases}$$

then $\mathcal{T}$ is $\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$–a nonexpansive but not nonexpansive mapping.

Proof. First we show that given mapping is not nonexpnasive. For this we take $x = 2$ and $y = \frac{5}{2}$, then it is not nonexpnasive. Now we prove that given mapping is $\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$–nonexpansive. For this we have different cases;

Case(I): If $x, y \in [0, 2]$, then

$$\begin{aligned} \alpha\|x-y\| + \beta\|x - \mathcal{T}x\| + \gamma\|x - \mathcal{T}y\| &= \frac{1}{3}\|x-y\| + \frac{1}{3}\|x - \frac{x+1}{2}\| + \frac{2}{3}\|x - \frac{y+1}{2}\| \\ &= \frac{1}{3}|x-y| + \frac{1}{6}|x-1| + \frac{1}{3}|2x-y-1| \\ &\geq |\mathcal{T}x - \mathcal{T}y| \end{aligned}$$

Case(II): If $x, y \in (2, 3]$, then

$$\begin{aligned} \alpha\|x-y\| + \beta\|x - \mathcal{T}x\| + \gamma\|x - \mathcal{T}y\| &= \frac{1}{3}\|x-y\| + \frac{1}{3}\|x-1\| + \frac{2}{3}\|x-1\| \\ &= \frac{1}{3}|x-y| + ||x-1| \\ &\geq |\mathcal{T}x - \mathcal{T}y| \end{aligned}$$

Case(III): If $x \in [0, 2]$ and $y \in (2, 3]$, then

$$\begin{aligned} \alpha\|x-y\| + \beta\|x - \mathcal{T}x\| + \gamma\|x - \mathcal{T}y\| &= \frac{1}{3}\|x-y\| + \frac{1}{3}\|x - \frac{x+1}{2}\| + \frac{2}{3}\|x-1\| \\ &= \frac{1}{3}|x-y| + \frac{1}{6}|x-1| + \frac{2}{3}|x-1| \\ &\geq |\mathcal{T}x - \mathcal{T}y| \end{aligned}$$

Case(IV): If $y \in [0, 2]$ and $x \in (2, 3]$ then

$$\begin{aligned} \alpha\|x-y\| + \beta\|x - \mathcal{T}x\| + \gamma\|x - \mathcal{T}y\| &= \frac{1}{3}\|x-y\| + \frac{1}{3}\|x-1\| + \frac{2}{3}\|x - \frac{y+1}{2}\| \\ &= \frac{1}{3}|x-y| + \frac{1}{3}|x-1| + \frac{1}{3}|x-y-1| \\ &\geq |\mathcal{T}x - \mathcal{T}y| \end{aligned}$$

Hence the map $\mathcal{T}$ is $\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$–nonexpansive. □

In the numerical example, we set $\alpha_n = 0.22$, $\beta_n = 0.66$, $\gamma_n = 0.25$, and the initial value $x_1 = 1.5$ for all the considered iteration schemes, i.e., the JK, Noor, Agarwal, and Mann iterations. We set the stopping criterion $\|x_n - x_{n+1}\| < 10^{-7}$. The obtained results are presented in Tab. 1 and Fig. 1.

From the obtained results, we see that after the first iteration, the value calculated using the JK (1.0745375) is closer to the fixed point (i.e., 1) as compared to the first iteration of the other iterative schemes. The closest fixed point approximation among the methods from the literature can be observed for the Agarwal iteration. In the subsequent iterations, we see that each iteration scheme gets closer to the fixed point but with various speeds. The fastest method is the proposed JK iteration, which found the fixed point in 8 iterations.

Table 1. Numerical results produced by JK, Noor Agarwal and Mann iterative schemes for $\mathcal{T}$ of the Example 3.1.

n	JK	Noor	Agarwal	Mann
1	1.5	1.5	1.5	1.5
2	1.0745375	1.4245812	1.2318500	1.4450000
3	1.0111116	1.3605384	1.1075088	1.3960500
4	1.0016564	1.3061558	1.0498518	1.3524844
5	1.0002469	1.2599759	1.0231163	1.3137112
6	1.0000368	1.2207618	1.0107190	1.2792029
7	1.0000054	1.1874626	1.0049704	1.2484906
8	1.0000008	1.1591862	1.0023048	1.2211566
9	1.0	1.1351750	1.0010687	1.1968294
10	1.0	1.1147856	1.0004956	1.1751782
11	1.0	1.0974716	1.0002298	1.1559085
12	1.0	1.0827692	1.0001066	1.1387586

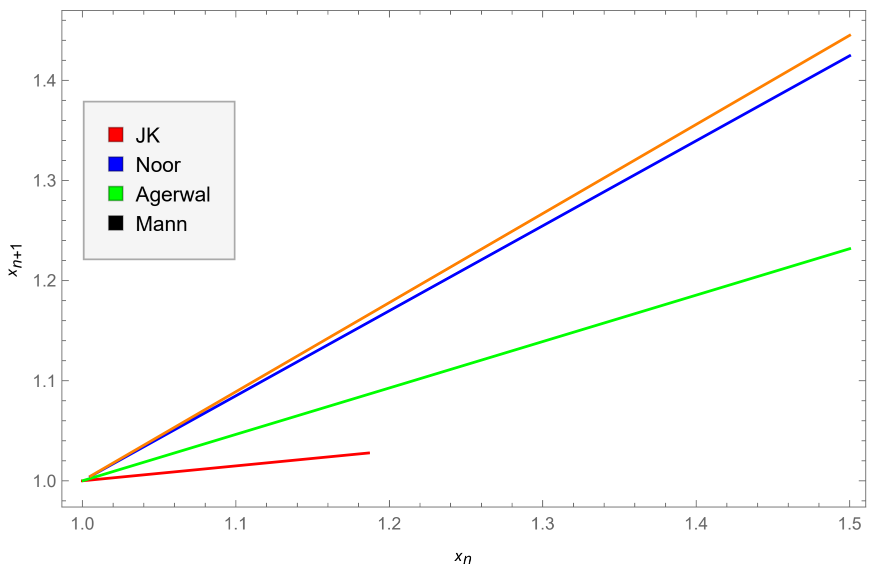

Figure 1. Convergence behavior of JK (1.5), Noor (1.4), Agarwal (1.3) and Mann (1.2) iteration processes corresponding to Tab. 1.

4. Conclusions

In this paper, the convergence behavior of the JK iteration scheme is investigated in association with (α, β, γ)-nonexpansive mappings. The JK iteration scheme provides a promising approach for fixed-point approximation of (α, β, γ)-nonexpansive mappings. Finally, we adopt $\|x_n - x_{n+1}\| < 10^{-7}$ as the stopping criterion and compare the convergence of different iterative methods to demonstrate the efficiency of the proposed scheme.

Data Availability: All data supporting the findings of this study are available within the paper.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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