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On Some Results Related to Non-Uniform Tight Wavelet Frames

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Abstract. The main objective of this paper is to develop some algorithms for the explicit construction of nonuniform tight wavelet frames using the unitary extension principles.

1. Introduction

Tight wavelet frames associated with nonuniform multiresolution analysis has been studied in this article. A tight wavelet frame adds redundancy to a wavelet system, therefore generalizing an orthonormal wavelet basis. Tight wavelet frames are simpler to construct than orthonormal wavelets since they are a reduntant wavelet system. Compared to orthonormal wavelet bases, tight wavelet frames have greater application flexibility. The unitary extension principle (UEP) was firstly introduced by Ron and Shen in [12,13], by which Construction of tight wavelet frames from refinable function can be done. The tight wavelet frames that are based on multiresolution analysis, and the generators are often referred to as mother framelets. The benefits of MRA-based tight wavelet frames and its possible features in applications have drawn a significant interest and effort in recent years. Readers are referred to [3–5,10,11,14–16], for additional information on tight wavelet frames.

The concept of multiresolution analysis has been extended in various ways. By replacing the dilation factor 2 by an integer $N \geq 2$, we construct $N-1$ wavelets to generate the whole space $L^2(\mathbb{R})$. But in all these cases, the translation set is always a group. In [6,7], Gabardo and Nashed took into consideration a generalization of Mallat's multiresolution analysis [9].

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The outline of the paper is as follows. We provide preliminary results on non-uniform multiresolution analysis in section 2. Section 3 is devoted to the construction of a nonuniform tight wavelet frames.

2. Preliminaries on Non-uniform Multiresolution Analysis

If $N \geq 1$ is an integer, we define the translation set $\Lambda_{r,N}$ as

$$\Lambda_{r,N} = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z} = \left\{\frac{rk}{N} + 2n : n \in \mathbb{Z}, k = 0, 1\right\}$$

where r is any odd integer with $1 \leq r \leq 2N-1$ such that r and N are relatively prime. For basic ideas, results on nonuniform multiresolution analysis and nonuniform wavelets we refer to [1,6–8].

For given $\Psi = \{\psi_1, \psi_2, ..., \psi_L\} \subset L^2(\mathbb{R})$, define the wavelet system [6,7,16,17]

$$X(\Psi) = \{\psi^l_{j,\lambda} : 1 \leq l \leq L; j \in \mathbb{Z}, \lambda \in \Lambda\} \tag{2.1}$$

where $\psi^l_{j,\lambda} = (2N)^{-j/2}\psi^l\Big((2N)^j x - \lambda\Big)$. The wavelet system $X(\Psi)$ is referred to as a non-uniform wavelet frame, if there exist positive values $0 < A \leq B < \infty$ such that

$$A\|h\|^2 \leq \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda} |< h, \psi^l_{j,\lambda} >|^2 \leq B\|h\|^2, \text{ for all } h \in L^2(\mathbb{R}). \tag{2.2}$$

A frame is referred to as a tight frame if A and B can be selected, such that $A = B$. Moreover, if $A = B = 1$, the wavelet frame is referred to as a Parseval frame, i.e.

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda} |< h, \psi^l_{j,\lambda} >|^2 = \|h\|^2, \text{ for all } h \in L^2(\mathbb{R}) \tag{2.3}$$

and in this case, every function $h \in L^2(\mathbb{R})$ can be expressed as

$$h(x) = \sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda} < h, \psi^l_{j,\lambda} > \psi^l_{j,\lambda}(x). \tag{2.4}$$

Function $\varphi \in L^2(\mathbb{R})$ is a refinable function, if it provides an equation of the following form

$$\varphi\left(\frac{x}{2N}\right) = \sum_{\lambda\in\Lambda} a_\lambda \varphi(x - \lambda) \tag{2.5}$$

where $\sum_{\lambda\in\Lambda} |a_\lambda|^2 < \infty$. The Fourier transform of (2.5) is given by

$$\hat{\varphi}(2N\omega) = b_0(\omega)\hat{\varphi}(\omega), \tag{2.6}$$

and b_0 has the form

$$b_0(\omega) = b_0^1(\omega) + e^{-2\pi i\omega\frac{r}{N}} b_0^2(\omega) \tag{2.7}$$

for some locally L^2, $\frac{1}{2}$ periodic functions b_0^1 and b_0^2. It is proved in [1] that a function $\varphi \in L^2(\mathbb{R})$ generates a NUMRA in $L^2(\mathbb{R})$ if and only if

$$\sum_{p\in\mathbb{Z}} |\hat{\varphi}(\omega + p/2)|^2 = 2 \text{ for a.e. } \omega \in [0, 1/2) \tag{2.8}$$

and

$$\sum_{p\in\mathbb{Z}} e^{-i\pi\frac{r}{N}p} |\hat{\varphi}(\omega + p/2)|^2 = 0 \text{ for a.e. } \omega \in [0, 1/2) \tag{2.9}$$

and

$$\lim_{j\to\infty} \left|\hat{\varphi}\left(\frac{\omega}{(2N)^j}\right)\right| = 1 \text{ for a.e. } \omega \in \mathbb{R}. \tag{2.10}$$

Let the refinable function $\varphi \in L^2(\mathbb{R})$ generates an MRA $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mathbb{R})$ and $\Psi = \{\psi_1, \psi_2, ..., \psi_L\} \subset V_1$, then

$$\frac{1}{2N}\psi^l\left(\frac{x}{2N}\right) = \sum_{\lambda\in\Lambda} a^l_\lambda \varphi(x - \lambda),\ l = 1, 2, ..., L. \tag{2.11}$$

After applying the Fourier transform, we get

$$\hat{\psi}^l(2N\omega) = b_l(\omega)\hat{\varphi}(\omega) \tag{2.12}$$

where

$$b_l(\omega) = \sum_{\lambda\in\Lambda} a^l_\lambda e^{-2\pi i\lambda\omega} \tag{2.13}$$

The functions b_l, $0 \le l \le L$ are locally L^2. As $\Lambda = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z}$, we are able to write

$$b_l(\omega) = b_l^1(\omega) + e^{-2\pi i\omega\frac{r}{N}} b_l^2(\omega),\ 0 \le l \le L, \tag{2.14}$$

where $b_l^1(\omega)$ and $b_l^2(\omega)$ are locally L^2, $\frac{1}{2}$-periodic functions.

