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Discrepancies in Euclidean Operator Radii in Hilbert C^*-Modules

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Abstract. In this research, we establish precise limits for the Euclidean operator radius of two bounded linear operators operating within a Hilbert C^*-module over $\mathfrak{A}$. Furthermore, our work establishes a connection between these limits and recent research findings that provide accurate upper and lower bounds for the numerical radius of linear operators. The primary objective of this investigation is to explore various specific scenarios of interest and extend existing inequalities found in the literature to encompass the Euclidean radius of two operators in a Hilbert $\mathfrak{A}$-module. Additionally, our study presents conclusions that reveal relationships between the operator norm, the typical numerical radius of a composite operator, and the Euclidean operator radius. Furthermore, we introduce several new inequalities involving the Euclidean numerical radius and Euclidean operator norm of 2-tuple operators. These inequalities offer both lower and upper bounds for the Euclidean numerical radius of 2-tuple operators, as well as for the sum and product of 2-tuple operators. We also delve into the study of Euclidean numerical radius inequalities for 2×2 operator matrices whose entries consist of 2-tuple operators, leading to the derivation of some Euclidean operator radius inequalities. Additionally, we establish an inequality for the Euclidean operator norm of 2×2 operator matrices.

1. Introduction

A frequently employed tool within operator theory and operator algebra is the concept of a Hilbert C^*-module. These modules encompass a substantial category within operator C^*-module theory. Moreover, the field of Hilbert C^*-modules is inherently intriguing, with its profound interplay with operator algebra theory and the incorporation of various essential concepts. It emerges from the realm of non-commutative geometry, giving rise to both novel discoveries and intriguing challenges that capture significant attention.

Expanding upon the notion of a Hilbert space, the concept of a Hilbert C^*-module was initially introduced by Kaplansky [6]. The study of Hilbert C^*-modules took its initial steps with the work of Rieffel's induced representations of C^*-algebras [20] and Paschke's pioneering PhD dissertation

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[13], see also [17–19]. These modules find utility across diverse domains, benefiting the theory of operator algebras, operator K-theory, group representation theory, and the theory of operator spaces. Furthermore, they play a pivotal role in investigating Morita equivalence among C^*-algebras, the C^*-algebra quantum group, and C^*-algebra K-theory, as exemplified by Lance [10] and Wegge-Olsen [21].

This research significantly advances operator theory within Hilbert C^*-modules, with applications spanning quantum mechanics, functional analysis, numerical analysis, signal processing, and operator algebras. Its findings deepen our understanding of operator norms and inequalities, impacting diverse mathematical and physical domains.

A pre-Hilbert module over a C^*-algebra $\mathfrak{A}$ constitutes a complex linear space denoted as $\mathfrak{E}$. This space functions as a right $\mathfrak{A}$-module, adhering to the properties $\lambda(ax) = (\lambda a)x = a(\lambda x)$, where $\lambda \in \mathbb{C}$, $a \in \mathfrak{A}$, and $x \in \mathfrak{E}$. It is equipped with an inner product $\langle \cdot, \cdot \rangle : \mathfrak{E} \times \mathfrak{E} \longrightarrow \mathfrak{A}$ that satisfies the following conditions:

(i) $\langle x, x \rangle \geq 0$, and $\langle x, x \rangle = 0$ if and only if $x = 0$.
(ii) $\langle x, \alpha y + \beta z \rangle = \alpha \langle x, y \rangle + \beta \langle x, z \rangle$.
(iii) $\langle x, ya \rangle = \langle x, y \rangle a$.
(iv) $\langle x, y \rangle = \langle y, x \rangle^*$,

where $x, y, z \in \mathfrak{E}$, $a \in \mathfrak{A}$, and $\alpha, \beta \in \mathbb{C}$."

A pre-Hilbert module over $\mathfrak{A}$ is designated as a Hilbert $\mathfrak{A}$-module, or alternatively, a Hilbert C^*-module over $\mathfrak{A}$, if it possesses completeness relative to the norm $\|x\| = \|\langle x, x \rangle\|^{\frac{1}{2}}$. If we consider $\mathfrak{E}$ and $\mathfrak{F}$ as Hilbert C^*-modules, we introduce the set $\mathscr{L}(\mathfrak{E}, \mathfrak{F})$, which comprises all mappings $t : \mathfrak{E} \longrightarrow \mathfrak{F}$ for which there exists a corresponding map $t^* : \mathfrak{F} \longrightarrow \mathfrak{E}$. This map t^* satisfies the property $\langle tx, y \rangle = \langle x, t^* y \rangle$ for all x in $\mathfrak{E}$ and y in $\mathfrak{F}$.

In the context of $\mathfrak{E}$, $\mathscr{L}(\mathfrak{E}, \mathfrak{E})$ is denoted simply as $\mathscr{L}(\mathfrak{E})$. It is well-established that $\mathscr{L}(\mathfrak{E})$ constitutes a C^*-algebra. Within the domain of C^*-algebras, a state is defined as a positive linear functional on $\mathfrak{A}$ possessing a norm of one. The state space of $\mathfrak{A}$ is symbolized as $\varpi(\mathfrak{A})$.

The primary objective of this research is to explore additional intriguing scenarios and expand upon various inequalities found in existing literature concerning the Euclidean radius of two operators within a Hilbert $\mathfrak{A}$-module. Furthermore, this study provides insights that establish connections between the operator norm, the standard numerical radius of a composite operator, and the Euclidean operator radius.

2. Terminology and Supplementary Findings

In this section, we will elucidate and define key terms, concepts, and notations essential for a comprehensive understanding of Hilbert C^*-modules. Furthermore, we will delve into supplementary findings and insights that complement and enrich our comprehension of this specialized area within functional analysis. This section aims to equip readers with the necessary linguistic tools and supplementary knowledge required to navigate the intricate terrain of Hilbert C^*-modules with clarity and confidence.

Definition 2.1. *Let $\mathbf{t} = (\mathbf{x}, \mathbf{y}) \in \mathscr{L}^2(\mathfrak{E})$. The Euclidean operator radius is defined by*

$$\begin{aligned} w_e(\mathbf{t}) : &= w_e(\mathbf{x}, \mathbf{y}) \\ &= \sup\left\{\left(\left|\psi\left(\langle\xi, \mathbf{x}\xi\rangle\right)\right|^2 + \left|\psi\left(\langle\xi, \mathbf{y}\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} : \xi \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}) \text{ and } \psi\left(|\xi|\right) = 1\right\}. \end{aligned} \tag{2.1}$$

And the Euclidean operator norm is defined by

$$\begin{aligned} \|\mathbf{t}\| : &= \sqrt{\|\mathbf{x}^*\mathbf{x} + \mathbf{y}^*\mathbf{y}\|} \\ &= \sup\left\{\left(\psi\left(|\mathbf{x}\xi|^2\right) + \psi\left(|\mathbf{y}\xi|^2\right)\right)^{\frac{1}{2}} : \xi \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}) \text{ and } \psi\left(|\xi|\right) = 1\right\}. \end{aligned} \tag{2.2}$$

Lemma 2.1. *$\|\cdot\|$ is a norm on $\mathfrak{E}$.*

Proof. If $\mathbf{t} = (t_1, t_2) = 0$, then obviously $\|\mathbf{t}\| = 0$. If $\|\mathbf{t}\| = 0$, then for every $\psi \in \varpi(\mathfrak{A})$ and each $\xi \in \mathfrak{E}$ such that $\psi\left(|\xi|\right) = 1$, we have $\psi\left(|t_j\xi|^2\right) = 0$ for all $j = 1, 2$. We want to show that $\mathbf{t}\xi = 0$ for each $\xi \in \mathfrak{E}$. Fix $\xi \in \mathfrak{E}$.

(1) If $\psi\left(|\xi|\right) = 0$, then by the Cauchy-Schwartz inequality we have

$$\psi\left(\left\langle t_j\xi, t_j\xi\right\rangle\right) = \psi\left(\left\langle t_j^*t_j\xi, \xi\right\rangle\right) \leq \psi\left(\left\langle t_j^*t_j\xi, t_j^*t_j\xi\right\rangle\right)^{\frac{1}{2}} \psi\left(|\xi|\right) = 0,$$

and so $t_j = 0$ for all $j = 1, 2$, i.e., $\mathbf{t} = 0$.

(2) If $\psi\left(|\xi|\right) \neq 0$, then by taking $\zeta = \frac{\xi}{\psi(|\xi|)}$, then $\psi\left(|\zeta|\right) = 1$. By Definition 2.2, $\psi\left(\left|t_j\zeta\right|\right) = 0$ for all $j = 1, 2$ and so $\frac{1}{\phi(|\xi|)}\psi\left(\left|t_j\xi\right|\right) = 0$ for all $j = 1, 2$. Thus $\psi\left(\left|t_j\xi\right|\right) = 0$ for all $j = 1, 2$. Since for every $\psi \in \varpi(\mathfrak{A})$, we have $\psi\left(|t_j\xi|\right) = 0$ for all $j = 1, 2$. We conclude that $|t_j\xi| = 0$ for each $\xi \in \mathfrak{E}$ and for all $j = 1, 2$. So $t_j = 0$ for all $j = 1, 2$ and so $\mathbf{t} = 0$.

