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Inequalities for the Davis-Wielandt Radius of Operators in Hilbert C^*-Modules Space

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Abstract. The content of this paper presents a fresh method of studying the Davis–Wielandt radius of bounded operators on Hilbert C^*-modules. Using this method, we arrive at new results that improve upper and lower bounds for the Davis-Wielandt radius and generalize known theorems for bounded operators on Hilbert spaces to bounded adjointable operators on Hilbert C^*-module spaces.

1. Introduction

Let $(\mathcal{H};\langle .,.\rangle)$ be a complex Hilbert space and $B(\mathcal{H})$ by the C^*-algebra of all bounded linear operators on $\mathcal{H}$. For $T \in B(\mathcal{H})$, let T^*, $\|T\|$ and $|T|$ denote the adjoint, the usual operator norm and the absolute value of T (*i.e.* $|T|^2 = T^*T$), respectively. The numerical radius of $T \in B(\mathcal{H})$ is defined by

$$w(T) = \sup\{|\langle Tx, x\rangle| : \ x \in \mathcal{H}, \ \|x\| = 1\}.$$

This concept plays an important role in many different areas, especially mathematics and physics (see [1], [2], [3]). For more on numerical radius inequalities, we refer the reader to a few articles and books (see [4], [5], [6], [7], [8], [2]).

The Davis-Wielandt radius of an operator is an important generalization of the numerical radius. One of the most interesting of these generalizations is the Davis–Wielandt shell and Davis–Wielandt radius of an operator $T \in B(\mathcal{H})$ which are given by

$$DW(T) := \left\{\Big(\langle Tx, x\rangle, \langle Tx, Tx\rangle\Big) : \ x \in \mathcal{H}, \ \|x\| = 1\right\},$$

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and

$$dw(T) = \sup\left\{\left(|\langle Tx, x\rangle|^2 + \|T\|^4\right)^{\frac{1}{2}} : x \in \mathcal{H}, \|x\| = 1\right\},$$

respectively. For more details about These concepts, we refer the reader to a few articles (see [9], [10], [11], [12], [13]). It is easy to verify that the Davis-Wielandt radius $dw(.)$ cannot define a norm on $B(\mathcal{H})$. As a direct consequence, one can easily observe that

$$\max\{w(T), \|T\|^2\} \leq dw(T) \leq \sqrt{w^2(T) + \|T\|^4}, \tag{1.1}$$

for all $T \in B(\mathcal{H})$.

Hilbert C^*-modules are frequently utilized in operator and operator algebra theory, representing a crucial class of examples within the study of the operator C^*-modules. Additionally, the theory of Hilbert C^*-modules is intriguing by itself, engaging with operator algebra theory and integrating various concepts.

A Hilbert C^*-module is an extension of the concept of a Hilbert space. These techniques were first applied by Kaplansky (see [14]). The field of Hilbert C^*-modules was introduced by Rieffel (see [15]) with his induced mathematical models for C^*-algebras and by Paschke (see [16]). Hilbert C^*-modules are helpful in group representation theory, the theory of operator spaces, and operator algebras. Additionally, they are used to investigate the Morita equivalence of C^*-algebras, the C^*-algebra quantum group, and C^*-algebra K-theory (Lance [17]; Wegge-Olsen [18]).

The unique structure of Hilbert C^*-modules allows some inequalities in Hilbert C^*-module spaces to be proven using standard methods. However, varying definitions of some concepts, which extend standard definitions, are often required for studying certain inequalities in Hilbert C^*-modules.

Let's review the definition of a Hilbert module over a C^*-algebra $\mathcal{A}$, as outlined in [17].

Definition 1.1. *([17]). Let $\mathcal{A}$ be a C^*-algebra. An inner-product $\mathcal{A}$-module is a linear space E which is a right $\mathcal{A}$-module (with compatible scalar multiplication:*

$$\lambda(xa) = (\lambda x)a = x(\lambda a) \text{ for all } x \in E,\ a \in \mathcal{A} \text{ and } \lambda \in \mathbb{C}),$$

together with a map $\langle .,. \rangle : E \times E \longrightarrow \mathcal{A}$, which has the following properties:

(i) $\langle x, x\rangle \geqslant 0$, *if* $\langle x, x\rangle = 0$ *then* $x = 0$,

(ii) $\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$,

(iii) $\langle x, ya\rangle = \langle x, y\rangle a$,

(iv) $\langle x, y\rangle^* = \langle y, x\rangle$,

for all $x, y, z \in E$, $a \in \mathcal{A}$, $\alpha, \beta \in \mathbb{C}$.

We can define a norm on E by $\|x\| = \|\langle x, x\rangle\|^{\frac{1}{2}}$. An inner-product $\mathcal{A}$-module that is complete concerning its norm is called a Hilbert $\mathcal{A}$-module, or a Hilbert C^*-module over the C^*-algebra $\mathcal{A}$. Suppose that E, F are Hilbert C^*-modules. We define $\mathcal{L}(E, F)$ to be the set of all maps $T : E \longrightarrow F$

for which there is a map $T^*: F \longrightarrow E$ which satisfies $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x \in E$, $y \in F$. $\mathcal{L}(E,E)$ is simply denoted by $\mathcal{L}(E)$. It is known that $\mathcal{L}(E)$ is a C^*-algebra.

Definition 1.2. *([19, page 89]). A state on a C^*-algebra $\mathcal{A}$ is a positive linear functional on $\mathcal{A}$ of norm one. We denote the state space of $\mathcal{A}$ by $S(\mathcal{A})$.*

Definition 1.3. *([20]). Suppose that E is a Hilbert right $\mathcal{A}$-module. We define the numerical radius of $T \in \mathcal{L}(E)$ by*

$$w_{\mathcal{A}}(T) = \sup\{|\varrho\langle x, Tx\rangle| : \ x \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = 1\}.$$

Definition 1.4. *([21]). Suppose that E is a Hilbert right $\mathcal{A}$-module. We define the Euclidean operator radius of $B, C \in \mathcal{L}(E)$ by*

$$w_{\mathcal{A},e}(B,C) = \sup\left\{\left(|\varrho\langle x, Bx\rangle|^2 + |\varrho\langle x, Cx\rangle|^2\right)^{\frac{1}{2}} : \ x \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = 1\right\}.$$

To establish our main results, we need the following lemmas:

Lemma 1.1. *([20]). $w_{\mathcal{A}}(T) = \|T\|$ for every self-adjoint element of $\mathcal{L}(E)$.*

Lemma 1.2. *([16]). For $T \in \mathcal{L}(E)$, we have*

$$\langle Tx, Tx\rangle \leq \|T\|^2\langle x, x\rangle \ for \ every \ x \in E.$$

Remark 1.1. *It follows from Lemma 1.2 that for every positive linear functional ϱ,*

$$\varrho\langle Tx, Tx\rangle \leq \|T\|^2\varrho\langle x, x\rangle \ for \ every \ x \in E.$$

Lemma 1.3. *([19, page. 88, Theorem 3.3.2]). Let $\mathcal{A}$ be a C^*-algebra. If ϱ is a positive linear functional on $\mathcal{A}$, then*

$$\varrho(a^*) = \overline{\varrho(a)}, \ for \ all \ a \in \mathcal{A}.$$

Lemma 1.4. *([20]). Let $T \in \mathcal{L}(E)$ and $\varrho \in S(\mathcal{A})$. The following statements are equivalent:*
a) $\varrho\langle x, Tx\rangle = 0$ for every $x \in E$ with $\varrho\langle x, x\rangle = 1$,
b) $\varrho\langle x, Tx\rangle = 0$ for every $x \in E$.

Lemma 1.5. *([20]). Let $T \in \mathcal{L}(E)$, then $T = 0$ if and only if $\varrho\langle x, Tx\rangle = 0$ for every $x \in E$ and $\varrho \in S(\mathcal{A})$.*

For $T \in \mathcal{L}(E)$, then T is self-adjoint if and only $\varrho\langle x, Tx\rangle$ is positive for every $x \in E$ and $\varrho \in S(\mathcal{A})$.