With $b_l(\omega)$, $l = 0, 1, ..., L$ as the nonuniform wavelet masks, we construct the matrix $W(\omega)$ of the order $4N \times (2L + 2)$ with the entries $W_{mn}(\omega)$, $0 \le m \le 4N - 1$ and $0 \le n \le 2L + 1$, defined by

$$\begin{aligned}
&b_n^1\left(\omega + \frac{m}{4N}\right),\ 0 \le m \le 2N - 1,\ 0 \le n \le L\\
&b_n^2\left(\omega + \frac{m-2N}{4N}\right),\ 2N \le m \le 4N - 1,\ 0 \le n \le L\\
&\alpha^m b^1_{n-(L+1)}\left(\omega + \frac{m}{4N}\right),\ 0 \le m \le 2N - 1,\ L + 1 \le n \le 2L + 1\\
&\alpha^m b^2_{n-(L+1)}\left(\omega + \frac{m-2N}{4N}\right),\ 2N \le m \le 4N - 1,\ L + 1 \le n \le 2L + 1,
\end{aligned}$$

where $\alpha = e^{-i\pi r/N}$. The matrix $W(\omega)$ is significant for constructing tight frames via MRA. In fact,

$$W(\omega)W^*(\omega) = I_{4N} \tag{2.15}$$

is analogous to that for any function $h \in L^2(\mathbb{R})$, there exists exact formulae of decomposition and reconstruction.

3. Non-Uniform Tight Wavelet Frames

Theorem 3.1. *Suppose that the refinable function φ and the framelet symbols b_0, $b_1, ..., b_L$ satisfy (2.6)-(2.14). Moreover, the wavelet system $X(\Psi)$ defined by (2.1) is a nonuniform tight wavelet frame for $L^2(\mathbb{R})$ if the matrix $W(\omega)$ satisfies (2.15).*

We divided the proof of Theorem 3.1 into multiple lemmas.

Lemma 3.1. *If the condition (2.15) is satisfied by the framelet symbols b_l, $l = 0, 1, ..., L$. Then for any $\omega \in \mathbb{R}$, we obtain*

$$\sum_{m=0}^{2N-1} \left| b_l^1 \left(\omega + \frac{m}{4N} \right) \right|^2 \leq \frac{1}{2} \text{ and } \sum_{m=0}^{2N-1} \left| b_l^2 \left(\omega + \frac{m}{4N} \right) \right|^2 \leq \frac{1}{2}. \tag{3.1}$$

Proof. Without the loss of generality it suffices to prove inequality (3.1) only for $l = 0$. Let $W_0(\omega)$ be the matrix of the order $4N \times 2L$ with the entries $W_{mn}(\omega)$, $0 \leq m \leq 4N-1$, $1 \leq n \leq L$ and $L+2 \leq n \leq 2L+1$, given by

$$\begin{aligned}
& b_n^1 \left(\omega + \frac{m}{4N} \right), \ 0 \leq m \leq 2N-1, \ 1 \leq n \leq L \\
& b_n^2 \left(\omega + \frac{m-2N}{4N} \right), \ 2N \leq m \leq 4N-1, \ 1 \leq n \leq L \\
& \alpha^m b_{n-(L+1)}^1 \left(\omega + \frac{m}{4N} \right), \ 0 \leq m \leq 2N-1, \ L+2 \leq n \leq 2L+1 \\
& \alpha^m b_{n-(L+1)}^2 \left(\omega + \frac{m-2N}{4N} \right), \ 2N \leq m \leq 4N-1, \ L+2 \leq n \leq 2L+1
\end{aligned} \tag{3.2}$$

Taking

$$A = \begin{pmatrix}
b_0^1(\omega) & b_0^1(\omega) \\
b_0^1\left(\omega + \frac{1}{4N}\right) & \alpha b_0^1\left(\omega + \frac{1}{4N}\right) \\
\vdots & \vdots \\
b_0^1\left(\omega + \frac{2N-1}{4N}\right) & \alpha^{2N-1} b_0^1\left(\omega + \frac{2N-1}{4N}\right) \\
b_0^2(\omega) & \alpha^{2N} b_0^2(\omega) \\
\vdots & \vdots \\
b_0^2\left(\omega + \frac{2N-2}{4N}\right) & \alpha^{4N-2} b_0^2\left(\omega + \frac{2N-2}{4N}\right) \\
b_0^2\left(\omega + \frac{2N-1}{4N}\right) & \alpha^{4N-1} b_0^2\left(\omega + \frac{2N-1}{4N}\right),
\end{pmatrix}$$

we can rewrite (2.15) as

$$\mathbb{W}(\omega) = W_0(\omega) W_0^*(\omega) = I_{4N} - AA^T = (\beta_1, \beta_2, ..., \beta_{4N}), \tag{3.3}$$

where

$$\begin{aligned}
\beta_1 &= \Big(1-2|b_0^1(\omega)|^2, -(1+\alpha)b_0^1\left(\omega+\tfrac{1}{4N}\right)\overline{b_0^1(\omega)}, \cdots, -(1+\alpha^{(2N-1)}) \\
&\quad b_0^1\left(\omega+\tfrac{2N-1}{4N}\right)\overline{b_0^1(\omega)}, -2b_0^2(\omega)\,\overline{b_0^1(\omega)}, \cdots, -(1+\alpha^{(2N-1)}) \\
&\quad b_0^2\left(\omega+\tfrac{2N-1}{4N}\right)\overline{b_0^1(\omega)}\Big)^T
\end{aligned}$$

