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Analytical Solutions for a Nonlinear Quadratic Delayed Functional Integral Inclusion With Feedback Control on the Real Half-Line

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Abstract. Let $\mathcal{E}$ be a reflexive Banach space. In this research, we are interested in the solvability of the nonlinear quadratic delayed functional integral inclusion (NQDFII) with a feedback control condition on the real half-axis. Our examination is found within the space $\mathcal{BC}(\mathcal{R}_+,\mathcal{E})$ of bounded continuous functions on the real half-axis $\mathcal{R}_+$ and takes values in a reflexive Banach space $\mathcal{E}$ beneath the assumption that the set-valued function G satisfy Lipschitz condition in $\mathcal{E}$. The base we depend on in this study is the procedure related to a measure of noncompactness in the space $\mathcal{BC}(\mathcal{R}_+,\mathcal{E})$ by a given norm of continuity and applying Darbo's fixed point theorem. Moreover, the asymptotic stability of the solution and the asymptotic dependency of the solution on the set of selections S_G will be examined. Also, we give an example to illustrate the adequacy and esteem of our comes about.

1. Introduction

Let $\mathcal{R}$ be the set of all real numbers and $\mathcal{R}_+ = [0,\infty)$ and let $\mathcal{E}$ be a reflexive Banach space with norm $\|.\|_{\mathcal{E}}$. Denote by $\mathcal{BC}(\mathcal{R}_+,\mathcal{E})$ the Banach space of all functions defined, continuous and bounded on the real half-axis $\mathcal{R}_+$ and taking values in a given Banach space $\mathcal{E}$.
The norm of $f \in \mathcal{BC}(\mathcal{R}_+,\mathcal{E})$ is defined by

$$\|f\|_{\mathcal{BC}} = sup_{t\in\mathcal{R}_+}\|f(t)\|_{\mathcal{E}}.$$

The theory of measures of noncompactness plays an important role in applications to the nonlinear analysis and to the theories of differential and integral equations (see [1] and [2]- [3]). Also applies to control theory and the operator theory (see [4]). Investigation on the real half-axis of the integral equations on different spaces of functions has gotten an awesome consideration (see [5]-[10]). Measures of noncompactness in the space of functions which are defined, continuous and

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bounded on the real half-axis and taking values in an arbitrary Banach space $\mathcal{E}$ denote $\mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ are discussed in [7] and [11].

The nonlinear integral equations recently studied by many authors for example (see [12]- [14]), where authors examine the solvability of non-linear 2D Volterra integral equations through Petryshyn fixed point theorem in Banach space, two systems of nonlinear Volterra integral equation and Volterra integro-differential equation through Banach's contraction principle and a nonlinear integral equation with multiple variable time delays and a nonlinear integro-differential equation without delay by the fixed point method using progressive contractions. Also, consider some properties of this solution.

Moreover, the nonlinear functional integral equations with feedback control were studied by P. Nasertayoob, using the measure of noncompactness in conjunction with Darbo's fixed point theorem see ([15]). Typically, in [16] authors studied a nonlinear neutral delay population system with feedback control and investigated a positive periodic solution using the strict set contraction operators fixed point theorem.

In [17], A. M. A. El-Sayed and M. A. H. Alrashdi are concerned with a nonlinear functional integral equation with a nonlinear functional equation constraint merge a control parameter function. Other results exist in [18], where researchers care about a nonlinear functional integral equation restricted by a functional equation with a parameter.

Furthermore, in [19] the authors discussed the existence of solutions for integral inclusions with fractal feedback control and they are seeking solutions for these inclusions based on the procedure associated with measures of noncompactness by a given modulus of continuity in the space in $\mathcal{BC}(\mathcal{R}_+)$. However, in this article, we establish our results utilizing the procedure of measures of noncompactness by a given norm of continuity in the space $\mathcal{BC}(\mathcal{R}_+, \mathcal{E})$.

Functional differential inclusions on the real-half line have been broadly examined by several creators and there are numerous curiously comes about concerning these issues (see [20]- [23]) and a functional integral inclusion was studied by B.C. Dhage (see [24]- [27]). The Lipschitz selections of the set-valued functions were explored by some authors (see [28]- [29]).

Here, in this article we are going to apply the theory of measure of noncompactness with a given norm of continuity and using Darbo's fixed point theorem to study the solution of the nonlinear quadratic delayed functional integral inclusion (NQDFII)

$$x(t) \in G(t, g_1(t, x(t)) \int_0^{\varphi(t)} g_2(t, s, u(s))ds), \ t \in \mathcal{R}_+ \tag{1.1}$$

with the feedback control

$$\frac{du(t)}{dt} = -\varrho u(t) + \psi(t, x(t)), \ \varrho > 0, \ u(0) = u_0 \in \mathcal{E} \tag{1.2}$$

in a reflexive Banach space $\mathcal{E}$ on the real half-axis, assuming the set-valued function G satisfies a Lipschitz condition in $\mathcal{E}$. The study is conducted within the space of bounded continuous functions $\mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ where $G : \mathcal{R}_+ \times \mathcal{E} \to \Omega(\mathcal{E})$ is a Lipschitzian set-valued map and $\Omega(\mathcal{E})$ denote the family of nonempty subsets of the Banach space $\mathcal{E}$. Moreover, the asymptotic stability and the asymptotic dependency of the solution for the nonlinear quadratic delayed functional integral inclusion NQDFII (1.1) with the feedback control (1.2) on the set of selections S_G will be analyzed. An example is provided to demonstrate the significance and validity of the results obtained.

2. Preliminaries

Here, we show a few documentation and assistance comes about that will be required in this work.

Let us take an arbitrary and bounded set X, $X \subseteq \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$. Next for an arbitrary fixed function $x \in X$ and for $\varepsilon > 0$, we denote by $\omega^{\mathcal{T}}(x, \varepsilon)$ the norm of continuity, $\mathcal{T} > 0$ of the function x on the interval $\mathcal{I} = [0, \mathcal{T}]$ and defined by

$$\omega^{\mathcal{T}}(x, \varepsilon) = sup\{\|x(t) - x(s)\|_{\mathcal{E}} : \ t, s \in \mathcal{I}, \ |t - s| \le \varepsilon\}$$

and

$$\omega^{\mathcal{T}}(X, \varepsilon) = sup\{\omega^{\mathcal{T}}(x, \varepsilon), \ x \in X\}.$$

Also,

$$\omega_0^{\mathcal{T}}(X) = lim_{\varepsilon \to 0} \omega^{\mathcal{T}}(X, \varepsilon) \ and \ \omega_0(X) = lim_{\mathcal{T} \to \infty} \omega_0^{\mathcal{T}}(X).$$

Next, for $t \in \mathcal{R}_+$ let us define

$$diam X(t) = sup\{\|x(t) - y(t)\|_{\mathcal{E}} : \ x, y \in X\}$$

and

$$C(X) = lim_{t \to \infty} sup \ diam X(t).$$

Finally, the measure of noncompactness on $\mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ is given by

$$\begin{aligned} \mu(X) &= \omega_0(X) + C(X) \\ &= \omega_0(X) + lim_{t \to \infty} sup \ diam X(t). \end{aligned}$$

Now, we state the Darbo's fixed point theorem [30].

Theorem 2.1. *Assume that $\mathcal{A} : \chi \to \chi$ is continuous operator and χ is a nonempty closed bounded convex subset of the space $\mathcal{E}$ with $\mu(\mathcal{A}X) \le K\mu(X)$ for any nonempty subset X of χ, where the constant $K \in [0, 1)$. Then $\mathcal{A}$ has a fixed point in the set χ.*

Now, let $x : \mathcal{I} \to \mathcal{E}$.

Definition 2.1. *[21] A set-valued map G from $\mathcal{I} \times \mathcal{E}$ to the family of all nonempty closed subsets of $\mathcal{E}$ is called Lipschitzian if there exists $L > 0$ such that for all $t_1, t_2 \in \mathcal{I}$ and all $x_1, x_2 \in \mathcal{E}$, we have*

$$\mathcal{H}(G(t_1, x_1), G(t_2, x_2)) \le L\{|t_1 - t_2| + \|x_1 - x_2\|_{\mathcal{E}}\}$$

where $\mathcal{H}(\ell, \jmath)$ is the Hausdorff metric between the two subsets $\ell, \jmath \in \mathcal{I} \times \mathcal{E}$.

Denote $S_G = Lip(\mathcal{I}, \mathcal{E})$ be the set of Lipschitz selections of G.

3. Main Result

In this section, we display our fundamental result by demonstrating the existence of solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ for the NQDFII (1.1) with the feedback control (1.2) in the reflexive Banach space $\mathcal{E}$ on the real half-axis utilizing the procedure related with a measure of noncompactness in $\mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ and using Darbo's fixed point theorem beneath the assumption that the set-valued function G satisfy Lipschitz condition.