Lemma 1.6. *([22]). For $a, b \geq 0$ and $0 \leq \alpha \leq 1$,*

$$a^\alpha b^{1-\alpha} \leq \alpha a + (1-\alpha)b \leq (\alpha a^r + (1-\alpha)b^r)^{\frac{1}{r}} \ for \ r \geq 1.$$

Lemma 1.7. *([20]). Let $T \in \mathcal{L}(E)$, $T \geq 0$ and $x \in E$, then for every $\varrho \in S(\mathcal{A})$*

(i) $(\varrho\langle x, Tx\rangle)^r \leq \|x\|^{2(1-r)}\varrho\langle x, T^r x\rangle$ *for* $r \geq 1$,
(ii) $(\varrho\langle x, Tx\rangle)^r \geq \|x\|^{2(1-r)}\varrho\langle x, T^r x\rangle$ *for* $0 < r \leq 1$.

Lemma 1.8. *([20, page.7]). Let $T \in \mathcal{L}(E)$, then*

$$w_{\mathcal{A}}(T^2) \leq w^2_{\mathcal{A}}(T).$$

Lemma 1.9. *([23, Cauchy-Schwarz inequality]). Let $T \in B(\mathcal{H})$ and $0 \leq \alpha \leq 1$, then*

$$|\langle x, Ty\rangle|^2 \leq \langle x, |T|^{2\alpha} x\rangle\langle y, |T^*|^{2(1-\alpha)} y\rangle,$$

for all any $x, y \in \mathcal{H}$.

The following result is a consequence of Lemma 1.9.

Corollary 1.1. *Let $x \in E$ and $\varrho \in S(\mathcal{A})$, $\varrho\langle ., .\rangle$ is a semi-inner product. Suppose that $T \in \mathcal{L}(E)$ and $0 \leq \alpha \leq 1$, then*

$$|\varrho\langle x, Ty\rangle|^2 \leq \varrho\langle x, |T|^{2\alpha} x\rangle\varrho\langle y, |T^*|^{2(1-\alpha)} y\rangle,$$

for all any $x, y \in E$.

In this section, we offer fresh definitions of the Davis–Wielandt shell and Davis–Wielandtl radius for bounded adjointable operators on Hilbert C^*-modules, which are the natural generalizations of these concepts to operators on Hilbert spaces. Using these definitions, we improve the upper and lower bounds for the Davis-Wielandt radius of bounded adjointable operators on Hilbert C^*-modules. In addition, we also prove some new inequalities on the Davis–Wielandtl radius of bounded operators on Hilbert C^*-modules, which extend some known inequalities in the space of bounded operators on Hilbert spaces.

2. Main results

We start our work with the following definitions.

Definition 2.1. *Suppose that E is a Hilbert right $\mathcal{A}$-module. We define the Davis-Wielandt shell of $T \in \mathcal{L}(E)$ by*

$$DW_{\mathcal{A}}(T) := \left\{\left(\varrho\langle x, Tx\rangle, \varrho\langle x, T^*Tx\rangle\right) : \ x \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = 1\right\}.$$

We also define the Davis–Wielandt radius of $T \in \mathcal{L}(E)$ by

$$dw_{\mathcal{A}}(T) = \sup\left\{\left(|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, T^*Tx\rangle|^2\right)^{\frac{1}{2}} : \ x \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = 1\right\}.$$

Note that our definition is a natural extension of the definition for Davis-Wielandt shell and Davis–Wielandt radius of bounded operators on Hilbert spaces. In fact, in this case, the C^*-algebra $\mathcal{A}$ is the set of complex numbers and $S(\mathcal{A})$ contains only the identity function on the set of complex numbers (see [20]).

Moreover, we assume that $\mathcal{A}$ is a C^*-algebra and E is an inner product $\mathcal{A}$-module.

The following theorem summarizes some basic properties of the Davis-Wielandt radius of bounded adjointable operators on Hilbert C^*-modules.

Theorem 2.1. *Let $T, S \in \mathcal{L}(E)$. Then, the following properties hold:*

(1) $dw_{\mathcal{A}}(T) = 0$ *if and only if* $T = 0$;
(2) $dw_{\mathcal{A}}(U^*TU) = dw_{\mathcal{A}}(T)$ *for any unitary operator* $U : E \longrightarrow E$;
(3) $w_{\mathcal{A},e}(T, T^*T) = dw_{\mathcal{A}}(T)$;
(4) *For all* $\lambda \in \mathbb{C}$ *we have*

$$dw_{\mathcal{A}}(\lambda T) \begin{cases} \leq |\lambda| dw_{\mathcal{A}}(T) \ if \ |\lambda| < 1, \\ = |\lambda| dw_{\mathcal{A}}(T) \ if \ |\lambda| = 1, \\ \geq |\lambda| dw_{\mathcal{A}}(T) \ if \ |\lambda| > 1; \end{cases}$$

(5) $\max\{w_{\mathcal{A}}(T), \|T\|^2\} \leq dw_{\mathcal{A}}(T) \leq \sqrt{w^2_{\mathcal{A}}(T) + \|T\|^4}$;
(6) $dw_{\mathcal{A}}(.)$ *satisfies*

$$dw_{\mathcal{A}}(T+S) \leq \sqrt{2(dw_{\mathcal{A}}(T) + dw_{\mathcal{A}}(S)) + 4(dw_{\mathcal{A}}(T) + dw_{\mathcal{A}}(S))^2}.$$

Proof. (1) Let $dw_{\mathcal{A}}(T) = 0$. Then for every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle Tx, Tx\rangle|^2 = 0.$$

Hence $\varrho\langle x, Tx\rangle = 0$ for every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$. By Lemma 1.4, $\varrho\langle x, Tx\rangle = 0$ for every $x \in E$ and $\varrho \in S(\mathcal{A})$, and by Lemma 1.5, $T = 0$.
If $T = 0$, then clearly $dw_{\mathcal{A}}(T) = 0$.

(2)

$$\begin{aligned} dw_{\mathcal{A}}(U^*TU) &= \sup\left\{\left(|\varrho\langle x, U^*TUx\rangle|^2 + |\varrho\langle U^*TUx, U^*TUx\rangle|^2\right)^{\frac{1}{2}} : x \in E, \varrho \in S(\mathcal{A}), \varrho\langle x, x\rangle = 1\right\} \\ &= \sup\left\{\left(|\varrho\langle Ux, TUx\rangle|^2 + |\varrho\langle TUx, UU^*TUx\rangle|^2\right)^{\frac{1}{2}} : x \in E, \varrho \in S(\mathcal{A}), \varrho\langle x, x\rangle = 1\right\} \\ &= \sup\left\{\left(|\varrho\langle y, Ty\rangle|^2 + |\varrho\langle Ty, Ty\rangle|^2\right)^{\frac{1}{2}} : y \in E, \varrho \in S(\mathcal{A}), \varrho\langle y, y\rangle = 1\right\} = dw_{\mathcal{A}}(T). \end{aligned}$$

(3)

$$\begin{aligned} w_{\mathcal{A},e}(T, T^*T) &= \sup\left\{\left(|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, T^*Tx\rangle|^2\right)^{\frac{1}{2}} : x \in E, \varrho \in S(\mathcal{A}), \varrho\langle x, x\rangle = 1\right\} \\ &= \sup\left\{\left(|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle Tx, Tx\rangle|^2\right)^{\frac{1}{2}} : x \in E, \varrho \in S(\mathcal{A}), \varrho\langle x, x\rangle = 1\right\} \\ &= dw_{\mathcal{A}}(T). \end{aligned}$$

The proof for (4), (5) and (6) is the same as in the classical case of Hilbert spaces (see [11]). □

The following theorem generalizes the Theorem in [13, Theorem 4] for operators on Hilbert C^*-modules.

