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Multiplicity of Positive Solutions for a Fractional Elliptic System With Strongly Coupled Critical Terms

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Abstract. By the Nehari method and variational method, two positive solutions are obtained for a fractional elliptic system with strongly coupled critical terms and concave-convex nonlinearities. Recent results from the literature are extended to the case of the integral fractional Laplacian form.

1. Introduction

In present paper, we consider the following fractional elliptic system with strongly coupled critical terms and concave-convex nonlinearities

$$\begin{cases} (-\Delta)^s u = \frac{\eta_1\alpha_1}{2_s^*}|u|^{\alpha_1-2}|v|^{\beta_1}u + \frac{\eta_2\alpha_2}{2_s^*}|u|^{\alpha_2-2}|v|^{\beta_2}u + \lambda\frac{|u|^{q-2}u}{|x|^\gamma}, & x\in\Omega,\\ (-\Delta)^s v = \frac{\eta_1\beta_1}{2_s^*}|u|^{\alpha_1}|v|^{\beta_1-2}v + \frac{\eta_2\beta_2}{2_s^*}|u|^{\alpha_2}|v|^{\beta_2-2}v + \mu\frac{|v|^{q-2}v}{|x|^\gamma}, & x\in\Omega,\\ u=v=0, & x\in\mathbb{R}^N\backslash\Omega, \end{cases} \tag{1.1}$$

where $s\in(0,1)$, η_1,η_2, λ,μ, are positive, $2_s^* := \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent, $N>2s$, $\alpha_1+\beta_1=2_s^*$, $\alpha_2+\beta_2=2_s^*$, Ω is an open bounded domain in $\mathbb{R}^N$ with Lipschitz boundary, and the fractional Laplacian operator $(-\Delta)^s$ is defined, up to a normalization factor, by

$$-(-\Delta)^s u(x) = \frac{1}{2}\int_{\mathbb{R}^N}\frac{u(x+y)+u(x-y)-2u(x)}{|y|^{N+2s}}\,dy \quad ,x\in\mathbb{R}^N.$$

We assume that

$$(\mathcal{H}):\quad 1<q<2 \ \text{ and } \ 0\le\gamma<N+sq-\frac{qN}{2}.$$

$|u|^{\alpha_i-2}u|v|^{\beta_i}$ and $|u|^{\alpha_i}|v|^{\beta_i-2}v, i=1,2$ are called strongly coupled terms . We now recall some known results concerning the elliptic system involving the strongly coupled critical terms. When $s=1$,

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$\eta_1 = \eta_2 = 1$, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$ and $\gamma = 0$, problem (1.1) becomes the following Laplacian elliptic system:

$$\begin{cases} -\Delta u = \frac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} + \lambda |u|^{q-2}u & \text{in } \Omega, \\ -\Delta v = \frac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v + \mu |v|^{q-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.2}$$

Using variational methods and decomposition of Nehari manifold, the authors in [13] proved that the system admits at least two positive solutions when (λ, μ) belongs to certain subset of $\mathbb{R}^2$. Later, Hsu [12] obtained the same results for the p-Laplacian elliptic system. There are other multiplicity results for critical elliptic equations involving concave–convex nonlinearities, see for example [1,2]. A natural question is whether or not these results obtained in the classical context can be extended to the nonlocal framework of the fractional Laplacian type operators. In this direction, another important contribution to the elliptic system involving the strongly-coupled critical terms has been given in [5] , in which the authors obtain multiple positive solutions of (1.2) by replacing the Laplacian with the fractional Laplacian. A similar result has been obtained in [11] for system (1.2) involving spectral fractional Laplacian. Here, we are interested in the works [7,17], where the fibering and Nehari manifold methods are applicable to obtain two positive solutions for

$$\begin{cases} \mathcal{L}u = \frac{\eta_1\alpha_1}{2^*}|u|^{\alpha_1-2}|v|^{\beta_1}u + \frac{\eta_2\alpha_2}{2^*}|u|^{\alpha_2-2}|v|^{\beta_2}u + \lambda\frac{|u|^{q-2}u}{|x|^{\gamma}}, & x \in \Omega, \\ \mathcal{L}v = \frac{\eta_1\beta_1}{2^*}|u|^{\alpha_1}|v|^{\beta_1-2}v + \frac{\eta_2\beta_2}{2^*}|u|^{\alpha_2}|v|^{\beta_2-2}v + \mu\frac{|v|^{q-2}v}{|x|^{\gamma}}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

where, $\mathcal{L} = -\Delta$ or $\mathcal{L}$ is $(-\Delta)^s$, the spectral fractional Laplacian operator. To the best of our knowledge, there seems to be no result on the multiplicity of positive solutions for (1.3) concerning the integral fractional Laplacian. This is the main purpose of this paper. Our main results are:

Theorem 1.1. *Assume that $(\mathcal{H})$ holds. Then for any $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$ system (1.1) has a positive ground state solution, where Θ_1 is defined by (2.19).*

Theorem 1.2. *Assume that $(\mathcal{H})$ holds, $N > 4s$ and $N - (N-2s)q \le \gamma < N + sq - \frac{qN}{2}$. Then there exists $\Lambda > 0$ such that for any $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Lambda)$, system (1.1) has at least two positive solutions, and one of the solutions is a positive ground state solution.*

This paper is organized as follows. In section 2, we present some notations and prove some useful preliminary lemmas. We give the proof of Theorem 1.1 concerning the existence of a positive ground state solution of system (1.1) in the Section 3. The proof of Theorem 1.2 is contained in Section 4.

2. Some Preliminary Results

In this section, we introduce the functional space that we shall use later, and analyse fibering maps.

2.1. **Functional space.** Let Ω be an open bounded set of $\mathbb{R}^N$ with Lipschitz boundary. We introduce the following functional space, which was introduced by Servadei and Valdinoci in [14], as follows:

$$X := \{u \mid u : \mathbb{R}^N \to \mathbb{R} \text{ is measurable}, u|_\Omega \in L^2(\Omega) \text{ and } \int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y < \infty\},$$

where $Q := \mathbb{R}^{2N} \backslash (C\Omega \times C\Omega)$ with $C\Omega = \mathbb{R}^N \backslash \Omega$. The space X is endowed with the norm defined by

$$\|u\|_X := \|u\|_{L^2(\Omega)} + \left(\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}. \tag{2.1}$$

Let the functional space Z be the closure of $C_0^\infty(\Omega)$ in X. By lemma 4 in [9], the space Z is a Hilbert space that can be endowed with both the scalar product defined for any $u, v \in Z$ as

$$\langle u, v\rangle_Z := \int_Q \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \tag{2.2}$$

and the norm

$$\|u\|_Z := \left(\int_Q \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}. \tag{2.3}$$

Since $u = 0$ almost everywhere (a.e.) in $\mathbb{R}^N \backslash \Omega$, we have that the integrations in (2.1)-(2.3) can be extended to all $\mathbb{R}^N$.

Lemma 2.1. *([15]).*
(i) We have that $C_0^2(\Omega) \subseteq Z$, and so X and Z are non-empty. Moreover, $X \subset H^s(\Omega)$ and $Z \subset H^s\left(\mathbb{R}^N\right)$, where $H^s(\Omega)$ is the usual fractional Sobolev space endowed with the (Gagliardo) norm

$$\|u\|_{H^s(\Omega)} := \|u\|_{L^2(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\right)^{1/2}.$$

(ii) Let $s \in (0,1)$ and $N > 2s$. Then there exists a constanl $C > 0$ such that

$$\|u\|_{L^{2_s^*}(\Omega)} = \|u\|_{L^{2_s^*}(\mathbb{R}^N)} \leqslant C\|u\|_Z$$

for every $u \in Z$, where $2_s^* = 2N/(N-2s)$ is the fractional critical exponent. Moreover, the embedding $Z \hookrightarrow L^r(\Omega)$ is continuous for any $r \in [2, 2_s^*]$ and compact whenever $r \in [2, 2_s^*)$.

For further details on X and Z and their properties we refer the reader to [6] and the references therein.
Let $E := Z \times Z$ be the Cartesian product of two Hilbert spaces, which is a reflexive Banach space endowed with the norm

$$\|(u,v)\|^2 := \|u\|_Z^2 + \|v\|_Z^2. \tag{2.4}$$

Definition 2.1. *We say that $(u,v)\in E$ is a weak solution of problem (1.1) if $(u,v)\in E$, one has*

$$
\begin{aligned}
&\int_Q \frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y+\int_Q \frac{(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&=\int_\Omega\left(\frac{\eta_1\alpha_1}{2_s^*}|u|^{\alpha_1-2}|v|^{\beta_1}u\phi+\frac{\eta_2\alpha_2}{2_s^*}|u|^{\alpha_2-2}|v|^{\beta_2}u\phi\right)dx\\
&+\int_\Omega\left(\frac{\eta_1\beta_1}{2_s^*}|u|^{\alpha_1}|v|^{\beta_1-2}v\psi+\frac{\eta_2\beta_2}{2_s^*}|u|^{\alpha_2}|v|^{\beta_2-2}v\psi\right)dx\\
&+\int_\Omega\left(\lambda\frac{|u|^{q-2}u}{|x|^\gamma}\phi+\mu\frac{|v|^{q-2}v}{|x|^\gamma}\psi\right)dx \quad \textit{for all } (\phi,\psi)\in E.
\end{aligned}
\tag{2.5}
$$

The fact that (u,v) is a weak solution is equivalent to it being a critical point of the following functional:

$$
\begin{aligned}
\mathcal{J}_{\lambda,\mu}(u,v):=&\frac{1}{2}\int_Q\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y+\frac{1}{2}\int_Q\frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&-\frac{1}{2_s^*}Q(u,v)-\frac{1}{q}K(u,v),
\end{aligned}
\tag{2.6}
$$

where

$$Q(u,v):=\int_\Omega\left(\eta_1|u|^{\alpha_1}|v|^{\beta_1}+\eta_2|u|^{\alpha_2}|v|^{\beta_2}\right)dx$$

and

$$K(u,v):=\int_\Omega\left(\lambda\frac{|u|^q}{|x|^\gamma}+\mu\frac{|v|^q}{|x|^\gamma}\right)dx.$$