Definition 3.1. *By a solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of the NQDFII (1.1) with the feedback control (1.2) in the reflexive Banach space $\mathcal{E}$ on the real half-axis we mean a single-valued function $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$, we mean a single-valued function, which fulfills (1.1) and (1.2).*

Consider presently the NQDFII (1.1) with the feedback control (1.2) beneath the taking after assumptions:
(H1) The set $G(t, y)$ is compact and convex for all $(t, y) \in \mathcal{R}_+ \times \mathcal{E}$.
(H2) The set-valued map G is Lipschitzian with a Lipschitz constant $L > 0$ such that

$$\mathcal{H}(G(t_1, y), G(t_2, z)) \leq L(|t_1 - t_2| + \|y - z\|_{\mathcal{E}})$$

for all $t_1, t_2 \in \mathcal{R}_+$ and $y, z \in \mathcal{E}$, where $\mathcal{H}(\ell, \jmath)$ is the Hausdorff metric between the two subsets $\ell, \jmath \in \mathcal{R}_+ \times \mathcal{E}$.
(H3) The set of Lipschitz selections S_G of the set valued function G is nonempty.
(H4) $g_1 : \mathcal{R}_+ \times \mathcal{E} \to \mathcal{E}$ satisfy Lipschitz condition and there exists a function $a_1(t)$ and a constant L_1 such that

$$\|g_1(t, x(t))\|_{\mathcal{E}} \leq \|a_1(t)\|_{\mathcal{E}} + L_1\|x(t)\|_{\mathcal{E}}.$$

(H5) $g_2 : \mathcal{R}_+ \times \mathcal{R}_+ \times \mathcal{E} \to \mathcal{R}_+$ is continuous and there exist continuous functions $k(t, s) : \mathcal{R}_+ \times \mathcal{R}_+ \to \mathcal{R}_+$ and $b(s) : \mathcal{R}_+ \to \mathcal{R}_+$ such that

$$|g_2(t, s, u(s))| \leq |k(t, s)| + |b(s)|\|x(s)\|_{\mathcal{E}}, \ \forall t, s \in \mathcal{R}_+,$$

where

$$lim_{t\to\infty} \int_0^t |k(t, s)| ds = 0, \ sup_{t\in\mathcal{R}_+} \int_0^t |k(t, s)| ds = K$$

and

$$lim_{t\to\infty} \int_0^t |b(s)| ds = 0, \ sup_{t\in\mathcal{R}_+} \int_0^t |b(s)| ds = B.$$

(H6) The function $\varphi : \mathcal{I} \to \mathcal{I}$ is continuous nondecreasing function and $\varphi(t) \leq t$.
(H7) The function $\psi : \mathcal{R}_+ \times \mathcal{E} \to \mathcal{E}$ is Caratheodory function that is measurable in $t \in \mathcal{R}_+$, $\forall\, x \in \mathcal{E}$ and continuous in $x \in \mathcal{E}$, $\forall\, t \in \mathcal{R}_+$ and there are two integrable functions $d_1, d_2 : \mathcal{R}_+ \to \mathcal{R}$ such that

$$\|\psi(t, x(t))\| \leq |d_1(t)| + |d_2(t)|\|x(t)\|_{\mathcal{E}}, \ t \in \mathcal{R}_+.$$

(H8) There exists a positive real number r of the algebraic equation

$$LL_1U_2r^2B + (LL_1K + L\|a_1\|_{\mathcal{BC}}U_2B + LL_1[U + U_1]B - 1) + \|a\|_{\mathcal{BC}} + L\|a_1\|_{\mathcal{BC}}K + L\|a_1\|_{\mathcal{BC}}[U + U_1]B = 0.$$

Remark 3.1. *For any function x belonging to $\mathcal{BC}(\mathcal{R}_+, \mathcal{E})$, the solution to a feedback control equation*

$$\frac{du(t)}{dt} = -\varrho u(t) + \psi(t, x(t)),\ \varrho > 0,\ u(0) = u_0 \in \mathcal{E}$$

which denoted by $u(t)$ and it is given by

$$u(t) = e^{-\varrho t}u(0) + \int_0^t e^{-\varrho(t-s)}\psi(s, x(s))ds.$$

According to the positivity of the initial condition $u(0) > 0$, then the solution $u(t)$ is globally attractive and bounded above by positive constants which proved by F. Chen (see [31]), then

$$\begin{aligned}
\|u(t)\|_{\mathcal{E}} &= \|e^{-\varrho t}u(0) + \int_0^t e^{-\varrho(t-s)}\psi(s, x(s))ds\|_{\mathcal{E}} \\
&\leq e^{-\varrho t}\|u(0)\|_{\mathcal{E}} + \int_0^t e^{-\varrho(t-s)}\|\psi(s, x(s))\|_{\mathcal{E}}ds \\
&\leq e^{-\varrho t}\|u(0)\|_{\mathcal{E}} + \int_0^t e^{-\varrho(t-s)}\{|d_1(s)| + |d_2(s)|\|x(s)\|_{\mathcal{E}}\}ds \\
&\leq e^{-\varrho t}\|u(0)\|_{\mathcal{E}} + \int_0^t e^{-\varrho(t-s)}|d_1(s)|ds + \int_0^t e^{-\varrho(t-s)}|d_2(s)|\|x(s)\|_{\mathcal{E}}ds \\
&\leq sup_{t\in\mathcal{R}_+}\ e^{-\varrho t}\|u(0)\|_{\mathcal{E}} + sup_{t\in\mathcal{R}_+}\ e^{-\varrho t}\int_0^t e^{\varrho s}|d_1(s)|ds \\
&+ \|x\|_{\mathcal{BC}} sup_{t\in\mathcal{R}_+} e^{-\varrho t}\int_0^t e^{\varrho s}|d_2(s)|ds \\
&\leq sup_{t\in\mathcal{R}_+}\ e^{-\varrho t}\|u(0)\|_{\mathcal{E}} + sup_{t\in\mathcal{R}_+}\ \int_0^t e^{-\varrho\vartheta}|d_1(t-\vartheta)|d\vartheta \\
&+ \|x\|_{\mathcal{BC}} sup_{t\in\mathcal{R}_+}\int_0^t e^{-\varrho\vartheta}|d_2(t-\vartheta)|d\vartheta \\
&\leq U + U_1 + U_2\|x\|_{\mathcal{BC}},
\end{aligned}$$

where

$$sup_{t\in\mathcal{R}_+}\ e^{-\varrho t}\|u(0)\|_{\mathcal{E}} = U,\ sup_{t\in\mathcal{R}_+}\ \int_0^t e^{-\varrho\vartheta}|d_1(t-\vartheta)|d\vartheta = U_1$$

and

$$sup_{t\in\mathcal{R}_+}\int_0^t e^{-\varrho\vartheta}|d_2(t-\vartheta)|d\vartheta = U_2.$$

Hence

$$\|u\|_{\mathcal{BC}} = U + U_1 + U_2\|x\|_{\mathcal{BC}}.$$

Now, let $X \subseteq \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ and $\chi = \{x : x \in X\}$.
Define the following norm of continuity:

$$\omega^{\mathcal{T}}(x, \varepsilon) = sup\{\|x(t) - x(\tau)\|_{\mathcal{E}} :\ t, \tau \in \mathcal{I},\ |t-\tau| \leq \varepsilon\}$$

and

$$\omega^{\mathcal{T}}(X, \varepsilon) = sup\{\omega^{\mathcal{T}}(x, \varepsilon),\ x \in X\}.$$

Also,

$$\omega_0^{\mathcal{T}}(X) = lim_{\varepsilon\to 0}\omega^{\mathcal{T}}(X,\varepsilon) \; and \; \omega_0(X) = lim_{\mathcal{T}\to\infty}\omega_0^{\mathcal{T}}(X).$$

In addition for $t \in \mathcal{R}_+$

$$diam X(t) = sup\{\|x_1(t) - x_2(t)\|_{\mathcal{E}} : \; x_1, x_2 \in X\}$$

and

$$C(X) = lim_{t\to\infty} sup \; diam X(t)$$

Remark 3.2. *From assumptions (H1)-(H3), there exists a Lipschitz selection $g \in S_G$; $g : \mathcal{R}_+ \times \mathcal{E} \to \mathcal{E}$ such that*

$$\|g(t, y(t))\|_{\mathcal{E}} \leq \|a(t)\|_{\mathcal{E}} + L\|y(t)\|_{\mathcal{E}}$$

this selection satisfy the nonlinear quadratic delayed functional integral equation (NQDFIE)

$$x(t) = g(t, g_1(t, x(t)) \int_0^{\varphi(t)} g_2(t, s, u(s))ds), \; t \in \mathcal{R}_+ \tag{3.1}$$

Then the solution of the NQDFIE (3.1) with feedback control (1.2), if it exists, is the solution of the NQDFII (1.1) with feedback control (1.2).

Now, we study the existence of the solution of the NQDFIE (3.1) with feedback control (1.2).