Theorem 2.2. *Let $T \in \mathcal{L}(E)$. Then*

$$\frac{1}{\sqrt{2}} w_{\mathcal{A}}(T + |T|^2) \leq dw_{\mathcal{A}}(T) \leq \frac{1}{2}\left\|(|T| + |T^*|)^2 + 4|T|^4\right\|^{\frac{1}{2}}. \tag{2.1}$$

Proof. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned} |\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, T^*Tx\rangle|^2 &\geq \frac{1}{2}\Big(|\varrho\langle x, Tx\rangle| + |\varrho\langle x, T^*Tx\rangle|\Big)^2 \\ &\geq \frac{1}{2}|\varrho\langle x, (T + T^*T)x\rangle|^2, \end{aligned}$$

and so

$$\begin{aligned} dw^2_{\mathcal{A}}(T) &= \sup\left\{|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle Tx, Tx\rangle|^2 : \ x \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = 1\right\} \\ &\geq \frac{1}{2}\sup\left\{|\varrho\langle x, (T + T^*T)x\rangle|^2 : \ x \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = 1\right\} \\ &= \frac{1}{2} w^2_{\mathcal{A}}(T + |T|^2). \end{aligned}$$

Now we prove the second inequality. Let $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, then

$$\begin{aligned} |\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, T^*Tx\rangle|^2 &\leq \left(\varrho\langle x, |T|x\rangle^{\frac{1}{2}}\varrho\langle x, |T^*|x\rangle^{\frac{1}{2}}\right)^2 + \varrho\langle x, |T^*T|x\rangle^2 (\textit{by Corollary } 1.1) \\ &\leq \varrho\left\langle x, \frac{|T| + |T^*|}{2}x\right\rangle^2 + \varrho\langle x, |T^*T|x\rangle^2 (\textit{by Lemma } 1.6) \\ &\leq \varrho\left\langle x, \left(\frac{|T| + |T^*|}{2}\right)^2 x\right\rangle + \varrho\langle x, |T^*T|^2 x\rangle (\textit{by Lemma } 1.7) \\ &\leq \varrho\left\langle x, \left[\left(\frac{|T| + |T^*|}{2}\right)^2 + |T^*T|^2\right]x\right\rangle \\ &\leq \frac{1}{4}\left\|(|T| + |T^*|)^2 + 4|T|^4\right\|. \end{aligned}$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$dw_{\mathcal{A}}(T) \leq \frac{1}{2}\left\|(|T| + |T^*|)^2 + 4|T|^4\right\|^{\frac{1}{2}}.$$

□

In the following result, we generalize the above lower bound for the Davis-Wielandt radius for operators on Hilbert C^*-modules.

Theorem 2.3. *Let $T \in \mathcal{L}(E)$. Then*

$$\max_{0\le\alpha\le1} w_{\mathcal{A}}(\sqrt{\alpha}T + \sqrt{1-\alpha}|T|^2) \le dw_{\mathcal{A}}(T). \tag{2.2}$$

Proof. Let $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x,x\rangle = 1$. Then applying the inequality $(ab+cd)^2 \le (a^2+c^2)(b^2+d^2))$ for real numbers a,b,c,d, we have

$$\begin{aligned}\sqrt{\alpha}|\varrho\langle x,Tx\rangle| + \sqrt{1-\alpha}|\varrho\langle x,|T|^2x\rangle| &\le \left(|\varrho\langle x,Tx\rangle|^2 + |\varrho\langle x,|T|^2x\rangle|^2\right)^{\frac{1}{2}}\left((\sqrt{\alpha})^2 + (\sqrt{1-\alpha})^2\right)^{\frac{1}{2}} \\ &= \left(|\varrho\langle x,Tx\rangle|^2 + |\varrho\langle x,|T|^2x\rangle|^2\right)^{\frac{1}{2}}.\end{aligned}$$

Therefore,

$$\begin{aligned}\left(|\varrho\langle x,Tx\rangle|^2 + |\varrho\langle x,|T|^2x\rangle|^2\right)^{\frac{1}{2}} &\ge \sqrt{\alpha}|\varrho\langle x,Tx\rangle| + \sqrt{1-\alpha}|\varrho\langle x,|T|^2x\rangle| \\ &= |\varrho\langle x,\sqrt{\alpha}Tx\rangle| + |\varrho\langle x,\sqrt{1-\alpha}|T|^2x\rangle| \\ &\ge |\varrho\langle x,\sqrt{\alpha}Tx\rangle + \varrho\langle x,\sqrt{1-\alpha}|T|^2x\rangle| \\ &= |\varrho\langle x,(\sqrt{\alpha}T + \sqrt{1-\alpha}|T|^2)x\rangle|.\end{aligned}$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x,x\rangle = 1$, we get

$$dw_{\mathcal{A}}(T) \ge w_{\mathcal{A}}(\sqrt{\alpha}T + \sqrt{1-\alpha}|T|^2).$$

This holds for all $\alpha \in [0,1]$, so we get the desired inequality. □

Remark 2.1. *(i) The inequality (2.2) is a refinement of the lower bound in (2.1) obtained in Theorem 2.2. To see that the refinement is proper, consider $\mathcal{A} = \mathbb{C}$ and $T = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix}$. Then we get,*

$$\frac{1}{\sqrt{2}} w_{\mathcal{A}}(T + |T|^2) = 4.242640687119285 < 4.472121749462181 = \max_{0\le\alpha\le1} w_{\mathcal{A}}(\sqrt{\alpha}T + \sqrt{1-\alpha}|T|^2).$$

(ii) In the same example, Employing the sharp lower bound in (1.1) we get that $dw(T) \ge 4$. By applying the lower bound in (2.2), we get $dw_{\mathcal{A}}(T) \ge 4.47$, which means that the lower bound in (2.2) is better than the one given in (1.1).

We next obtain some lower bounds for the Davis-Wielandt radius of operators in Hilbert C^*-modules space.

Theorem 2.4. *Let $T \in \mathcal{L}(E)$. Then*

(1) $dw^2_{\mathcal{A}}(T) \ge \max\{w^2_{\mathcal{A}}(T) + c^2_{\mathcal{A}}(|T|^2), w^2_{\mathcal{A}}(|T|^2) + c^2_{\mathcal{A}}(T)\}$,
(2) $dw^2_{\mathcal{A}}(T) \ge 2\max\{w_{\mathcal{A}}(T)c_{\mathcal{A}}(|T|^2), w_{\mathcal{A}}(|T|^2)c_{\mathcal{A}}(T)\}$,

where

$$c_{\mathcal{A}}(T) = \inf\{|\varrho\langle x,Tx\rangle| : x \in E, \varrho \in S(\mathcal{A}), \varrho\langle x,x\rangle = 1\}.$$

Proof. 1. Let $T \in \mathcal{L}(E)$. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned} dw^2_{\mathcal{A}}(T) &\geq |\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2 \\ &\geq |\varrho\langle x, Tx\rangle|^2 + c^2_{\mathcal{A}}(|T|^2). \end{aligned}$$

Therefore, taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$dw^2_{\mathcal{A}}(T) \geq w^2_{\mathcal{A}}(T) + c^2_{\mathcal{A}}(|T|^2).$$

Again from $dw^2_{\mathcal{A}}(T) \geq |\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2 \geq c^2_{\mathcal{A}}(T) + |\varrho\langle x, |T|^2x\rangle|^2$.

Taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$dw^2_{\mathcal{A}}(T) \geq c^2_{\mathcal{A}}(T) + w^2_{\mathcal{A}}(|T|^2).$$

This completes the proof of 1.

2. Let $T \in \mathcal{L}(E)$. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2 \geq 2|\varrho\langle x, Tx\rangle||\varrho\langle x, |T|^2x\rangle|,$$

and so,

$$dw^2_{\mathcal{A}}(T) \geq 2|\varrho\langle x, Tx\rangle||\varrho\langle x, |T|^2x\rangle| \geq 2|\varrho\langle x, Tx\rangle|c_{\mathcal{A}}(|T|^2).$$

Taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$dw^2_{\mathcal{A}}(T) \geq 2w_{\mathcal{A}}(T)c_{\mathcal{A}}(|T|^2).$$

Again from $|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2 \geq 2|\varrho\langle x, Tx\rangle||\varrho\langle x, |T|^2x\rangle|$ we have

$$dw^2_{\mathcal{A}}(T) \geq 2c_{\mathcal{A}}(T)|\varrho\langle x, |T|^2x\rangle|.$$

Taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$dw^2_{\mathcal{A}}(T) \geq 2c_{\mathcal{A}}(T)w_{\mathcal{A}}(|T|^2).$$

This completes the proof. □

Remark 2.2. (i) *It is clear that the inequality obtained in Theorem 2.4 (1) improves on the first inequality in Theorem 2.1 (5).*

(ii) *Also, both the inequalities in [10, Theorem 2.4] follow from Theorem 2.4 by considering $\mathcal{A} = \mathbb{C}$.*

The following result shows that Theorem 2.1 in [12] is true for bounded operators on Hilbert C*-modules.