We can see that $\mathcal{J}_{\lambda,\mu}\in C^1(E,\mathbb{R})$ and

$$
\begin{aligned}
\left\langle\mathcal{J}'_{\lambda,\mu}(u,v),(\phi,\psi)\right\rangle=&\int_Q\frac{(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y+\int_Q\frac{(v(x)-v(y))(\psi(x)-\psi(y))}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y\\
&-\int_\Omega\left(\frac{\eta_1\alpha_1}{2_s^*}|u|^{\alpha_1-2}|v|^{\beta_1}u\phi+\frac{\eta_2\alpha_2}{2_s^*}|u|^{\alpha_2-2}|v|^{\beta_2}u\phi\right)dx\\
&-\int_\Omega\left(\frac{\eta_1\beta_1}{2_s^*}|u|^{\alpha_1}|v|^{\beta_1-2}v\psi+\frac{\eta_2\beta_2}{2_s^*}|u|^{\alpha_2}|v|^{\beta_2-2}v\psi\right)dx\\
&-\int_\Omega\left(\lambda\frac{|u|^{q-2}u}{|x|^\gamma}\phi+\mu\frac{|v|^{q-2}v}{|x|^\gamma}\psi\right)dx.
\end{aligned}
$$

For the critical case, since the embedding $Z\hookrightarrow L^{2_s^*}\left(\mathbb{R}^N\right)$ is not compact, the energy functional does not satisfy the Palais Smale condition globally, but it is true for the energy functional in a suitable range related to the best fractional critical Sobolev constant in the embedding $Z\hookrightarrow L^{2_s^*}\left(\mathbb{R}^N\right)$. To do this, let us define the best fractional critical Sobolev constant S as

$$S:=\inf_{u\in Z\setminus\{0\}}\frac{\displaystyle\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y}{\left(\displaystyle\int_{\mathbb{R}^N}|u|^{2_s^*}dx\right)^{2/2_s^*}} \tag{2.7}$$

and

$$S_{\eta,\alpha,\beta} := \inf_{(u,v)\in E\setminus\{0\}} \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y + \int_{\mathbb{R}^{2N}} \frac{|v(x)-v(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y}{\left(\displaystyle\int_{\mathbb{R}^N} \left(\eta_1|u|^{\alpha_1}|v|^{\beta_1} + \eta_2|u|^{\alpha_2}|v|^{\beta_2}\right)dx\right)^{2/2_s^*}} \tag{2.8}$$
$$= \inf_{(u,v)\in E\setminus\{0\}} \|(u,v)\|^2 \left(\int_{\mathbb{R}^N} \left(\eta_1|u|^{\alpha_1}|v|^{\beta_1} + \eta_2|u|^{\alpha_2}|v|^{\beta_2}\right)dx\right)^{-2/2_s^*}.$$

Then it is easy to obtain that

$$\int_{\mathbb{R}^N} \left(\eta_1|u|^{\alpha_1}|v|^{\beta_1} + \eta_2|u|^{\alpha_2}|v|^{\beta_2}\right)dx \leq (S_{\eta,\alpha,\beta})^{-2_s^*/2}\|(u,v)\|^{2_s^*}. \tag{2.9}$$

As well known [3], the function

$$U_\varepsilon(x) := \varepsilon^{\frac{2s-N}{2}} U(\frac{x}{\varepsilon}), \tag{2.10}$$

where

$$U(x) := \frac{C_{N,s}}{(1+|x|^2)^{\frac{N-2s}{2}}}, \quad x \in \mathbb{R}^N, \quad \text{with} \quad C_{N,s} := \left(\int_{\mathbb{R}^N} \frac{1}{(1+|x|^2)^N}\right)^{-2_s^*}$$

is an extremal function for the minimization problem (2.7), that is, it is a positive solution of the following problem

$$(-\triangle)^s u = |u|^{2_s^*-2}u, \quad \forall x \in \mathbb{R}^N.$$

Moreover,

$$\int_{\mathbb{R}^{2N}} \frac{|U_\varepsilon(x)-U_\varepsilon(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^N} |U_\varepsilon|^{2_s^*}dx = S^{\frac{N}{2s}}.$$

We define

$$f(\tau) := \frac{1+\tau^2}{(\eta_1\tau^{\beta_1} + \eta_2\tau^{\beta_2})^{\frac{2}{2_s^*}}}, \quad \tau > 0. \tag{2.11}$$

Since f is continuous on $(0,\infty)$ such that $\lim_{\tau\to 0^+} f(\tau) = \lim_{\tau\to+\infty} f(\tau) = +\infty$, then there exists $\tau_0 > 0$ such that

$$f(\tau_0) := \min_{\tau>0} f(\tau) > 0. \tag{2.12}$$

Lemma 2.2. *($S_{\eta,\alpha,\beta}$ versus S). Suppose that $(\mathcal{H})$ holds, $f(\tau)$ is defined as in (2.11) and $U_\varepsilon(x)$ is the minimizer of S defined as in (2.10), we have $S_{\eta,\alpha,\beta} = f(\tau_0)\, S$ and has the minimizers $(U_\varepsilon(x), \tau_0 U_\varepsilon(x))$.*

Proof. Suppose $w \in Z\setminus\{0\}$. Choosing $(u,v) = (w, \tau_0 w)$ in (2.8) we have

$$\frac{1+\tau_0^2}{\left(\eta_1\tau_0^{\beta_1} + \eta_2\tau_0^{\beta_2}\right)^{\frac{2}{2_s^*}}} \frac{\displaystyle\int_{\mathbb{R}^{2N}} \frac{|w(x)-w(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y}{\left(\displaystyle\int_{\mathbb{R}^N} |w|^{2_s^*}dx\right)^{2/2_s^*}} \geq S_{\eta,\alpha,\beta}. \tag{2.13}$$

Taking the infimum as $w \in Z\setminus\{0\}$ in (2.13), we have

$$f(\tau_0)\, S \geq S_{\eta,\alpha,\beta}. \tag{2.14}$$

Let $\{(u_n, v_n)\} \subset E$ be a minimizing sequence of $S_{\eta,\alpha,\beta}$ and define $z_n = s_n v_n$, where

$$s_n := \left(\left(\int_{\mathbb{R}^N} |v_n|^{2_s^*} \, dx\right)^{-1} \int_{\mathbb{R}^N} |u_n|^{2_s^*} \, dx\right)^{\frac{1}{2_s^*}}.$$

Then

$$\int_{\mathbb{R}^N} |z_n|^{2_s^*} \, dx = \int_{\mathbb{R}^N} |u_n|^{2_s^*} \, dx. \tag{2.15}$$

From the Young inequality and (2.14) it follows that

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n|^{\alpha_i} |z_n|^{\beta_i} \, dx &\le \frac{\alpha_i}{2_s^*} \int_{\mathbb{R}^N} |u_n|^{2_s^*} \, dx + \frac{\beta_i}{2_s^*} \int_{\mathbb{R}^N} |z_n|^{2_s^*} \, dx \\ &= \int_{\mathbb{R}^N} |u_n|^{2_s^*} \, dx = \int_{\mathbb{R}^N} |z_n|^{2_s^*} \, dx, \quad i = 1, 2. \end{aligned} \tag{2.16}$$

Consequently,

$$\begin{aligned} \frac{\|(u_n, v_n)\|^2}{\left(\int_{\mathbb{R}^N} \left(\eta_1 |u_n|^{\alpha_1} |v_n|^{\beta_1} + \eta_2 |u_n|^{\alpha_2} |v_n|^{\beta_2}\right) dx\right)^{2/2_s^*}} &\ge \frac{\int_{\mathbb{R}^{2N}} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy}{\left(\left(\eta_1 s_n^{-\beta_1} + \eta_2 s_n^{-\beta_2}\right) \int_{\mathbb{R}^N} |u_n|^{2_s^*}\right)^{\frac{2}{2_s^*}}} \\ &\quad + \frac{s_n^{-2} \int_{\mathbb{R}^{2N}} \frac{|z_n(x) - z_n(y)|^2}{|x-y|^{N+2s}} \, dx \, dy}{\left(\left(\eta_1 s_n^{-\beta_1} + \eta_2 s_n^{-\beta_2}\right) \int_{\mathbb{R}^N} |z_n|^{2_s^*} \, dx\right)^{\frac{2}{2_s^*}}} \\ &\ge f\left(s_n^{-1}\right) S \\ &\ge f\left(\tau_0\right) S. \end{aligned}$$

As $n \to \infty$ we have

$$S_{\eta,\alpha,\beta} \ge f\left(\tau_0\right) S,$$

which together with (2.14) implies that

$$S_{\eta,\alpha,\beta} = f\left(\tau_0\right) S.$$

By (2.8) and (2.10), $S_{\eta,\alpha,\beta}$ has the minimizers $(U_\varepsilon(x), \tau_0 U_\varepsilon(x))$. □

Let $R_0 > 0$ be a constant such that $\Omega \subset B(0, R_0)$, where $B(0, R_0) = \{x \in \mathbb{R}^N : |x| < R_0\}$. By Hölder's inequality and (2.7), for all $(u, v) \in E$ and $1 < q < 2, 0 \le \gamma < N + sq - \frac{qN}{2}$, we get

$$\begin{aligned}\int_\Omega \frac{u^q}{|x|^\gamma}dx &\le \left(\int_\Omega |u|^{q\cdot\frac{2_s^*}{q}}dx\right)^{\frac{q}{2_s^*}}\left(\int_\Omega \left(\frac{1}{|x|^\gamma}\right)^{\frac{2_s^*}{2_s^*-q}}dx\right)^{\frac{2_s^*-q}{2_s^*}}\\
&\le S^{-\frac{q}{2}}\|u\|_Z^q\left(\int_{B(0,R_0)}\left(\frac{1}{|x|^\gamma}\right)^{\frac{2_s^*}{2_s^*-q}}dx\right)^{\frac{2_s^*-q}{2_s^*}}\\
&\le S^{-\frac{q}{2}}\|u\|_Z^q\left(\int_0^{R_0}\frac{r^{N-1}}{|r|^{\frac{2_s^*\gamma}{2_s^*-q}}}dr\right)^{\frac{2_s^*-q}{2_s^*}}\\
&= S^{-\frac{q}{2}}\|u\|_Z^q\left(\frac{2N-qN+2sq}{2N\left(N-\gamma-\frac{qN}{2}+sq\right)}\right)^{\frac{2_s^*-q}{2_s^*}} R_0^{N-\gamma-\frac{qN}{2}+sq},\end{aligned}\tag{2.17}$$

$$\int_\Omega \frac{v^q}{|x|^\gamma}dx \le S^{-\frac{q}{2}}\|v\|_Z^q\left(\frac{2N-qN+2sq}{2N\left(N-\gamma-\frac{qN}{2}+sq\right)}\right)^{\frac{2_s^*-q}{2_s^*}} R_0^{N-\gamma-\frac{qN}{2}+sq}.\tag{2.18}$$