On the other hand $\mathfrak{A}$ is an abelian C^*-algebra, then by [5, Theorem 3.6], $|\xi + \zeta| \leq |\xi| + |\zeta|$ for each $\xi, \zeta \in \mathfrak{E}$. Thus $|t\xi + s\xi| \leq |t\xi| + |s\xi|$ for all $\xi \in \mathfrak{E}$. Hence

$$\psi\left(|t\xi + s\xi|\right) \leq \psi\left(|t\xi|\right) + \psi\left(|s\xi|\right).$$

Now by Minkowski's inequality, we have

$$\begin{aligned} \left(\left(\psi\left|(t_1 + q_1)\xi\right|\right)^2 + \left(\psi\left|(t_2 + q_2)\xi\right|\right)^2\right)^{\frac{1}{2}} &\leq \left(\left(\psi\left(|t_1\xi|\right) + \psi\left(|q_1\xi|\right)\right)^2 + \left(\psi\left(|t_2\xi|\right) + \psi\left(|q_2\xi|\right)\right)^2\right)^{\frac{1}{2}} \\ &\leq \left(\psi\left(|t_1\xi|^2\right) + \psi\left(|t_2\xi|^2\right)\right)^{\frac{1}{2}} + \left(\psi\left(|q_1\xi|^2\right) + \psi\left(|q_2\xi|^2\right)\right)^{\frac{1}{2}} \end{aligned}$$

Taking the supremum over all $\xi \in \mathfrak{E}$ with $\psi\left(|\xi|\right) = 1$ and $\psi \in \varpi(\mathfrak{A})$, we have

$$\|\mathbf{t} + \mathbf{q}\| \leq \|\mathbf{t}\| + \|\mathbf{q}\|.$$

Finally, if $\mathbf{t} = (t_1, t_2)$, then we have

$$\|c\mathbf{t}\| = \left(\psi\left(|ct_1\xi|^2\right) + \psi\left(|ct_2\xi|^2\right)\right)^{\frac{1}{2}} = |c|\left(\psi\left(|t_1\xi|^2\right) + \psi\left(|t_2\xi|^2\right)\right)^{\frac{1}{2}} = |c|\|\mathbf{t}\|$$

for all $c \in \mathbb{C}$. □

Theorem 2.1. *Let* $\mathbf{t} = (t_1, t_2) \in \mathscr{L}^2(\mathfrak{E})$. *If* $\mathfrak{E}$ *is a Hilbert* $\mathfrak{A}$*-Modules, then*

$$\|\mathbf{t}\| = \sup\left\{\left(\left|\psi\left(\langle\zeta, t_1\xi\rangle\right)\right|^2 + \left|\psi\left(\langle\zeta, t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} : \xi, \zeta \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \textit{and } \psi\left(|\xi|\right) = \psi\left(|\zeta|\right) = 1\right\}.$$

Proof. Let

$$\varrho = \sup\left\{\left(\left|\psi\left(\langle\zeta, t_1\xi\rangle\right)\right|^2 + \left|\psi\left(\langle\zeta, t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} : \xi, \zeta \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}), \text{and } \psi\left(|\xi|\right) = \psi\left(|\zeta|\right) = 1\right\}.$$

It is sufficient to prove that $\|\mathbf{t}\| = \varrho$. If $\psi \in \varpi(\mathfrak{A})$ and $\xi, \zeta \in \mathfrak{E}$, such that $\psi\left(|\xi|\right) = \psi\left(|\zeta|\right) = 1$, then by using the Cauchy-Schwartz inequality, we get

$$\begin{aligned}\left(\left|\psi\left(\langle\zeta, t_1\xi\rangle\right)\right|^2 + \left|\psi\left(\langle\zeta, t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} &\leq \left(\psi\left(\langle t_1\xi, t_1\xi\rangle\right)\psi\left(\langle\zeta,\zeta\rangle\right) + \psi\left(\langle t_2\xi, t_2\xi\rangle\right)\psi\left(\langle\zeta,\zeta\rangle\right)\right)^{\frac{1}{2}} \\ &\leq \left(\psi\left(|t_1\xi|^2\right)\psi\left(|\zeta|^2\right) + \psi\left(|t_2\xi|^2\right)\psi\left(|\zeta|^2\right)\right)^{\frac{1}{2}} \\ &= \left(\psi\left(|t_1\xi|^2\right) + \psi\left(|t_2\xi|^2\right)\right)^{\frac{1}{2}} \leq \|\mathbf{t}\|\end{aligned}$$

and so $\varrho \leq \|\mathbf{t}\|$.
For $\psi \in \varpi(\mathfrak{A})$ and $\xi \in \mathfrak{E}$ with $\psi\left(|\xi|\right) = 1$, we have

$$\psi\left(|t_j\xi|\right)^4 = \psi\left(|t_j\xi|^2\right) = \psi\left(\left\langle t_j\xi, t_j\xi\right\rangle\right) = \psi\left(|t_j\xi|\right)^2 \psi\left(\left\langle \frac{t_j\xi}{\psi\left(|t_j\xi|\right)}\right\rangle, t_j\xi\right)^2,$$

where assume that $\psi\left(\left|t_j\xi\right|\right) \neq 0$ for all $j = 1, 2$. Thus

$$\begin{aligned}\left(\left|\psi\left(t_1\xi\right)\right|^2 + \left|\psi\left(t_2\xi\right)\right|^2\right)^{\frac{1}{2}} &= \left(\psi\left(\left\langle \frac{t_1\xi}{\psi\left(|t_1\xi|\right)}, t_1\xi\right\rangle\right)^2 + \psi\left(\left\langle \frac{t_2\xi}{\psi\left(|t_2\xi|\right)}, t_2\xi\right\rangle\right)^2\right)^{\frac{1}{2}} \\ &= \left(\left|\psi\left(\zeta, t_1\xi\right)\right|^2 + \left|\psi\left(\zeta, t_2\xi\right)\right|^2\right)^{\frac{1}{2}} \leq \varrho\end{aligned}$$

and hence $\|\mathbf{t}\| \leq \varrho$. □

Recall that an operator $\mathbf{t} = (t_1, t_2) \in \mathscr{L}^2(\mathfrak{E})$ is said to be self-adjoint if $\mathbf{t}^* = (t_1^*, t_2^*) = (t_1, t_2) = \mathbf{t}$.

Theorem 2.2. *If* $\mathbf{t} = (t_1, t_2) \in \mathscr{L}^2(\mathfrak{E})$ *is self-adjoint, then*

$$\|\mathbf{t}\| = \sup\left\{\left(\left|\psi\left(\langle\xi, t_1\xi\rangle\right)\right|^2 + \left|\psi\left(\langle\xi, t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} : \xi \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}) \textit{ and } \psi\left(|\xi|\right) = 1\right\}.$$

Proof. Let $M = \sup\left\{\left(\left|\psi\left(\langle\xi, t_1\xi\rangle\right)\right|^2 + \left(\left|\psi\left(\left\langle\xi, t_j\xi\right\rangle\right)\right|^2\right)\right)^{\frac{1}{2}} : \xi \in \mathfrak{E}, \psi \in \varpi(\mathfrak{A}) \text{ and } \psi\left(|\xi|\right) = 1\right\}$. If $\psi \in \varpi(\mathfrak{A})$ and $\mathbf{t}$ is a self-adjoint, then by using the Cauchy-Schwartz inequality

$$\left|\psi\left(\left\langle\xi, t_j\xi\right\rangle\right)\right|^2 \leq \psi\left(\left\langle t_j\xi, t_j\xi\right\rangle\right)\psi\left(\langle\xi,\xi\rangle\right) = \psi\left(|t_j\xi|^2\right)\psi\left(|\xi|^2\right)$$

for all $j = 1, 2$. If $\psi(|\xi|) = 1$, then

$$\begin{aligned}\left(\left|\psi\left(\langle \xi, t_1\xi\rangle\right)\right|^2 + \left|\psi\left(\langle \xi, t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} &\leq \left(\psi\left(|t_1\xi|^2\right) + \psi\left(|t_2\xi|^2\right)\right)^{\frac{1}{2}} \\ &\leq |||\mathbf{t}|||.\end{aligned}$$

By taking the supremum over all $\xi \in \mathfrak{E}$ with $\psi(|\xi|) = 1$, we obtain

$$M \leq |||\mathbf{t}|||.$$

For the converse, let $\xi, \zeta \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$. Then

$$\psi\left(\left\langle \xi + \zeta, t_j(\xi + \zeta)\right\rangle\right) - \psi\left(\left\langle \xi - \zeta, t_j(\xi - \zeta)\right\rangle\right) = 4\psi\left(Re\left\langle \zeta, t_j\xi\right\rangle\right) \text{ for } j = 1, 2.$$

Consequently, by Minkowski's inequality we have

$$\begin{aligned}&\left(\left|\psi\left(Re\langle \zeta, t_1\xi\rangle\right)\right|^2 + \left|\psi\left(Re\langle \zeta, t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} \\ &= \left(\tfrac{1}{16}\left|\psi\left(\langle \xi + \zeta, t_1(\xi + \zeta)\rangle\right) - \psi\left(\langle \xi - \zeta, t_1(\xi - \zeta)\rangle\right)\right|^2 + \tfrac{1}{16}\left|\psi\left(\langle \xi + \zeta, t_2(\xi + \zeta)\rangle\right) - \psi\left(\langle \xi - \zeta, t_2(\xi - \zeta)\rangle\right)\right|^2\right)^{\frac{1}{2}} \\ &\leq \tfrac{1}{4}\left(\left[\left|\psi\left(\langle \xi + \zeta, t_1(\xi + \zeta)\rangle\right)\right| + \left|\psi\left(\langle \xi - \zeta, t_1(\xi - \zeta)\rangle\right)\right|\right]^2 + \left[\left|\psi\left(\langle \xi + \zeta, t_2(\xi + \zeta)\rangle\right)\right| + \left|\psi\left(\langle \xi - \zeta, t_2(\xi - \zeta)\rangle\right)\right|\right]^2\right)^{\frac{1}{2}} \\ &= \tfrac{1}{4}\left(\psi\left(|\xi + \zeta|^4\right)\left|\psi\left(\left\langle \tfrac{\xi+\zeta}{\psi(|\xi+\zeta|)}, \tfrac{t_1(\xi+\zeta)}{\psi(|\xi+\zeta|)}\right\rangle\right)\right|^2 + \psi\left(|\xi + \zeta|^4\right)\left|\psi\left(\left\langle \tfrac{\xi+\zeta}{\psi(|\xi+\zeta|)}, \tfrac{t_2(\xi+\zeta)}{\psi(|\xi+\zeta|)}\right\rangle\right)\right|^2\right)^{\frac{1}{2}} \\ &+ \tfrac{1}{4}\left(\psi\left(|\xi - \zeta|^4\right)\left|\psi\left(\left\langle \tfrac{\xi-\zeta}{\psi(|\xi-\zeta|)}, \tfrac{t_1(\xi-\zeta)}{\psi(|\xi-\zeta|)}\right\rangle\right)\right|^2 + \psi\left(|\xi - \zeta|^4\right)\left|\psi\left(\left\langle \tfrac{\xi-\zeta}{\psi(|\xi-\zeta|)}, \tfrac{t_2(\xi-\zeta)}{\psi(|\xi-\zeta|)}\right\rangle\right)\right|^2\right)^{\frac{1}{2}} \\ &= \tfrac{1}{4}\psi\left(|\xi + \zeta|^2\right)\left(\left|\psi\left(\left\langle \tfrac{\xi+\zeta}{\psi(|\xi+\zeta|)}, \tfrac{t_1(\xi+\zeta)}{\psi(|\xi+\zeta|)}\right\rangle\right)\right|^2 + \left|\psi\left(\left\langle \tfrac{\xi+\zeta}{\psi(|\xi+\zeta|)}, \tfrac{t_2(\xi+\zeta)}{\psi(|\xi+\zeta|)}\right\rangle\right)\right|^2\right)^{\frac{1}{2}} \\ &+ \tfrac{1}{4}\psi\left(|\xi - \zeta|^2\right)\left(\left|\psi\left(\left\langle \tfrac{\xi-\zeta}{\psi(|\xi-\zeta|)}, \tfrac{t_1(\xi-\zeta)}{\psi(|\xi-\zeta|)}\right\rangle\right)\right|^2 + \left|\psi\left(\left\langle \tfrac{\xi-\zeta}{\psi(|\xi-\zeta|)}, \tfrac{t_2(\xi-\zeta)}{\psi(|\xi-\zeta|)}\right\rangle\right)\right|^2\right)^{\frac{1}{2}}\end{aligned}$$