$$\begin{aligned}
\beta_2 &= \Big(-(1+\overline{\alpha})b_0^1(\omega)\overline{b_0^1\left(\omega+\tfrac{1}{4N}\right)}, 1-2|b_0^1\left(\omega+\tfrac{1}{4N}\right)|^2, \cdots, -(1+\alpha^{(2N-2)}) \\
&\quad b_0^1\left(\omega+\tfrac{2N-1}{4N}\right)\overline{b_0^1\left(\omega+\tfrac{1}{4N}\right)}, -(1+\alpha^{(2N-1)})b_0^2(\omega)\,\overline{b_0^1\left(\omega+\tfrac{1}{4N}\right)}, \cdots, \\
&\quad -(1+\alpha^{(2N-2)})b_0^2\left(\omega+\tfrac{2N-1}{4N}\right)\overline{b_0^1\left(\omega+\tfrac{1}{4N}\right)}\Big)^T \\
&\vdots \\
\beta_{2N} &= \Big(-(1+\overline{\alpha}^{2N-1})b_0^1(\omega)\overline{b_0^1\left(\omega+\tfrac{2N-1}{4N}\right)}, -(1+\overline{\alpha}^{2N-2})b_0^1\left(\omega+\tfrac{1}{4N}\right) \\
&\quad \overline{b_0^1\left(\omega+\tfrac{2N-1}{4N}\right)}, \cdots, 1-2|b_0^1\left(\omega+\tfrac{2N-1}{4N}\right)|^2, -(1+\alpha)b_0^2(\omega) \\
&\quad \overline{b_0^1\left(\omega+\tfrac{2N-1}{4N}\right)}, \cdots, -2b_0^2\left(\omega+\tfrac{2N-1}{4N}\right)\overline{b_0^1\left(\omega+\tfrac{2N-1}{4N}\right)}\Big)^T
\end{aligned}$$

$$\begin{aligned}
\beta_{2N+1} &= \Big(-2b_0^1(\omega)\overline{b_0^2(\omega)}, -(1+\overline{\alpha})b_0^1\left(\omega+\tfrac{1}{4N}\right)\overline{b_0^2(\omega)}, \\
&\quad \cdots, -(1+\overline{\alpha})b_0^1\left(\omega+\tfrac{2N-1}{4N}\right)\overline{b_0^2(\omega)}, 1-2|b_0^2(\omega)|^2, \cdots, \\
&\quad -(1+\alpha^{(2N-1)})b_0^2\left(\omega+\tfrac{2N-1}{4N}\right)\overline{b_0^2(\omega)}\Big)^T \\
&\vdots \\
\beta_{4N} &= \Big(-(1+\overline{\alpha}^{2N-1})b_0^1(\omega)\overline{b_0^2\left(\omega+\tfrac{2N-1}{4N}\right)}, -(1+\overline{\alpha}^{2N-2})b_0^1\left(\omega+\tfrac{1}{4N}\right) \\
&\quad \overline{b_0^2\left(\omega+\tfrac{2N-1}{4N}\right)}, \cdots, -2b_0^1\left(\omega+\tfrac{2N-1}{4N}\right)\overline{b_0^2\left(\omega+\tfrac{2N-1}{4N}\right)}, \\
&\quad -(1+\alpha^{2N-1})b_0^2(\omega)\,\overline{b_0^2\left(\omega+\tfrac{2N-1}{4N}\right)}, \cdots, 1-2|b_0^2\left(\omega+\tfrac{2N-1}{4N}\right)|^2\Big)^T.
\end{aligned}$$

The Hermitian matrix $\mathbb{W}(\omega)$ has $4N$-eigenvalues, which are provided by

$$\gamma_1(\omega)=\gamma_2(\omega)=\cdots=\gamma_{2N-1}(\omega)=1,\ \gamma_{2N}(\omega)=1-2\sum_{m=0}^{2N-1}\left|b_0^1\left(\omega+\frac{m}{4N}\right)\right|^2$$

$$\gamma_{2N+1}(\omega)=\gamma_{2N+2}(\omega)=\cdots=\gamma_{4N-1}(\omega)=1,$$

$$\gamma_{4N}(\omega)=1-2\sum_{m=0}^{2N-1}\left|b_0^2\left(\omega+\frac{m}{4N}\right)\right|^2. \tag{3.4}$$

Since $\mathbb{W}(\omega)$ is a positive definite matrix, therefore $\gamma_{2N}(\omega)\geq 0$ and $\gamma_{4N}(\omega)\geq 0$, which is (3.1) for $l=0$.

□

Lemma 3.2. *Let $\varphi \in L^2(\mathbb{R})$ be a refinable function with refinement mask $m_0(\omega)$ such that condition (3.1) for $l = 0$ is satisfied. Then $Q_j = \sum_{\lambda\in\Lambda} | < u, \varphi_{j,\lambda} > |^2 < \infty$, for any function $u \in L^2(\mathbb{R})$ and*

$$(a) \lim_{j\to\infty} Q_j = \|u\|^2; \ (b) \lim_{j\to-\infty} Q_j = 0,$$

where $\varphi_{j,\lambda}(x) = (2N)^{j/2}\varphi((2N)^j x - \lambda),\ j \in \mathbb{Z},\ \lambda \in \Lambda_{r,N}$.