Set

$$\begin{aligned}\Theta &:= \left(\frac{2N-qN+2sq}{2N\left(N-\gamma-\frac{qN}{2}+sq\right)}\right)^{\frac{2_s^*-q}{2_s^*}} R_0^{N-\gamma-\frac{qN}{2}+sq}S^{-\frac{q}{2}},\\
\Theta_1 &:= \left[\frac{2_s^*-2}{\Theta\left(2_s^*-q\right)}\right]^{\frac{2}{2-q}}\left(\frac{2-q}{2_s^*-q}\right)^{\frac{N-2s}{2}}\left(S_{\eta,\alpha,\beta}\right)^{\frac{N}{2s}}.\end{aligned}\tag{2.19}$$

2.2. **Analysis of fibering maps.** We shall consider critical points of the function $\mathcal{J}_{\lambda,\mu}$ on the Hilbert space E. Consider the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} := \{(u,v)\in E\backslash\{(0,0)\} : \left\langle \mathcal{J}'_{\lambda,\mu}(u,v),(u,v)\right\rangle = 0\}.$$

Thus, $(u,v)\in\mathcal{N}_{\lambda,\mu}$ if and only if

$$\|(u,v)\|^2 - Q(u,v) - K(u,v) = 0.\tag{2.20}$$

Let $z=(u,v)$, then $\|z\|_E = \|(u,v)\| = \left(\|u\|_Z^2+\|v\|_Z^2\right)^{\frac{1}{2}}$. Define $\Phi(z) := \left\langle \mathcal{J}'_{\lambda,\mu}(z),z\right\rangle$, then for all $z=(u,v)\in\mathcal{N}_{\lambda,\mu}$, we have

$$\begin{aligned}\langle\Phi'(z),z\rangle &= 2\|z\|_E^2 - 2_s^*Q(z) - qK(z)\\
&= (2-q)\|z\|_E^2 - (2_s^*-q)\,Q(z)\\
&= (2-2_s^*)\,\|z\|_E^2 + (2_s^*-q)\,K(z).\end{aligned}\tag{2.21}$$

Thus, it is natural to split $\mathcal{N}_{\lambda,\mu}$ into three parts corresponding to local minima, local maxima and points of inflection, i.e.

$$\begin{aligned} \mathcal{N}^{+}_{\lambda,\mu} &:= \left\{ z \in \mathcal{N}_{\lambda,\mu} : \langle \Phi'(z), z \rangle > 0 \right\}, \\ \mathcal{N}^{-}_{\lambda,\mu} &:= \left\{ z \in \mathcal{N}_{\lambda,\mu} : \langle \Phi'(z), z \rangle < 0 \right\}, \\ \mathcal{N}^{0}_{\lambda,\mu} &:= \left\{ z \in \mathcal{N}_{\lambda,\mu} : \langle \Phi'(z), z \rangle = 0 \right\}. \end{aligned} \tag{2.22}$$

When $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$, we will prove that $\mathcal{N}^{\pm}_{\lambda,\mu} \neq \emptyset$ and $\mathcal{N}^{0}_{\lambda,\mu} = \emptyset$.

Lemma 2.3. *For each $z \in E$ such that $Q(z) > 0$, we have the following:*
(i) If $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$ (Θ_1 is defined by (2.19)), then there exist unique $0 < t^{+} < \bar{t}_{max} < t^{-}$ such that $t^{+}z \in \mathcal{N}^{+}_{\lambda,\mu}$, $t^{-}z \in \mathcal{N}^{-}_{\lambda,\mu}$ and

$$\mathcal{J}_{\lambda,\mu}\left(t^{+}z\right) = \inf_{0 \le t \le \bar{t}_{\max}} \mathcal{J}_{\lambda,\mu}(tz), \quad \mathcal{J}_{\lambda,\mu}\left(t^{-}z\right) = \sup_{t \ge \bar{t}_{\max}} \mathcal{J}_{\lambda,\mu}(tz),$$

that is, $\mathcal{N}^{\pm}_{\lambda,\mu} \neq \emptyset$;
(ii) If $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$, then $\mathcal{N}^{0}_{\lambda,\mu} = \emptyset$ and $\mathcal{N}^{-}_{\lambda,\mu}$ is a closed set.

Proof. (i) For each $z \in E$ such that $Q(z) > 0$, and for all $t \ge 0$, we have

$$\left\langle \mathcal{J}'_{\lambda,\mu}(tz), tz \right\rangle = t^{2}\|z\|^{2}_{E} - t^{2^{*}_{s}} Q(z) - t^{q} K(z).$$

We define $g, h : \mathbb{R}^{+} \to \mathbb{R}$ by

$$\begin{aligned} g(t) &: = t^{2-q}\|z\|^{2}_{E} - t^{2^{*}_{s}-q} Q(z) - K(z), \\ h(t) &: = t^{2-q}\|z\|^{2}_{E} - t^{2^{*}_{s}-q} Q(z). \end{aligned}$$

Clearly, we obtain $h(0) = 0$, and $h(t) \to -\infty$ as $t \to \infty$. Because

$$h'(t) = t^{1-q}\left[(2-q)\|z\|^{2}_{E} - \left(2^{*}_{s} - q\right) t^{2^{*}_{s}-2} Q(z)\right], \quad \text{for all } t > 0,$$

solving $h'(t) = 0$, we obtain

$$\bar{t}_{\max} = \left[\frac{(2-q)\|z\|^{2}_{E}}{\left(2^{*}_{s} - q\right) Q(z)}\right]^{\frac{1}{2^{*}_{s}-2}} > 0.$$

Easy computations show that $h'(t) > 0$ for all $0 < t < \bar{t}_{\max}$ and $h'(t) < 0$ for all $t > \bar{t}_{\max}$. Thus $h(t)$ attains its maximum at $\bar{t}_{\max}$, that is,

$$h\left(\bar{t}_{\max}\right) = \left[\frac{(2-q)\|z\|^{2}_{E}}{\left(2^{*}_{s} - q\right) Q(z)}\right]^{\frac{2-q}{2^{*}_{s}-2}} \frac{2^{*}_{s} - 2}{2^{*}_{s} - q}\|z\|^{2}_{E}.$$

Then from (2.19), (2.17) and (2.18), by the Hölder inequality, one gets

$$\begin{aligned} g\left(\bar{t}_{\max}\right) &= h\left(\bar{t}_{\max}\right) - K(u,v) \\ &= \left[\frac{(2-q)\|z\|_E^2}{\left(2_s^*-q\right)Q(z)}\right]^{\frac{2-q}{2_s^*-2}} \frac{2_s^*-2}{2_s^*-q}\|z\|_E^2 - \int_\Omega \left(\lambda\frac{u^q}{|x|^\gamma} + \mu\frac{v^q}{|x|^\gamma}\right)dx \\ &\geq \left[\frac{(2-q)\|z\|_E^2}{\left(2_s^*-q\right)\|z\|_E^{2_s^*}\left(S_{\eta,\alpha,\beta}\right)^{-\frac{2_s^*}{2}}}\right]^{\frac{2-q}{2_s^*-2}} \frac{2_s^*-2}{2_s^*-q}\|z\|_E^2 - \left(\lambda\|u\|_Z^q + \mu\|v\|_Z^q\right)\Theta \qquad (2.23) \\ &\geq \left(\frac{2-q}{2_s^*-q}\right)^{\frac{2-q}{2_s^*-2}} \left(S_{\eta,\alpha,\beta}\right)^{\frac{2_s^*(2-q)}{2(2_s^*-2)}} \frac{2_s^*-2}{2_s^*-q}\|z\|_E^q - \left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \|z\|_E^q\Theta \\ &> 0, \end{aligned}$$

where Θ is as in (2.19) and the last inequality holds for every $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$. It follows that there exist $t^+$and t^-such that

$$g\left(t^+\right) = g\left(t^-\right) \text{ and } \quad g'\left(t^+\right) > 0 > g'\left(t^-\right),$$

for $0 < t^+ < \bar{t}_{\max} < t^-$. We have $t^+z \in \mathcal{N}_{\lambda,\mu}^+$, $t^-z \in \mathcal{N}_{\lambda,\mu}^-$and

$$\mathcal{J}_{\lambda,\mu}\left(t^-z\right) \geq \mathcal{J}_{\lambda,\mu}(tz) \geq \mathcal{J}_{\lambda,\mu}\left(t^+z\right),$$

for each $t \in [t^+, t^-]$, and $\mathcal{J}_{\lambda,\mu}\left(t^+z\right) \leq \mathcal{J}_{\lambda,\mu}(tz)$ for each $t \in [0, t^+]$. Thus

$$\mathcal{J}_{\lambda,\mu}\left(t^+z\right) = \inf_{0\leq t\leq \bar{t}_{\max}} \mathcal{J}_{\lambda,\mu}(tz), \quad \mathcal{J}_{\lambda,\mu}\left(t^-z\right) = \sup_{t\geq \bar{t}_{\max}} \mathcal{J}_{\lambda,\mu}(tz).$$

(ii) From (i) we have that there exist exactly two numbers t^+ and t^- such that $0 < t^+ < t^-$ and $g\left(t^+\right) = g\left(t^-\right) = 0$. Furthermore, $g'\left(t^+\right) > 0 > g'\left(t^-\right)$, If, by contradiction, $z \in \mathcal{N}_{\lambda,\mu}^0$, then we have that $g(1) = 0$ with $g'(1) = 0$. Then, either $t^+ = 1$ or $t^- = 1$. In turn, either $g'(1) > 0$ or $g'(1) < 0$, which is a contradiction. Thus, $\mathcal{N}_{\lambda,\mu}^0 = \emptyset$ for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$.