Theorem 3.1. *Let the assumptions (H1)-(H8) be satisfied. Then the NQDFIE (3.1) with feedback control (1.2) has at least one solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$.*

Proof. Define the operator $\mathcal{A}$ by

$$\mathcal{A}x(t) = g(t, g_1(t, x(t)) \int_0^{\varphi(t)} g_2(t, s, u(s))ds), \; t \in \mathcal{R}_+.$$

Let the set χ_r defined by

$$\chi_r = \{x : \; x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E}), \; \|x\|_{\mathcal{BC}(\mathcal{R}_+,\mathcal{E})} \leq r\};$$

$$r = \|a\|_{\mathcal{BC}} + L\|a_1\|_{\mathcal{BC}}K + LL_1rK + L\|a_1\|_{\mathcal{BC}}[U + U_1 + U_2r]B + LL_1r[U + U_1 + U_2r]B.$$

Then, it is clear that it is nonempty, closed, bounded and convex subset of the space $\mathcal{E}$.
Let $x \in \chi_r$ be arbitrary, then

$$\begin{aligned}
\| \quad & \mathcal{A}x(t)\|_{\mathcal{E}} \\
= \quad & \|g(t, g_1(t, x(t)) \int_0^{\varphi(t)} g_2(t, s, u(s))ds)\|_{\mathcal{E}} \\
\leq \quad & \|a(t)\|_{\mathcal{E}} + L\|g_1(t, x(t)) \int_0^{\varphi(t)} g_2(t, s, u(s))ds\|_{\mathcal{E}} \\
\leq \quad & \|a(t)\|_{\mathcal{E}} + L\|g_1(t, x(t))\|_{\mathcal{E}} \int_0^{\varphi(t)} |g_2(t, s, u(s))|ds \\
\leq \quad & \|a(t)\|_{\mathcal{E}} + L\|g_1(t, x(t))\|_{\mathcal{E}} \int_0^{t} |g_2(t, s, u(s))|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq \|a(t)\|_{\mathcal{E}} + L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x(t)\|_{\mathcal{E}}\} \int_0^t \{|k(t,s)| + |b(s)|\|u(s)\|_{\mathcal{E}}\} ds \\
&\leq \|a(t)\|_{\mathcal{E}} + L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x(t)\|_{\mathcal{E}}\} \int_0^t |k(t,s)| ds \\
&+ L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x(t)\|_{\mathcal{E}}\} \int_0^t |b(s)|\|u(s)\|_{\mathcal{E}} ds \\
&\leq sup_{t\in\mathcal{R}_+}\|a(t)\|_{\mathcal{E}} + L\{sup_{t\in\mathcal{R}_+}\|a_1(t)\|_{\mathcal{E}} + L_1 sup_{t\in\mathcal{R}_+}\|x(t)\|_{\mathcal{E}}\} sup_{t\in\mathcal{R}_+} \int_0^t |k(t,s)| ds \\
&+ L\{sup_{t\in\mathcal{R}_+}\|a_1(t)\|_{\mathcal{E}} + L_1 sup_{t\in\mathcal{R}_+}\|x(t)\|_{\mathcal{E}}\} \int_0^t sup_{s\in\mathcal{R}_+}|b(s)|\; sup_{s\in\mathcal{R}_+}\|u(s)\|_{\mathcal{E}} ds \\
&\leq \|a\|_{\mathcal{BC}} + L\{\|a_1\|_{\mathcal{BC}} + L_1\|x\|_{\mathcal{BC}}\}K + L\{\|a_1\|_{\mathcal{BC}} + L_1\|x\|_{\mathcal{BC}}\}\|u\|_{\mathcal{BC}} B \\
&\leq \|a\|_{\mathcal{BC}} + L\{\|a_1\|_{\mathcal{BC}} + L_1\|x\|_{\mathcal{BC}}\}K + L\{\|a_1\|_{\mathcal{BC}} + L_1\|x\|_{\mathcal{BC}}\}[U + U_1 + U_2\|x\|_{\mathcal{BC}}]B \\
&\leq \|a\|_{\mathcal{BC}} + L\|a_1\|_{\mathcal{BC}}K + LL_1\|x\|_{\mathcal{BC}}K + L\|a_1\|_{\mathcal{BC}}[U + U_1 + U_2\|x\|_{\mathcal{BC}}]B \\
&+ LL_1\|x\|_{\mathcal{BC}}[U + U_1 + U_2\|x\|_{\mathcal{BC}}]B \\
&\leq \|a\|_{\mathcal{BC}} + L\|a_1\|_{\mathcal{BC}}K + LL_1 rK + L\|a_1\|_{\mathcal{BC}}[U + U_1 + U_2 r]B + LL_1 r[U + U_1 + U_2 r]B.
\end{aligned}
$$

Therefore

$$
\|\mathcal{A}x\|_{\mathcal{BC}} \leq \|a\|_{\mathcal{BC}} + L\|a_1\|_{\mathcal{BC}}K + LL_1 rK + L\|a_1\|_{\mathcal{BC}}[U + U_1 + U_2 r]B + LL_1 r[U + U_1 + U_2 r]B = r.
$$

Then

$$
\|\mathcal{A}x\|_{\mathcal{BC}} \leq r.
$$

Hence, $\mathcal{A}x \in \chi_r$, which proves that $\mathcal{A}\,\chi_r \subset \chi_r$ and $\mathcal{A} : \chi_r \to \chi_r$.
Now, we will show that $\mathcal{A}$ is continuous on the ball χ_r.
Let $\{x_n\}$ be a sequence in χ_r converges to x $\forall t \in \mathcal{R}_+$ in χ_r, i.e. $x_n \to x,\ \forall t \in \mathcal{R}_+$.
Now

$$
\|g(t, g_1(t, x_n(t)) \int_0^{\varphi(t)} g_2(t, s, u_n(s)) ds)\|_{\mathcal{E}} \leq \|a(t)\|_{\mathcal{E}} + L\|g_1(t, x_n(t)) \int_0^{\varphi(t)} g_2(t, s, u_n(s)) ds\|_{\mathcal{E}}
$$

and $x_n \to x$, then $g(t, g_1(t, x_n(t)) \int_0^{\varphi(t)} g_2(t, s, u_n(s)) ds) \to g(t, g_1(t, x(t)) \int_0^{\varphi(t)} g_2(t, s, u(s)) ds)$.
Since

$$
\mathcal{A}x_n(t) = g(t, g_1(t, x_n(t)) \int_0^{\varphi(t)} g_2(t, s, u_n(s)) ds),\ t \in \mathcal{R}_+.
$$

Then

$$
\begin{aligned}
\| \quad & \mathcal{A}x_n(t) - \mathcal{A}x(t)\|_{\mathcal{E}} \\
= \quad & \|g(t, g_1(t, x_n(t)) \int_0^{\varphi(t)} g_2(t, s, u_n(s)) ds) - g(t, g_1(t, x(t)) \int_0^{\varphi(t)} g_2(t, s, u(s)) ds)\|_{\mathcal{E}} \\
\leq \quad & L\|g_1(t, x_n(t)) \int_0^{\varphi(t)} g_2(t, s, u_n(s)) ds - g_1(t, x(t)) \int_0^{\varphi(t)} g_2(t, s, u(s)) ds\|_{\mathcal{E}}
\end{aligned}
$$

$$\begin{aligned}
&\leq L\|g_1(t,x_n(t))\int_0^{\varphi(t)} g_2(t,s,u_n(s))ds - g_1(t,x(t))\int_0^{\varphi(t)} g_2(t,s,u_n(s))ds\|_{\mathcal{E}} \\
&+ L\|g_1(t,x(t))\int_0^{\varphi(t)} g_2(t,s,u_n(s))ds - g_1(t,x(t))\int_0^{\varphi(t)} g_2(t,s,u(s))ds\|_{\mathcal{E}} \\
&\leq L\|g_1(t,x_n(t)) - g_1(t,x(t))\|_{\mathcal{E}}\int_0^{\varphi(t)} |g_2(t,s,u_n(s))|ds \\
&+ L\|g_1(t,x(t))\|_{\mathcal{E}}\int_0^{\varphi(t)} |g_2(t,s,u_n(s)) - g_2(t,s,u(s))|ds \\
&\leq L\|g_1(t,x_n(t)) - g_1(t,x(t))\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u_n(s))|ds \\
&+ L\|g_1(t,x(t))\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u_n(s)) - g_2(t,s,u(s))|ds \\
&\leq LL_1\|x_n(t) - x(t)\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u_n(s))|ds \\
&+ L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x(t)\|_{\mathcal{E}}\}\int_0^{t} |g_2(t,s,u_n(s)) - g_2(t,s,u(s))|ds \qquad (3.2) \\
&\leq LL_1\|x_n(t) - x(t)\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u_n(s))|ds \\
&+ 2L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x(t)\|_{\mathcal{E}}\}\int_0^{t} |g_2(t,s,u(s))|ds \\
&\leq LL_1\|x_n(t) - x(t)\|_{\mathcal{E}}\int_0^{t} \{|k(t,s)| + |b(s)|\|u_n(s)\|_{\mathcal{E}}\}ds \\
&+ 2L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x(t)\|_{\mathcal{E}}\}\int_0^{t} \{|k(t,s)| + |b(s)|\|u(s)\|_{\mathcal{E}}\}ds \\
&\leq LL_1\|x_n - x\|_{\mathcal{BC}}K + LL_1\|x_n - x\|_{\mathcal{BC}}\|u\|_{\mathcal{BC}}B \\
&+ 2L\{\|a_1\|_{\mathcal{BC}} + L_1\|x\|_{\mathcal{BC}}\}\int_0^{t} |k(t,s)|ds + 2L\{\|a_1\|_{\mathcal{BC}} + L_1\|x\|_{\mathcal{BC}}\}\|u\|_{\mathcal{BC}}\int_0^{t} |b(s)|ds \\
&\leq LL_1\|x_n - x\|_{\mathcal{BC}}K + LL_1\|x_n - x\|_{\mathcal{BC}}[U + U_1 + U_2\|x\|_{\mathcal{BC}}]B \\
&+ 2L\{\|a_1\|_{\mathcal{BC}} + L_1\|x\|_{\mathcal{BC}}\}\int_0^{t} |k(t,s)|ds + 2L\{\|a_1\|_{\mathcal{BC}} + L_1\|x\|_{\mathcal{BC}}\}[U + U_1 + U_2\|x\|_{\mathcal{BC}}]\int_0^{t} |b(s)|ds \\
&\leq LL_1\varepsilon_1 K + LL_1\varepsilon_1[U + U_1 + U_2 r]B + 2L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}\int_0^{t} |k(t,s)|ds \\
&+ 2L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}[U + U_1 + U_2 r]\int_0^{t} |b(s)|ds.
\end{aligned}$$

select $\mathcal{T} > 0$ such that the following inequalities hold for $t > \mathcal{T}$,

$$2L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}\int_0^{t} |k(t,s)|ds \leq \frac{\varepsilon_2}{2} \text{ and } 2L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}[U + U_1 + U_2 r]\int_0^{t} |b(s)|ds \leq \frac{\varepsilon_2}{2}.$$