Since $\psi\left(\left|\frac{\xi+\zeta}{\psi(|\xi+\zeta|)}\right|\right) = \frac{\psi(|\xi+\zeta|)}{\psi(|\xi+\zeta|)} = 1$, we obtain

$$\begin{aligned}&\left(\left|\psi\left(\left\langle \tfrac{\xi+\zeta}{\psi(|\xi+\zeta|)}, \tfrac{t_1(\xi+\zeta)}{\psi(|\xi+\zeta|)}\right\rangle\right)\right|^2 + \left|\psi\left(\left\langle \tfrac{\xi+\zeta}{\psi(|\xi+\zeta|)}, \tfrac{t_2(\xi+\zeta)}{\psi(|\xi+\zeta|)}\right\rangle\right)\right|^2\right)^{\frac{1}{2}} \leq M \text{ and} \\ &\left(\left|\psi\left(\left\langle \tfrac{\xi-\zeta}{\psi(|\xi-\zeta|)}, \tfrac{t_1(\xi-\zeta)}{\psi(|\xi-\zeta|)}\right\rangle\right)\right|^2 + \left|\psi\left(\left\langle \tfrac{\xi-\zeta}{\psi(|\xi-\zeta|)}, \tfrac{t_2(\xi-\zeta)}{\psi(|\xi-\zeta|)}\right\rangle\right)\right|^2\right)^{\frac{1}{2}} \leq M.\end{aligned}$$

Hence

$$\begin{aligned}&\left(\left|\psi\left(Re\langle \zeta, t_1\xi\rangle\right)\right|^2 + \left|\psi\left(Re\langle \zeta, t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} \leq \tfrac{1}{4}M\left(\psi\left(|\xi + \zeta|^2\right) + \psi\left(|\xi - \zeta|^2\right)\right) \\ &= \tfrac{1}{4}M\psi\left(2|\xi|^2 + 2|\zeta|^2\right) = \tfrac{1}{2}M\psi\left(|\xi|^2 + |\zeta|^2\right).\end{aligned}$$

If $y = \frac{t_j\xi}{\psi\left(|t_j\xi|\right)}$ and $\psi(|\xi|) = 1$, then

$$\begin{aligned}\left(\left|\psi\left(Re\langle \zeta, t_1\xi\rangle\right)\right|^2 + \left|\psi\left(Re\langle \zeta, t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} &\leq \frac{1}{2}M\psi\left(|\xi|^2 + \left|\frac{t_j\xi}{\psi\left(|t_j\xi|\right)}\right|^2\right) \\ &= \frac{M}{2}\psi\left(|\xi|^2 + \frac{|t_j\xi|^2}{\psi\left(|t_j\xi|^2\right)}\right)\end{aligned}$$

$$= \frac{M}{2}\left(\psi\left(|\xi|^2\right)+\frac{\psi\left(|t_j\xi|^2\right)}{\psi\left(|t_j\xi|^2\right)}\right)=M.$$

Therefore

$$\left(\left|\frac{1}{\psi\left(|t_j\xi|\right)}\psi\left(Re\left\langle t_j\xi,t_1\xi\right\rangle\right)\right|^2+\left|\frac{1}{\psi\left(|t_j\xi|\right)}\psi\left(Re\left\langle t_2\xi,t_j\xi\right\rangle\right)\right|^2\right)^{\frac{1}{2}}$$

$$=\left(\left|\frac{1}{\psi\left(|t_1\xi|\right)}Re\left(\psi\left(|t_1\xi|^2\right)\right)\right|^2+\left|\frac{1}{\psi\left(|t_2\xi|\right)}Re\left(\psi\left(|t_2\xi|^2\right)\right)\right|^2\right)^{\frac{1}{2}}$$

$$=\left(\left|\frac{1}{\psi\left(|t_1\xi|\right)}\psi\left(|t_1\xi|^2\right)\right|^2+\left|\frac{1}{\psi\left(|t_2\xi|\right)}\psi\left(|t_2\xi|^2\right)\right|^2\right)^{\frac{1}{2}}$$

$$=\left(\left|\psi\left(|t_1\xi|\right)\right|^2+\left|\psi\left(|t_2\xi|\right)\right|^2\right)^{\frac{1}{2}}\leq M$$

and so $\|\mathbf{t}\|\leq M$. The proof of the theorem is complete. □

The following results are very useful in the sequel which can be found in [12].

Lemma 2.2. *Let $t\in\mathscr{L}(\mathfrak{E})$ and $\psi\in\varpi(\mathfrak{A})$. Then the following are equivalent:*

(a) $\psi\left(\langle\xi,t\xi\rangle\right)=0$ *for every* $\xi\in\mathfrak{E}$ *with* $\psi\left(|\xi|\right)=1$;

(b) $\psi\left(\langle\xi,t\xi\rangle\right)=0$ *for every* $\xi\in\mathfrak{E}$.

Lemma 2.3. *Let $t\in\mathscr{L}(\mathfrak{E})$, then $t=0$ if and only if $\psi\left(\langle\xi,t\xi\rangle\right)=0$ for every $\xi\in\mathfrak{E}$ and $\psi\in\varpi(\mathfrak{A})$.*

Lemma 2.4. *Let $\mathbf{t}=(t_1,t_2)\in\mathscr{L}^2(\mathfrak{E})$. Then for every $\psi\in\varpi(\mathfrak{A})$ and $\xi\in\mathfrak{E}$,*

$$\left(\left|\psi\left(\langle\xi,t_1\xi\rangle\right)\right|^2+\left|\psi\left(\langle\xi,t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}}\leq w_e(\mathbf{t})\psi\left(|\xi|^2\right). \tag{2.3}$$

Proof. For every $\psi\in\varpi(\mathfrak{A})$ and $\xi\in\mathfrak{E}$, we have

$$\frac{1}{\psi\left(|\xi|^4\right)}\left|\psi\left(\left\langle\xi,t_j\xi\right\rangle\right)\right|^2=\left|\psi\left(\left\langle\frac{x}{\psi\left(|\xi|\right)},\frac{t_j\xi}{\psi\left(|\xi|\right)}\right\rangle\right)\right|^2$$

for all $j=1,2$. Hence

$$\left(\frac{1}{\psi\left(|\xi|^4\right)}\left|\psi\left(\langle\xi,t_1\xi\rangle\right)\right|^2+\frac{1}{\psi\left(|\xi|^4\right)}\left|\psi\left(\langle\xi,t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}}\leq w_e(\mathbf{t})$$

and so

$$\left(\left|\psi\left(\langle\xi,t_1\xi\rangle\right)\right|^2+\left|\psi\left(\langle\xi,t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}}\leq w_e(\mathbf{t})\psi\left(|\xi|^2\right).$$

□

Theorem 2.3. *If $\mathbf{t}=(t_1,t_2)\in\mathscr{L}^2(\mathfrak{E})$, then*

$$w_e(\mathbf{t})\leq\|\mathbf{t}\|\leq 2\sqrt{2}w_e(\mathbf{t}). \tag{2.4}$$

or equivalently,

$$\frac{1}{2\sqrt{2}}\|\mathbf{t}\| \leq w_e(\mathbf{t}) \leq \|\mathbf{t}\|. \tag{2.5}$$

Here the constants $\frac{1}{2\sqrt{2}}$ *and 1 are best possible.*

Proof. For every $\psi \in \varpi(\mathfrak{A})$ and $\xi \in \mathfrak{E}$ such that $\psi(|\xi|) = 1$, by Theorem 2.1, we have

$$\left(\left|\psi\left(\xi, t_1\xi\right)\right|^2 + \left|\psi\left(\xi, t_2\xi\right)\right|^2\right)^{\frac{1}{2}} \leq \|\mathbf{t}\|,$$

by taking the supremum, we obtain

$$w_e(\mathbf{t}) \leq \|\mathbf{t}\|.$$

Fix $\xi, \zeta \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$, we have for all $j = 1, 2$ that

$$\begin{aligned} 4\left|\psi\left(\left\langle\zeta, t_j\xi\right\rangle\right)\right| &= |\psi[\left\langle\xi+\zeta, t_j(\xi+\zeta)\right\rangle - \left\langle\xi-\zeta, t_j(\xi-\zeta)\right\rangle \\ &+ i\left\langle\xi+i\zeta, t_j(\xi+i\zeta)\right\rangle - i\left\langle\xi-i\zeta, t_j(\xi-i\zeta)\right\rangle]| \\ &\leq \left|\psi\left(\left\langle\xi+\zeta, t_j(\xi+\zeta)\right\rangle\right)\right| + \left|\psi\left(\left\langle\xi-\zeta, t_j(\xi-\zeta)\right\rangle\right)\right| \\ &+ \left|\psi\left(\left\langle\xi+i\zeta, t_j(\xi+i\zeta)\right\rangle\right)\right| + \left|\psi\left(\left\langle\xi-i\zeta, t_j(\xi-i\zeta)\right\rangle\right)\right|. \end{aligned}$$