Finally, we prove that $\mathcal{N}_{\lambda,\mu}^-$is a closed set for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$. Assume that $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}^-$such that $z_n \to z$ in E as $n \to +\infty$, then we need prove that $z \in \mathcal{N}_{\lambda,\mu}^-$. As $z_n \in \mathcal{N}_{\lambda,\mu}^-$, from the definition of $\mathcal{N}_{\lambda,\mu}^-$, one has

$$(2-q)\|z_n\|_E^2 - \left(2_s^*-q\right)Q\left(z_n\right) < 0. \qquad (2.24)$$

Consequently, as $z_n \to z$ in E as $n \to +\infty$, it follows from (2.24) that

$$(2-q)\|z\|_E^2 - \left(2_s^*-q\right)Q(z) \leq 0,$$

thus $z \in \mathcal{N}_{\lambda,\mu}^- \cup \mathcal{N}_{\lambda,\mu}^0$, then $z \in \mathcal{N}_{\lambda,\mu}^-$ because $\mathcal{N}_{\lambda,\mu}^0 = \emptyset$ for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$. Therefore, $\mathcal{N}_{\lambda,\mu}^-$is a closed set in E for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$. The proof is complete. □

Lemma 2.4. *For each $z \in E$ such that $K(z) > 0$, if $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$, where Θ_1 is as in (2.19), then there exist t^+, t^- with $0 < t^+ < t_{\max} < t^-$ such that $t^+ z \in \mathcal{N}^+_{\lambda,\mu}$ and $t^- z \in \mathcal{N}^-_{\lambda,\mu}$. We have*

$$t_{\max} = \left[\frac{(2^*_s - q)\, K(z)}{(2^*_s - 2)\, \|z\|^2_E} \right]^{\frac{1}{2-q}} > 0,$$

$$\mathcal{J}_{\lambda,\mu}\left(t^+ z\right) = \inf_{0 \le t \le t_{\max}} \mathcal{J}_{\lambda,\mu}(tz), \quad \mathcal{J}_{\lambda,\mu}\left(t^- z\right) = \sup_{t \ge t_{\max}} \mathcal{J}_{\lambda,\mu}(tz).$$

Proof. The proof is almost the same as that of the Lemma 2.3 and is omitted here. □

Lemma 2.5. *(Coercivity). The functional $\mathcal{J}_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. Suppose that $z = (u, v) \in \mathcal{N}_{\lambda,\mu}$. From (2.17), (2.18) and (2.20) by the Hölder inequality, we get

$$\begin{aligned} \mathcal{J}_{\lambda,\mu}(z) &= \left(\frac{1}{2} - \frac{1}{2^*_s}\right)\|z\|^2_E - \left(\frac{1}{q} - \frac{1}{2^*_s}\right)K(z) \\ &\ge \frac{s}{N}\|z\|^2_E - \left(\frac{1}{q} - \frac{1}{2^*_s}\right)\left(\lambda\|u\|^q_Z + \mu\|v\|^q_Z\right)\Theta \\ &\ge \frac{s}{N}\|z\|^2_E - \left(\frac{1}{q} - \frac{1}{2^*_s}\right)\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \|z\|^q_E \Theta, \end{aligned} \tag{2.25}$$

where Θ is given by (2.19). Thus, $\mathcal{J}_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$. The proof is complete. □

Since $\mathcal{N}_{\lambda,\mu} = \mathcal{N}^+_{\lambda,\mu} \cup \mathcal{N}^-_{\lambda,\mu}$, then from Lemma 2.3 and Lemma 2.5, the following quantities are well defined

$$m = \inf_{z \in \mathcal{N}_{\lambda,\mu}} \mathcal{J}_{\lambda,\mu}(z); \quad m^{\pm} = \inf_{z \in \mathcal{N}^{\pm}_{\lambda,\mu}} \mathcal{J}_{\lambda,\mu}(z).$$

Lemma 2.6. *($m^+ < 0$ and $m^- > 0$). (i) $m \le m^+ < 0$ for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$.*
(ii) There exists a positive constant d_0 depending on $\lambda, \mu, q, N, s, S_{\eta,\alpha,\beta}$ and Θ such that $m^- > d_0$ for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in \left(0, \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Theta_1\right)$.

Proof. (i) Let $z = (u, v) \in \mathcal{N}^+_{\lambda,\mu}$. By (2.20), (2.21) and (2.22), it follows that

$$\frac{2-q}{2^*_s - q}\|z\|^2_E > Q(z). \tag{2.26}$$

According to (2.20) and (2.26), we have that

$$\begin{aligned} \mathcal{J}_{\lambda,\mu}(z) &= \left(\frac{1}{2} - \frac{1}{q}\right)\|z\|^2_E + \left(\frac{1}{q} - \frac{1}{2^*_s}\right)Q(z) \\ &< \left[\left(\frac{1}{2} - \frac{1}{q}\right) + \left(\frac{1}{q} - \frac{1}{2^*_s}\right)\frac{2-q}{2^*_s - q}\right]\|z\|^2_E \\ &= -\frac{(2-q)s}{qN}\|z\|^2_E < 0, \end{aligned}$$

which implies that $m \le m^+ < 0$.

(ii) Suppose that $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in \left(0, \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Theta_1\right)$ and $z = (u, v) \in \mathcal{N}^-_{\lambda,\mu}$. By (2.8), (2.21) and (2.22), one has

$$\frac{2-q}{2^*_s - q}\|z\|^2_E < Q(z) \le S_{\eta,\alpha,\beta}^{-\frac{2^*_s}{2}}\|z\|^{2^*_s}_E,$$

which implies that

$$\|z\|_E > \left(\frac{2-q}{2^*_s - q}\right)^{\frac{1}{2^*_s-2}} S_{\eta,\alpha,\beta}^{\frac{2^*_s}{2(2^*_s-2)}}. \tag{2.27}$$

It follows from (2.25) and (2.27) that

$$\mathcal{J}_{\lambda,\mu}(z) \ge \|z\|^q_E \left[\frac{s}{N}\|z\|^{2-q}_E - \left(\frac{2^*_s - q}{2^*_s q}\right)\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}} \Theta\right] \ge d_0,$$

where $d_0 = d_0\left(\lambda, \mu, q, N, s, S_{\eta,\alpha,\beta}, \Theta\right)$ is a positive constant. □

Lemma 2.7. *(Natural Constraint). Suppose that $z_0 \in E$ is a local minimizer of $\mathcal{J}_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ and that $z_0 \notin \mathcal{N}^0_{\lambda,\mu}$, then $\mathcal{J}'_{\lambda,\mu}(z_0) = 0$ in E^{-1}.*

Proof. Suppose that $z_0 = (u_0, v_0) \in E$ is a local minimizer of $\mathcal{J}_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, then $\mathcal{J}_{\lambda,\mu}(z_0) = \min_{z\in\mathcal{N}_{\lambda,\mu}} \mathcal{J}_{\lambda,\mu}(z)$ and (2.21) holds. Furthermore, by the theory of Lagrange multipliers, there exists $\theta \in \mathbb{R}$ such that $\mathcal{J}'_{\lambda,\mu}(z_0) = \theta\Phi'(z_0)$. As $z_0 \in \mathcal{N}_{\lambda,\mu}$, we get

$$0 = \left\langle \mathcal{J}'_{\lambda,\mu}(z_0), z_0\right\rangle = \theta \left\langle \Phi'(z_0), z_0\right\rangle.$$

Since $z_0 \notin \mathcal{N}^0_{\lambda,\mu}$, $\langle \Phi'(z_0), z_0\rangle \neq 0$. Consequently, $\theta = 0$ and $\mathcal{J}'_{\lambda,\mu}(z_0) = 0$ in E^{-1}. □

3. Proof of Theorem 1.1

Definition 3.1. *Let $c \in \mathbb{R}$ and $\mathcal{J}_{\lambda,\mu} \in C^1(E, \mathbb{R})$. (i) $\{z_n\}$ is a $(PS)_c$-sequence in E for $\mathcal{J}_{\lambda,\mu}$ if $\mathcal{J}_{\lambda,\mu}(z_n) = c + o(1)$ and $\mathcal{J}'_{\lambda,\mu}(z_n) = o(1)$ strongly in E^{-1} as $n \to \infty$.*
(ii) We say that $\mathcal{J}_{\lambda,\mu}$ satisfies the $(PS)_c$ condition if any $(PS)_c$-sequence $\{z_n\}$ for $\mathcal{J}_{\lambda,\mu}$ has a convergent subsequence in E.

Lemma 3.1. *($(PS)_m$-Sequences). If $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$, then the functional $\mathcal{J}_{\lambda,\mu}$ has a $(PS)_m$-sequence $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}$, where Θ_1 is defined by (2.19).*

Proof. The proof is similar to that of [16] and is omitted here. □

Proposition 3.1. *(Existence of first solution). Assume that $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Theta_1)$ and $0 \le \gamma < N + sq - \frac{qN}{2}$. Then $\mathcal{J}_{\lambda,\mu}$ has a minimizer $z_1 = (u_1, v_1) \in \mathcal{N}^+_{\lambda,\mu}$ such that z_1 is a positive solution of system (1.1) and $\mathcal{J}_{\lambda,\mu}(z_1) = m = m^+ < 0$.*

Proof. By Lemma 3.1, there exists a $(PS)_m$-sequence $\{z_n\} = \{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ of $\mathcal{J}_{\lambda,\mu}$ such that

$$\mathcal{J}_{\lambda,\mu}(z_n) = m + o_n(1), \quad \mathcal{J}'_{\lambda,\mu}(z_n) = o_n(1). \tag{3.1}$$

Combining with Lemma 2.5, it follows that $\{z_n\}$ is bounded in E. Passing to a subsequence (still denoted by $\{z_n\}$), there exists $z_1 = (u_1, v_1) \in E$ such that

$$\begin{cases} u_n \rightharpoonup u_1, \quad v_n \rightharpoonup v_1, & \text{weakly in } Z, \\ u_n \to u_1, \quad v_n \to v_1, & \text{strongly in } L^r(\Omega)\,(1 \le r < 2_s^*)\,, \\ u_n(x) \to u_1(x), \quad v_n(x) \to v_1(x), & \text{a.e. in } \Omega. \end{cases} \tag{3.2}$$

From (3.1), we have $\left\langle \mathcal{J}'_{\lambda,\mu}(z_n), \varphi \right\rangle \to 0$ as $n \to \infty$ for all $\varphi \in E$. By (3.1) and (3.2), it is easy to see that z_1 is a solution of system (1.1). Because $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}$, we deduce that

$$K(z_n) = -\frac{2_s^* q}{2_s^* - q}\mathcal{J}_{\lambda,\mu}(z_n) + \frac{q\,(2_s^* - 2)}{2\,(2_s^* - q)}\|z_n\|_E^2. \tag{3.3}$$