Theorem 2.5. *Let $T \in \mathcal{L}(E)$, then*

$$dw^2_{\mathcal{A}}(T) \leqslant w^2_{\mathcal{A}}(T - |T|^2) + 2\|T\|^2 w_{\mathcal{A}}(T). \tag{2.3}$$

Proof. Let $T \in \mathcal{L}(E)$. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$,

$$\begin{aligned} |\varrho\langle x, Tx\rangle|^2 - 2Re(\varrho\langle x, Tx\rangle\overline{\varrho\langle x, |T|^2x\rangle}) + |\varrho\langle x, |T|^2x\rangle|^2 &= |\varrho\langle x, Tx\rangle - \varrho\langle x, |T|^2x\rangle|^2 \\ &= |\varrho\langle x, (T - |T|^2)x\rangle|^2 \\ &\leqslant w^2_{\mathcal{A}}(T - |T|^2). \end{aligned}$$

Thus,

$$\begin{aligned} |\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2 &\leqslant w^2_{\mathcal{A}}(T - |T|^2) + 2\varrho\langle Tx, Tx\rangle Re(\varrho\langle x, Tx\rangle) \\ &\leqslant w^2_{\mathcal{A}}(T - |T|^2) + 2\|T\|^2\varrho\langle x, x\rangle|\varrho\langle x, Tx\rangle| \\ &\leqslant w^2_{\mathcal{A}}(T - |T|^2) + 2\|T\|^2 w_{\mathcal{A}}(T). \end{aligned}$$

By taking supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$dw^2_{\mathcal{A}}(T) \leqslant w^2_{\mathcal{A}}(T - |T|^2) + 2\|T\|^2 w_{\mathcal{A}}(T).$$

□

In the following theorem, we obtain an upper bound for the Davis-Wielandt radius of operators in $\mathcal{L}(E)$.

Theorem 2.6. *Let $T \in \mathcal{L}(E)$. Then*

$$dw^2_{\mathcal{A}}(T) \leq \sup_{\theta\in\mathbb{R}} \|e^{i\theta}T + |T|^2\|^2 - 2c_{\mathcal{A}}(T)m^2_{\mathcal{A}}(T), \tag{2.4}$$

where $m_{\mathcal{A}} = \inf\{\varrho\langle Tx, Tx\rangle^{\frac{1}{2}} : \ x \in E, \ \varrho \in S(\mathcal{A}), \ \varrho\langle x, x\rangle = 1\}$

Proof. Let $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$. Then there exists $\theta \in \mathbb{R}$ such that $|\varrho\langle x, Tx\rangle| = e^{i\theta}\varrho\langle x, Tx\rangle$. Now,

$$\begin{aligned} |\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2 &= \varrho\langle x, e^{i\theta}Tx\rangle^2 + \varrho\langle x, |T|^2x\rangle^2 \\ &= (\varrho\langle x, e^{i\theta}Tx\rangle + \varrho\langle x, |T|^2x\rangle)^2 - 2\varrho\langle x, e^{i\theta}Tx\rangle\varrho\langle x, |T|^2x\rangle \\ &= \varrho\langle x, (e^{i\theta}T + |T|^2)x\rangle^2 - 2\varrho\langle x, e^{i\theta}Tx\rangle\varrho\langle x, |T|^2x\rangle \\ &\leq \|e^{i\theta}T + |T|^2\|^2 - 2|\varrho\langle x, Tx\rangle|\varrho\langle x, |T|^2x\rangle. \end{aligned}$$

Therefore,

$$2|\varrho\langle x, Tx\rangle|\varrho\langle Tx, Tx\rangle + |\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2 \leq \sup_{\theta\in\mathbb{R}} \|e^{i\theta}T + |T|^2\|^2,$$

and so

$$2c_{\mathcal{A}}(T)m^2_{\mathcal{A}}(T) + |\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2 \leq \sup_{\theta\in\mathbb{R}} \|e^{i\theta}T + |T|^2\|^2.$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$2c_{\mathcal{A}}(T)m^2_{\mathcal{A}}(T) + dw^2_{\mathcal{A}}(T) \leq \sup_{\theta\in\mathbb{R}} \|e^{i\theta}T + |T|^2\|^2.$$

□

Remark 2.3. *If we consider $\mathcal{A} = \mathbb{C}$ and $T = \begin{pmatrix} -1 & 0 \\ 0 & -2 \end{pmatrix}$, then it follows from Theorem 2.6 that $dw^2_{\mathcal{A}}(T) \leq 35.99$, whereas Theorem 2.5 gives $dw^2_{\mathcal{A}}(T) \leq 52$. This shows that the upper bound of $dw_{\mathcal{A}}(.)$ obtained in Theorem 2.6 is better than that obtained in Theorem 2.5.*

Next, we obtain the following upper and lower bounds for the Davis-Wielandt radius of operators in $\mathcal{L}(E)$.

Theorem 2.7. *Let $T \in \mathcal{L}(E)$. Then*

$$\frac{1}{2}\Big(w^2_{\mathcal{A}}(T+|T|^2) + c^2_{\mathcal{A}}(T-|T|^2)\Big) \leq dw^2_{\mathcal{A}}(T) \leq \frac{1}{2}\Big(w^2_{\mathcal{A}}(T+|T|^2) + w^2_{\mathcal{A}}(T-|T|^2)\Big). \tag{2.5}$$

Proof. Let $T \in \mathcal{L}(E)$. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned} |\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2 &\leq \frac{1}{2}|\varrho\langle x, Tx\rangle + \varrho\langle x, |T|^2x\rangle|^2 + \frac{1}{2}|\varrho\langle x, Tx\rangle - \varrho\langle x, |T|^2x\rangle|^2 \\ &\leq \frac{1}{2}|\varrho\langle x, (T+|T|^2)x\rangle|^2 + \frac{1}{2}|\varrho\langle x, (T-|T|^2)x\rangle|^2 \\ &\leq \frac{1}{2}\Big(w^2_{\mathcal{A}}(T+|T|^2) + w^2_{\mathcal{A}}(T-|T|^2)\Big). \end{aligned}$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$dw^2_{\mathcal{A}}(T) \leq \frac{1}{2}\Big(w^2_{\mathcal{A}}(T+|T|^2) + w^2_{\mathcal{A}}(T-|T|^2)\Big).$$

Again

$$\begin{aligned} |\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2 &= \frac{1}{2}\Big(|\varrho\langle x, Tx\rangle| + |\varrho\langle x, |T|^2x\rangle|\Big)^2 + \frac{1}{2}\Big(|\varrho\langle x, Tx\rangle| - |\varrho\langle x, |T|^2x\rangle|\Big)^2 \\ &\geq \frac{1}{2}\Big|\varrho\langle x, Tx\rangle + \varrho\langle x, |T|^2x\rangle\Big|^2 + \frac{1}{2}\Big|\varrho\langle x, Tx\rangle - \varrho\langle x, |T|^2x\rangle\Big|^2 \\ &= \frac{1}{2}\Big|\varrho\langle x, (T+|T|^2)x\rangle\Big|^2 + \frac{1}{2}\Big|\varrho\langle x, (T-|T|^2)x\rangle\Big|^2 \\ &\geq \frac{1}{2}\Big|\varrho\langle x, (T+|T|^2)x\rangle\Big|^2 + \frac{1}{2}c^2_{\mathcal{A}}(T-|T|^2). \end{aligned}$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$dw^2_{\mathcal{A}}(T) \geq \frac{1}{2}\Big(w^2_{\mathcal{A}}(T+|T|^2) + c^2_{\mathcal{A}}(T-|T|^2)\Big).$$

Hence, it completes the proof. □

Remark 2.4. *We remark that the inequality obtained in Theorem 2.7 generalizes the inequality in [10, Theorem 2.8] for operators on Hilbert C^*- modules.*

We are ready to state an extension of Theorem 2.16 in [12].