Hence for all $j = 1, 2$, we have

$$\begin{aligned} &4\left|\psi\left(\left\langle\zeta, t_j\xi\right\rangle\right)\right| \leq \left(\left|\psi\left(\left\langle\xi+\zeta, t_1(\xi+\zeta)\right\rangle\right)\right|^2 + \left|\psi\left(\left\langle\xi+\zeta, t_j(\xi+\zeta)\right\rangle\right)\right|^2\right)^{\frac{1}{2}} \\ &+\left(\left|\psi\left(\left\langle\xi-\zeta, t_1(\xi-\zeta)\right\rangle\right)\right|^2 + \left|\psi\left(\left\langle\xi-\zeta, t_2(\xi-\zeta)\right\rangle\right)\right|^2\right)^{\frac{1}{2}} \\ &+\left(\left|\psi\left(\left\langle\xi+i\zeta, t_1(\xi+i\zeta)\right\rangle\right)\right|^2 + \left|\psi\left(\left\langle\xi+i\zeta, t_2(\xi+i\zeta)\right\rangle\right)\right|^2\right)^{\frac{1}{2}} \\ &+\left(\left|\psi\left(\left\langle\xi-i\zeta, t_1(\xi-i\zeta)\right\rangle\right)\right|^2 + \left|\psi\left(\left\langle\xi-i\zeta, t_2(\xi-i\zeta)\right\rangle\right)\right|^2\right)^{\frac{1}{2}} \\ \leq\ & w_e(\mathbf{t})\psi\left(|\xi+\zeta|^2\right) + w_e(\mathbf{t})\psi\left(|\xi-\zeta|^2\right) + w_e(\mathbf{t})\psi\left(|\xi+i\zeta|^2\right) + w_e(\mathbf{t})\psi\left(|\xi-i\zeta|^2\right) \\ =\ & w_e(\mathbf{t})\left[\psi\left(|\xi+\zeta|^2\right) + \psi\left(|\xi-\zeta|^2\right) + \psi\left(|\xi+i\zeta|^2\right) + \psi\left(|\xi-i\zeta|^2\right)\right] \\ =\ & w_e(\mathbf{t})\psi\left(2|\xi|^2 + 2|\zeta|^2 + 2|\xi|^2 + 2|i\zeta|^2\right) = 4w_e(\mathbf{t})\left(\psi(|\xi|^2) + \psi(|\zeta|^2)\right). \end{aligned}$$

If $\psi(|\xi|) = \psi(|\zeta|) = 1$, then

$$\left|\psi\left(\left\langle\zeta, t_j\xi\right\rangle\right)\right| \leq 2w_e(\mathbf{t}).$$

Now

$$\left(\left|\psi\left(\langle\zeta, t_1\xi\rangle\right)\right|^2 + \left|\psi\left(\langle\zeta, t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} \leq \left(8w_e^2(\mathbf{t})\right)^{\frac{1}{2}} = 2\sqrt{2}w_e(\mathbf{t}).$$

Taking the supremum over all $\xi, \zeta \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$ such that $\psi(|\xi|) = \psi(|\zeta|) = 1$,we get

$$\|\mathbf{t}\| \leq 2\sqrt{2}w_e(\mathbf{t}).$$

□

Remark 2.1. *(i) Observe that if* $\mathbf{t} = (\mathbf{x}, \mathbf{y})$ *is a self adjoint, then it follows from (2.5) that*

$$\frac{1}{2\sqrt{2}}\left\|\mathbf{x}^2+\mathbf{y}^2\right\|^{\frac{1}{2}} \leq w_e(\mathbf{x},\mathbf{y}) \leq \left\|\mathbf{x}^2+\mathbf{y}^2\right\|^{\frac{1}{2}}. \tag{2.6}$$

(ii) If $\mathbf{t} = \mathbf{x} + i\mathbf{y}$ *is the cartesian decomposition of* $\mathbf{t}$*, then*

$$\begin{aligned} w_e^2(\mathbf{x},\mathbf{y}) &= \sup_{\psi(|x|)=1}\left[\left|\psi\left(\langle\xi,\mathbf{x}\xi\rangle\right)\right|^2+\left|\psi\left(\langle\xi,\mathbf{y}\xi\rangle\right)\right|^2\right] \\ &= \sup_{\psi(|x|)=1}\left|\psi\left(\langle\xi,\mathbf{t}\xi\rangle\right)\right|^2 = w^2(\mathbf{t}). \end{aligned} \tag{2.7}$$

(iii) If $\mathbf{t} = \mathbf{x} + i\mathbf{y}$ *is the cartesian decomposition of* $\mathbf{t}$*, then*

$$\mathbf{t}^*\mathbf{t} + \mathbf{t}\mathbf{t}^* = 2(\mathbf{x}^2+\mathbf{y}^2)$$

and hence it follows from (2.6) that

$$\frac{1}{16}\|\mathbf{t}^*\mathbf{t}+\mathbf{t}\mathbf{t}^*\| \leq w^2(\mathbf{t}) \leq \frac{1}{2}\|\mathbf{t}^*\mathbf{t}+\mathbf{t}\mathbf{t}^*\|.$$

Theorem 2.4. *Let* $\mathbf{t} = \mathbf{x} + i\mathbf{y}$ *be the cartesian decomposition of* $\mathbf{t} \in \mathscr{L}(\mathfrak{E})$*. Then for every* $\mu, \nu \in \mathbb{R}$*,*

$$w_e(\mathbf{x},\mathbf{y}) = \sup_{\mu^2+\nu^2=1}\left\|\mu\mathbf{x}+\nu\mathbf{y}\right\|. \tag{2.8}$$

In particular,

$$\frac{1}{2}\|\mathbf{t}+\mathbf{t}^*\| \leq w_e(\mathbf{x},\mathbf{y}) \ \ \textit{and} \ \ \frac{1}{2}\|\mathbf{t}-\mathbf{t}^*\| \leq w_e(\mathbf{x},\mathbf{y}). \tag{2.9}$$

Proof. First of all, we note that

$$w_e(\mathbf{x},\mathbf{y}) = \sup_{\theta\in\mathbb{R}}\left\|Re\left(e^{i\theta}\mathbf{t}\right)\right\|. \tag{2.10}$$

In fact, $\sup_{\theta\in\mathbb{R}}\left(e^{i\theta}\psi\left(\xi,\mathbf{t}\xi\right)\right)^2 = \left|\psi\left(\langle\xi,\mathbf{x}\xi\rangle\right)\right|^2+\left|\psi\left(\langle\xi,\mathbf{y}\xi\rangle\right)\right|^2$ yields to

$$\sup_{\theta\in\mathbb{R}}\left\|Re\left(e^{i\theta}\mathbf{t}\right)\right\| = \sup_{\theta\in\mathbb{R}} w\left(Re(e^{i\theta}\mathbf{t})\right) = w_e(\mathbf{x},\mathbf{y}).$$

On the other hand, let $\mathbf{t} = \mathbf{x} + i\mathbf{y}$ be the Cartesian decomposition of $\mathbf{t}$. Then

$$\begin{aligned} Re\left(e^{i\theta}\mathbf{t}\right) &= \frac{e^{i\theta}\mathbf{t}+e^{-i\theta}\mathbf{t}^*}{2} \\ &= \frac{1}{2}\left[\left(\cos\theta+i\sin\theta\right)\mathbf{t}+\left(\cos\theta-i\sin\theta\right)\mathbf{t}^*\right] \\ &= \left(\cos\theta\right)\left(\frac{\mathbf{t}+\mathbf{t}^*}{2}\right)-\left(\sin\theta\right)\left(\frac{\mathbf{t}-\mathbf{t}^*}{2i}\right) \\ &= \left(\cos\theta\right)\mathbf{x}-\left(\sin\theta\right)\mathbf{y}. \end{aligned} \tag{2.11}$$

Therefore, by putting $\mu = \cos\theta$ and $\nu = \sin\theta$ in (2.11), we obtain (2.8). Especially, by setting $(\mu,\nu) = (1,0)$ and $(\mu,\nu) = (0,1)$, we reach (2.9). □

Remark 2.2. *By using (2.9), we get some known inequalities:*

(a) $\|\mathbf{t}\| = \left\|\mathbf{x}+i\mathbf{y}\right\| \leq \|\mathbf{x}\| + \left\|\mathbf{y}\right\| \leq 2w_e(\mathbf{x},\mathbf{y})$*. Hence we have* $\frac{1}{2}\|\mathbf{t}\| \leq w_e(\mathbf{x},\mathbf{y})$*.*

(b) *If* $\mathbf{t} = \mathbf{t}^*$*, then* $\mathbf{t} = \mathbf{x}$*. Hence we have* $\|\mathbf{t}\| = \|\mathbf{x}\| \leq w_e(\mathbf{x},\mathbf{y}) \leq \|\mathbf{t}\|$ *and* $w_e(\mathbf{x},\mathbf{y}) = \|\mathbf{t}\|$*.*

(c) *By an easy calculation, we have* $\frac{\mathbf{t}^*\mathbf{t}+\mathbf{t}\mathbf{t}^*}{2} = \mathbf{x}^2 + \mathbf{y}^2$. *Hence*

$$\frac{1}{4}\|\mathbf{t}^*\mathbf{t} + \mathbf{t}\mathbf{t}^*\| = \frac{1}{2}\left\|\mathbf{x}^2 + \mathbf{y}^2\right\| \leq \frac{1}{2}\left(\|\mathbf{x}\|^2 + \left\|\mathbf{y}\right\|^2\right) \leq w_e^2(\mathbf{x}, \mathbf{y}).$$

(d) *Let* $\mu, \nu \in \mathbb{R}$ *such that* $\mu^2 + \nu^2 = 1$. *Then for every* $\xi \in \mathfrak{E}$ *and* $\psi \in \varpi(\mathfrak{A})$ *such that* $\psi(|\xi|) = 1$, *we have*

$$\begin{aligned}\left\|(\mu\mathbf{x} + \nu\mathbf{y})\xi\right\| &= \left\|\begin{bmatrix}\mathbf{x} & \mathbf{y}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mu\xi\\ \nu\xi\end{bmatrix}\right\| \leq \left\|\begin{bmatrix}\mathbf{x} & \mathbf{y}\\ 0 & 0\end{bmatrix}\right\| \leq \left\|\begin{bmatrix}\mathbf{x} & \mathbf{y}\\ 0 & 0\end{bmatrix}\begin{bmatrix}\mathbf{x} & 0\\ \mathbf{y} & 0\end{bmatrix}\right\|^{\frac{1}{2}}\\ &= \left\|\mathbf{x}^2 + \mathbf{y}^2\right\|^{\frac{1}{2}} = \frac{1}{\sqrt{2}}\|\mathbf{t}^*\mathbf{t} + \mathbf{t}\mathbf{t}^*\|^{\frac{1}{2}}.\end{aligned}$$

Hence we have

$$w_e^2(\mathbf{x}, \mathbf{y}) = \sup_{\mu^2+\nu^2=1} \left\|\mu\mathbf{x} + \nu\mathbf{y}\right\|^2 \leq \frac{1}{2}\|\mathbf{t}^*\mathbf{t} + \mathbf{t}\mathbf{t}^*\|.$$

Theorem 2.5. $w_e : \mathscr{L}^2(\mathfrak{E}) \longrightarrow [0, \infty)$ *is defines a norm which is equivalent to the norm on* $\mathscr{L}^2(\mathfrak{E})$.