Consider the two ideas:
(1) If $t \geq \mathcal{T}$, we obtain

$$\|\mathcal{A}x_n(t) - \mathcal{A}x(t)\|_{\mathcal{E}} \leq LL_1\varepsilon_1 K + LL_1\varepsilon_1[U + U_1 + U_2 r]B + \varepsilon_2 = \varepsilon.$$

(2) If $t \leq \mathcal{T}$, let us take a function $\omega = \omega(\varepsilon)$ given by

$$\omega(\varepsilon) = sup\{|g_2(t,s,u_n(s)) - g_2(t,s,u(s))| : t,s \in \mathcal{I},\ u,u_n \in [-r,r],\ \|u_n(s) - u(s)\|_{\mathcal{E}} < \varepsilon\}.$$

Then from the uniform continuity of the function $g_2(t,s,u(s))$ on the set $\mathcal{I} \times \mathcal{I} \times [-r,r]$, we deduce that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, from (3.2)

$$\|\mathcal{A}x_n(t) - \mathcal{A}x(t)\|_{\mathcal{E}} \leq LL_1\varepsilon_1 K + LL_1\varepsilon_1[U + U_1 + U_2 r]B + L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}\omega(\varepsilon)\mathcal{T}.$$

Hence

$$\|\mathcal{A}x_n - \mathcal{A}x\|_{\mathcal{BC}} \leq LL_1\varepsilon_1 K + LL_1\varepsilon_1[U + U_1 + U_2 r]B + L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}\omega(\varepsilon)\mathcal{T}.$$

Finally, according to cases (1) and (2) and the previous facts, we conclude that $\mathcal{A}x_n \to \mathcal{A}x,\ \forall x_n \to x$ and the operator $\mathcal{A}$ is continuous on χ_r.
Now, for any $x_1, x_2 \in X \subseteq \chi_r$ and fixed $t \geq 0$, we obtain

$$\begin{aligned}
\| & \mathcal{A}x_1(t) - \mathcal{A}x_2(t)\|_{\mathcal{E}} \\
= & \|g(t, g_1(t,x_1(t))\int_0^{\varphi(t)} g_2(t,s,u_1(s))ds) - g(t, g_1(t,x_2(t))\int_0^{\varphi(t)} g_2(t,s,u_2(s))ds)\|_{\mathcal{E}} \\
\leq & L\|g_1(t,x_1(t))\int_0^{\varphi(t)} g_2(t,s,u_1(s))ds - g_1(t,x_2(t))\int_0^{\varphi(t)} g_2(t,s,u_2(s))ds\|_{\mathcal{E}} \\
\leq & L\|g_1(t,x_1(t))\int_0^{\varphi(t)} g_2(t,s,u_1(s))ds - g_1(t,x_2(t))\int_0^{\varphi(t)} g_2(t,s,u_1(s))ds\|_{\mathcal{E}} \\
+ & L\|g_1(t,x_2(t))\int_0^{\varphi(t)} g_2(t,s,u_1(s))ds - g_1(t,x_2(t))\int_0^{\varphi(t)} g_2(t,s,u_2(s))ds\|_{\mathcal{E}} \\
\leq & L\|g_1(t,x_1(t)) - g_1(t,x_2(t))\|_{\mathcal{E}}\int_0^{\varphi(t)} |g_2(t,s,u_1(s))|ds \\
+ & L\|g_1(t,x_2(t))\|_{\mathcal{E}}\int_0^{\varphi(t)} |g_2(t,s,u_1(s)) - g_2(t,s,u_2(s))|ds \\
\leq & L\|g_1(t,x_1(t)) - g_1(t,x_2(t))\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u_1(s))|ds \\
+ & L\|g_1(t,x_2(t))\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u_1(s)) - g_2(t,s,u_2(s))|ds \\
\leq & LL_1\|x_1(t) - x_2(t)\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u_1(s))|ds \\
+ & 2L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x_2(t)\|_{\mathcal{E}}\}\int_0^{t} |g_2(t,s,u_2(s))|ds
\end{aligned}$$

$$
\begin{aligned}
&\leq LL_1\|x_1(t)-x_2(t)\|_{\mathcal{E}}\int_0^t\{|k(t,s)|+|b(s)|\|u_1(s)\|_{\mathcal{E}}\}ds\\
&+ 2L\{\|a_1(t)\|_{\mathcal{E}}+L_1\|x_2(t)\|_{\mathcal{E}}\}\int_0^t\{|k(t,s)|+|b(s)|\|u_2(s)\|_{\mathcal{E}}\}ds\\
&\leq LL_1\|x_1(t)-x_2(t)\|_{\mathcal{E}}\int_0^t|k(t,s)|ds+LL_1\|x_1(t)-x_2(t)\|_{\mathcal{E}}\int_0^t|b(s)|\|u_1(s)\|_{\mathcal{E}}ds\\
&+ 2L\{\|a_1(t)\|_{\mathcal{E}}+L_1\|x_2(t)\|_{\mathcal{E}}\}\int_0^t|k(t,s)|ds+2L\{\|a_1(t)\|_{\mathcal{E}}+L_1\|x_2(t)\|_{\mathcal{E}}\}\int_0^t|b(s)|\|u_2(s)\|_{\mathcal{E}}ds\\
&\leq LL_1\, diam X(t)\, K+LL_1\, diam X(t)\|u_1\|_{\mathcal{BC}}B\\
&+ 2L\{\|a_1\|_{\mathcal{BC}}+L_1\|x_2\|_{\mathcal{BC}}\}\int_0^t|k(t,s)|ds+2L\{\|a_1\|_{\mathcal{BC}}+L_1\|x_2\|_{\mathcal{BC}}\}\|u_2\|_{\mathcal{BC}}\int_0^t|b(s)|ds\\
&\leq LL_1\, diam X(t)\, K+LL_1\, diam X(t)\,[U+U_1+U_2\|x_1\|_{\mathcal{BC}}]B\\
&+ 2L\{\|a_1\|_{\mathcal{BC}}+L_1\|x_2\|_{\mathcal{BC}}\}\int_0^t|k(t,s)|ds\\
&+ 2L\{\|a_1\|_{\mathcal{BC}}+L_1\|x_2\|_{\mathcal{BC}}\}[U+U_1+U_2\|x_1\|_{\mathcal{BC}}]\int_0^t|b(s)|ds\\
&\leq LL_1\, diam X(t)\, K+LL_1\, diam X(t)\,[U+U_1+U_2 r]B\\
&+ 2L\{\|a_1\|_{\mathcal{BC}}+L_1 r\}\int_0^t|k(t,s)|ds\\
&+ 2L\{\|a_1\|_{\mathcal{BC}}+L_1 r\}[U+U_1+U_2 r]\int_0^t|b(s)|ds.
\end{aligned}
$$

Hence, we obtain

$$
\begin{aligned}
diam(\mathcal{A}X)(t) &\leq (LL_1K+LL_1[U+U_1+U_2r]B)diam X(t)+2L\{\|a_1\|_{\mathcal{BC}}+L_1r\}\int_0^t|k(t,s)|ds\\
&+ 2L\{\|a_1\|_{\mathcal{BC}}+L_1r\}[U+U_1+U_2r]\int_0^t|b(s)|ds
\end{aligned}
$$

and

$$
lim_{t\to\infty}sup\; diam(\mathcal{A}X)(t)\leq (LL_1K+LL_1[U+U_1+U_2r]B)lim_{t\to\infty}sup\; diam X(t).
$$

Then

$$
lim_{t\to\infty}sup\; diam(\mathcal{A}X)(t)\leq C\; lim_{t\to\infty}sup\; diam X(t). \tag{3.3}
$$