Theorem 2.8. *Let $T \in \mathcal{L}(E)$, then*

$$dw^2_{\mathcal{A}}(T) \leqslant \max\left\{w_{\mathcal{A}}(T), w_{\mathcal{A}}(|T|^2)\right\}\left(w_{\mathcal{A}}(|T|^2+|T|^4)+2w_{\mathcal{A}}(T^*|T|^2)\right)^{\frac{1}{2}}. \tag{2.6}$$

Proof. Firstly, let us observe that for any $y, u, v \in E$ and $\varrho \in S(\mathcal{A})$ we have successively

$$\begin{aligned}
&\varrho\left(\left\langle \varrho(\langle u,y\rangle)u+\varrho(\langle v,y\rangle)v, \varrho(\langle u,y\rangle)u+\varrho(\langle v,y\rangle)v\right\rangle\right)\\
= \;& |\varrho(\langle y,u\rangle)|^2\varrho(\langle u,u\rangle)+|\varrho(\langle y,v\rangle)|^2\varrho(\langle v,v\rangle)\\
&+2Re\left(\varrho(\langle y,u\rangle)\overline{\varrho(\langle y,v\rangle)}\varrho(\langle u,v\rangle)\right)\\
\leq \;& |\varrho(\langle y,u\rangle)|^2\varrho(\langle u,u\rangle)+|\varrho(\langle y,v\rangle)|^2\varrho(\langle v,v\rangle)\\
&+2\left(|\varrho(\langle y,u\rangle)||\varrho(\langle y,v\rangle)||\varrho(\langle u,v\rangle)|\right)\\
\leq \;& |\varrho(\langle y,u\rangle)|^2\varrho(\langle u,u\rangle)+|\varrho(\langle y,v\rangle)|^2\varrho(\langle v,v\rangle)+|\varrho(\langle y,u\rangle)|^2|\varrho(\langle u,v\rangle)|\\
&+|\varrho(\langle y,v\rangle)|^2|\varrho(\langle u,v\rangle)|\\
\leq \;& \max\left\{|\varrho(\langle y,u\rangle)|^2, |\varrho(\langle y,v\rangle)|^2\right\}\left(\varrho(\langle u,u\rangle)+\varrho(\langle v,v\rangle)+2|\varrho(\langle u,v\rangle)|\right).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
&\left(|\varrho(\langle y,u\rangle)|^2+|\varrho(\langle y,v\rangle)|^2\right)^2\\
= \;& \left(\varrho(\langle y,u\rangle)\varrho(\langle u,y\rangle)+\varrho(\langle y,v\rangle)\varrho(\langle v,y\rangle)\right)^2\\
= \;& \left(\varrho(\langle y, \varrho(\langle u,y\rangle)u+\varrho(\langle v,y\rangle)v\rangle)\right)^2\\
\leq \;& \varrho(\langle y,y\rangle)\varrho\left(\left\langle \varrho(\langle u,y\rangle)u+\varrho(\langle v,y\rangle)v, \varrho(\langle u,y\rangle)u+\varrho(\langle v,y\rangle)v\right\rangle\right)\\
\leq \;& \varrho(\langle y,y\rangle)\max\left\{|\varrho(\langle y,u\rangle)|^2, |\varrho(\langle y,v\rangle)|^2\right\}\left(\varrho(\langle u,u\rangle)+\varrho(\langle v,v\rangle)+2|\varrho(\langle u,v\rangle)|\right),
\end{aligned}$$

for any $y, u, v \in E$.

Thus,

$$|\varrho(\langle y,u\rangle)|^2+|\varrho(\langle y,v\rangle)|^2 \leq (\varrho\langle y,y\rangle)^{\frac{1}{2}}\max\left\{|\varrho(\langle y,u\rangle)|, |\varrho(\langle y,v\rangle)|\right\}\left(\varrho(\langle u,u\rangle)+\varrho(\langle v,v\rangle)+2|\varrho(\langle u,v\rangle)|\right)^{\frac{1}{2}}, \tag{2.7}$$

for any $y, u, v \in E$.

Now, if we apply the inequality (2.7) for $y = x$, $u = Tx$ and $v = |T|^2x$ with $\varrho\langle x, x\rangle = 1$ then we can state that

$$\begin{aligned}
|\varrho(\langle x, Tx\rangle)|^2 + |\varrho(\langle x, |T|^2x\rangle)|^2 &\leq (\varrho\langle x, x\rangle)^{\frac{1}{2}} \max\left\{|\varrho(\langle x, Tx\rangle)|, |\varrho(\langle x, |T|^2x\rangle)|\right\} \\
&\quad \left(\varrho(\langle Tx, Tx\rangle) + \varrho(\langle |T|^2x, |T|^2x\rangle) + 2|\varrho(\langle Tx, |T|^2x\rangle)|\right)^{\frac{1}{2}} \\
&\leq \max\left\{w_{\mathcal{A}}(T), w_{\mathcal{A}}(|T|^2)\right\}\left(\varrho(\langle x, (|T|^2 + |T|^4)x\rangle) + 2|\varrho(\langle x, T^*|T|^2x\rangle)|\right)^{\frac{1}{2}} \\
&\leq \max\left\{w_{\mathcal{A}}(T), w_{\mathcal{A}}(|T|^2)\right\}\left(w_{\mathcal{A}}(|T|^2 + |T|^4) + 2w_{\mathcal{A}}(T^*|T|^2)\right)^{\frac{1}{2}}.
\end{aligned}$$

By taking supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$dw^2_{\mathcal{A}}(T) \leqslant \max\left\{w_{\mathcal{A}}(T), w_{\mathcal{A}}(|T|^2)\right\}\left(w_{\mathcal{A}}(|T|^2 + |T|^4) + 2w_{\mathcal{A}}(T^*|T|^2)\right)^{\frac{1}{2}}.$$

□

We derive the following result from the inequality (2.7).

Theorem 2.9. *Let $T \in \mathcal{L}(E)$, then*

$$dw^2_{\mathcal{A}}(T) \leqslant \|T\| \max\left\{w_{\mathcal{A}}(T), w_{\mathcal{A}}(|T|^2)\right\}\left(1 + \|T\|^2 + 2w_{\mathcal{A}}(T)\right)^{\frac{1}{2}}. \tag{2.8}$$

Proof. Let $T \in \mathcal{L}(E)$. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$. Choosing in (2.7) $y = Tx$, $u = x$ and $v = Tx$, we have

$$\begin{aligned}
&|\varrho(\langle Tx, x\rangle)|^2 + |\varrho(\langle Tx, Tx\rangle)|^2 \\
= \ &|\varrho(\langle x, Tx\rangle)|^2 + |\varrho(\langle Tx, Tx\rangle)|^2 \\
\leq \ &(\varrho\langle Tx, Tx\rangle)^{\frac{1}{2}} \max\left\{|\varrho(\langle Tx, x\rangle)|, |\varrho(\langle Tx, Tx\rangle)|\right\}\left(\varrho(\langle x, x\rangle) + \varrho(\langle Tx, Tx\rangle) + 2|\varrho(\langle x, Tx\rangle)|\right)^{\frac{1}{2}} \\
\leq \ &\|T\|(\varrho\langle x, x\rangle)^{\frac{1}{2}} \max\left\{|\varrho(\langle x, Tx\rangle)|, |\varrho(\langle x, |T|^2x\rangle)|\right\}\left(1 + \varrho(\langle x, |T|^2x\rangle) + 2|\varrho(\langle x, Tx\rangle)|\right)^{\frac{1}{2}} \\
\leq \ &\|T\| \max\left\{w_{\mathcal{A}}(T), w_{\mathcal{A}}(|T|^2)\right\}\left(1 + \|T\|^2 + 2w_{\mathcal{A}}(T)\right)^{\frac{1}{2}}.
\end{aligned}$$

By taking supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$dw^2_{\mathcal{A}}(T) \leqslant \|T\| \max\left\{w_{\mathcal{A}}(T), w_{\mathcal{A}}(|T|^2)\right\}\left(1 + \|T\|^2 + 2w_{\mathcal{A}}(T)\right)^{\frac{1}{2}}.$$

□

In the following theorem, we obtain an upper bound for the Davis-Wielandt radius of bounded adjointable operators on Hilbert C^*-module spaces.