Taking $n \to \infty$ in (3.3), by (3.1), (3.2) and the fact $m < 0$, we obtain

$$K(z_1) \ge -\frac{2_s^* q}{2_s^* - q} m > 0.$$

Therefore, $z_1 \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial solution of system (1.1). Next, we prove that $z_n \to z_1$ strongly in E and $\mathcal{J}_{\lambda,\mu}(z_1) = m$. Similar to (2.17) and (2.18), for some $q < r < 2_s^*$, by the Hölder inequality, one gets,

$$\begin{aligned} K(z_n) = &\int_\Omega \left(\lambda \frac{|u_n|^q}{|x|^\gamma} + \mu \frac{|v_n|^q}{|x|^\gamma}\right) dx \\ \le &\lambda \left(\int_\Omega |u_n|^{q \cdot \frac{r}{q}} dx\right)^{\frac{q}{r}} \left(\int_\Omega \left(\frac{1}{|x|^\gamma}\right)^{\frac{r}{r-q}} dx\right)^{\frac{r-q}{r}} \\ &+ \mu \left(\int_\Omega |v_n|^{q \cdot \frac{r}{q}} dx\right)^{\frac{q}{r}} \left(\int_\Omega \left(\frac{1}{|x|^\gamma}\right)^{\frac{r}{r-q}} dx\right)^{\frac{r-q}{r}} \\ \le &C|u_n|^{\frac{q}{r}} + \tilde{C}|v_n|^{\frac{q}{r}}, \end{aligned}$$

where $C, \tilde{C} > 0$ are constants. By (3.2) and the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} K(z_n) = K(z_1). \tag{3.4}$$

Noting $z_1 \in \mathcal{N}_{\lambda,\mu}$ and applying the Fatou lemma and (3.4), one has

$$\begin{aligned} m \le \mathcal{J}_{\lambda,\mu}(z_1) &= \frac{s}{N}\|z_1\|_E^2 - \frac{2_s^* - q}{2_s^* q} K(z_1) \\ &\le \liminf_{n\to\infty} \left(\frac{s}{N}\|z_n\|_E^2 - \frac{2_s^* - q}{2_s^* q} K(z_n)\right) \\ &= \liminf_{n\to\infty} \mathcal{J}_{\lambda,\mu}(z_n) = m, \end{aligned}$$

which implies that $\mathcal{J}_{\lambda,\mu}(z_1) = m$ and $\lim_{n\to\infty}\|z_n\|_E^2 = \|z_1\|_E^2$. Combining with (3.2), $z_n \rightharpoonup z_1$ as $n \to \infty$ in E, it shows that $z_n \to z_1$. Moreover, we have $z_1 \in \mathcal{N}^+_{\lambda,\mu}$. Otherwise, if $z_1 \in \mathcal{N}^-_{\lambda,\mu}$, then by

Lemma 2.3 there exist unique $t_0^{\pm}$such that $t_0^{\pm} z_1 \in \mathcal{N}^{\pm}_{\lambda,\mu}$ and $t_0^+ < t_0^- = 1$. Because of

$$\frac{d}{dt}\mathcal{J}_{\lambda,\mu}\left(t_0^+ z_1\right) = 0 \text{ and } \quad \frac{d^2}{dt^2}\mathcal{J}_{\lambda,\mu}\left(t_0^+ z_1\right) > 0,$$

there exists $\bar{t} \in \left(t_0^+, t_0^-\right)$ such that $\mathcal{J}_{\lambda,\mu}\left(t_0^+ z_1\right) < \mathcal{J}_{\lambda,\mu}\left(\bar{t} z_1\right)$. According Lemma 2.3, one obtains

$$\mathcal{J}_{\lambda,\mu}\left(t_0^+ z_1\right) < \mathcal{J}_{\lambda,\mu}\left(\bar{t} z_1\right) \leq \mathcal{J}_{\lambda,\mu}\left(t_0^- z_1\right) = \mathcal{J}_{\lambda,\mu}\left(z_1\right),$$

which is a contradiction. Thus, by (i) in Lemma 2.6, $\mathcal{J}_{\lambda,\mu}\left(z_1\right) = m$, and $z_1 \in \mathcal{N}^+_{\lambda,\mu}$. Consequently, we get that $\mathcal{J}_{\lambda,\mu}\left(z_1\right) = m = m^+ < 0$. Finally, we prove that z_1 is a positive solution of system (1.1). In particular $u_1 \not\equiv 0, v_1 \not\equiv 0$. Indeed, without loss of generality, we may assume that $v_1 \equiv 0$. Then u_1 is a nontrivial nonnegative solution of

$$\begin{cases} (-\Delta u)^s = \lambda \frac{|u|^{q-2}u}{|x|^{\gamma}}, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega. \end{cases}$$

By the standard regularity theory, we have $u_1 > 0$ in Ω and

$$\left\|(u_1, 0)\right\|^2 = K\left(u_1, 0\right) > 0.$$

Moreover, we may choose $\omega \in Z \backslash \{0\}$ such that

$$\|(0, \omega)\|^2 = K(0, \omega) > 0.$$

Now,

$$K\left(u_1, \omega\right) = K\left(u_1, 0\right) + K(0, \omega) > 0.$$

Consequently, by Lemma 2.4 there is a unique $0 < t^+ < t_{\max}$ such that $\left(t^+ u_1, t^+ \omega\right) \in \mathcal{N}^+_{\lambda,\mu}$. Moreover,

$$t_{\max} = \left[\frac{\left(2_s^* - q\right) K\left(u_1, \omega\right)}{\left(2_s^* - 2\right)\left\|\left(u_1, \omega\right)\right\|^2}\right]^{\frac{1}{2-q}} = \left(\frac{2_s^* - q}{2_s^* - 2}\right)^{\frac{1}{2-q}} > 1$$

and

$$\mathcal{J}_{\lambda,\mu}\left(t^+ u_1, t^+ \omega\right) = \inf_{0 \leq t \leq t_{\max}} \mathcal{J}_{\lambda,\mu}\left(t u_1, t \omega\right).$$

This implies

$$m^+ \leq \mathcal{J}_{\lambda,\mu}\left(t^+ u_1, t^+ \omega\right) \leq \mathcal{J}_{\lambda,\mu}\left(u_1, \omega\right) < \mathcal{J}_{\lambda,\mu}\left(u_1, 0\right) = m^+,$$

which is a contradiction. Finally, by Lemma 2.7 and the strong maximum principle, we deduce that $u_1, v_1 > 0$ in Ω and z_1 is a positive solution of system (1.1). □

4. Proof of Theorem 1.2

In this section, we want to obtain the second positive solution of system (1.1). Firstly, due to lacking of compactness, we will prove that the corresponding energy function satisfies the $(PS)_c$ condition.

Lemma 4.1. *(Uniform Lower Bound). If $\{z_n\} \subset E$ is a $(PS)_c$-sequence for $\mathcal{J}_{\lambda,\mu}$ with $z_n \rightharpoonup z$ in E, then $\mathcal{J}'_{\lambda,\mu}(z) = 0$, and there exists a positive constant C_0 such that*

$$\mathcal{J}_{\lambda,\mu}(z) \geq -C_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right), \tag{4.1}$$

where

$$C_0 = \frac{2-q}{2}\left[\left(\frac{2N-qN+2sq}{4s}\right)\Theta\right]^{\frac{2}{2-q}}.$$

Proof. Let $z_n = (u_n, v_n)$ and $z = (u, v)$. If $\{z_n\}$ is a $(PS)_c$-sequence for $\mathcal{J}_{\lambda,\mu}$ such that

$$\mathcal{J}_{\lambda,\mu}(z_n) = c + o_n(1), \quad \mathcal{J}'_{\lambda,\mu}(z_n) = o_n(1). \tag{4.2}$$

We claim that $\{z_n\}$ is bounded in E. In fact, for n large enough, one has

$$\begin{aligned}
c + o(1) + \|z_n\|_E &\geq \mathcal{J}_{\lambda,\mu}(z_n) - \frac{1}{2_s^*}\left\langle \mathcal{J}'_{\lambda,\mu}(z_n), z_n\right\rangle \\
&= \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\|z_n\|_E^2 - \left(\frac{1}{q} - \frac{1}{2_s^*}\right)K(z_n) \\
&\geq \frac{s}{N}\|z_n\|_E^2 - \left(\frac{1}{q} - \frac{1}{2_s^*}\right)\left(\lambda\|u_n\|_Z^q + \mu\|v_n\|_Z^q\right)\Theta \\
&\geq \frac{s}{N}\|z_n\|_E^2 - \left(\frac{1}{q} - \frac{1}{2_s^*}\right)\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right)^{\frac{2-q}{2}}\|z_n\|_E^q\,\Theta,
\end{aligned}$$

which implies that $\{z_n\}$ is bounded in E. So, our claim is true. Passing to a subsequence (still denoted by $\{z_n\}$), there exists $z = (u, v) \in \mathcal{J}_{\lambda,\mu}$ such that $z_n \to z$ in E and

$$\begin{cases}
u_n \rightharpoonup u, \quad v_n \rightharpoonup v, & \text{weakly in } Z, \\
u_n \to u, \quad v_n \to v, & \text{strongly in } L^r(\Omega) \quad (1 \leq r < 2_s^*), \\
u_n(x) \to u(x), \quad v_n(x) \to v(x), & \text{a.e. in } \Omega.
\end{cases} \tag{4.3}$$

By taking $\varphi = (\phi_1, \phi_2) \in E$. Combining with (2.17), (2.18) and (4.3), one gets

$$\lim_{n\to\infty}\int_\Omega \frac{|u_n|^{q-2}u_n}{|x|^\gamma}\phi_1 dx = \int_\Omega \frac{|u|^{q-2}u}{|x|^\gamma}\phi_1 dx, \lim_{n\to\infty}\int_\Omega \frac{|v_n|^{q-2}v_n}{|x|^\gamma}\phi_2 dx = \int_\Omega \frac{|v|^{q-2}v}{|x|^\gamma}\phi_2 dx. \tag{4.4}$$