Where we denote $C=(LL_1K+LL_1[U+U_1+U_2r]B)$.
Let $\mathcal{T}>0$ and $\varepsilon>0$ be given. Let $x\in X\subseteq\chi_r$ and $t,\tau\in\mathcal{I}$ such that $\tau\leq t$ and $|t-\tau|\leq\varepsilon$, then

$$
\begin{aligned}
\| &\mathcal{A}x(t)-\mathcal{A}x(\tau)\|_{\mathcal{E}}\\
&= \|g(t,g_1(t,x(t))\int_0^{\varphi(t)}g_2(t,s,u(s))ds)-g(\tau,g_1(\tau,x(\tau))\int_0^{\varphi(\tau)}g_2(\tau,s,u(s))ds)\|_{\mathcal{E}}\\
&\leq L\{|t-\tau|+\|g_1(t,x(t))\int_0^{\varphi(t)}g_2(t,s,u(s))ds-g_1(\tau,x(\tau))\int_0^{\varphi(\tau)}g_2(\tau,s,u(s))ds\|_{\mathcal{E}}\}
\end{aligned}
$$

$$
\begin{aligned}
&\leq L\{|t-\tau| + \|g_1(t,x(t))\int_0^{\varphi(t)} g_2(t,s,u(s))ds - g_1(\tau,x(\tau))\int_0^{\varphi(t)} g_2(t,s,u(s))ds\|_{\mathcal{E}} \\
&+ \|g_1(\tau,x(\tau))\int_0^{\varphi(t)} g_2(t,s,u(s))ds - g_1(\tau,x(\tau))\int_0^{\varphi(\tau)} g_2(\tau,s,u(s))ds\|_{\mathcal{E}}\} \\
&\leq L|t-\tau| + L\|g_1(t,x(t)) - g_1(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{\varphi(t)} |g_2(t,s,u(s))|ds \\
&+ L\|g_1(\tau,x(\tau))\|_{\mathcal{E}}\|\int_0^{\varphi(t)} g_2(t,s,u(s))ds - \int_0^{\varphi(\tau)} g_2(\tau,s,u(s))ds\|_{\mathcal{E}} \\
&\leq L|t-\tau| + L\|g_1(t,x(t)) - g_1(t,x(\tau)) + g_1(t,x(\tau)) - g_1(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{\varphi(t)} |g_2(t,s,u(s))|ds \\
&+ L\|g_1(\tau,x(\tau))\|_{\mathcal{E}}\|\int_0^{\varphi(\tau)} g_2(t,s,u(s))ds + \int_{\varphi(\tau)}^{\varphi(t)} g_2(t,s,u(s))ds - \int_0^{\varphi(\tau)} g_2(\tau,s,u(s))ds\|_{\mathcal{E}} \\
&\leq L|t-\tau| + L\|g_1(t,x(t)) - g_1(t,x(\tau))\|_{\mathcal{E}}\int_0^{\varphi(t)} |g_2(t,s,u(s))|ds \\
&+ L\|g_1(t,x(\tau)) - g_1(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{\varphi(t)} |g_2(t,s,u(s))|ds \\
&+ L\|g_1(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{\varphi(\tau)} |g_2(t,s,u(s)) - g_2(\tau,s,u(s))|ds \\
&+ L\|g_1(\tau,x(\tau))\|_{\mathcal{E}}\int_{\varphi(\tau)}^{\varphi(t)} |g_2(t,s,u(s))|ds \\
&\leq L|t-\tau| + L\|g_1(t,x(t)) - g_1(t,x(\tau))\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u(s))|ds \\
&+ L\|g_1(t,x(\tau)) - g_1(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u(s))|ds \\
&+ L\|g_1(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{\tau} |g_2(t,s,u(s)) - g_2(\tau,s,u(s))|ds \\
&+ L\|g_1(\tau,x(\tau))\|_{\mathcal{E}}\int_{\varphi(\tau)}^{\varphi(t)} |g_2(t,s,u(s))|ds \\
&\leq L|t-\tau| + LL_1\|x(t) - x(\tau))\|_{\mathcal{E}}\int_0^{t} \{|k(t,s)| + |b(s)|\|u(s)\|_{\mathcal{E}}\}ds \\
&+ L\|g_1(t,x(\tau)) - g_1(\tau,x(\tau))\|_{\mathcal{E}}\int_0^{t} \{|k(t,s)| + |b(s)|\|u(s)\|_{\mathcal{E}}\}ds \\
&+ L\{\|a_1(\tau)\|_{\mathcal{E}} + L_1\|x(\tau)\|_{\mathcal{E}}\}\int_0^{\tau} |g_2(t,s,u(s)) - g_2(\tau,s,u(s))|ds \\
&+ L\{\|a_1(\tau)\|_{\mathcal{E}} + L_1\|x(\tau)\|_{\mathcal{E}}\}\int_{\varphi(\tau)}^{\varphi(t)} \{|k(t,s)| + |b(s)|\|u(s)\|_{\mathcal{E}}\}ds \\
&\leq L|t-\tau| + LL_1\|x(t) - x(\tau))\|_{\mathcal{E}}\int_0^{t} |k(t,s)|ds + LL_1\|x(t) - x(\tau))\|_{\mathcal{E}}\int_0^{t} |b(s)|\|u(s)\|_{\mathcal{E}}ds
\end{aligned}
$$

$$
\begin{aligned}
&+ \quad L\|g_1(t,x(\tau)) - g_1(\tau,x(\tau))\|_{\mathcal{E}} \int_0^t |k(t,s)|ds + L\|g_1(t,x(\tau)) - g_1(\tau,x(\tau))\|_{\mathcal{E}} \int_0^t |b(s)|\|u(s)\|_{\mathcal{E}} ds \\
&+ \quad L\{\|a_1(\tau)\|_{\mathcal{E}} + L_1\|x(\tau)\|_{\mathcal{E}}\} \int_0^\tau |g_2(t,s,u(s)) - g_2(\tau,s,u(s))|ds \\
&+ \quad L\{\|a_1(\tau)\|_{\mathcal{E}} + L_1\|x(\tau)\|_{\mathcal{E}}\} \int_{\varphi(\tau)}^{\varphi(t)} |k(t,s)|ds + L\{\|a_1(\tau)\|_{\mathcal{E}} + L_1\|x(\tau)\|_{\mathcal{E}}\} \int_{\varphi(\tau)}^{\varphi(t)} |b(s)|\|u(s)\|_{\mathcal{E}} ds \\
&\leq \quad L\varepsilon + LL_1\omega^{\mathcal{T}}(x,\varepsilon)K + LL_1\omega^{\mathcal{T}}(x,\varepsilon)\|u\|_{\mathcal{BC}}B + L\omega^{\mathcal{T}}(g_1,\varepsilon)K \\
&+ \quad L\omega^{\mathcal{T}}(g_1,\varepsilon)\|u\|_{\mathcal{BC}}B + L\{\|a_1\|_{\mathcal{BC}} + L_1\|x\|_{\mathcal{BC}}\}\omega^{\mathcal{T}}(g_2,\varepsilon)\mathcal{T} \\
&+ \quad L\{\|a_1\|_{\mathcal{BC}} + L_1\|x\|_{\mathcal{BC}}\} \int_{\varphi(\tau)}^{\varphi(t)} |k(t,s)|ds + L\{\|a_1\|_{\mathcal{BC}} + L_1\|x\|_{\mathcal{BC}}\}\|u\|_{\mathcal{BC}} \int_{\varphi(\tau)}^{\varphi(t)} |b(s)|ds \\
&\leq \quad L\varepsilon + LL_1\omega^{\mathcal{T}}(x,\varepsilon)K + LL_1\omega^{\mathcal{T}}(x,\varepsilon)[U + U_1 + U_2 r]B + L\omega^{\mathcal{T}}(g_1,\varepsilon)K \\
&+ \quad L\omega^{\mathcal{T}}(g_1,\varepsilon)[U + U_1 + U_2 r]B + L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}\omega^{\mathcal{T}}(g_2,\varepsilon)\mathcal{T} \\
&+ \quad L\{\|a_1\|_{\mathcal{BC}} + L_1 r\} \int_{\varphi(\tau)}^{\varphi(t)} |k(t,s)|ds + L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}[U + U_1 + U_2 r] \int_{\varphi(\tau)}^{\varphi(t)} |b(s)|ds,
\end{aligned}
$$

where

$$
\omega^{\mathcal{T}}(g_1,\varepsilon) = sup\{|g_1(t,x(\tau)) - g_1(\tau,x(\tau))| : \ t,\tau \in \mathcal{I}, \ |t-\tau| \leq \varepsilon, \ \|x\|_{\mathcal{BC}} \leq r\}
$$

and

$$
\omega^{\mathcal{T}}(g_2,\varepsilon) = sup\{|g_2(t,s,u(s)) - g_2(\tau,s,u(s))| : \ t,\tau \in \mathcal{I}, \ |t-\tau| \leq \varepsilon, \ \|u\|_{\mathcal{BC}} \leq [U + U_1 + U_2 r]\}.
$$

Hence

$$
\begin{aligned}
\omega^{\mathcal{T}}(\mathcal{A}x,\varepsilon) \quad &\leq \quad L\varepsilon + LL_1\omega^{\mathcal{T}}(x,\varepsilon)K + LL_1\omega^{\mathcal{T}}(x,\varepsilon)[U + U_1 + U_2 r]B + L\omega^{\mathcal{T}}(g_1,\varepsilon)K \\
&+ \quad L\omega^{\mathcal{T}}(g_1,\varepsilon)[U + U_1 + U_2 r]B + L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}\omega^{\mathcal{T}}(g_2,\varepsilon)\mathcal{T} \\
&+ \quad L\{\|a_1\|_{\mathcal{BC}} + L_1 r\} \int_{\varphi(\tau)}^{\varphi(t)} |k(t,s)|ds \\
&+ \quad L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}[U + U_1 + U_2 r] \int_{\varphi(\tau)}^{\varphi(t)} |b(s)|ds.
\end{aligned}
$$