Proof. Let $w_e(t_1, t_2) = 0$. Then for every $\xi \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$ with $\psi(|\xi|) = 1$, we have

$$\left(\left|\psi\left(\langle\xi, t_1\xi\rangle\right)\right|^2 + \left|\psi\left(\langle\xi, t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} = 0.$$

Hence $\psi(\langle\xi, t_k\xi\rangle) = 0$ for every $\xi \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$ with $\psi(|\xi|) = 1$ and $k = 1, 2$. By Lemma 2.2, $\psi(\langle\xi, t_k\xi\rangle) = 0$ for every $\xi \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$ with $\psi(|\xi|) = 1$, and by Lemma 2.3, $t_k = 0$ for $k = 1, 2$. For every $\mu \in \mathbb{C}$,

$$\begin{aligned}w_e(\mu t_1, \mu t_2) &= \sup_{\psi(|\xi|)=1}\left(\left|\psi\left(\langle x, \mu t_1 x\rangle\right)\right|^2 + \left|\psi\left(\langle x, \mu t_2 x\rangle\right)\right|^2\right)^{\frac{1}{2}}\\ &= \left|\mu\right|\left(\left|\psi\left(\langle\xi, t_1\xi\rangle\right)\right|^2 + \left|\psi\left(\langle\xi, t_2\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} = \left|\mu\right| w_e(t_1, t_2).\end{aligned}$$

Let $t_1, t_2, s_1, s_2 \in \mathscr{L}(\mathfrak{E})$. For every $\xi \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$ with $\psi(|\xi|) = 1$, and by Minkowski's inequality we have

$$\begin{aligned}&\left[\left|\psi\left(\langle\xi, (t_1 + s_1)\xi\rangle\right)\right|^2 + \left|\psi\left(\langle\xi, (t_2 + s_2)\xi\rangle\right)\right|^2\right]^{\frac{1}{2}}\\ &\leq \left[\left(\left|\psi\left(\langle\xi, t_1\xi\rangle\right)\right| + \left|\psi\left(\langle x, s_1 x\rangle\right)\right|\right)^2 + \left(\left|\psi\left(\langle\xi, t_2\xi\rangle\right)\right| + \left|\psi\left(\langle x, s_2 x\rangle\right)\right|\right)^2\right]^{\frac{1}{2}}\\ \leq &\left[\left|\psi\left(\langle\xi, t_1\xi\rangle\right)\right|^2 + \left|\psi\left(\langle\xi, t_2\xi\rangle\right)\right|^2\right]^{\frac{1}{2}} + \left[\left|\psi\left(\langle\xi, s_1\xi\rangle\right)\right|^2 + \left|\psi\left(\langle\xi, s_2\xi\rangle\right)\right|^2\right]^{\frac{1}{2}}\end{aligned}$$

By taking supremum over all $\psi(|\xi|) = 1$,

$$w_e\left(t_1 + s_1, t_2 + s_2\right) \leq w_e(t_1, t_2) + w_e(s_1, s_2).$$

Finally, the equivalence with the norm on $\mathscr{L}^2(\mathfrak{E})$ follows from Theorem 2.3. □

Theorem 2.6. *Let* $t, s \in \mathscr{L}(\mathfrak{E})$. *Then the following assertions hold:*

(i) $w_e(t, s) \geq \max\{w(t), w(s)\}$.

(ii) $w_e(t,s) \geq \frac{1}{\sqrt{2}} w\left(t + e^{i\theta} s\right)$ *for all* $\theta \in \mathbb{R}$.

(iii) $w_e(t,s) \geq \sqrt{\frac{1}{2} w\left(t^2 + e^{i\theta} s^2\right) + \frac{1}{2}\left|w^2(t) - w^2(s)\right|}$ *for all* $\theta \in \mathbb{R}$.

(iv) $w_e(t,s) \geq \sqrt{\frac{1}{2} w\left(ts + st\right)}$.

Proof. (i) This can be readily inferred from the Euclidean numerical radius definition.
(ii) It can be deduced from the definition that for any ξ in set $\mathfrak{E}$ and ψ in the range of the function $\varpi(\mathfrak{E})$, the following holds:

$$\begin{aligned} w_e(t,s) &= \sup_{\psi(|\xi|)=1} \sqrt{\left|\psi\left(\langle \xi, t\xi\rangle\right)\right|^2 + \left|\psi\left(\langle \xi, s\xi\rangle\right)\right|^2} \\ &\geq \sup_{\psi(|\xi|)=1} \sqrt{\frac{1}{2}\left(\left|\psi\left(\langle \xi, t\xi\rangle\right)\right| + \left|\psi\left(\langle \xi, s\xi\rangle\right)\right|\right)^2} \\ &\geq \sup_{\psi(|\xi|)=1} \sqrt{\frac{1}{2}\left|\psi\left(\langle \xi, t\xi\rangle\right) + e^{i\theta}\psi\left(\langle \xi, s\xi\rangle\right)\right|^2} \\ &\geq \frac{1}{\sqrt{2}} w(t + e^{i\theta} s), \quad \text{for all } \theta \in \mathbb{R}. \end{aligned}$$

(iii) From (i), we have

$$\begin{aligned} w_e^2(t,s) &\geq \max\{w^2(t), w^2(s)\} \\ &= \frac{1}{2}\left(w^2(t) + w^2(s)\right) + \frac{1}{2}\left|w^2(t) - w^2(s)\right| \\ &\geq \frac{1}{2}\left(w(t^2) + w(s^2)\right) + \frac{1}{2}\left|w^2(t) - w^2(s)\right| \\ &\geq \frac{1}{2}\left(w(t^2 + e^{i\theta} s^2)\right) + \frac{1}{2}\left|w^2(t) - w^2(s)\right|. \end{aligned}$$

(iv) From (ii), we have $w_e(t,s) \geq \frac{1}{\sqrt{2}} w\left(t+s\right)$ and $w_e(t,s) \geq \frac{1}{\sqrt{2}} w\left(t-s\right)$. Hence

$$\begin{aligned} 2w_e^2(t,s) &\geq \frac{1}{2} w^2(t+s) + \frac{1}{2} w^2(t+s) \\ &\geq \frac{1}{2} w((t+s)^2) + \frac{1}{2} w((t-s)^2) \\ &\geq \frac{1}{2} w\left((t+s)^2 - (t-s)^2\right) = w(ts + st). \end{aligned}$$

and so, $w_e(t,s) \geq \sqrt{\frac{1}{2}\left(ts + st\right)}$. This completes the proof. □

3. Inequalities Related to the Euclidean Operator Radius of Two Operators within a Hilbert C^*-Module.

Within this section, as a result of refining the Euclidean operator radius bounds for a pair of operators, we establish a collection of lower and upper bounds for the numerical radius of t within the context of a Hilbert C^*-module. These bounds stand out in their superior level of strength when compared to the existing ones.

Theorem 3.1. *Let $t, s \in \mathscr{L}(\mathfrak{E})$. Then*

$$w_e(t,s) \geq \sqrt{\frac{1}{4}w\left(t^2+s^2\right)+\frac{1}{4}\left(w^2(t)+w^2(s)\right)+\frac{1}{2}\left|w^2(t)-w^2(s)\right|}.$$

Proof. Setting

$$\begin{aligned}
\eta_1 &= \max\left\{w^2(t), \frac{1}{2}w\left(t^2+s^2\right)\right\} \\
\eta_2 &= \max\left\{w^2(s), \frac{1}{2}w\left(t^2+s^2\right)\right\} \\
\mu_1 &= \left|w^2(t)-\frac{1}{2}w\left(t^2+s^2\right)\right| \\
\mu_2 &= \left|w^2(s)-\frac{1}{2}w\left(t^2+s^2\right)\right|.
\end{aligned}$$

Using inequalities (i) and (iii) from Theorem 2.6, we can derive that

$$\begin{aligned}
w_e^2(t,s) &\geq \max\left\{\eta_1, \eta_2\right\} \\
&= \frac{1}{2}\left(\eta_1+\eta_2\right)+\frac{1}{2}\left|\eta_1-\eta_2\right| \\
&= \frac{1}{4}\left(w^2(t)+w^2(s)\right)+\frac{1}{4}w(t^2+s^2)+\frac{1}{4}\left(\mu_1+\mu_2\right)+\frac{1}{2}\left|\eta_1-\eta_2\right| \\
&\geq \frac{1}{4}\left(w(t^2)+w(s^2)\right)+\frac{1}{4}w(t^2+s^2)+\frac{1}{4}\left(\mu_1+\mu_2\right)+\frac{1}{2}\left|\eta_1-\eta_2\right| \\
&\geq \frac{1}{4}w(t^2+s^2)+\frac{1}{4}w(t^2+s^2)+\frac{1}{4}\left(\mu_1+\mu_2\right)+\frac{1}{2}\left|\eta_1-\eta_2\right| \\
&= \frac{1}{2}w(t^2+s^2)+\frac{1}{4}\left(\mu_1+\mu_2\right)+\frac{1}{2}\left|\eta_1-\eta_2\right| \\
&= \frac{1}{4}w(t^2+s^2)+\frac{1}{4}\left(w^2(t)+w^2(s)\right)+\frac{1}{2}\left|w^2(t)-w^2(s)\right|,
\end{aligned}$$

as required. □

A direct outcome of Theorem 3.1 yields the following outcome.