Proof. Let $H(\omega) = \sum_{j\in\mathbb{Z}} \|\hat{\varphi}(\omega + Nj)\|^2$

$$\begin{aligned} H(2N\omega) &= \sum_{j\in\mathbb{Z}} \left|\hat{\varphi}\left(2N\left(\omega + j/2\right)\right)\right|^2 \\ &= \sum_{j\in\mathbb{Z}} \left|b_0\left(\omega + j/2\right)\right|^2 \left|\hat{\varphi}\left(\omega + j/2\right)\right|^2 \\ &= \sum_{j\in\mathbb{Z}} \left\{b_0^1\left(\omega + j/2\right) + e^{-2\pi i \frac{r}{N}(\omega+j/2)} b_0^2\left(\omega + j/2\right)\right\} \\ &\quad \left\{\overline{b_0^1\left(\omega + j/2\right)} + e^{2\pi i\omega r/N}\overline{b_0^2\left(\omega + j/2\right)}\right\} \left|\hat{\varphi}\left(\omega + j/2\right)\right|^2 \end{aligned}$$

$$\begin{aligned} &= \left(\left|b_0^1(\omega)\right|^2 + \left|b_0^2(\omega)\right|^2\right) \sum_{j\in\mathbb{Z}} \left|\hat{\varphi}\left(\omega + j/2\right)\right|^2 \\ &\quad + b_0^1(\omega)\overline{b_0^2(\omega)} \sum_{j\in\mathbb{Z}} e^{2\pi i(\omega+j/2)r/N} \left|\hat{\varphi}\left(\omega + j/2\right)\right|^2 \\ &\quad + b_0^2(\omega)\overline{b_0^1(\omega)} \sum_{j\in\mathbb{Z}} e^{-2\pi i(\omega+j/2)r/N} \left|\hat{\varphi}\left(\omega + j/2\right)\right|^2 . \end{aligned} \tag{3.5}$$

Using (2.8), (2.9) and (3.5), we have

$$H(2N\omega) = 2\left\{\left|b_0^1(\omega)\right|^2 + \left|b_0^2(\omega)\right|^2\right\}$$

$$\sum_{j\in\mathbb{Z}} \left|\hat{\varphi}\left(2N\left(\omega + j/2\right)\right)\right|^2 = 2\left\{\left|b_0^1(\omega)\right|^2 + \left|b_0^2(\omega)\right|^2\right\}$$

$$\sum_{p=0}^{2N-1} \sum_{j\in\mathbb{Z}} \left|\hat{\varphi}\left(\omega + p/2 + jN\right)\right|^2 = 2\sum_{p=0}^{2N-1} \left\{\left|b_0^1\left(\omega + \frac{p}{4N}\right)\right|^2 + \left|b_0^2\left(\omega + \frac{p}{4N}\right)\right|^2\right\}.$$

By using condition (3.1) for $l = 0$, we have

$$\sum_{q\in\mathbb{Z}} \left|\hat{\varphi}\left(\omega + \frac{q}{2}\right)\right|^2 < 1 \tag{3.6}$$

Since $\Lambda_{r,N} = \left\{0, \frac{r}{N}\right\} + 2\mathbb{Z}$. Then, we get

$$\sum_{\lambda\in\Lambda} \left|< u, \varphi_{j,\lambda} >\right|^2 = \sum_{\lambda\in 2\mathbb{Z}} \left|< u, \varphi_{j,\lambda} >\right|^2 + \sum_{\lambda\in\left(\frac{r}{N}+2\mathbb{Z}\right)} \left|< u, \varphi_{j,\lambda} >\right|^2 \tag{3.7}$$

By Plancherel and Parseval Identity, we get

$$\begin{aligned}\sum_{\lambda\in 2\mathbb{Z}}\left|< u,\varphi_{j,\lambda} >\right|^2 &= (2N)^{-j}\sum_{\lambda\in 2\mathbb{Z}}\left|\int_{\mathbb{R}}\hat{u}(\omega)\overline{\hat{\varphi}((2N)^{-j}\omega)}e^{\frac{2\pi i\lambda\omega}{(2N)^j}}d\omega\right|^2\\ &= \tfrac{1}{2}\sum_{m\in\mathbb{Z}}\Big|\int_0^{\frac{(2N)^j}{2}}\Big[\sum_{\sigma\in .4Z}\hat{u}\Big(\omega+(2N)^j\tfrac{\sigma}{2}\Big)\overline{\hat{\varphi}\Big((2N)^{-j}\Big(\omega+(2N)^j\tfrac{\sigma}{2}\Big)\Big)}\Big]\\ &\qquad\times\sqrt{2}(2N)^{-j/2}e^{\frac{2\pi i(2m)\omega}{(2N)^j}}d\omega\Big|^2\\ &= \tfrac{1}{2}\int_0^{\frac{(2N)^j}{2}}\Big|\sum_{\sigma\in\mathbb{Z}}\hat{u}\Big(\omega+(2N)^j\tfrac{\sigma}{2}\Big)\overline{\hat{\varphi}\Big((2N)^{-j}\omega+\tfrac{\sigma}{2}\Big)}\Big|^2 d\omega=\tfrac{1}{2}\|U_j^1\|^2,\end{aligned}\tag{3.8}$$

where

$$U_j^1(\omega)=\sum_{\sigma\in\mathbb{Z}}\hat{u}\Big(\omega+(2N)^j\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\Big((2N)^{-j}\omega+\frac{\sigma}{2}\Big)}.$$

By Plancherel and Parseval Identity again, we get

$$\begin{aligned}&\sum_{\lambda\in\left(\frac{r}{N}+2\mathbb{Z}\right)}\left|< u,\varphi_{j,\lambda} >\right|^2\\ &= \frac{1}{2}\sum_{m\in\mathbb{Z}}\Big|\int_0^{\frac{(2N)^j}{2}}\Big[\sum_{\sigma\in\mathbb{Z}}\hat{u}\Big(\omega+(2N)^j\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\Big((2N)^{-j}\Big(\omega+(2N)^j\frac{\sigma}{2}\Big)\Big)}\Big]\\ &\qquad\times e^{\frac{\pi ir\sigma}{N}}\sqrt{2}(2N)^{-j/2}e^{\frac{2\pi i\left(\frac{r}{N}+2m\right)\omega}{(2N)^j}}d\omega\Big|^2\\ &= \frac{1}{2}\int_0^{\frac{(2N)^j}{2}}\Big|\sum_{\sigma\in\mathbb{Z}}\hat{u}\Big(\omega+(2N)^j\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\Big((2N)^{-j}\omega+\frac{\sigma}{2}\Big)}e^{\frac{\pi ir\sigma}{N}}\Big|^2 d\omega=\frac{1}{2}\|U_j^2\|^2,\end{aligned}\tag{3.9}$$

where

$$U_j^2(\omega)=\sum_{\sigma\in\mathbb{Z}}\hat{u}\Big(\omega+(2N)^j\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\Big((2N)^{-j}\omega+\frac{\sigma}{2}\Big)}e^{\frac{\pi ir\sigma}{N}}.$$