And

$$
\begin{aligned}
\omega^{\mathcal{T}}(\mathcal{A}X,\varepsilon) \quad &\leq \quad L\varepsilon + LL_1\omega^{\mathcal{T}}(X,\varepsilon)K + LL_1\omega^{\mathcal{T}}(X,\varepsilon)[U + U_1 + U_2 r]B + L\omega^{\mathcal{T}}(g_1,\varepsilon)K \\
&+ \quad L\omega^{\mathcal{T}}(g_1,\varepsilon)[U + U_1 + U_2 r]B + L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}\omega^{\mathcal{T}}(g_2,\varepsilon)\mathcal{T} \\
&+ \quad L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}\{\varepsilon sup\{|k(t,s)| : s \in \mathcal{I}\}\} \\
&+ \quad L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}[U + U_1 + U_2 r]\{\varepsilon sup\{|b(s)| : s \in \mathcal{I}\}\}
\end{aligned}
$$

From the uniform continuity of the function $g_1(t,x)$, $g_2(t,s,u(s))$ on the sets $\mathcal{I} \times[-r,r]$ and $\mathcal{I} \times\mathcal{I} \times[-r,r]$ respectively, we deduce that $\omega^{\mathcal{T}}(g_1,\varepsilon)$, $\omega^{\mathcal{T}}(g_2,\varepsilon) \to 0$ as $\varepsilon \to 0$. Consequently, we obtain

$$\begin{aligned}\omega_0^{\mathcal{T}}(\mathcal{A}X) &\leq LL_1\omega_0^{\mathcal{T}}(X)K + LL_1\omega_0^{\mathcal{T}}(X)[U+U_1+U_2r]B\\ &\leq (LL_1K + LL_1[U+U_1+U_2r]B)\omega_0^{\mathcal{T}}(X)\\ &\leq C\omega_0^{\mathcal{T}}(X).\end{aligned}$$

As $\mathcal{T} \to \infty$, we have

$$\omega_0(\mathcal{A}X) \leq C\omega_0(X). \tag{3.4}$$

Now, from the estimations (3.3) and (3.4) and the definition of the measure of noncompactness μ on X (see [4]), we obtain

$$\mu(\mathcal{A}X) \leq C\mu(X). \tag{3.5}$$

Since all conditions of Darbo fixed point theorem are satisfied, then the operator $\mathcal{A}$ has at least one fixed point $x \in \chi_r$, then there exists at least one solution $x \in \mathcal{BC}(\mathcal{R}_+,\mathcal{E})$ of the NQDFIE (3.1) with the feedback control (1.2).
Consequently, there exists at least one solution $x \in \mathcal{BC}(\mathcal{R}_+,\mathcal{E})$ of the NQDFII (1.1) with the feedback control (1.2).

4. Asymptotic Stability

Here we study the Asymptotic stability of the solution for the NQDFII (1.1) with the feedback control (1.2).

Definition 4.1. *The solution $x \in \mathcal{BC}(\mathcal{R}_+,\mathcal{E})$ of the NQDFII (1.1) with the feedback control (1.2) is asymptotically stable, that is, $\forall\varepsilon > 0$ there exists $\mathcal{T}(\varepsilon) > 0$ and $r > 0$ such that if any two solutions to the nonlinear quadratic delayed functional integral inclusion (1.1) are $x, x_1 \in \mathcal{BC}(\mathcal{R}_+,\mathcal{E})$, then $||x(t) - x_1(t)||_{\mathcal{E}} \leq \varepsilon,\ t \geq \mathcal{T}(\varepsilon)$.*

Theorem 4.1. *Let the assumptions (H1)-(H8) be realized, Then the solution $x \in \mathcal{BC}(\mathcal{R}_+,\mathcal{E})$ of the NQDFII (1.1) with the feedback control (1.2) is asymptotically stable.*

Proof. Let $\varepsilon > 0$ be given, take $x, x_1 \in \mathcal{BC}(\mathcal{R}_+,\mathcal{E})$ be any two solutions of (1.1), Then

$$\begin{aligned}|| &\quad x(t) - x_1(t)||_{\mathcal{E}}\\ = &\quad ||g(t, g_1(t,x(t)))\int_0^{\varphi(t)} g_2(t,s,u(s))ds) - g(t, g_1(t,x_1(t))\int_0^{\varphi(t)} g_2(t,s,u_1(s))ds)||_{\mathcal{E}}\\ \leq &\quad L||g_1(t,x(t))\int_0^{\varphi(t)} g_2(t,s,u(s))ds - g_1(t,x_1(t))\int_0^{\varphi(t)} g_2(t,s,u_1(s))ds||_{\mathcal{E}}\\ \leq &\quad L||g_1(t,x(t))\int_0^{\varphi(t)} g_2(t,s,u(s))ds - g_1(t,x_1(t))\int_0^{\varphi(t)} g_2(t,s,u(s))ds||_{\mathcal{E}}\end{aligned}$$

$$
\begin{aligned}
&+ \quad L\|g_1(t,x_1(t))\int_0^{\varphi(t)} g_2(t,s,u(s))ds - g_1(t,x_1(t))\int_0^{\varphi(t)} g_2(t,s,u_1(s))ds\|_{\mathcal{E}} \\
&\leq \quad L\|g_1(t,x(t)) - g_1(t,x_1(t))\|_{\mathcal{E}}\int_0^{\varphi(t)} |g_2(t,s,u(s))|ds \\
&+ \quad L\|g_1(t,x_1(t))\|_{\mathcal{E}}\int_0^{\varphi(t)} |g_2(t,s,u(s)) - g_2(t,s,u_1(s))|ds \\
&\leq \quad L\|g_1(t,x(t)) - g_1(t,x_1(t))\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u(s))|ds \\
&+ \quad L\|g_1(t,x_1(t))\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u(s)) - g_2(t,s,u_1(s))|ds \\
&\leq \quad LL_1\|x(t) - x_1(t)\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u(s))|ds \\
&+ \quad L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x_1(t)\|_{\mathcal{E}}\}\int_0^{t} |g_2(t,s,u(s)) - g_2(t,s,u_1(s))|ds \qquad (4.1)\\
&\leq \quad LL_1\|x(t) - x_1(t)\|_{\mathcal{E}}\int_0^{t} |g_2(t,s,u(s))|ds \\
&+ \quad 2L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x_1(t)\|_{\mathcal{E}}\}\int_0^{t} |g_2(t,s,u_1(s))|ds \\
&\leq \quad LL_1\|x(t) - x_1(t)\|_{\mathcal{E}}\int_0^{t} \{|k(t,s)| + |b(s)|\|u(s)\|_{\mathcal{E}}\}ds \\
&+ \quad 2L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x_1(t)\|_{\mathcal{E}}\}\int_0^{t} \{|k(t,s)| + |b(s)|\|u_1(s)\|_{\mathcal{E}}\}ds \\
&\leq \quad LL_1\|x - x_1\|_{\mathcal{BC}} sup_{t\in R_+}\int_0^{t} |k(t,s)|ds + LL_1\|x - x_1\|_{\mathcal{BC}}\|u\|_{\mathcal{BC}} sup_{t\in\mathcal{R}_+}\int_0^{t} |b(s)|ds \\
&+ \quad 2\{L\|a_1\|_{\mathcal{BC}} + LL_1\|x_1\|_{\mathcal{BC}}\}\int_0^{t} |k(t,s)|ds + 2\{L\|a_1\|_{\mathcal{BC}} + LL_1\|x_1\|_{\mathcal{BC}}\}\|u_1\|_{\mathcal{BC}}\int_0^{t} |b(s)|ds \\
&\leq \quad LL_1\varepsilon K + LL_1\varepsilon[U + U_1 + U_2 r]B + 2\{L\|a_1\|_{\mathcal{BC}} + LL_1 r\}\int_0^{t} |k(t,s)|ds \\
&+ \quad 2\{L\|a_1\|_{\mathcal{BC}} + LL_1 r\}[U + U_1 + U_2 r]\int_0^{t} |b(s)|ds
\end{aligned}
$$

select $\mathcal{T} > 0$ such that the following inequality holds for $t > \mathcal{T}$,

$$2\{L\|a_1\|_{\mathcal{BC}} + LL_1 r\}\int_0^{t} |k(t,s)|ds \leq \frac{\varepsilon}{2} \text{ and } 2\{L\|a_1\|_{\mathcal{BC}} + LL_1 r\}[U + U_1 + U_2 r]\int_0^{t} |b(s)|ds \leq \frac{\varepsilon}{2}.$$

Take into account the following two situations:

(i) If $t \geq \mathcal{T}$, we obtain

$$\|x(t) - x_1(t)\|_{\mathcal{E}} \leq LL_1\varepsilon K + LL_1\varepsilon[U + U_1 + U_2 r]B + \varepsilon,\ t \geq \mathcal{T}(\varepsilon).$$

Hence

$$\|x - x_1\|_{\mathcal{BC}} \leq LL_1\varepsilon K + LL_1\varepsilon[U + U_1 + U_2 r]B + \varepsilon = \varepsilon.$$