Theorem 2.10. *Let $T \in \mathcal{L}(E)$. Then*

$$dw^2_{\mathcal{A}}(T) \leq \max\{\|T\|^2, \|T\|^4\} + w_{\mathcal{A}}(T^*|T|^2). \tag{2.9}$$

Proof. Firstly, let us observe that for any $y, u, v \in E$ and $\varrho \in S(\mathcal{A})$ we have successively

$$\begin{aligned}
&\varrho\Big(\Big\langle \varrho(\langle u, y\rangle)u + \varrho(\langle v, y\rangle)v, \varrho(\langle u, y\rangle)u + \varrho(\langle v, y\rangle)v\Big\rangle\Big)\\
= \;& |\varrho(\langle y, u\rangle)|^2\varrho(\langle u, u\rangle) + |\varrho(\langle y, v\rangle)|^2\varrho(\langle v, v\rangle) + 2Re\Big(\varrho(\langle y, u\rangle)\overline{\varrho(\langle y, v\rangle)}\varrho(\langle u, v\rangle)\Big)\\
\leq \;& |\varrho(\langle y, u\rangle)|^2\varrho(\langle u, u\rangle) + |\varrho(\langle y, v\rangle)|^2\varrho(\langle v, v\rangle) + 2\Big(|\varrho(\langle y, u\rangle)||\varrho(\langle y, v\rangle)||\varrho(\langle u, v\rangle)|\Big)\\
\leq \;& |\varrho(\langle y, u\rangle)|^2\varrho(\langle u, u\rangle) + |\varrho(\langle y, v\rangle)|^2\varrho(\langle v, v\rangle) + \Big(|\varrho(\langle y, u\rangle)|^2 + |\varrho(\langle y, v\rangle)|^2\Big)|\varrho(\langle u, v\rangle)|\\
\leq \;& \Big(|\varrho(\langle y, u\rangle)|^2 + |\varrho(\langle y, v\rangle)|^2\Big)\Big(\max\{\varrho(\langle u, u\rangle), \varrho(\langle v, v\rangle)\} + |\varrho(\langle u, v\rangle)|\Big).
\end{aligned} \tag{2.10}$$

On the other hand,

$$\begin{aligned}
&\Big(|\varrho(\langle y, u\rangle)|^2 + |\varrho(\langle y, v\rangle)|^2\Big)^2\\
= \;& \Big(\varrho(\langle y, u\rangle)\varrho(\langle u, y\rangle) + \varrho(\langle y, v\rangle)\varrho(\langle v, y\rangle)\Big)^2\\
= \;& \Big(\varrho(\langle y, \varrho(\langle u, y\rangle)u + \varrho(\langle v, y\rangle)v\rangle)\Big)^2\\
\leq \;& \varrho(\langle y, y\rangle)\varrho\Big(\Big\langle \varrho(\langle u, y\rangle)u + \varrho(\langle v, y\rangle)v, \varrho(\langle u, y\rangle)u + \varrho(\langle v, y\rangle)v\Big\rangle\Big),
\end{aligned} \tag{2.11}$$

for any $y, u, v \in E$.

Making use of (2.10) and (2.11) we deduce that

$$|\varrho(\langle y, u\rangle)|^2 + |\varrho(\langle y, v\rangle)|^2 \leq \varrho(\langle y, y\rangle)\Big(\max\{\varrho(\langle u, u\rangle), \varrho(\langle v, v\rangle)\} + |\varrho(\langle u, v\rangle)|\Big), \tag{2.12}$$

for any $y, u, v \in E$.

Now, if we apply the inequality (2.12) for $y = x$, $u = Tx$ and $v = |T|^2x$ with $\varrho\langle x, x\rangle = 1$ then we can state that

$$\begin{aligned}
|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2 \;&\leq\; \varrho\langle x, x\rangle\Big(\max\{\varrho\langle Tx, Tx\rangle, \varrho\langle |T|^2x, |T|^2x\rangle\} + |\varrho\langle Tx, |T|^2x\rangle|\Big)\\
&\leq\; \max\{\|T\|^2\varrho\langle x, x\rangle, \||T|^2\|\varrho\langle x, x\rangle\} + |\varrho\langle Tx, |T|^2x\rangle| \;(\textit{by Remark } 1.1)\\
&\leq\; \max\{\|T\|^2, \|T\|^4\} + |\varrho\langle x, T^*|T|^2x\rangle|\\
&\leq\; \max\{\|T\|^2, \|T\|^4\} + w_{\mathcal{A}}(T^*|T|^2),
\end{aligned}$$

for any $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$.
By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$dw^2_{\mathcal{A}}(T) \le \max\{\|T\|^2, \|T\|^4\} + w_{\mathcal{A}}(T^*|T|^2).$$

□

Remark 2.5. *It is clear that the inequality obtained in Theorem 2.10 improves on the first inequality in [12, Theorem 2.13].*

Example 2.1. *If we consider $\mathcal{A} = \mathbb{C}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then it follows from Theorem 2.10 that $dw^2_{\mathcal{A}}(T) \le 5.999$, whereas Theorem 2.13 in [12] gives $dw^2(T) \le 6.828$. This shows that the upper bound of $dw_{\mathcal{A}}(.)$ obtained in Theorem 2.10 is better than that obtained in Theorem 2.13 in [12].*

The results providing other bound for $dw_{\mathcal{A}}(.)$ may be stated as follows:

Theorem 2.11. *Let $T \in \mathcal{L}(E)$. Then*

$$dw^2_{\mathcal{A}}(T) \le \frac{1}{2}\Big(w_{\mathcal{A}}(|T|^2 + |T|^4) + w_{\mathcal{A}}(|T|^2 - |T|^4)\Big) + w_{\mathcal{A}}(T^*|T|^2). \tag{2.13}$$

Proof. Let $T \in \mathcal{L}(E)$. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$. Choosing in (2.12) $y = x$, $u = Tx$ and $v = |T|^2x$, we have

$$\begin{aligned}
|\varrho\langle x, Tx\rangle|^2 &+ |\varrho\langle x, |T|^2x\rangle|^2 \le \varrho\langle x, x\rangle\Big(\max\{\varrho\langle Tx, Tx\rangle, \varrho\langle |T|^2x, |T|^2x\rangle\} + |\varrho\langle Tx, |T|^2x\rangle|\Big) \\
&= \frac{1}{2}\Big(\varrho\langle Tx, Tx\rangle + \varrho\langle |T|^2x, |T|^2x\rangle + \Big|\varrho\langle Tx, Tx\rangle - \varrho\langle |T|^2x, |T|^2x\rangle\Big|\Big) + |\varrho\langle x, T^*|T|^2x\rangle| \\
&= \frac{1}{2}\Big(\varrho\langle x, (|T|^2 + |T|^4)x\rangle + \Big|\varrho\langle x, (|T|^2 - |T|^4)x\rangle\Big|\Big) + |\varrho\langle x, T^*|T|^2x\rangle| \\
&\le \frac{1}{2}\Big(w_{\mathcal{A}}(|T|^2 + |T|^4) + w_{\mathcal{A}}(|T|^2 - |T|^4)\Big) + w_{\mathcal{A}}(T^*|T|^2).
\end{aligned}$$

By taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$dw^2_{\mathcal{A}}(T) \le \frac{1}{2}\Big(w_{\mathcal{A}}(|T|^2 + |T|^4) + w_{\mathcal{A}}(|T|^2 - |T|^4)\Big) + w_{\mathcal{A}}(T^*|T|^2).$$

□

We derive the following result from the inequality (2.12).