Since $\left\{|u_n|^{\alpha_i-2}|v_n|^{\beta_i}u_n\right\}$ and $\left\{|u_n|^{\alpha_i}|v_n|^{\beta_i-2}v_n\right\}$ for $i = 1, 2$ are uniformly bounded in $\left(L^{2_s^*}(\Omega)\right)'$ and converge pointwisely to $|u|^{\alpha_i-2}|v|^{\beta_i}u$ and $|u|^{\alpha_i}|v|^{\beta_i-2}v$ respectively, we obtain

$$|u_n|^{\alpha_i-2}|v_n|^{\beta_i}u_n \rightharpoonup |u|^{\alpha_i-2}|v|^{\beta_i}u, \quad |u_n|^{\alpha_i}|v_n|^{\beta_i-2}v_n \rightharpoonup |u|^{\alpha_i}|v|^{\beta_i-2}v$$

weakly in $\left(L^{2_s^*}(\Omega)\right)' \times \left(L^{2_s^*}(\Omega)\right)'$ for $i = 1,2$ as $n \to \infty$. Consequently, it follows from (4.2) and (4.4) that

$$\begin{aligned}
&\lim_{n\to\infty} \left\langle \mathcal{J}'_{\lambda,\mu}(z_n), \varphi \right\rangle = \left\langle \mathcal{J}'_{\lambda,\mu}(z), \varphi \right\rangle \\
&= \int_Q \frac{(u(x)-u(y))(\phi_1(x)-\phi_1(y))}{|x-y|^{N+2s}} \, dx\, dy + \int_Q \frac{(v(x)-v(y))(\phi_2(x)-\phi_2(y))}{|x-y|^{n+2x}} \, dx\, dy \\
&\quad - \int_\Omega \left(\frac{\eta_1\alpha_1}{2_s^*} |u|^{\alpha_1-2}|v|^{\beta_1} u\phi_1 + \frac{\eta_2\alpha_2}{2_s^*} |u|^{\alpha_2-2}|v|^{\beta_2} u\phi_1 \right) dx \\
&\quad - \int_\Omega \left(\frac{\eta_1\beta_1}{2_s^*} |u|^{\alpha_1}|v|^{\beta_1-2} v\phi_2 + \frac{\eta_2\beta_2}{2_s^*} |u|^{\alpha_2}|v|^{\beta_2-2} v\phi_2 \right) dx \\
&\quad - \int_\Omega \left(\lambda \frac{|u|^{q-2}u}{|x|^\gamma}\phi_1 + \mu \frac{|v|^{q-2}v}{|x|^\gamma}\phi_2 \right) dx = 0.
\end{aligned} \tag{4.5}$$

Particularly, choosing $\varphi = z$ in (4.5), one has $\left\langle \mathcal{J}'_{\lambda,\mu}(z), z \right\rangle = 0$ and (2.20) is true. Consequently,

$$\mathcal{J}_{\lambda,\mu}(z) = \left(\frac{1}{2} - \frac{1}{2_s^*}\right)\|z\|_E^2 - \left(\frac{1}{q} - \frac{1}{2_s^*}\right)K(z). \tag{4.6}$$

Combining (2.17), (2.18) and the Young inequality, we have

$$\begin{aligned}
K(z) &\le \left(\lambda\|u\|_Z^q + \mu\|v\|_Z^q\right)\Theta \\
&= \left(\left[\frac{2}{q}\frac{s}{N}\left(\frac{1}{q}-\frac{1}{2_s^*}\right)^{-1}\right]^{\frac{q}{2}} \|u\|_Z^q\right)\left(\left[\frac{2}{q}\frac{s}{N}\left(\frac{1}{q}-\frac{1}{2_s^*}\right)^{-1}\right]^{-\frac{q}{2}} \lambda\Theta\right) \\
&\quad + \left(\left[\frac{2}{q}\frac{s}{N}\left(\frac{1}{q}-\frac{1}{2_s^*}\right)^{-1}\right]^{\frac{q}{2}} \|v\|_Z^q\right)\left(\left[\frac{2}{q}\frac{s}{N}\left(\frac{1}{q}-\frac{1}{2_s^*}\right)^{-1}\right]^{-\frac{q}{2}} \mu\Theta\right) \\
&\le \frac{s}{N}\left(\frac{1}{q}-\frac{1}{2_s^*}\right)^{-1}\left(\|u\|_Z^2 + \|v\|_Z^2\right) + \widehat{C}\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right) \\
&= \frac{s}{N}\left(\frac{1}{q}-\frac{1}{2_s^*}\right)^{-1}\|(u,v)\|^2 + \widehat{C}\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right),
\end{aligned} \tag{4.7}$$

with

$$\widehat{C} = \frac{2-q}{2}\left(\left[\frac{2}{q}\frac{s}{N}\left(\frac{1}{q}-\frac{1}{2_s^*}\right)^{-1}\right]^{-\frac{q}{2}}\Theta\right)^{\frac{2}{2-q}} = \frac{2-q}{2}\left[\left(\frac{2N-qN+2sq}{4s}\right)^{\frac{q}{2}}\Theta\right]^{\frac{2}{2-q}}.$$

Then (4.1) follows from (4.6) and (4.7) with $C_0 = \left(\frac{1}{q}-\frac{1}{2_s^*}\right)\widehat{C}$. □

Lemma 4.2. *Suppose that $(\mathcal{H})$ holds and $0 \le \gamma < N + sq - \frac{qN}{2}$, then $\mathcal{J}_{\lambda,\mu}$ satisfies the $(PS)_c$ condition in E, with c satisfying*

$$-\infty < c < c_\infty = \frac{s}{N} S_{\eta,\alpha,\beta}^{\frac{N}{2s}} - C_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right),$$

where C_0 is given by Lemma 4.1.

Proof. Let $\{z_n\} \subset E$ be a $(PS)_c$-sequence satisfying $\mathcal{J}_{\lambda,\mu}(z_n) = c + o(1)$ and $\mathcal{J}'_{\lambda,\mu}(z_n) = o(1)$, where $z_n = (u_n, v_n)$. The same to Lemma 4.1, one has $\{z_n\}$ is bounded in E. Furthermore, we can obtain (3.2) for some $z = (u, v) \in E$. Set $\widetilde{u}_n = u_n - u, \widetilde{v}_n = v_n - v$ and $\widetilde{z}_n = (\widetilde{u}_n, \widetilde{v}_n)$. From Brézis-Lieb's lemma [4], it follows that

$$\left\|\widetilde{z}_n\right\|_E^2 = \|z_n\|_E^2 - \|z\|_E^2 + o(1) \tag{4.8}$$

and by Lemma 2.3 in [10] one has

$$\int_\Omega |\widetilde{u}_n|^{\alpha_i}|\widetilde{v}_n|^{\beta_i}dx = \int_\Omega |u_n|^{\alpha_i}|v_n|^{\beta_i}dx - \int_\Omega |u|^{\alpha_i}|v|^{\beta_i}dx + o(1), \quad i = 1, 2. \tag{4.9}$$

Consequently, from (3.4), one gets

$$\left\|\widetilde{z}_n\right\|_E^2 + \|z\|_E^2 - Q(\widetilde{z}_n) - Q(z) - K(z) = o(1)$$

and

$$\lim_{n\to\infty} \left\langle \mathcal{J}'_{\lambda,\mu}(z_n), z \right\rangle = \|z\|_E^2 - Q(z) - K(z) = 0. \tag{4.10}$$

Since $\mathcal{J}_{\lambda,\mu}(z_n) = c + o(1)$, $\mathcal{J}'_{\lambda,\mu}(z_n) = o(1)$ and by (4.8) to (4.10), we can deduce that

$$\frac{1}{2}\left\|\widetilde{z}_n\right\|_E^2 - \frac{1}{2_s^*}Q(\widetilde{z}_n) = c - \mathcal{J}_{\lambda,\mu}(z) + o(1) \tag{4.11}$$

and

$$\left\|\widetilde{z}_n\right\|_E^2 - Q(\widetilde{z}_n) = o(1).$$

Now, we can assume that

$$\lim_{n\to\infty}\left\|\widetilde{z}_n\right\|_E^2 = \lim_{n\to\infty} Q(\widetilde{z}_n) = l. \tag{4.12}$$

If $l = 0$, the proof is complete. Suppose $l > 0$, then from (4.12) and the definition of $S_{\eta,\alpha,\beta}$, we have

$$\left\|\widetilde{z}_n\right\|_E^2 \geq S_{\eta,\alpha,\beta} Q^{\frac{2}{2_s^*}}(\widetilde{z}_n),$$

which implies that

$$l \geq S_{\eta,\alpha,\beta}^{\frac{N}{2s}}. \tag{4.13}$$

Using (4.10) to (4.13) and Lemma 4.1, we get

$$c = \left(\frac{1}{2} - \frac{1}{2_s^*}\right)l + \mathcal{J}_{\lambda,\mu}(z) \geq \frac{s}{N}S_{\eta,\alpha,\beta}^{\frac{N}{2s}} - C_0\left(\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}}\right) = c_\infty,$$

which contradicts the definition of c. Therefore, $l = 0$ and $(u_n, v_n) \to (u, v)$ strongly in E. The proof is complete. □

Next, we establish the existence of a local minimum for $\mathcal{J}_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}$. Assume, without loss of generality, that $0 \in \Omega$ and there exists $\rho_0 > 0$ such that $B(0, \rho_0) \subset \Omega$.
Also, let us introduce a cut-off function $\psi \in C_0^\infty(\Omega)$ such that $\psi(x) = 1$ for $|x| < \frac{\rho_0}{2}$, $\psi(x) = 0$ for

$|x| > \rho_0$, $0 \leq \psi(x) \leq 1$ for $\frac{\rho_0}{2} \leq |x| \leq \rho_0$ and $|\nabla\psi| \leq C_1$.
Define

$$u_\varepsilon(x) := U_\varepsilon(x)\,\psi(x) = \frac{C_{N,s}\varepsilon^{\frac{N-2s}{2}}}{\left(\varepsilon^2+|x|^2\right)^{\frac{N-2s}{2}}}\psi(x),$$

where $U_\varepsilon(x)$ is defined as (2.10).
From the estimates of proposition 21 in [15], we have the following result.

Lemma 4.3. *Let $s \in (0,1)$ and $N > 2s$. Then the following estimates hold:*

$$\|u_\varepsilon\|_Z^2 = \int_{\mathbb{R}^{2N}} \frac{|u_\varepsilon(x)-u_\varepsilon(y)|^2}{|x-y|^{N+2s}}\,\mathrm{d}x\,\mathrm{d}y \leqslant S^{N/2s} + O\left(\varepsilon^{N-2s}\right)$$

and

$$\int_\Omega |u_\varepsilon(x)|^{2_s^*}\,\mathrm{d}x = S^{N/2s} + O\left(\varepsilon^N\right)$$

as $\varepsilon \to 0$.