(ii) If $t \le \mathcal{T}$, let us take a function $\omega = \omega(\varepsilon)$ given by

$$\omega(\varepsilon) = sup\{|g_2(t,s,u(s)) - g_2(t,s,u_1(s))| : t,s \in \mathcal{I},\ x,x_1 \in [-r,r],\ \|u(s) - u_1(s)\|_{\mathcal{E}} < \varepsilon\}.$$

Then from the uniform continuity of the function $g_2(t,s,u(s))$ on the set $\mathcal{I} \times \mathcal{I} \times [-r,r]$, we deduce that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, from (4.1)

$$\|x(t) - x_1(t)\|_{\mathcal{E}} \le LL_1\varepsilon K + LL_1\varepsilon[U + U_1 + U_2 r]B + L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}\omega(\varepsilon)\mathcal{T},\ t \ge \mathcal{T}(\varepsilon).$$

Hence

$$\|x - x_1\|_{\mathcal{BC}} \le LL_1\varepsilon K + LL_1\varepsilon[U + U_1 + U_2 r]B + L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}\omega(\varepsilon)\mathcal{T}.$$

5. Asymptotic Dependency on S_G

Definition 5.1. *The solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of the NQDFII (1.1) with the feedback control (1.2) is depends asymptotically on the set S_G, if for every $\varepsilon > 0$, and any two functions $g,\ h \in S_G$, there exists $\delta > 0$ such that $\|g - h\|_{\mathcal{E}} < \delta,\ t > \mathcal{T}(\varepsilon)$ implies $\|x_g - x_h\|_{\mathcal{BC}} < \varepsilon$.*

Theorem 5.1. *Let the assumptions (H1)-(H8) be fulfilled, Then the solution $x \in \mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ of the NQDFII (1.1) with the feedback control (1.2) depends asymptotically on S_G.*

Proof. Let $g,\ h \in S_G$ such that

$$\|g(t, g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s)ds)) - h(t, g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s)ds))\|_{\mathcal{E}} < \delta, \delta > 0, t \in \mathcal{R}_+.$$

Then

$$\begin{aligned}
\| \quad & x_g(t) - x_h(t) \|_{\mathcal{E}} \\
= \quad & \|g(t, g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s))ds) - h(t, g_1(t, x_h(t)) \int_0^{\varphi(t)} g_2((t),s,u_h(s))ds)\|_{\mathcal{E}} \\
\le \quad & \|g(t, g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s))ds) - h(t, g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s))ds)\|_{\mathcal{E}} \\
+ \quad & \|h(t, g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s))ds) - h(t, g_1(t, x_h(t)) \int_0^{\varphi(t)} g_2(t,s,u_h(s))ds)\|_{\mathcal{E}} \\
\le \quad & \|g(t, g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s))ds) - h(t, g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s))ds)\|_{\mathcal{E}} \\
+ \quad & L\|g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s))ds - g_1(t, x_h(t)) \int_0^{\varphi(t)} g_2(t,s,u_h(s))ds\|_{\mathcal{E}} \\
\le \quad & \|g(t, g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s))ds) - h(t, g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s))ds)\|_{\mathcal{E}} \\
+ \quad & L\|g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s))ds - g_1(t, x_h(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s))ds\|_{\mathcal{E}} \\
+ \quad & L\|g_1(t, x_h(t)) \int_0^{\varphi(t)} g_2(t,s,u_g(s))ds - g_1(t, x_h(t)) \int_0^{\varphi(t)} g_2(t,s,u_h(s))ds\|_{\mathcal{E}}
\end{aligned}$$

$$
\begin{aligned}
&\le \|g(t, g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t, s, u_g(s))ds) - h(t, g_1(t, x_g(t)) \int_0^{\varphi(t)} g_2(t, s, u_g(s))ds)\|_{\mathcal{E}} \\
&+ L\|g(t, x_g(t)) - g_1(t, x_h(t))\|_{\mathcal{E}} \int_0^{\varphi(t)} |g_2(t, s, u_g(s))|ds \\
&+ L\|g_1(t, x_h(t))\|_{\mathcal{E}} \int_0^{\varphi(t)} |g_2(t, s, u_g(s)) - g_2(t, s, u_h(s))|ds \\
&\le \delta + L\|g_1(t, x_g(t)) - g_1(t, x_h(t))\|_{\mathcal{E}} \int_0^{t} |g_2(t, s, u_g(s))|ds \\
&+ L\|g_1(t, x_h(t))\|_{\mathcal{E}} \int_0^{t} |g_2(t, s, u_g(s)) - g_2(t, s, u_h(s))|ds \\
&\le \delta + LL_1\|x_g(t) - x_h(t)\|_{\mathcal{E}} \int_0^{t} |g_2(t, s, u_g(s))|ds \\
&+ L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x_h(t)\|_{\mathcal{E}}\} \int_0^{t} |g_2(t, s, u_g(s)) - g_2(t, s, u_h(s))|ds \qquad (5.1) \\
&\le \delta + LL_1\|x_g(t) - x_h(t)\|_{\mathcal{E}} \int_0^{t} |g_2(t, s, u_g(s))|ds \\
&+ 2L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x_h(t)\|_{\mathcal{E}}\} \int_0^{t} |g_2(t, s, u_h(s))|ds \\
&\le \delta + LL_1\|x_g(t) - x_h(t)\|_{\mathcal{E}} \int_0^{t} \{|k(t, s)| + |b(s)|\|u_g(s)\|_{\mathcal{E}}\}ds \\
&+ 2L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x_h(t)\|_{\mathcal{E}}\} \int_0^{t} \{|k(t, s)| + |b(s)|\|u_h(s)\|_{\mathcal{E}}\}ds \\
&\le \delta + LL_1\|x_g(t) - x_h(t)\|_{\mathcal{E}}\{\int_0^{t} |k(t, s)|ds + \int_0^{t} |b(s)|\|u_g(s)\|_{\mathcal{E}}ds\} \\
&+ 2L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x_h(t)\|_{\mathcal{E}}\}\{\int_0^{t} |k(t, s)|ds + \int_0^{t} |b(s)|\|u_h(s)\|_{\mathcal{E}}ds\} \\
&\le \delta + LL_1\|x_g - x_h\|_{\mathcal{BC}}\{K + [U + U_1 + U_2 r]B\} + 2L\{\|a_1\|_{\mathcal{BC}} + L_1 r\} \int_0^{t} |k(t, s)|ds \\
&+ 2L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}[U + U_1 + U_2 r] \int_0^{t} |b(s)|ds
\end{aligned}
$$

select $T > 0$ such that the two relations hold for $t > \mathcal{T}$,

$$
2L\{\|a_1\|_{\mathcal{BC}} + L_1 r\} \int_0^{t} |k(t, s)|ds \le \frac{\varepsilon}{2} \textit{ and } 2L\{\|a_1\|_{\mathcal{BC}} + L_1 r\}[U + U_1 + U_2 r] \int_0^{t} |b(s)|ds \le \frac{\varepsilon}{2}.
$$

Now we have two situations:

(1) If $t \ge \mathcal{T}$, we obtain

$$
\|x_g(t) - x_h(t)\|_{\mathcal{E}} \le \delta + LL_1\|x_g - x_h\|_{\mathcal{BC}}\{K + [U + U_1 + U_2 r]B\} + \varepsilon = \varepsilon.
$$

Then

$$
\|x_g - x_h\|_{\mathcal{BC}} \le \frac{\delta + \varepsilon}{1 - LL_1\{K + [U + U_1 + U_2 r]B\}} = \varepsilon.
$$

(2) If $t \leq \mathcal{T}$, let us take a function $\omega = \omega(\varepsilon)$ given by

$$\omega(\varepsilon) = sup\{|g_2(t,s,u_g(s)) - g_2(t,s,u_h(s))| : t,s \in \mathcal{I},\ u_g,\ u_h \in [-r,r],\ \|u_g(s) - u_h(s)\|_{\mathcal{E}} < \varepsilon\}.$$

Then from the uniform continuity of the function $g_2(t,s,u(s))$ on the set $\mathcal{I} \times \mathcal{I} \times [-r,r]$, we deduce that $\omega(\varepsilon) \to 0$ as $\varepsilon \to 0$. Thus, from (5.1),

$$\|x_g(t) - x_h(t)\|_{\mathcal{E}} \leq \delta + LL_1\|x_g - x_h\|_{\mathcal{BC}}\{K + [U + U_1 + U_2r]B\} + L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x_h(t)\|_{\mathcal{E}}\}\omega(\varepsilon)\mathcal{T}.$$

Hence

$$\|x_g(t) - x_h(t)\|_{\mathcal{BC}} \leq \frac{\delta + L\{\|a_1(t)\|_{\mathcal{E}} + L_1\|x_h(t)\|_{\mathcal{E}}\}\omega(\varepsilon)\mathcal{T}}{1 - LL_1\{K + [U + U_1 + U_2r]B\}} = \varepsilon.$$

This complete the prove of our investigation.