Theorem 2.12. *Let $T \in \mathcal{L}(E)$. Then*

$$dw^2_{\mathcal{A}}(T) \le \|T\|^2\Big(\max\{1, \|T\|^2\} + w_{\mathcal{A}}(T)\Big). \tag{2.14}$$

Proof. Let $T \in \mathcal{L}(E)$. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x,x\rangle = 1$. Choosing in (2.12) $y = Tx$, $u = x$ and $v = Tx$, we have

$$\begin{aligned}
|\varrho\langle Tx,x\rangle|^2 + |\varrho\langle Tx,Tx\rangle|^2 &\leq \varrho\langle Tx,Tx\rangle\left(\max\{\varrho\langle x,x\rangle,\varrho\langle Tx,Tx\rangle\} + |\varrho\langle x,Tx\rangle|\right)\\
&\leq \|T\|^2\varrho\langle x,x\rangle\left(\max\{1,\|T\|^2\varrho\langle x,x\rangle\} + |\varrho\langle x,Tx\rangle|\right)\\
&= \|T\|^2\left(\max\{1,\|T\|^2\} + |\varrho\langle x,Tx\rangle|\right)\\
&\leq \|T\|^2\left(\max\{1,\|T\|^2\} + w_{\mathcal{A}}(T)\right).
\end{aligned}$$

Thus,

$$|\varrho\langle x,Tx\rangle|^2 + |\varrho\langle x,|T|^2x\rangle|^2 \leq \|T\|^2\left(\max\{1,\|T\|^2\} + w_{\mathcal{A}}(T)\right).$$

Taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x,x\rangle = 1$ yields that

$$dw^2_{\mathcal{A}}(T) \leq \|T\|^2\left(\max\{1,\|T\|^2\} + w_{\mathcal{A}}(T)\right).$$

□

The following theorem generalizes Theorem 6 in [13] for operators on Hilbert C^*-modules.

Theorem 2.13. *Let $T \in \mathcal{L}(E)$, $r \geqslant 1$ and $0 \leqslant \alpha \leqslant 1$, then*

$$dw^{2r}_{\mathcal{A}}(T) \leqslant 2^{r-1}\left\|\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r} + |T^*T|^{2r}\right\|. \tag{2.15}$$

Proof. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x,x\rangle = 1$, we have

$$\begin{aligned}
|\varrho\langle x,Tx\rangle|^2 &\leqslant \varrho\langle x,|T|^{2\alpha}x\rangle\varrho\langle x,|T^*|^{2(1-\alpha)}x\rangle \ (\textit{by Corollary } 1.1)\\
&\leqslant \varrho\langle x,|T|^2x\rangle^{\alpha}\varrho\langle x,|T^*|^2x\rangle^{1-\alpha} \ (\textit{by Lemma } 1.7)\\
&\leqslant \left(\alpha\varrho\langle x,|T|^2x\rangle^r + (1-\alpha)\varrho\langle x,|T^*|^2x\rangle^r\right)^{\frac{1}{r}} \ (\textit{by Lemma } 1.6)\\
&\leqslant \left(\alpha\varrho\langle x,|T|^{2r}x\rangle + (1-\alpha)\varrho\langle x,|T^*|^{2r}x\rangle\right)^{\frac{1}{r}} \ (\textit{by Lemma } 1.7)\\
&= \left(\varrho\langle x,(\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r})x\rangle\right)^{\frac{1}{r}}.
\end{aligned}$$

Thus,

$$|\varrho\langle x,Tx\rangle|^{2r} \leqslant \varrho\langle x,(\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r})x\rangle,$$

Also, since T^*T is selfadjoint then we have

$$|\varrho\langle x,T^*Tx\rangle|^{2r} \leqslant \varrho\langle x,|T^*T|^{2r}x\rangle.$$

By the convexity of the function $f(t) = t^r$ on $[0, \infty)$, we have

$$\begin{aligned}(|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, T^*Tx\rangle|^2)^r &\leqslant 2^{r-1}(|\varrho\langle x, Tx\rangle|^{2r} + |\varrho\langle x, T^*Tx\rangle|^{2r}) \\ &\leqslant 2^{r-1}\varrho\langle x, (\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r} + |T^*T|^{2r})x\rangle.\end{aligned}$$

By taking supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$dw^{2r}_{\mathcal{A}}(T) \leqslant 2^{r-1}\left\|\alpha|T|^{2r} + (1-\alpha)|T^*|^{2r} + |T^*T|^{2r}\right\|.$$

□

The next result shows that Theorem 5 in [13] is true for operators on Hilbert C^*-modules.

Theorem 2.14. *Let $T \in \mathcal{L}(E)$, $r \geqslant 2$ and $0 \leqslant \alpha \leqslant 1$, then*

$$dw^{2r}_{\mathcal{A}}(T) \leqslant \frac{2^{\frac{r}{2}}}{4}\left\||T|^{2r\alpha} + |T^*|^{2r(1-\alpha)} + |T^*T|^{2r\alpha} + |T^*T|^{2r(1-\alpha)}\right\|. \tag{2.16}$$

Proof. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned}|\varrho\langle x, Tx\rangle| &\leqslant \varrho\langle x, |T|^{2\alpha}x\rangle^{\frac{1}{2}}\varrho\langle x, |T^*|^{2(1-\alpha)}x\rangle^{\frac{1}{2}} \ (by\ Corollary\ 1.1) \\ &\leqslant \left(\frac{\varrho\langle x, |T|^{2\alpha}x\rangle^r + \varrho\langle x, |T^*|^{2(1-\alpha)}x\rangle^r}{2}\right)^{\frac{1}{r}} \ (by\ Lemma\ 1.6) \\ &\leqslant \left(\frac{\varrho\langle x, |T|^{2r\alpha}x\rangle + \varrho\langle x, |T^*|^{2r(1-\alpha)}x\rangle}{2}\right)^{\frac{1}{r}} \ (by\ Lemma\ 1.7) \\ &\leqslant 2^{-\frac{1}{r}}\Big(\varrho\langle x, (|T|^{2r\alpha} + |T^*|^{2r(1-\alpha)})x\rangle\Big)^{\frac{1}{r}}.\end{aligned}$$

Thus,

$$|\varrho\langle x, Tx\rangle|^r \leqslant \frac{1}{2}\varrho\langle x, (|T|^{2r\alpha} + |T^*|^{2r(1-\alpha)})x\rangle, \tag{2.17}$$

and

$$\begin{aligned}|\varrho\langle x, T^*Tx\rangle| &\leqslant \varrho\langle x, |T^*T|^{2\alpha}x\rangle^{\frac{1}{2}}\varrho\langle x, |T^*T|^{2(1-\alpha)}x\rangle^{\frac{1}{2}} \ (by\ Corollary\ 1.1) \\ &\leqslant \left(\frac{\varrho\langle x, |T^*T|^{2\alpha}x\rangle^r + \varrho\langle x, |T^*T|^{2(1-\alpha)}x\rangle^r}{2}\right)^{\frac{1}{r}} \ (by\ Lemma\ 1.6) \\ &\leqslant \left(\frac{\varrho\langle x, |T^*T|^{2r\alpha}x\rangle + \varrho\langle x, |T^*T|^{2r(1-\alpha)}x\rangle}{2}\right)^{\frac{1}{r}} \ (by\ Lemma\ 1.7) \\ &\leqslant 2^{-\frac{1}{r}}\Big(\varrho\langle x, (|T^*T|^{2r\alpha} + |T^*T|^{2r(1-\alpha)})x\rangle\Big)^{\frac{1}{r}}.\end{aligned}$$

Thus,

$$|\varrho\langle x, T^*Tx\rangle|^r \leqslant \frac{1}{2}\varrho\langle x, (|T^*T|^{2r\alpha} + |T^*T|^{2r(1-\alpha)})x\rangle. \tag{2.18}$$

Adding (2.17) and (2.18) and by the convexity of the function $f(t) = t^{\frac{r}{2}}$ on $[0, \infty)$, we get

$$\begin{aligned}\Big(|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, T^*Tx\rangle|^2\Big)^{\frac{r}{2}} &\leq 2^{\frac{r}{2}-1}\Big[\Big(|\varrho\langle x, Tx\rangle|^2\Big)^{\frac{r}{2}} + \Big(|\varrho\langle x, T^*Tx\rangle|^2\Big)^{\frac{r}{2}}\Big] \\ &= 2^{\frac{r}{2}-1}\Big(|\varrho\langle x, Tx\rangle|^r + |\varrho\langle x, T^*Tx\rangle|^r\Big) \\ &\leq 2^{\frac{r}{2}-2}\Big(\varrho\langle x, (|T|^{2r\alpha} + |T^*|^{2r(1-\alpha)} + |T^*T|^{2r\alpha} + |T^*T|^{2r(1-\alpha)})x\rangle\Big) \\ &\leq 2^{\frac{r}{2}-2}\Big\||T|^{2r\alpha} + |T^*|^{2r(1-\alpha)} + |T^*T|^{2r\alpha} + |T^*T|^{2r(1-\alpha)}\Big\|.\end{aligned}$$

By taking supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$dw^r_{\mathcal{A}}(T) \leqslant 2^{\frac{r}{2}-2}\Big\||T|^{2r\alpha} + |T^*|^{2r(1-\alpha)} + |T^*T|^{2r\alpha} + |T^*T|^{2r(1-\alpha)}\Big\|,$$

as required. □

In the next theorem we obtain upper bounds for the Davis-Wielandt radius of $T \in \mathcal{L}(E)$. First, we need the following lemma.