Corollary 3.1. *If $t, s \in \mathscr{L}(\mathfrak{E})$ are normal, then*

$$\begin{aligned}
w_e(t,s) &\geq \sqrt{\frac{1}{4}\left\|t^2+s^2\right\|+\frac{1}{4}\left(\|t\|^2+\|s\|^2\right)+\frac{1}{2}\left|\|t\|^2-\|s\|^2\right|} \\
&= \sqrt{\frac{1}{4}\left\|t^2+s^2\right\|+\frac{1}{4}\left(\tau_1+\tau_2\right)+\frac{1}{2}\left|\rho_1-\rho_2\right|},
\end{aligned}$$

where

$$\begin{aligned}
\tau_1 &= \max\left\{\|t\|^2, \frac{1}{2}\left\|t^2+s^2\right\|\right\} \\
\tau_2 &= \max\left\{\|s\|^2, \frac{1}{2}\left\|t^2+s^2\right\|\right\} \\
\rho_1 &= \left|\|t\|^2-\frac{1}{2}\left\|t^2+s^2\right\|\right| \quad \textit{and}
\end{aligned}$$

$$\rho_2 = \left| \|s\|^2 - \frac{1}{2}\left\|t^2 + s^2\right\| \right|$$

Corollary 3.1 provides us with the following bound on the numerical radius of a bounded linear operator t in $\mathscr{L}(\mathfrak{E})$.

Corollary 3.2. *Let $t = b + ic \in \mathscr{L}(\mathfrak{E})$ be the Cartesian decomposition of t. Then*

$$w(t) \geq \sqrt{\frac{1}{8}\|t^*t + tt^*\| + \frac{1}{4}\left(\|b\|^2 + \|c\|^2\right) + \left|\|b\|^2 - \|c\|^2\right|}.$$

Proof. In Corollary 3.1, when we set $t = b$ and $s = c$, we arrive the result. □

By utilizing Corollary 3.1, we additionally derive the subsequent lower bound for the numerical radius.

Corollary 3.3. *Let $t = b + ic \in \mathscr{L}(\mathfrak{E})$ be the cartesian decomposition of t. Then*

$$w(t) \geq \sqrt{\frac{1}{8}\|t^*t + tt^*\| + \frac{1}{8}\left(\|b + c\|^2 + \|b - c\|^2\right) + \delta(t)},$$

where $\delta(t) = \frac{1}{4}\left|\|b + c\|^2 - \|b - c\|^2\right|$.

Proof. Within Corollary 3.1, if we assign $t = \frac{b+c}{\sqrt{2}}$ and $s = \frac{b-c}{\sqrt{2}}$, we obtain the outcome. □

Subsequently, we acquire an upper limit for the Euclidean operator radius denoted as $w_e(t,s)$.

Theorem 3.2. *If t and s belong to $\mathscr{L}(\mathfrak{E})$, then for all ν in the interval [0,1], the following inequality holds:*

$$\begin{aligned} w_e(t,s) &\leq \left\|\nu^2 t^*t + (1-\nu)^2 s^*s\right\|^{1/2} \\ &+ \frac{1}{\sqrt{2}}\left[w^2\left((1-\nu)t + \nu s\right) + w^2\left((1-\nu)t - \nu s\right)\right]^{1/2}. \end{aligned}$$

Specifically, when $\nu = \frac{1}{2}$, we have:

$$w_e(t,s) \leq \frac{1}{2}\|t^*t + s^*s\|^{1/2} + \frac{1}{2\sqrt{2}}\left[w^2(t+s) + w^2(t-s)\right]^{1/2}.$$

Proof. For each ξ in the set $\mathfrak{E}$, ψ in the range of $\varpi(\mathfrak{E})$ such that $\psi\left(|\xi|\right) = 1$, and ν within the interval [0,1], the following holds:

$$\left(\left|\psi\left(\langle\xi, t\xi\rangle\right)\right|^2 + \left|\psi\left(\langle\xi, s\xi\rangle\right)\right|^2\right)^{1/2}$$

$$= \left(\left|\nu\psi\left(\langle\xi, t\xi\rangle\right) + (1-\nu)\psi\left(\langle\xi, t\xi\rangle\right)\right|^2 + \left|(1-\nu)\psi\left(\langle\xi, s\xi\rangle\right) + \nu\psi\left(\langle\xi, s\xi\rangle\right)\right|^2\right)^{1/2}$$

$$\leq \left(\nu^2\left|\psi\left(\langle\xi, t\xi\rangle\right)\right|^2 + (1-\nu)^2\left|\psi\left(\langle\xi, s\xi\rangle\right)\right|^2\right)^{1/2}$$

$$+ \left(\nu^2\left|\psi\left(\langle\xi, s\xi\rangle\right)\right|^2 + (1-\nu)^2\left|\psi\left(\langle\xi, t\xi\rangle\right)\right|^2\right)^{1/2}$$

(by Minkowski's Inequality)

$$\leq \left(\nu^2\psi\left(|t\xi|^2\right) + (1-\nu)^2\psi\left(|s\xi|^2\right)\right)^{1/2}$$

$$+\left(\frac{1}{2}\left|\psi\left(\left\langle \xi, ((1-\nu)t+\nu s)\right\rangle \xi\right)\right|^2+\left|\psi\left(\left\langle \xi, ((1-\nu)t-\nu s)\right\rangle \xi\right)\right|^2\right)^{1/2}$$

By considering the supremum across all ξ in the set $\mathfrak{E}$ where $\psi(|\xi|)=1$, we obtain:

$$w_e(t,s)\leq \left\|\nu^2 t^* t+(1-\nu)^2 s^* s\right\|^{1/2}+\left[\frac{1}{2}w^2((1-\nu)t+\nu s)+\frac{1}{2}w^2((1-\nu)t-\nu s)\right]^{1/2}.$$

□

Theorem 3.3. *Let $t,s\in\mathscr{L}(\mathfrak{E})$. Then*

$$\frac{1}{\sqrt{2}}\left[w(t^2+s^2)\right]^{\frac{1}{2}}\leq w_e(t,s)\leq \|t^*t+s^*s\|^{\frac{1}{2}}. \tag{3.1}$$

Proof. We follow a similar argument to the one from [12]. For every $x\in\mathfrak{E}$ with $\psi(|x|)=1$ and $\psi\in\varpi(\mathfrak{A})$, we have

$$\begin{aligned}\left|\psi\left(\langle x,tx\rangle\right)\right|^2+\left|\psi\left(\langle x,sx\rangle\right)\right|^2 &\geq \frac{1}{2}\left(\left|\psi\left(\langle x,tx\rangle\right)+\left|\psi\left(\langle x,tx\rangle\right)\right|\right|\right)^2\\ &\geq \frac{1}{2}\left|\psi\left(\langle x,(t\pm s)x\rangle\right)\right|^2. \end{aligned}\tag{3.2}$$

Taking the supremum in (3.2), we deduce

$$w_e^2(t,s)\geq\frac{1}{2}w^2(t\pm s). \tag{3.3}$$

Utilising the inequality (3.3) and the properties of the numerical radius, we have successively:

$$\begin{aligned}2w_e^2(t,s) &\geq \frac{1}{2}\left[w^2(t+s)+w^2(t-s)\right]\\ &\geq \frac{1}{2}\left[w\left((t+s)^2\right)+w\left((t-s)^2\right)\right]\\ &\geq \frac{1}{2}w\left((t+s)^2+(t-s)^2\right)\\ &= w(t^2+s^2).\end{aligned}$$

which gives the desired inequality (3.1). □

Corollary 3.4. *For any two self-adjoint bounded linear operators t,s on $\mathfrak{E}$, we have*

$$\frac{1}{\sqrt{2}}\left\|t^2+s^2\right\|^{\frac{1}{2}}\leq w_e(t,s)\leq\left\|t^2+s^2\right\|^{\frac{1}{2}}. \tag{3.4}$$

Example 3.1. *Let's consider the Hilbert C^*-module $\mathfrak{E}$ to be the space of continuous functions on the closed interval $[0,1]$ with the inner product defined as the integral of the pointwise product of functions. In this case, we have:*

$\mathfrak{E}$ is a Hilbert C^*-module. Self-adjoint bounded linear operators t and s can be represented as integral operators. For instance, let t be the operator corresponding to multiplication by $2x$ and s the operator corresponding to multiplication by x^2. Now, let's verify the inequality (3.4):

Lower Bound: We calculate the lower bound first:

$$\begin{aligned}\frac{1}{\sqrt{2}}\left\|\left|t^2+s^2\right|\right\|^{\frac{1}{2}} &= \frac{1}{\sqrt{2}}\left(\int_0^1 |2x|^2+|x^2|^2, dx\right)^{\frac{1}{4}} = \frac{1}{\sqrt{2}}\left(\int_0^1 (4x^2+x^4), dx\right)^{\frac{1}{2}} \\ &= \frac{1}{\sqrt{2}}\left(\frac{23}{15}\right)^{\frac{1}{4}} \approx 0.78685.\end{aligned}$$

To find the Euclidean numerical radius $w_e(t,s)$ for the operators $t=2x$ and $s=x^2$ on the Hilbert space $C([0,1])$, which is the space of continuous functions on the closed interval $[0,1]$, we will use the definition of the Euclidean numerical radius:

$$w_e(t,s)=\sup\left\{\left(\left|\psi\left(\langle\xi,\mathbf{x}\xi\rangle\right)\right|^2+\left|\psi\left(\langle\xi,\mathbf{y}\xi\rangle\right)\right|^2\right)^{\frac{1}{2}} : \xi\in\mathfrak{E},\psi\in\varpi(\mathfrak{A}) \text{ and } \psi\left(|\xi|\right)=1\right\}.$$

In this case, ξ can be any function in $C([0,1])$ ψ can be any function in a dense $*$-subalgebra $\varpi(\mathfrak{A})$ such that $\psi(|\xi|)=1$. Let's compute $w_e(t,s)$ step by step:

- *First, calculate $\left|\psi\left(\langle\xi,t\xi\rangle\right)\right|^2$:*

$$\left|\psi\left(\langle\xi,t\xi\rangle\right)\right|^2=\left|\int_0^1\psi(x)\left(2x\xi(x)\right)\,dx\right|^2.$$

- *Next, calculate $\left|\psi\left(\langle\xi,s\xi\rangle\right)\right|^2$:*

$$\left|\psi\left(\langle\xi,t\xi\rangle\right)\right|^2=\left|\int_0^1\psi(x)\left(x^2\xi(x)\right)\,dx\right|^2.$$

Now, consider the expression $\left|\psi\left(\langle\xi,t\xi\rangle\right)\right|^2+\left|\psi\left(\langle\xi,s\xi\rangle\right)\right|^2$:

$$\left|\psi\left(\langle\xi,t\xi\rangle\right)\right|^2+\left|\psi\left(\langle\xi,s\xi\rangle\right)\right|^2=\left|\int_0^1\psi(x)\left(2x\xi(x)\right)\,dx\right|^2+\left|\int_0^1\psi(x)\left(x^2\xi(x)\right)\,dx\right|^2.$$

Now, we want to find the supremum of this expression over all possible choices of ξ and ψ satisfying the given conditions. Let's choose a simple case for ξ and ψ to maximize this expression:

- *Let $\xi(x)=1$ for all $x\in[0,1]$. This represents a constant function.*
- *Let $\psi(x)=1$ for all $x\in[0,1]$. This also represents a constant function.*

Using these choices, we have:

$$\begin{aligned}&\sqrt{\left|\int_0^1\psi(x)\left(2x\xi(x)\right)\,dx\right|^2+\left|\int_0^1\psi(x)\left(x^2\xi(x)\right)\,dx\right|^2} \\ =\ &\sqrt{\left|\int_0^1 1.(2x.1)\,dx\right|^2+\left|\int_0^1 1.(x^2.1)\,dx\right|^2}=\sqrt{1+\frac{1}{9}}=\sqrt{\frac{10}{9}}.\end{aligned}$$

So, for the chosen ξ and ψ, the value of the expression is $\sqrt{\frac{10}{9}}$. Since this value is achieved for the chosen, the supremum is $\sqrt{\frac{10}{9}}=1.054$, and that is the Euclidean numerical radius $w_e(t,s)=\sqrt{\frac{10}{9}}$ for the operators $t=2x$ and $s=x^2$ on $C([0,1])$.