6. An Example

In this section our aim is to illustrate the main result contained in Theorem 3.1.
Let $\overline{\chi} = \{x \in \mathcal{E} :\ \|x\|_{\mathcal{E}} \leq 1\}$ and $\mathcal{R}_+ = [0,\infty)$. Consider the multi-valued function $G : \mathcal{R}_+ \times \overline{\chi} \to \Omega(\mathcal{E})$ defined by

$$G(t,y(t)) = (a(t) + Ly(t))\overline{\chi},\ t \in \mathcal{R}_+.$$

Then G is Lipschitz. In fact, for the norm in the Banach space we have

$$\begin{aligned} \|G(t,y(t))\|_{\mathcal{E}} &= sup\{\|g\|_{\mathcal{E}} :\ g \in G(t,y(t))\} \\ &= \|(a(t) + Ly(t))\overline{\chi}\|_{\mathcal{E}} \\ &= \|a(t) + Ly(t)\|_{\mathcal{E}} \\ &\leq \|a(t)\|_{\mathcal{E}} + L\|y(t)\|_{\mathcal{E}}. \end{aligned}$$

Now let $g(t,y(t)) = a(t) + Ly(t) \in G(t,y(t))$.
Hence, we can apply our results to the NQDFIE

$$x(t) = t + \frac{t}{1+t^2}x(t)\int_0^{\varphi(t)} \frac{4\pi(t-s)s(s+1) + (1+t^4)(u(s))}{2\pi(1+t^4)s(s+1)}ds,\ t \in \mathcal{R}_+ \tag{6.1}$$

with the feedback control

$$\frac{du(t)}{dt} = -0.1u(t) + \frac{1}{4}e^{-0.1t} + t^2e^{-0.1t}x(t). \tag{6.2}$$

Now, we investigate the solvability of the NQDFIE (6.1) on the space $\mathcal{BC}(\mathcal{R}_+,\mathcal{E})$. This equation is a particular case of the equation (3.1) with $g_1(t,x(t)) = \frac{t}{1+t^2}x(t)$,
$g_2(t,s,u(s)) = \frac{4\pi(t-s)s(s+1)+(1+t^4)(u(s))}{2\pi(1+t^4)s(s+1)}$, $\varphi(t) \leq t$ and $\psi(t,x(t)) = \frac{1}{4}e^{-0.1t} + t^2e^{-\frac{3}{2}t}x(t)$.

Now

$$
\begin{aligned}
\|x(t)\|_{\mathcal{E}} &= \|t+\frac{t}{1+t^2}x(t)\int_0^{\varphi(t)}\frac{4\pi(t-s)s(s+1)+(1+t^4)(u(s))}{2\pi(1+t^4)s(s+1)}ds\|_{\mathcal{E}} \\
&\leq t+\frac{t}{1+t^2}\|x(t)\|_{\mathcal{E}}\int_0^{\varphi(t)}|\frac{4\pi(t-s)s(s+1)+(1+t^4)(u(s))}{2\pi(1+t^4)s(s+1)}|ds \\
&\leq t+\frac{t}{1+t^2}\|x(t)\|_{\mathcal{E}}\int_0^{\varphi(t)}|\frac{2(t-s)}{1+t^4}+\frac{1}{2\pi s(s+1)}(u(s))|ds \\
&\leq t+\frac{t}{1+t^2}\|x(t)\|_{\mathcal{E}}\int_0^{t}|\frac{2(t-s)}{1+t^4}+\frac{1}{2\pi s(s+1)}(u(s))|ds \\
&\leq t+\frac{t}{1+t^2}\|x(t)\|_{\mathcal{E}}[\int_0^{t}\frac{2(t-s)}{1+t^4}ds+\int_0^{t}\frac{1}{2\pi s(s+1)}\|u(s)\|_{\mathcal{E}}ds] \\
&\leq t+\frac{t}{1+t^2}\|x\|_{\mathcal{BC}}[\int_0^{t}\frac{2t}{1+t^4}ds-\int_0^{t}\frac{2s}{1+t^4}ds+\frac{1}{2\pi}\|u\|_{\mathcal{BC}}\int_0^{t}\frac{1}{s(s+1)}ds] \\
&\leq t+\frac{t}{1+t^2}\|x\|_{\mathcal{BC}}[\frac{t^2}{1+t^4}+\frac{1}{2\pi}\|u\|_{\mathcal{BC}}ln|\frac{t}{t+1}|] \\
&\leq 1+\frac{1}{2}r[\frac{1}{2}+\frac{1}{2\pi}r(ln\frac{1}{2})] \\
&\leq 1+\frac{1}{4}r+\frac{1}{4\pi}(ln\frac{1}{2})r^2.
\end{aligned}
$$

Then

$$
\|x\|_{\mathcal{BC}} \leq 1+\frac{1}{4}r+\frac{1}{4\pi}(ln\frac{1}{2})r^2.
$$

The assumptions (H1)-(H8) of Theorem 3.1 are satisfied with $a(t)=a_1(t)=t, L=L_1=0.1$. Obviously, the function $g_1(t,x(t))$ and $\psi(t,x(t))$ are continuous functions. Currently, for any $x_1,x_2\in\mathcal{E}$ and $t\in\mathcal{R}_+$, we have

$$
\begin{aligned}
\|g_1(t,x_1(t))-g_1(t,x_2(t))\|_{\mathcal{E}} &= \|\frac{t}{1+t^2}x_1(t)-\frac{t}{1+t^2}x_2(t)\|_{\mathcal{E}} \\
&\leq |\frac{t}{1+t^2}|\ \|x_1(t)-x_2(t)\|_{\mathcal{E}} \\
&\leq |\frac{1}{2}|\ \|x_1(t)-x_2(t)\|_{\mathcal{E}},
\end{aligned}
$$

$$
\|g_1(t,x(t))\|_{\mathcal{E}}\leq\|a_1(t)\|_{\mathcal{E}}+L_1\|x(t)\|_{\mathcal{E}}
$$

And

$$
\begin{aligned}
\|\psi(t,x(t))\|_{\mathcal{E}} &= \|\frac{1}{4}e^{-0.1t}+t^2e^{-0.1t}x(t)\|_{\mathcal{E}} \\
&\leq |\frac{1}{4}e^{-0.1t}|+|t^2e^{-0.1t}|\|x(t)\|_{\mathcal{E}}.
\end{aligned}
$$

Where, $|d_1(t)|=|\frac{1}{4}e^{-0.1t}|$, $|d_2(t)|=|t^2e^{-0.1t}|$. Also, $U=0.09,\ U_1=.0225$ and $U_2=0.3$.
Further, we also have $g_2(t,s,u(s))$ fulfills condition (H5) with

$$\begin{aligned} |g_2(t,s,u(s))| &= |\frac{4\pi(t-s)s(s+1)+(1+t^4)(u(s))}{2\pi(1+t^4)s(s+1)}| \\ &= |\frac{2(t-s)}{1+t^4}+\frac{1}{2\pi s(s+1)}(u(s))| \\ &\leq \frac{2(t-s)}{1+t^4}+\frac{1}{2\pi s(s+1)}\|u(s)\|_{\mathcal{E}}. \end{aligned}$$

This indicates that we can insert $k(t,s)=\frac{2(t-s)}{1+t^4}$ and $b(s)=\frac{1}{2\pi s(s+1)}$.
To verify the assumption (H5), notice that

$$\begin{aligned} & lim_{t\to\infty} \int_0^t k(t,s)ds \\ = \quad & lim_{t\to\infty}\int_0^t \frac{2(t-s)}{1+t^4}ds \\ = \quad & lim_{t\to\infty}[\int_0^t \frac{2t}{1+t^4}ds-\int_0^t \frac{2s}{1+t^4}ds] \\ = \quad & lim_{t\to\infty}\frac{t^2}{(1+t^4)}=0 \end{aligned}$$

and

$$\begin{aligned} & lim_{t\to\infty} \int_0^t b(s)ds \\ = \quad & lim_{t\to\infty}\int_0^t \frac{1}{2\pi s(s+1)}ds \\ = \quad & \frac{1}{2\pi}lim_{t\to\infty}ln\frac{t}{t+1}=0. \end{aligned}$$

Moreover, we have

$$K=sup_{t\in R_+}\int_0^t k(t,s)ds=sup_{t\in R_+}\frac{t^2}{(1+t^4)}=\frac{1}{2}$$

and

$$B=sup_{t\in R_+}\int_0^t b(s)ds=sup_{t\in R_+}ln\frac{t}{t+1}=\frac{1}{2}.$$

Finally, let us pay attention to the fact that the inequality of Theorem 3.1 has the form $C=LL_1K+LL_1[U+U_1+U_2r]B<1$.
Consequently, all the requirements of Theorem 3.1 have been met. As a result the NQDFIE (6.1) with the feedback control (6.2) has at least one asymptotically stable solution in the space $\mathcal{BC}(\mathcal{R}_+,\mathcal{E})$.

7. Conclusions

This research paper has investigated the solvability of the nonlinear quadratic delayed functional integral inclusion (NQDFII) with the feedback control on the real half-axis in a reflexive Banach space. In the main result, we introduced sufficient conditions by applying the theory of measure of noncompactness by a given norm of continuity and used Darbo's fixed point theorem to study the solution, the asymptotic stability and the asymptotic dependency of the solution for that (NQDFII) with the feedback control condition on the real half-axis in the reflexive Banach space $\mathcal{BC}(\mathcal{R}_+, \mathcal{E})$ beneath the assumption that the set-valued function G satisfy Lipschitz condition in $\mathcal{E}$. Finally, we gave an example to illustrate the adequacy and esteem of our comes about.

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