Lemma 4.4. *Under the assumptions of Theorem 1.1, there exist $\tilde{z} \in E\backslash\{0\}$ and $\Lambda^* > 0$ such that for all $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0,\Lambda^*)$ there holds*

$$\sup_{t\geq 0} \mathcal{J}_{\lambda,\mu}(t\tilde{z}) < c_\infty.$$

Proof. Set $z_\varepsilon = (u_\varepsilon, \tau_0 u_\varepsilon)$, where $\varepsilon > 0$ small enough and τ_0 is defined by Lemma 2.2. For any $t \geq 0$, we denote

$$\begin{aligned}
\Phi_\varepsilon(t) &= \mathcal{J}_{\lambda,\mu}(tz_\varepsilon)\\
&= \mathcal{J}_{\lambda,\mu}(tu_\varepsilon, t\tau_0 u_\varepsilon)\\
&= \frac{t^2}{2}\left(1+\tau_0^2\right)\|u_\varepsilon\|^2 - \frac{t^{2_s^*}}{2_s^*}\left(\eta_1\tau_0^{\beta_1}+\eta_2\tau_0^{\beta_2}\right)\int_\Omega u_\varepsilon^{2_s^*}dx - \left(\lambda+\mu\tau_0^q\right)\frac{t^q}{q}\int_\Omega \frac{u_\varepsilon^q}{|x|^\gamma}dx\\
&= \Phi_{\varepsilon,1}(t) - \left(\lambda+\mu\tau_0^q\right)\Phi_{\varepsilon,2}(t).
\end{aligned}$$

Notice that $\Phi_\varepsilon(0) = 0$, $\lim\limits_{t\to+\infty}\Phi_\varepsilon(t) = -\infty$, and $\lim\limits_{t\to 0^+}\Phi_\varepsilon(t) = 0$ uniformly for all ε. If $\inf\limits_{0<\varepsilon\leq 1}\sup\limits_{t\geq 0}\Phi_\varepsilon(t) \leq 0$ then $\mathcal{J}_{\lambda,\mu}(tz_\varepsilon) \leq 0 < c_\infty$, for any $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \frac{S_{\eta,\alpha,\beta}^{\frac{N}{2s}}}{NC_0}$. Thus, for any $0 < \lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} < \frac{S_{\eta,\alpha,\beta}^{\frac{N}{2s}}}{NC_0}$, one obtains

$$\sup_{t\geq 0}\mathcal{J}_{\lambda,\mu}(tz_\varepsilon) \leq c_\infty.$$

On the other hand, if $\inf\limits_{0<\varepsilon\leq 1}\sup\limits_{t\geq 0}\Phi_\varepsilon(t) > 0$, then $\sup\limits_{t\geq 0}\Phi_\varepsilon(t) > 0$ and it attains for some $t_\varepsilon > 0$. So, there exist two constants $t_1, t_2 > 0$ such that $t_1 < t_\varepsilon < t_2$.
Note that $\Phi_{\varepsilon,1}$ is increasing in $(0, t_{\max})$ and decreasing in $(t_{\max},\infty)$, where $t_{\max}$ satisfies $\Phi'_{\varepsilon,1}(t_{\max}) = 0$, one has

$$t_{\max} = \left[\frac{\left(1+\tau_0^2\right)\|u_\varepsilon\|_Z^2}{\left(\eta_1\tau_0^{\beta_1}+\eta_2\tau_0^{\beta_2}\right)\int_\Omega u_\varepsilon^{2_s^*}dx}\right]^{\frac{N-2s}{4s}}.$$

Then, according to Lemma 2.2 and Lemma 4.3 , we obtain

$$\begin{aligned}\Phi_{\varepsilon,1}(t) &\leq \Phi_{\varepsilon,1}\left(t_{\max}\right)\\ &\leq \frac{s}{N}\left[\frac{\left(1+\tau_0^2\right)\|u_\varepsilon\|_Z^2}{\left(\left(\eta_1\tau_0^{\beta_1}+\eta_2\tau_0^{\beta_2}\right)\int_\Omega u_\varepsilon^{2_s^*}dx\right)^{\frac{2}{2_s^*}}}\right]^{\frac{N}{2s}}\\ &\leq \frac{s}{N}\left[f\left(\tau_0\right)\frac{S^{\frac{N}{2s}}+O\left(\varepsilon^{N-2s}\right)}{\left(S^{\frac{N}{2s}}+O\left(\varepsilon^N\right)\right)^{\frac{2}{2_s^*}}}\right]^{\frac{N}{2s}}\\ &\leq \frac{s}{N}\left[f\left(\tau_0\right)S\right]^{\frac{N}{2s}}+C_2\varepsilon^{N-2s}\\ &= \frac{s}{N}S_{\eta,\alpha,\beta}^{\frac{N}{2s}}+C_2\varepsilon^{N-2s}.\end{aligned} \tag{4.14}$$

Now, we estimate $\Phi_{\varepsilon,2}\left(t_\varepsilon\right)$.

$$\begin{aligned}\Phi_{\varepsilon,2}\left(t_\varepsilon\right) =& \frac{t_\varepsilon^q}{q}\int_\Omega \frac{u_\varepsilon^q}{|x|^\gamma}dx\\ =& \frac{t_\varepsilon^q}{q}\int_\Omega \frac{\psi^q(x)C_{N,s}^q\varepsilon^{\frac{(N-2s)q}{2}}}{|x|^\gamma\left(\varepsilon^2+|x|^2\right)^{\frac{(N-2s)q}{2}}}dx\\ \geq& \frac{t_1^q}{q}\int_{|x|\leq\frac{\rho_0}{2}} \frac{C_{N,s}^q\varepsilon^{\frac{(N-2s)q}{2}}}{|x|^\gamma\left(\varepsilon^2+|x|^2\right)^{\frac{(N-2s)q}{2}}}dx\\ =& \frac{t_1^q}{q}C_{N,s}^q\int_0^{\frac{\rho_0}{2}} \frac{\varepsilon^{\frac{(N-2s)q}{2}}r^{N-1}}{|r|^\gamma\varepsilon^{(N-2s)q}\left[1+\left(\frac{r}{\varepsilon}\right)^2\right]^{\frac{(N-2s)q}{2}}}dr\\ =& \frac{t_1^q}{q}C_{N,s}^q\varepsilon^{N-\gamma+sq-\frac{qN}{2}}\int_0^{\frac{\rho_0}{2\varepsilon}} \frac{r^{N-1}}{r^\gamma\left(1+r^2\right)^{\frac{(N-2s)q}{2}}}dr\\ =& \frac{t_1^q}{q}C_{N,s}^q\varepsilon^{N-\gamma+sq-\frac{qN}{2}}\int_0^1 \frac{r^{N-1}}{r^\gamma\left(1+r^2\right)^{\frac{(N-2s)q}{2}}}dr\\ &+\frac{t_1^q}{q}C_{N,s}^q\varepsilon^{N-\gamma+sq-\frac{qN}{2}}\int_1^{\frac{\rho_0}{2\varepsilon}} \frac{r^{N-1}}{r^\gamma\left(1+r^2\right)^{\frac{(N-2s)q}{2}}}dr.\end{aligned} \tag{4.15}$$

From (4.15), we get

$$\Phi_{\varepsilon,2}\left(t_\varepsilon\right)=\frac{t_\varepsilon^q}{q}\int_\Omega\frac{u_\varepsilon^q}{|x|^\gamma}dx\geq\begin{cases}C_3\varepsilon^{N-\gamma+sq-\frac{qN}{2}}, & \gamma>N-(N-2s)q,\\ C_4\varepsilon^{\frac{qN}{2}-sq}|\ln\varepsilon|, & \gamma=N-(N-2s)q,\\ C_5\varepsilon^{\frac{qN}{2}-sq}, & \gamma<N-(N-2s)q,\end{cases}$$

where $C_i>0(i=3,4,5)$ are positive constants (C_i independent of ε). The case of $\gamma>N-(N-2s)q$, combining (4.14) with (4.15), one has

$$
\begin{aligned}
\sup_{t\geq 0}\mathcal{J}_{\lambda,\mu}\left(tz_{\varepsilon}\right) &= \Phi_{\varepsilon}\left(t_{\varepsilon}\right)\\
&= \Phi_{\varepsilon,1}\left(t_{\varepsilon}\right) - \left(\lambda+\mu\tau_0^q\right)\Phi_{\varepsilon,2}\left(t_{\varepsilon}\right)\\
&\leq \frac{s}{N}S_{\eta,\alpha,\beta}^{\frac{N}{2s}} + C_2\varepsilon^{N-2s} - C_3\left(\lambda+\mu\tau_0^q\right)\varepsilon^{N-\gamma+sq-\frac{qN}{2}}.
\end{aligned}
$$

Let $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}=\varepsilon^{N-2s}$, that is, $\varepsilon=\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{1}{N-2s}}$, then we can choose $\delta_1>0$ such that

$$
\begin{aligned}
&C_2\varepsilon^{N-2s} - C_3\left(\lambda+\mu\tau_0^q\right)\varepsilon^{N-\gamma+sq-\frac{qN}{2}}\\
=&C_2\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right) - C_3\left(\lambda+\mu\tau_0^q\right)\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{2N-2\gamma+2sq-qN}{2(N-2s)}}\\
<&-C_0\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right),
\end{aligned}
$$

for all $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in(0,\delta_1)$. Then, for $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in(0,\delta_1)$, one gets

$$
\sup_{t\geq 0}\mathcal{J}_{\lambda,\mu}\left(tz_{\varepsilon}\right) < c_{\infty}.
$$

The case of $\gamma=N-(N-2s)q$, it follows from (4.14) and (4.15) that

$$
\begin{aligned}
\sup_{t\geq 0}\mathcal{J}_{\lambda,\mu}\left(tz_{\varepsilon}\right) &= \Phi_{\varepsilon}\left(t_{\varepsilon}\right)\\
&= \Phi_{\varepsilon,1}\left(t_{\varepsilon}\right) - \left(\lambda+\mu\tau_0^q\right)\Phi_{\varepsilon,2}\left(t_{\varepsilon}\right)\\
&\leq \frac{s}{N}S_{\eta,\alpha,\beta}^{\frac{N}{2s}} + C_2\varepsilon^{N-2s} - C_4\left(\lambda+\mu\tau_0^q\right)\varepsilon^{\frac{qN}{2}-sq}|\ln\varepsilon|.
\end{aligned}
$$

Let $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}=\varepsilon^{N-2s}$, that is, $\varepsilon=\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{1}{N-2s}}$, choosing $\delta_2>0$ such that for all $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in(0,\delta_2)$, then one has

$$
\begin{aligned}
&C_2\varepsilon^{N-2s} - C_4\left(\lambda+\mu\tau_0^q\right)\varepsilon^{\frac{qN}{2}-sq}|\ln\varepsilon|\\
=&C_2\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right) - C_4\left(\lambda+\mu\tau_0^q\right)\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)^{\frac{q}{2}}\left|\ln\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right)\right|\\
<&-C_0\left(\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\right).
\end{aligned}
$$