Thus, from (3.7), (3.8) and (3.9), we have

$$Q_j=\sum_{\lambda\in\Lambda}\left|< u,\varphi_{j,\lambda} >\right|^2=\frac{1}{2}\left(\left\|U_j^1\right\|_2^2+\left\|U_j^2\right\|_2^2\right).\tag{3.10}$$

Next, consider the sequence of functions

$$\hat{g}_j(\omega)=\begin{cases}\hat{u}(\omega),\ \text{if }\omega\in[0,(2N)^j]\\ \\ 0,\ \text{if }\omega\notin[0,(2N)^j]\end{cases}\qquad h_j=u-g_j,\ j\in\Lambda_{r,N}$$

$$G_j^1(\omega)=\sum_{\sigma\in\Lambda}\hat{g}_j\Big(\omega+(2N)^j\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\Big((2N)^{-j}\Big(\omega+(2N)^j\frac{\sigma}{2}\Big)\Big)},$$

$$G_j^2(\omega) = \sum_{\sigma\in\Lambda} \hat{g}_j\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\left((2N)^{-j}\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\right)}e^{\pi i r\sigma/N},$$

$$H_j^1(\omega) = \sum_{\sigma\in\Lambda} \hat{h}_j\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\left((2N)^{-j}\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\right)}$$

$$H_j^2(\omega) = \sum_{\sigma\in\Lambda} \hat{h}_j\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\left((2N)^{-j}\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\right)}e^{\pi i r\sigma/N}.$$

Clearly, as $j \to \infty$, $\|G_j^1\|_2 = \|\hat{u}\|_2$ and $j \to \infty$, $\|G_j^2\|_2 = \|\hat{u}\|_2$. Furthermore, from Eq. (3.6), we get

$$\begin{aligned}
\|H_j^1\|^2 &= \int_0^{\frac{(2N)^j}{2}} \left|\sum_{\sigma\in\Lambda} \hat{h}_j\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\left((2N)^{-j}\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\right)}\right|^2 d\omega \\
&\le \int_0^{\frac{(2N)^j}{2}} \left|\sum_{\sigma\in\Lambda} \hat{h}_j\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\right|^2 \sum_{\sigma\in\Lambda}\left|\hat{\varphi}\left((2N)^{-j}\omega + \frac{\sigma}{2}\right)\right|^2 d\omega \\
&\le \int_0^{\frac{(2N)^j}{2}} \left|\sum_{\sigma\in\Lambda} \hat{h}_j\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\right|^2 d\omega \\
&\le \|\hat{h}_j\|^2 \to 0, \text{ as } j \to \infty,
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\|H_j^2\|^2 &= \int_0^{\frac{(2N)^j}{2}} \left|\sum_{\sigma\in\Lambda} \hat{h}_j\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\left((2N)^{-j}\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\right)}e^{\pi i r\sigma/N}\right|^2 d\omega \\
&\le \int_0^{\frac{(2N)^j}{2}} \left|\sum_{\sigma\in\Lambda} \hat{h}_j\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\left((2N)^{-j}\Big(\omega + (2N)^j\frac{\sigma}{2}\Big)\right)}\right|^2 d\omega \\
&\le \|\hat{h}_j\|^2 \to 0, \text{ as } j \to \infty,
\end{aligned} \tag{3.12}$$

since

$$\|G_j^1\|_2 - \|H_j^1\|_2 \le \|U_j^1\|_2 = \|G_j^1 + H_j^1\|_2 \le \|G_j^1\|_2 + \|H_j^1\|_2.$$

and

$$\|G_j^2\|_2 - \|H_j^2\|_2 \le \|U_j^2\|_2 = \|G_j^2 + H_j^2\|_2 \le \|G_j^2\|_2 + \|H_j^2\|_2.$$

From (3.11) and (3.12), we obtain

$$\|U_j^1\|_2^2 \to \|\hat{u}\|_2^2 = \|u\|_2^2, \text{ and } \|U_j^2\|_2^2 \to \|\hat{u}\|_2^2 = \|u\|_2^2 \text{ as } j \to \infty. \tag{3.13}$$

Now, from (3.10) and (3.13), we have

$$Q_j = \sum_{\lambda\in\Lambda} \left|< u, \varphi_{j,\lambda} >\right|^2 = \frac{1}{2}\left(\left\|U_j^1\right\|_2^2 + \left\|U_j^2\right\|_2^2\right) = \|u\|_2^2, \text{ as } j \to \infty.$$

Thus part (a) is proved.

For any $b \in \mathbb{R}$, assume

$$u_b(x) = \begin{cases} u(x), \text{ if } x \in [0, b] \\ 0, \text{ otherwise.} \end{cases}$$

Consequently, for given $\varepsilon > 0$ ($\varepsilon < \frac{1}{p}$), we get $\|u - u_b\|_2 < \varepsilon$. Since

$$\begin{aligned} \sum_{\lambda\in\Lambda} | < u, \varphi_{j,\lambda} > |^2 &= \sum_{\lambda\in\Lambda} | < u - u_b, \varphi_{j,\lambda} > + < u_b, \varphi_{j,\lambda} > |^2 \\ &\leq p\sum_{\lambda\in\Lambda} | < u_b, \varphi_{j,\lambda} > |^2 + p\sum_{\lambda\in\Lambda} | < u - u_b, \varphi_{j,\lambda} > |^2 \\ &\leq p\sum_{\lambda\in\Lambda} | < u_b, \varphi_{j,\lambda} > |^2 + \|u - u_b\|_2 \\ &\leq p\sum_{\lambda\in\Lambda} | < u_b, \varphi_{j,\lambda} > |^2 + \varepsilon. \end{aligned}$$