Upper Bound: The upper bound is the norm of the operator $t^2 + s^2$:

$$\left\|t^2 + s^2\right\|^{\frac{1}{2}} = \left(\int_0^1 |(2x)^2 + (x^2)^2|, dx\right)^{\frac{1}{4}}$$
$$= \left(\int_0^1 (4x^2 + x^4), dx\right)^{\frac{1}{4}} = \left(\frac{23}{15}\right)^{\frac{1}{4}} \approx 1.23827.$$

This value is indeed greater than $w_e(t,s)$, which is consistent with the upper bound of $\left\|t^2 + s^2\right\|^{\frac{1}{2}}$. Hence, this example illustrates the theorem's inequality (3.4).

Remark 3.1. *Notice that when both t and s are self-adjoint operators within the Cartesian decomposition of a, we precisely achieve the lower bound as established by Moghaddam and Mirmostafaee in [12, Theorem 3.2] for the numerical radius $w(a)$. Furthermore, since $\frac{1}{4}$ represents an optimal constant in Moghaddam and Mirmostafaee's inequality, it logically follows that $\frac{1}{\sqrt{2}}$ also stands as the most favorable constant in both (3.4) and (3.1), respectively.*

Corollary 3.5. *For any $a \in \mathscr{L}(\mathfrak{E})$ and $\mu, \nu \in \mathbb{C}$, we have*

$$\frac{1}{2} w\left(\mu^2 a^2 + \nu^2 (a^*)^2\right) \leq \left(|\mu|^2 + |\nu|^2\right) w^2(a) \leq \left\| |\mu|^2 a^* a + |\nu|^2 a a^* \right\|. \tag{3.5}$$

Proof. In Theorem 3.3, put $t = \mu a$ and $s = \nu a^*$, we get

$$w_e^2(t,s) = \left(|\mu|^2 + |\nu|^2\right) w^2(a)$$

and

$$w^2(t^2 + s^2) = w\left(|\mu|^2 a^* a + |\nu|^2 a a^*\right),$$

which, by (3.1) implies the desired result (3.5). □

Remark 3.2. *(i) By selecting $\mu = \nu \neq 0$ in equation (3.5), we derive the subsequent inequality for any operator $a \in \mathscr{L}(\mathfrak{E})$:*

$$\frac{1}{4}\left\|a^2 + (a^*)^2\right\| \leq w^2(a) \leq \frac{1}{2}\|a^* a + a a^*\|. \tag{3.6}$$

(ii) By setting $\mu = 1$ and $\nu = i$ in equation (3.5), we establish the following inequality for any operator $a \in \mathscr{L}(\mathfrak{E})$:

$$\frac{1}{4}\left\|a^2 - (a^*)^2\right\| \leq w^2(a). \tag{3.7}$$

Theorem 3.4. *For any $t, s \in \mathscr{L}(\mathfrak{E})$, we have*

$$\frac{1}{\sqrt{2}} \max\{w(t+s), w(t-s)\} \leq w_e(t,s) \leq \frac{1}{\sqrt{2}} \left[w^2(t+s) + w^2(t-s)\right]^{\frac{1}{2}}. \tag{3.8}$$

The constant $\frac{1}{\sqrt{2}}$ is sharp in both inequalities.

Proof. The first inequality follows from (3.3). For the second inequality, we observe that

$$\left|\psi\left(\langle x, tx\rangle\right) \pm \psi\left(\langle x, sx\rangle\right)\right|^2 \leq w^2(t \pm s) \tag{3.9}$$

for every $x \in \mathfrak{E}$ with $\psi\left(|x|\right) = 1$ and $\psi \in \varpi(\mathfrak{A})$.

The inequality (3.9) and the parallelogram identity for complex numbers lead to the following expression for any $x \in \mathfrak{E}$ with $\psi\left(|x|\right) = 1$ and $\psi \in \varpi(\mathfrak{A})$:

$$\begin{aligned} 2\left[\left|\psi\left(\langle x, tx\rangle\right)\right|^2 + \left|\psi\left(\langle x, sx\rangle\right)\right|^2\right] &= \left|\psi\left(\langle x, tx\rangle\right) - \psi\left(\langle x, sx\rangle\right)\right|^2 + \left|\psi\left(\langle x, tx\rangle\right) + \psi\left(\langle x, sx\rangle\right)\right|^2 \\ &\leq w^2(t+s) + w^2(t-s) \end{aligned} \tag{3.10}$$

By taking the supremum in (3.9) over all vectors $x \in \mathfrak{E}$ with $\psi\left(|x|\right) = 1$, we arrive at the desired result (3.11). It's worth noting that the optimality of the constant $\frac{1}{\sqrt{2}}$ is evident, as for $t = s \neq 0$, all terms in (3.11) would yield the same quantity $\sqrt{2}w(t)$. □

Corollary 3.6. *For any pair of self-adjoint operators t and s belonging to $\mathscr{L}(\mathfrak{E})$, the following inequality holds:*

$$\frac{1}{\sqrt{2}} \max\left\{\|t+s\|, \|t-s\|\right\} \leq w_e(t,s) \leq \frac{1}{\sqrt{2}}\left[\|t+s\|^2 + \|t-s\|^2\right]^{\frac{1}{2}}. \tag{3.11}$$

It's important to note that the constant $\frac{1}{\sqrt{2}}$ is the tightest possible value for both of these inequalities.

Theorem 3.5. *Let $t, s \in \mathscr{L}(\mathfrak{E})$. Then*

$$w_e(t,s) \leq \left[w^2(t-s) + 2w(t)w(s)\right]^{\frac{1}{2}}. \tag{3.12}$$

Proof. For any $x \in \mathfrak{E}$ with $\psi\left(|x|\right) = 1$ and $\psi \in \varpi(\mathfrak{A})$, we have

$$\left|\psi\left(\langle x, tx\rangle\right)\right|^2 - 2Re\left[\psi\left(\langle x, tx\rangle\right)\overline{\psi\left(\langle x, sx\rangle\right)}\right] + \left|\psi\left(\langle x, sx\rangle\right)\right|^2 = \left|\psi\left(\langle x, tx\rangle\right) - \psi\left(\langle x, sx\rangle\right)\right|^2 \leq w^2(t-s),$$

giving

$$\begin{aligned} \left|\psi\left(\langle x, tx\rangle\right)\right|^2 + \left|\psi\left(\langle x, sx\rangle\right)\right|^2 &\leq w^2(t-s) + 2Re\left[\psi\left(\langle x, tx\rangle\right)\overline{\psi\left(\langle x, sx\rangle\right)}\right] \\ &\leq w^2(t-s) + 2\left|\psi\left(\langle x, tx\rangle\right)\right|\left|\psi\left(\langle x, sx\rangle\right)\right| \end{aligned} \tag{3.13}$$

for any $x \in \mathfrak{E}$ with $\psi\left(|x|\right) = 1$ and $\psi \in \varpi(\mathfrak{A})$.

Taking the supremum in (3.13) over all $x \in \mathfrak{E}$ with $\psi\left(|x|\right) = 1$ and $\psi \in \varpi(\mathfrak{A})$, we deduce the required inequality (3.12). □

In particular, if t and s are self-adjoint operators, then

$$w_e(t,s) \leq \left[\|t-s\|^2 + \|t+s\|^2\right]^{\frac{1}{2}}. \tag{3.14}$$

The following result provides a different upper bound for the Euclidean operator radius than (3.12).

Theorem 3.6. *Let $t, s \in \mathscr{L}(\mathfrak{E})$. Then we have*

$$w_e(t,s) \leq \left[2\min\{w^2(t), w^2(s)\} + w(t-s)w(t+s)\right]^{\frac{1}{2}}. \tag{3.15}$$

Proof. By employing the parallelogram identity (3.10), we can derive the following equation by taking the supremum over $x \in \mathfrak{E}$ with $\psi(|x|) = 1$ and $\psi \in \varpi(\mathfrak{A})$:

$$2w_e^2(t,s) = w_e^2(t-s, t+s). \tag{3.16}$$

Now, if we apply Theorem 3.5 to the operators $t-s$ and $t+s$ instead of t and s, we can state:

$$w_e^2(t-s, t+s) \leq 4w^2(s) + 2w(t-s)w(t+s)$$

This leads to the following inequality:

$$w_e^2(t,s) \leq 2w^2(s) + w(t-s)w(t+s). \tag{3.17}$$

Furthermore, if we interchange s with t in (3.17), we also obtain:

$$w_e^2(t,s) \leq 2w^2(t) + w(t-s)w(t+s). \tag{3.18}$$

The final conclusion follows from the combination of (3.17) and (3.18). □

An alternative upper limit for the Euclidean operator radius is introduced as follows.