Consequently, for $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in(0,\delta_2)$, we obtain

$$
\sup_{t\geq 0}\mathcal{J}_{\lambda,\mu}\left(tz_{\varepsilon}\right) < c_{\infty}.
$$

Thus, there exists $\Lambda^*:=\min\left\{\frac{sS_{\eta,\alpha,\beta}^{\frac{N}{2s}}}{NC_0},\delta_1,\delta_2\right\}>0$ such that

$$
\sup_{t\geq 0}\mathcal{J}_{\lambda,\mu}\left(tz_{\varepsilon}\right) < c_{\infty},
$$

for any $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in(0,\Lambda^*)$. The proof is complete by taking $\tilde{z}=z_{\varepsilon}$. □

Lemma 4.5. *(Curves into $\mathcal{N}^-_{\lambda,\mu}$). For $z \in \mathcal{N}^-_{\lambda,\mu}$, then there exist $\eta > 0$ and a differentiable functional $\xi : B(0;\eta) \subset E \to \mathbb{R}^+$such that $\zeta(0) = 1, \xi(\nu)(z-\nu) \in \mathcal{N}^-_{\lambda,\mu}$ for any $\nu \in B(0;\eta)$, and*

$$\langle \xi'(0), \nu \rangle = \frac{\langle \Phi'(z), \nu \rangle}{\langle \Phi'(z), z \rangle} \tag{4.16}$$

for any $\nu \in B(0;\eta)$, where Φ is defined as (2.21).

Proof. The proof is almost the same as in [16]. For $z \in \mathcal{N}^-_{\lambda,\mu}$ and $w \in E$, define a function $F_z : \mathbb{R} \times E \to \mathbb{R}$ by

$$\begin{aligned} F_z(\xi, w) =& \left\langle \mathcal{J}'_{\lambda,\mu}(\xi(z-w)), \xi(z-w) \right\rangle \\ =& \xi^2 \|z-w\|^2_E - \xi^{2^*_s} Q(z-w) - \xi^q K(z-w). \end{aligned}$$

Then $F_z(1,0) = \left\langle \mathcal{J}'_{\lambda,\mu}(z), z \right\rangle = 0$ and

$$\begin{aligned} \frac{d}{d\xi} F_z(1,0) &= \langle \Phi'(z), z \rangle \\ &= 2\|z\|^2_E - 2^*_s Q(z) - qK(z) \\ &= (2-q)\|z\|^2_E - (2^*_s - q)\, Q(z) \\ &= (2-2^*_s)\, \|z\|^2_E + (2^*_s - q)\, K(z) < 0. \end{aligned}$$

In turn, by virtue of the Implicit Function Theorem, there exists $\eta > 0$ and a function $\xi : B(0;\eta) \subset E \to \mathbb{R}$ of class C^1 such that $\xi(0) = 1$ and formula (4.16) holds, via direct computation. Moreover,

$$\left\langle \mathcal{J}'_{\lambda,\mu}(\xi(\nu)(z-\nu)), \xi(\nu)(z-\nu) \right\rangle = 0 \quad \text{for all } \nu \in B(0;\eta),$$

namely $\xi(\nu)(z-\nu) \in \mathcal{N}^-_{\lambda,\mu}$. □

Proof of Theorem 1.2. Let $\Lambda = \min\left\{\Lambda^*, \left(\frac{q}{2}\right)^{\frac{2}{2-q}} \Theta_1\right\}$ and $\lambda^{\frac{2}{2-q}} + \mu^{\frac{2}{2-q}} \in (0, \Lambda)$. By Lemma 2.6, one has $m^- = \inf\limits_{z \in \mathcal{N}^-_{\lambda,\mu}} \mathcal{J}_{\lambda,\mu}(z) > 0$. Let $\{z_k\} \subset E$ be a minimizing sequence for m^-. According to Lemma 2.2, we get

$$0 < m^- \le \mathcal{J}_{\lambda,\mu}\left(t^-_\varepsilon z_\varepsilon\right) \le \sup_{t \ge 0} \mathcal{J}_{\lambda,\mu}\left(t z_\varepsilon\right) < c_\infty.$$

Now, we prove that $\{z_k\}$ is a $(PS)_{m^-}$ sequence for $\mathcal{J}_{\lambda,\mu}$. By Ekeland's Variational Principle (see [8]), there exists a subsequence $\{z_k\}$ (still denoted by $\{z_k\}$) such that

$$\mathcal{J}_{\lambda,\mu}(z_k) < m^- + \frac{1}{k}, \tag{4.17}$$

$$\mathcal{J}_{\lambda,\mu}(z_k) \le \mathcal{J}_{\lambda,\mu}(w) + \frac{\|w - z_k\|_E}{k}, \quad w \in \mathcal{N}^-_{\lambda,\mu}. \tag{4.18}$$

Thus, we only need prove that $\mathcal{J}'_{\lambda,\mu}(z_k) \to 0$ in E^{-1} as $k \to \infty$. By applying Lemma 4.4, there exist $\eta_k > 0$ and the differentiable functional $\zeta_k : B(0;\eta_k) \subset E \to \mathbb{R}^+$such that $\zeta_k(0) = 1, \zeta_k(w)(z_k - w) \in \mathcal{N}^-_{\lambda,\mu}$ for any $w \in B(0;\eta_k)$. Let $\varphi \in E$ with $\|\varphi\|_E = 1$, and $0 < \sigma < \eta_k$, choosing $w = \sigma\varphi$. Then

$w=\sigma\varphi\in B\left(0;\eta_k\right)$ and $\omega_{\sigma,k}=\zeta_k\left(\sigma\varphi\right)\left(z_k-\sigma\varphi\right)\in\mathcal{N}^-_{\lambda,\mu}$. From (4.18), by the mean value theorem, let $\sigma\to 0^+$, we obtain

$$\begin{aligned}\frac{\left\|\omega_{\sigma,k}-z_k\right\|_E}{k}\geq&\mathcal{J}_{\lambda,\mu}\left(z_k\right)-\mathcal{J}_{\lambda,\mu}\left(\omega_{\sigma,k}\right)\\ =&\left\langle\mathcal{J}'_{\lambda,\mu}\left(t_0z_k+\left(1-t_0\right)\omega_{\sigma,k}\right),z_k-\omega_{\sigma,k}\right\rangle\\ =&\left\langle\mathcal{J}'_{\lambda,\mu}\left(z_k\right),z_k-\omega_{\sigma,k}\right\rangle+o\left(\left\|z_k-\omega_{\sigma,k}\right\|_E\right)\\ =&\sigma\zeta_k\left(\sigma\varphi\right)\left\langle\mathcal{J}'_{\lambda,\mu}\left(z_k\right),\sigma\varphi\right\rangle\\ &+\left(1-\zeta_k\left(\sigma\varphi\right)\right)\left\langle\mathcal{J}'_{\lambda,\mu}\left(z_k\right),z_k\right\rangle+o\left(\left\|z_k,-\omega_{\sigma,k}\right\|_E\right)\\ =&\sigma\zeta_k\left(\sigma\varphi\right)\left\langle\mathcal{J}'_{\lambda,\mu}\left(z_k\right),\sigma\varphi\right\rangle+o\left(\left\|z_k-\omega_{\sigma,k}\right\|_E\right),\end{aligned}$$

where $0<t_0<1$. Hence, let $\sigma\to 0^+$, one has

$$\begin{aligned}\left\langle\mathcal{J}'_{\lambda,\mu}\left(z_k\right),\sigma\varphi\right\rangle\leq&\frac{\left\|\omega_{\sigma,k}-z_k\right\|_E\left(\frac{1}{k}+|o(1)|\right)}{\sigma|\zeta_k\left(\sigma\varphi\right)|}\\ \leq&\frac{\left\|z_k\left(\zeta_k\left(\sigma\varphi\right)-\zeta_k(0)\right)-\sigma\varphi\zeta_k\left(\sigma\varphi\right)\right\|_E\left(\frac{1}{k}+|o(1)|\right)}{\sigma|\zeta_k\left(\sigma\varphi\right)|}\\ \leq&\frac{\|z_k\|_E\left|\zeta_k\left(\sigma\varphi\right)-\zeta_k(0)\right|+\sigma\left\|\varphi\right\|_E\left|\zeta_k\left(\sigma\varphi\right)\right|}{\sigma|\zeta_k\left(\sigma\varphi\right)|}\left(\frac{1}{k}+|o(1)|\right)\\ \leq&C\left(1+\left\|\zeta'_k(0)\right\|_E\right)\left(\frac{1}{k}+|o(1)|\right).\end{aligned}$$

From the boundedness of $\{z_k\}$ and $\zeta'_k(0)$, we obtain $\mathcal{J}'_{\lambda,\mu}\left(z_k\right)\to 0$ in E^{-1} as $k\to\infty$. Hence $\{z_k\}$ is a $(PS)_{m^-}$-sequence for $\mathcal{J}_{\lambda,\mu}$ at the level m^-.

From Lemma 4.2 and Lemma 4.4, there exist a subsequence (still denoted by $\{z_k\}$) and $z_2=(u_2,v_2)\in\mathcal{N}^-_{\lambda,\mu}$such that $z_k\to z_2$ strongly in E and $\mathcal{J}_{\lambda,\mu}\left(z_2\right)=m^->0$ for all $\lambda^{\frac{2}{2-q}}+\mu^{\frac{2}{2-q}}\in(0,\Lambda)$. Since $\mathcal{J}_{\lambda,\mu}\left(u_2,v_2\right)=\mathcal{J}_{\lambda,\mu}\left(|u_2|,|v_2|\right)$ and $z_2\in\mathcal{N}^-_{\lambda,\mu}$, we get

$$Q\left(z_2\right)>\frac{2-q}{2^*_s-q}\left\|(u_2,v_2)\right\|^2>0.$$

This implies that $u_2\not\equiv 0$ and $v_2\not\equiv 0$. By the strong maximum principle, it follows that (u_2,v_2) is a positive solution of system (1.1). Since $\mathcal{N}^+_{\lambda,\mu}\cap\mathcal{N}^-_{\lambda,\mu}=\emptyset$, this implies that (u_1,v_1) and (u_2,v_2) are distinct. The proof is complete. □

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