Now, to show

$$\lim_{j\to-\infty} \sum_{\lambda\in\Lambda} | < u_b, \varphi_{j,\lambda} > |^2 = 0.$$

Since

$$\begin{aligned} \sum_{\lambda\in\Lambda} | < u_b, \varphi_{j,\lambda} > |^2 &= \sum_{\lambda\in\Lambda} \left| \int_{x\in[0,b]} u(x)\varphi_{j,\lambda}(x)dx \right|^2 \\ &= \sum_{\lambda\in\Lambda} \left| \int_{x\in[0,b]} u(x)(2N)^{j/2}\overline{\varphi((2N)^j x - \lambda)}dx \right|^2 \\ &\leq (2N)^j \sum_{\lambda\in\Lambda} \left(\int_{x\in[0,b]} |u(x)||\varphi((2N)^j x - \lambda)|dx \right)^2 \\ &\leq (2N)^j \|u\|^2 \sum_{\lambda\in\Lambda} \left(\int_{x\in[0,b]} |\varphi((2N)^j x - \lambda)|dx \right)^2 \\ &= \|u\|^2 \sum_{\lambda\in\Lambda} \int_{y+\lambda\in[0,(2N)^j b]} |\varphi(y)|^2 dy \\ &= \|u\|^2 \int_{\cup_{\lambda\in\Lambda}[0,(2N)^j b+\lambda]} |\varphi(y)|^2 dy \to 0, \text{ as } j \to -\infty. \end{aligned}$$

Hence, we obtain, $\lim_{j\to-\infty} Q_j = 0$. □

Lemma 3.3. *If (2.15) is true, then we get the following*

$$\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{\lambda\in\Lambda}|<u,\psi^{l}_{j,\lambda}>|^2=\sum_{\lambda\in\Lambda}|<u,\varphi_{J,\lambda}>|^2+\sum_{l=1}^{L}\sum_{j\geq J}\sum_{\lambda\in\Lambda}|<u,\psi^{l}_{j,\lambda}>|^2<\infty, \tag{3.14}$$

for any $u\in L^2(\mathbb{R})$ and $J\in\mathbb{Z}$.

Proof. From (2.15), we have

$$\sum_{l=0}^{L}|b^{\mu}_{l}(\omega+m/4N)|^2=\frac{1}{2},\ \forall\ m=0,1,2,...,2N-1 \text{ and } \mu\in\{1,2\}, \tag{3.15}$$

and

$$\sum_{l=0}^{L}b^{\mu}_{l}(\omega+n/4N)\,\overline{b^{\mu'}_{l}(\omega+m/4N)}=0, \tag{3.16}$$

for all $m=0,1,2,...,2N-1, m\neq N-n,$ if $0\leq n\leq N, m\neq 3N-n,$ if $N+1\leq n\leq 2N-1,$ and $\mu,\mu'\in\{1,2\}$,

By using (2.14), we get

$$\begin{aligned}|b_l(\omega+m/4N)|^2=&\Big(b^1_l(\omega+m/4N)+e^{-2\pi i\frac{r}{N}(\omega+m/4N)}b^2_l(\omega+m/4N)\Big)\\&\times\Big(\overline{b^1_l(\omega+m/4N)}+e^{2\pi i\frac{r}{N}(\omega+m/4N)}\overline{b^2_l(\omega+m/4N)}\Big).\end{aligned}$$

Thus, with the help of (3.15) and (3.16) for all $m=0,1,2,...,2N-1$, we have

$$\sum_{l=0}^{L}|b_l(\omega+m/4N)|^2=\sum_{l=0}^{L}|b^1_l(\omega+m/4N)|^2+\sum_{l=0}^{L}|b^2_l(\omega+m/4N)|^2=1 \tag{3.17}$$

and

$$\sum_{l=0}^{L}b_l(\omega+n/4N)\overline{b_l(\omega+m/4N)}=0 \tag{3.18}$$

for all $m=0,1,2,...,2N-1, m\neq N-n,$ if $0\leq n\leq N, m\neq 3N-n,$ if $N+1\leq n\leq 2N-1.$
Let

$$\begin{aligned}\Delta_q\ =\ &\sum_{\sigma\in\mathbb{Z}}\hat{u}\Big(\omega+(2N)^{r+1}\frac{\sigma}{2}+(q-1)(2N)^r\Big)\\&\times\overline{\hat{\varphi}\Big((2N)^{-r-1}\omega+\frac{\sigma}{2}+(q-1)(4N)^{-1}\Big)},\ 1\leq q\leq 2N\end{aligned}$$

and

$$\begin{aligned}\Delta^*_q\ =\ &\sum_{\sigma\in\mathbb{Z}}\hat{u}\Big(\omega+(2N)^{r+1}\frac{\sigma}{2}+(q-1)(2N)^r\Big)\\&\times\overline{\hat{\varphi}\Big((2N)^{-r-1}\omega+\frac{\sigma}{2}+(q-1)(4N)^{-1}\Big)}e^{\frac{\pi r\sigma}{N}},1\leq q\leq 2N.\end{aligned}$$

By analogy with (3.7), (3.8) and (3.9) for any $r \in \mathbb{Z}$, we derive

$$\sum_{\lambda\in\Lambda} | < u, \varphi_{r,\lambda} > |^2 + \sum_{l=1}^{L}\sum_{\lambda\in\Lambda} | < u, \psi^l_{r,\lambda} > |^2$$

$$= \frac{1}{2}\int_0^{\frac{(2N)^j}{2}} \left|\sum_{\sigma\in\mathbb{Z}} \hat{u}\Big(\omega + (2N)^r\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\Big((2N)^{-r}\Big(\omega+\frac{\sigma}{2}\Big)\Big)}\right|^2 d\omega$$

$$+\frac{1}{2}\int_0^{\frac{(2N)^j}{2}} \left|\sum_{\sigma\in\mathbb{Z}} \hat{u}\Big(\omega + (2N)^r\frac{\sigma}{2}\Big)\overline{\hat{\varphi}\Big((2N)^{-r}\Big(\omega+\frac{\sigma}{2}\Big)\Big)}e^{\frac{\pi i r\sigma}{N}}\right|^2 d\omega$$