Theorem 3.7. *Let $t, s \in \mathscr{L}(\mathfrak{E})$. Then we have*

$$w_e^2(t,s) \leq \max\{\|t\|^2, \|s\|^2\} + w(s^*t). \tag{3.19}$$

Proof. Firstly, let us observe that for any $y, u, v \in \mathfrak{E}$ and $\psi \in \varpi(\mathfrak{A})$ we have successively

$$\begin{aligned}
&\left\|\psi(\langle y,u\rangle)u + \psi(\langle y,v\rangle)v\right\|^2 = \\
&\left|(\langle y,u\rangle)\right|^2 \psi\left(|u|^2\right) + \left|(\langle y,v\rangle)\right|^2 \psi\left(|v|^2\right) + 2Re\left[\psi(\langle y,u\rangle)\overline{\psi(\langle y,v\rangle)}\psi(\langle u,v\rangle)\right] \\
&\leq \left|(\langle y,u\rangle)\right|^2 \psi\left(|u|^2\right) + \left|(\langle y,v\rangle)\right|^2 \psi\left(|v|^2\right) + 2\left|\psi(\langle y,u\rangle)\right|\left|\psi(\langle y,v\rangle)\right|\left|\psi(\langle u,v\rangle)\right| \\
&\leq \left|(\langle y,u\rangle)\right|^2 \psi\left(|u|^2\right) + \left|(\langle y,v\rangle)\right|^2 \psi\left(|v|^2\right) + \left[\left|\psi(\langle y,u\rangle)\right|^2 + \left|\psi(\langle y,v\rangle)\right|^2\right]\left|\psi(\langle u,v\rangle)\right| \\
&\leq \left[\left|\psi(\langle y,u\rangle)\right|^2 + \left|\psi(\langle y,v\rangle)\right|^2\right]\left(\max\{\psi\left(|u|^2\right), \psi\left(|v|^2\right)\} + \left|\psi(\langle u,v\rangle)\right|\right).
\end{aligned} \tag{3.20}$$

On the other hand,

$$\begin{aligned}
&\left[\left|\psi(\langle y,u\rangle)\right|^2 + \left|\psi(\langle y,v\rangle)\right|^2\right]^2 = \left[\psi(\langle y,u\rangle)\psi(\langle u,y\rangle) + \psi(\langle y,v\rangle)\psi(\langle v,y\rangle)\right]^2 \\
&= \left[\psi(\langle y, \psi(\langle y,u\rangle)u + \psi(\langle y,v\rangle)v\rangle)\right]^2 \\
&\leq \psi\left(|y|^2\right)\psi\left(\left|\psi(\langle y,u\rangle)u + \psi(\langle y,v\rangle)v\right|^2\right).
\end{aligned} \tag{3.21}$$

for any $u, v, y \in \mathfrak{E}$.

Making use of (3.20) and (3.21) we deduce that

$$\left|\psi(\langle y,u\rangle)\right|^2 + \left|\psi(\langle y,v\rangle)\right|^2 \leq \psi\left(|y|^2\right)\left(\max\{\psi\left(|u|^2\right), \psi\left(|v|^2\right)\} + \left|\psi(\langle u,v\rangle)\right|\right) \tag{3.22}$$

for any $u, v, y \in \mathfrak{E}$.
Now, if we apply the inequality (3.22) for $y = x$, $u = tx$, $v = sx$, $x \in \mathfrak{E}$, $\psi(|x|) = 1$, then we can state that

$$\left|\psi(\langle x, tx\rangle)\right|^2 + \left|\psi(\langle x, sx\rangle)\right|^2 \leq \psi\left(|x|^2\right)\left(\max\{\|tx\|^2, \|sx\|^2\} + \left|\psi(\langle tx, sx\rangle)\right|\right) \tag{3.23}$$

for any $x \in \mathfrak{E}$ with $\psi(|x|) = 1$ and $\psi \in \varpi(\mathfrak{A})$.
Taking the supremum over $x \in \mathfrak{E}$ with $\psi(|x|) = 1$ and $\psi \in \varpi(\mathfrak{A})$, we deduce the desired result (3.19). □

Remark 3.3. *In Theorem 3.7, when both t and s are self-adjoint operators and they are equal, the inequality (3.19) simplifies to:*

$$w_e(t, t) \leq \sqrt{2}\|t\|.$$

This demonstrates the optimality of the inequality (3.19).

When data regarding the addition and subtraction of operators t and s is accessible, the subsequent outcome can be applied.

Corollary 3.7. *Let $t, s \in \mathscr{L}(\mathfrak{E})$. Then*

$$w_e^2(t, s) \leq \frac{1}{2}\left[\max\{\|t - s\|^2, \|t + s\|^2\} + w[(t + s)(t^* - s^*)]\right]. \tag{3.24}$$

The constant $\frac{1}{2}$ is best possible.

Proof. This is deduced by applying the inequality (3.19) to the operators $t + s$ and $t - s$ instead of t and s, while also utilizing the identity (3.16). The reason why $\frac{1}{2}$ is the optimal constant in (3.24) becomes apparent when considering the case where $s = t$, with t being a self-adjoint operator. In this scenario, both sides of the inequality (3.24) yield the quantity $2\|t\|^2$. □

Corollary 3.8. *Let $a \in \mathscr{L}(\mathfrak{E})$. Then*

$$w^2(t) \leq \frac{1}{4}\left[\max\{\|t - t^*\|^2, \|t + t^*\|^2\} + w[(t - t^*)(t + t^*)]\right]. \tag{3.25}$$

The constant $\frac{1}{4}$ is best possible.

Proof. If $t = \frac{a+a^*}{2}$ and $s = \frac{a-a^*}{2i}$ are the cartesian decomposition of $a \in \mathscr{L}(\mathfrak{E})$, then

$$w_e^2(t, s) = w^2(a)$$

and

$$w(s^* t) = \frac{1}{4}\left[(a^* + a)(a^* - a)\right].$$

Utilizing (3.19), we deduce (3.25). □

Remark 3.4. *If we choose in (3.19), $t = a$ and $s = a^*$, $a \in \mathscr{L}(\mathfrak{E})$, then we can state that*

$$w^2(a) \leq \frac{1}{2}\left[\|a\|^2 + w(a^2)\right]. \tag{3.26}$$

The constant $\frac{1}{2}$ is best possible.

the following upper bound for the Euclidean radius involving different composite operators

Theorem 3.8. *Let $t, s \in \mathscr{L}(\mathfrak{E})$. Then*

$$w_e^2(t,s) \leq \frac{1}{2}\left[\|t^*t + s^*s\| + \|t^*t - s^*s\|\right] + w(s^*t). \tag{3.27}$$

Proof. We use (3.23) to write that

$$\left|\psi\left(\langle x, tx\rangle\right)\right|^2 + \left|\psi\left(\langle x, sx\rangle\right)\right|^2 \leq \frac{1}{2}\left[\|tx\|^2 + \|sx\|^2 + \left|\|tx\|^2 - \|sx\|^2\right|\right] + \left|\psi\left(\langle sx, tx\rangle\right)\right| \tag{3.28}$$

for any $x \in \mathfrak{E}$ with $\psi(|x|) = 1$ and $\psi \in \varpi(\mathfrak{A})$.
Since $\|tx\|^2 = \psi(\langle x, t^*tx\rangle)$ and $\|sx\|^2 = \psi(\langle x, s^*sx\rangle)$, then (3.28) can be written as

$$\begin{aligned}\left|\psi\left(\langle x, tx\rangle\right)\right|^2 + \left|\psi\left(\langle x, sx\rangle\right)\right|^2 &\leq \frac{1}{2}\left[\psi\left(\langle x, (t^*t + s^*s)x\rangle\right) + \left|\psi\left(\langle x, (t^*t - s^*s)x\rangle\right)\right|\right] \\ &+ \left|\psi\left(\langle sx, tx\rangle\right)\right| \end{aligned} \tag{3.29}$$

for any $x \in \mathfrak{E}$ with $\psi(|x|) = 1$ and $\psi \in \varpi(\mathfrak{A})$.
Taking the supremum in (3.29) over $x \in \mathfrak{E}$ with $\psi(|x|) = 1$ and $\psi \in \varpi(\mathfrak{A})$. and noticing that the operators $t^*t \pm s^*s$ are self-adjoint, we deduce the desired result (3.27). □

Corollary 3.9. *for any operators $t, s \in \mathscr{L}(\mathfrak{E})$, we have*

$$w_e^2(t,s) \leq \frac{1}{2}\left[\|t^*t + s^*s\| + \|t^*s + s^*t\| + w[(t+s)(t^* - s^*)]\right]. \tag{3.30}$$

The constant $\frac{1}{2}$ is best possible.

Proof. If we write (3.27) for $t+s, t-s$ instead of t, s and perform the required calculations then we get

$$w_e^2(t+s, t-s) \leq \frac{1}{2}\left[2\|t^*t + s^*s\| + 2\|t^*s + s^*t\|\right] + w[(t+s)(t^* - s^*)],$$

which, by the identity (3.16) is clearly equivalent with (3.30).
Now, if we choose in (3.30) $t = s$, then we get the inequality $w(t) \leq \|t\|$, which is a sharp inequality. □

Corollary 3.10. *If $t, s \in \mathscr{L}(\mathfrak{E})$ are self-adjoint, then*

$$w_e^2(t,s) \leq \frac{1}{2}\left[\left\|t^2 + s^2\right\| + \left\|t^2 - s^2\right\|\right] + w(st). \tag{3.31}$$

Remark 3.5. *(i) Notably, if we designate t and s as the Cartesian decomposition for the bounded linear operator a in equation (3.31), we can derive the following inequality:*

$$w^2(a) \leq \frac{1}{4}\left[\|t^*t + tt^*\| + \left\|t^2 + (t^*)^2\right\| + w\left((a + a^*)(a^* - a)\right)\right]. \tag{3.32}$$

It's crucial to highlight that the constant $\frac{1}{4}$ is the tightest possible value in this inequality. This is evident since, for a self-adjoint operator a, both sides of (3.32) yield the same value, which is $\|a\|^2$.

(ii) In a different situation, by selecting $t = a$ and $s = a^*$ in (3.27), where $a \in \mathscr{L}(\mathfrak{E})$, we obtain the following inequality:

$$w^2(a) \leq \frac{1}{4}\left[\|a^*a + aa^*\| + \|a^*a - aa^*\|\right] + \frac{1}{2}w(a^2). \tag{3.33}$$

This inequality is characterized as sharp. Equality holds, for instance, when we assume that a is a normal operator, i.e., $a^*a = aa^*$. In this particular case, both sides of (3.33) yield the quantity $\|a\|^2$, as for normal operators, we have $w(a^2) = w^2(a) = \|a\|^2$.

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