$$+\frac{1}{2}\sum_{l=1}^{L}\int_0^{\frac{(2N)^j}{2}} \left|\sum_{\sigma\in\mathbb{Z}} \hat{u}\Big(\omega + (2N)^r\frac{\sigma}{2}\Big)\overline{\hat{\psi}\Big((2N)^{-r}\Big(\omega+\frac{\sigma}{2}\Big)\Big)}\right|^2 d\omega$$

$$+\frac{1}{2}\sum_{l=1}^{L}\int_0^{\frac{(2N)^j}{2}} \left|\sum_{\sigma\in\mathbb{Z}} \hat{u}\Big(\omega + (2N)^r\frac{\sigma}{2}\Big)\overline{\hat{\psi}\Big((2N)^{-r}\Big(\omega+\frac{\sigma}{2}\Big)\Big)}e^{\frac{\pi i r\sigma}{N}}\right|^2 d\omega$$

$$= \frac{1}{2}\sum_{l=0}^{L}\int_0^{\frac{(2N)^j}{2}} \Big|\sum_{\sigma\in\mathbb{Z}} \hat{u}\Big(\omega + (2N)^r\frac{\sigma}{2}\Big)\overline{b_l\Big((2N)^{-r-1}\Big(\omega+(2N)^r\frac{\sigma}{2}\Big)\Big)}$$
$$\times\overline{\hat{\varphi}\Big((2N)^{-r-1}\Big(\omega+(2N)^r\frac{\sigma}{2}\Big)\Big)}\Big|^2 d\omega$$

$$+\frac{1}{2}\sum_{l=0}^{L}\int_0^{\frac{(2N)^j}{2}} \Big|\sum_{\sigma\in\mathbb{Z}} \hat{u}\Big(\omega + (2N)^r\frac{\sigma}{2}\Big)\overline{b_l\Big((2N)^{-r-1}\Big(\omega+(2N)^r\frac{\sigma}{2}\Big)\Big)}$$
$$\times\overline{\hat{\varphi}\Big((2N)^{-r-1}\Big(\omega+(2N)^r\frac{\sigma}{2}\Big)\Big)}e^{\frac{\pi i r\sigma}{N}}\Big|^2 d\omega$$

$$= \sum_{q=1}^{2N}\Big[\frac{1}{2}\sum_{l=0}^{L}\int_0^{\frac{(2N)^j}{2}} \Big|\Delta_q b_l\Big((2N)^{-r-1}\omega + \frac{q-1}{4N}\Big)\Big|^2 d\omega\Big]$$

$$+\sum_{q=1}^{2N}\sum_{s=1,s\neq q}^{2N}\Big[\frac{1}{2}\sum_{l=0}^{L}\int_0^{\frac{(2N)^j}{2}} \Big|\Delta_q \overline{b_l\Big((2N)^{-r-1}\omega + \frac{q-1}{4N}\Big)}$$
$$\times\overline{\Delta_s} b_l\Big((2N)^{-r-1}\omega + \frac{q-1}{4N}\Big)\Big|^2 d\omega\Big]$$

$$+\sum_{q=1}^{2N}\Big[\frac{1}{2}\sum_{l=0}^{L}\int_0^{\frac{(2N)^j}{2}} \Big|\Delta^*_q b_l\Big((2N)^{-r-1}\omega + \frac{q-1}{4N}\Big)\Big|^2 d\omega\Big]$$

$$+\sum_{q=1}^{2N}\sum_{s=1,s\neq q}^{2N}\Big[\frac{1}{2}\sum_{l=0}^{L}\int_0^{\frac{(2N)^j}{2}}\Big|\Delta_q^* b_l\Big((2N)^{-r-1}\omega+\frac{q-1}{4N}\Big)$$
$$\times\overline{\Delta_s^*} b_l\Big((2N)^{-r-1}\omega+\frac{q-1}{4N}\Big)\Big|^2 d\omega\Big]$$

$$=\sum_{q=1}^{2N}\Big[\frac{1}{2}\int_0^{\frac{(2N)^j}{2}}\Big|\Delta_q\Big|^2 d\omega\Big]+\sum_{q=1}^{2N}\Big[\frac{1}{2}\int_0^{\frac{(2N)^j}{2}}\Big|\Delta_q^*\Big|^2 d\omega\Big]$$

$$=\sum_{\lambda\in\Lambda}|<u,\varphi_{r+1,\lambda}>|^2<\infty.$$

We derive Lemma 3.3 from Lemma 3.2. □

Hence, Theorem 3.1 follows from Lemmas 3.1-3.3.

Theorem 3.1 provides the construction of wavelet frames by a given compactly supported refinable function on $\mathbb{R}$ is to find solutions $b_l(\omega),\ l=1,2,...,L$, satisfying equation (2.15).
We will present the solution of $b_l(\omega),\ l=1,2,...,L$, satisfying equation (2.15) in the following.

Let b_0 satisfy

$$\sum_{m=0}^{2N-1}\Big|b_0^1\Big(\omega+\frac{m}{4N}\Big)\Big|^2\leq\frac{1}{2}\text{ and }\sum_{m=0}^{2N-1}\Big|b_0^2\Big(\omega+\frac{m}{4N}\Big)\Big|^2\leq\frac{1}{2}$$

and $\mathbb{W}(\omega)$ (from 3.3) with $4N$-eigenvalues provided by (3.4), subsequently the unit eigen-vector of the matrix $\mathbb{W}(\omega)$ can be expressed as

$$\delta_1 = \frac{1}{\Omega_1}\Big(-(1+\overline{\alpha})\,\overline{b_0^1\Big(\omega+\frac{1}{4N}\Big)},-2\overline{b_0^1(\omega)},0,0,...,0\Big)^T$$

$$\delta_m = \frac{1}{\Omega_m}\Big(-2(1+\overline{\alpha}^m)b_0^1(\omega)\overline{b_0^1\Big(\omega+\frac{m}{4N}\Big)},...,-(1+\alpha^{m-1})(1+\overline{\alpha}^m)$$
$$b_0^1\Big(\omega+\frac{m-1}{4N}\Big)\overline{b_0^1\Big(\omega+\frac{m}{4N}\Big)},-2\sum_{t=0}^{m-1}\Big|b_0^1\Big(\omega+\frac{t}{4N}\Big)\Big|^2,0,0,...,0\Big)^T,$$
$$m=2,3,...,2N-1,$$

$$\delta_{2N} = \frac{1}{\Omega_{2N}}\Big(2b_0^1(\omega),(1+\alpha)b_0^1\Big(\omega+\frac{1}{4N}\Big),...,(1+\alpha^{2N-1})$$
$$b_0^1\Big(\omega+\frac{2N-1}{4N}\Big),0,0,...,0\Big)^T,$$

$$\delta_{2N+1} = \frac{1}{\Omega_{2N+1}}\Big(2b_0^1(\omega),(1+\alpha)b_0^1\Big(\omega+\frac{1}{4N}\Big),...,(1+\alpha^{2N-1})$$

$$
\begin{aligned}
& b_0^1\Big(\omega+\frac{2N-1}{4N}\Big),-(1+\overline{\alpha})\,\overline{b_0^2\Big(\omega+\frac{1}{4N}\Big)},-2\overline{b_0^2\,(\omega)},0,0,...,0\Big)^T, \\
\delta_{2N+m} &= \frac{1}{\Omega_{2N+m}}\Big(2b_0^1(\omega),(1+\alpha)b_0^1\Big(\omega+\frac{1}{4N}\Big),...,(1+\alpha^{2N-1}) \\
& b_0^1\Big(\omega+\frac{2N-1}{4N}\Big),-2(1+\overline{\alpha}^m)b_0^2(\omega)\overline{b_0^2\Big(\omega+\frac{m}{4N}\Big)},..., \\
& -(1+\alpha^{m-1})(1+\overline{\alpha}^m)b_0^2\Big(\omega+\frac{m-1}{4N}\Big)\overline{b_0^2\Big(\omega+\frac{m}{4N}\Big)}, \\
& -2\sum_{t=0}^{m-1}\Big|b_0^2\Big(\omega+\frac{t}{4N}\Big)\Big|^2,0,0,...,0\Big)^T,\ m=2,3,...,2N-1, \\
\delta_{4N} &= \frac{1}{\Omega_{4N}}\Big(2b_0^1(\omega),(1+\alpha)b_0^1\Big(\omega+\frac{1}{4N}\Big),...,(1+\alpha^{2N-1}) \\
& b_0^1\Big(\omega+\frac{2N-1}{4N}\Big),2b_0^2(\omega),(1+\alpha)b_0^2\Big(\omega+\frac{1}{4N}\Big),..., \\
& (1+\alpha^{2N-1})b_0^2\Big(\omega+\frac{2N-1}{4N}\Big)\Big)^T,
\end{aligned}
$$

where

$$
\begin{aligned}
\Omega_1^2 &= 4\Big|b_0^1(\omega)\Big|^2+(1+\overline{\alpha})^2\Big|b_0^1\Big(\omega+\frac{1}{4N}\Big)\Big|^2 \\
\Omega_m^2 &= (1+\overline{\alpha}^m)^2\Big|b_0^1\Big(\omega+\frac{m}{4N}\Big)\Big|^2\sum_{t=0}^{m-1}(1+\alpha^t)^2\Big|b_0^1\Big(\omega+\frac{t}{4N}\Big)\Big|^2 \\
& +4\Big(\sum_{t=0}^{m-1}\Big|b_0^1\Big(\omega+\frac{t}{4N}\Big)\Big|^2\Big)^2,\ m=2,...,2N-1 \\
\Omega_{2N}^2 &= \sum_{t=0}^{2N-1}(1+\alpha^t)^2\Big|b_0^1\Big(\omega+\frac{t}{4N}\Big)\Big|^2 \\
\Omega_{2N+1}^2 &= \sum_{t=0}^{2N-1}(1+\alpha^t)^2\Big|b_0^1\Big(\omega+\frac{t}{4N}\Big)\Big|^2+4\Big|b_0^2(\omega)\Big|^2+(1+\overline{\alpha})^2\Big|b_0^2\Big(\omega+\frac{1}{4N}\Big)\Big|^2 \\
\Omega_{2N+m}^2 &= \sum_{t=0}^{2N-1}(1+\alpha^t)^2\Big|b_0^1\Big(\omega+\frac{t}{4N}\Big)\Big|^2+(1+\overline{\alpha}^m)^2\Big|b_0^2\Big(\omega+\frac{m}{4N}\Big)\Big|^2
\end{aligned}
$$

$$\sum_{t=0}^{m-1}(1+\alpha^t)^2\Big|b_0^2\Big(\omega+\frac{t}{4N}\Big)\Big|^2+4\Big(\sum_{t=0}^{m-1}\Big|b_0^2\Big(\omega+\frac{t}{4N}\Big)\Big|^2\Big)^2,$$
$$m=2,...,2N-1$$
$$\Omega_{4N}^2 = \sum_{t=0}^{2N-1}(1+\alpha^t)^2\Big|b_0^1\Big(\omega+\frac{t}{4N}\Big)\Big|^2+\sum_{s=0}^{2N-1}(1+\alpha^s)^2\Big|b_0^2\Big(\omega+\frac{s}{4N}\Big)\Big|^2$$

Thus, we have

$$\mathbb{W}=\mathcal{P}(\omega)\Lambda(\omega)\mathcal{P}^*(\omega), \tag{3.19}$$

where $\mathcal{P}(\omega)=(\lambda_1,\lambda_2,...,\lambda_{2N},\lambda_{2N+1},...,\lambda_{4N})$ and $\Lambda(\omega)=\operatorname{diag}(\gamma_1,\gamma_2,...,\gamma_{2N},\gamma_{2N+1},...,\gamma_{4N})$.

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