Lemma 2.1. *([23]). Let $T \in B(H)$, and f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\varrho\langle x, Ty\rangle| \leqslant \|f(|T|)x\|\|g(|T^*|)y\|,$$

for all any $x, y \in H$.

The following result is a consequence of Lemma 2.1.

Corollary 2.1. *For $\varrho \in S(\mathcal{A})$, $\varrho\langle ., .\rangle$ is a semi-inner product. Suppose that $T \in \mathcal{L}(E)$, and f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$|\varrho\langle x, Ty\rangle| \leqslant \varrho\langle f(|T|)x, f(|T|)x\rangle^{\frac{1}{2}}\varrho\langle g(|T^*|)y, g(|T^*|)y\rangle^{\frac{1}{2}},$$

for all any $x, y \in E$.

Theorem 2.15. *Let $T \in \mathcal{L}(E)$ and let f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$dw^{2r}_{\mathcal{A}}(T) \leqslant 2^{r-1}\Big\|\alpha\Big(f^2(|T|)\Big)^{\frac{r}{\alpha}} + (1-\alpha)\Big(g^2(|T^*|)\Big)^{\frac{r}{1-\alpha}} + \alpha\Big(f^2(|T^*T|)\Big)^{\frac{r}{\alpha}} + (1-\alpha)\Big(g^2(|T^*T|)\Big)^{\frac{r}{1-\alpha}}\Big\|, \quad (2.19)$$

for $r \geqslant 1$ and $0 < \alpha < 1$.

Proof. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned} |\varrho\langle x, Tx\rangle|^2 &\leqslant \varrho\langle x, f^2(|T|)x\rangle \varrho\langle x, g^2(|T^*|)x\rangle \ (by\ Corollary\ 2.1) \\ &= \varrho\langle x, \left((f^2(|T|))^{\frac{1}{\alpha}}\right)^{\alpha} x\rangle \varrho\langle x, \left((g^2(|T^*|))^{\frac{1}{1-\alpha}}\right)^{1-\alpha} x\rangle \\ &\leqslant \varrho\langle x, \left(f^2(|T|)\right)^{\frac{1}{\alpha}} x\rangle^{\alpha} \varrho\langle x, \left(g^2(|T^*|)\right)^{\frac{1}{1-\alpha}} x\rangle^{1-\alpha} \ (by\ Lemma\ 1.7) \\ &\leqslant \left(\alpha\varrho\langle x, \left(f^2(|T|)\right)^{\frac{1}{\alpha}} x\rangle^r + (1-\alpha)\varrho\langle x, \left(g^2(|T^*|)\right)^{\frac{1}{1-\alpha}} x\rangle^r\right)^{\frac{1}{r}} \ (by\ Lemma\ 1.6) \\ &\leqslant \left(\alpha\varrho\langle x, \left(f^2(|T|)\right)^{\frac{r}{\alpha}} x\rangle + (1-\alpha)\varrho\langle x, \left(g^2(|T^*|)\right)^{\frac{r}{1-\alpha}} x\rangle\right)^{\frac{1}{r}} \ (by\ Lemma\ 1.7). \end{aligned}$$

Thus,

$$|\varrho\langle x, Tx\rangle|^{2r} \leqslant \varrho\langle x, [\alpha\left(f^2(|T|)\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(g^2(|T^*|)\right)^{\frac{r}{1-\alpha}}]x\rangle,$$

similarly, we have

$$|\varrho\langle x, T^*Tx\rangle|^{2r} \leqslant \varrho\langle x, [\alpha\left(f^2(|T^*T|)\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(g^2(|T^*T|)\right)^{\frac{r}{1-\alpha}}]x\rangle.$$

By convexity of the function $f(t) = t^r$ on $[0, \infty)$, we have

$$\begin{aligned} \left(|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, T^*Tx\rangle|^2\right)^r &\leqslant 2^{r-1}\left(|\varrho\langle x, Tx\rangle|^{2r} + |\varrho\langle x, T^*Tx\rangle|^{2r}\right) \\ &\leqslant 2^{r-1}\Big[\varrho\langle x, [\alpha\left(f^2(|T|)\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(g^2(|T^*|)\right)^{\frac{r}{1-\alpha}} \\ &\quad + \alpha\left(f^2(|T^*T|)\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(g^2(|T^*T|)\right)^{\frac{r}{1-\alpha}}]x\rangle\Big]. \end{aligned}$$

Taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$dw_{\mathcal{A}}^{2r}(T) \leqslant 2^{r-1}\left\|\alpha\left(f^2(|T|)\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(g^2(|T^*|)\right)^{\frac{r}{1-\alpha}} + \alpha\left(f^2(|T^*T|)\right)^{\frac{r}{\alpha}} + (1-\alpha)\left(g^2(|T^*T|)\right)^{\frac{r}{1-\alpha}}\right\|.$$

□

As a consequence of Theorem 2.15, we get the following corollary.

Choosing $\alpha = \frac{1}{2}$, we get:

Corollary 2.2. *Let $T \in \mathcal{L}(E)$ and let f and g be non-negative continuous functions on $[0, \infty)$ satisfying $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then*

$$dw_{\mathcal{A}}^{2r}(T) \leqslant 2^{r-2}\left\|\left(f^2(|T|)\right)^{2r} + \left(g^2(|T^*|)\right)^{2r} + \left(f^2(|T^*T|)\right)^{2r} + \left(g^2(|T^*T|)\right)^{2r}\right\|,$$

for $r \geqslant 1$.

Letting $f(t) = g(t) = t^{\frac{1}{2}}$, we get:

Corollary 2.3. *Let* $T \in \mathcal{L}(E)$. *Then*

$$dw_{\mathcal{A}}^{2r}(T) \leqslant 2^{r-2}\left\| |T|^{2r} + |T^*|^{2r} + 2|T^*T|^{2r} \right\|,$$

for $r \geqslant 1$. *In particular, if we choose* $r = 1$, *we have*

$$dw_{\mathcal{A}}^{2}(T) \leqslant \frac{1}{2}\left\| |T|^{2} + |T^*|^{2} + 2|T|^{4} \right\|.$$

Remark 2.6. *In particular, if we consider* $\mathcal{A} = \mathbb{C}$ *in Corollary 2.3 then we have the inequality in [10, Corollary 2.14].*

To prove the next upper bound we need the following lemma known as Jensen's inequality, obtained from more general results for superquadratic functions [24].

Lemma 2.2. *The following inequality*

$$\left(\frac{1}{n}\sum_{k=1}^{n} a_k\right)^p \leq \frac{1}{n}\sum_{k=1}^{n} a_k^p - \frac{1}{n}\sum_{k=1}^{n}|a_k - \frac{1}{n}\sum_{j=1}^{n} a_j|^p,$$

holds for $p \geq 2$ *and for every finite positive sequence of real numbers* $a_1, a_2, ..., a_n$.

Theorem 2.16. *Let* $T \in \mathcal{L}(E)$. *Then*

$$dw_{\mathcal{A}}^{2r}(T) \leqslant \frac{1}{2}w_{\mathcal{A}}^{2r}(T + |T|^2) + \frac{1}{2}w_{\mathcal{A}}^{2r}(T - |T|^2) - 2^r\|T\|^{2r} \inf_{\varrho\langle x,x\rangle=1, \varrho\in S(\mathcal{A})} |Re(\varrho\langle x, Tx\rangle)|^r, \tag{2.20}$$

holds for every $r \geqslant 2$.

Proof. Let $T \in \mathcal{L}(E)$. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we have

$$\begin{aligned}
&\left(|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2\right)^r \\
\leq\ &\left(\frac{1}{2}|\varrho\langle x, Tx\rangle + \varrho\langle x, |T|^2x\rangle|^2 + \frac{1}{2}|\varrho\langle x, Tx\rangle - \varrho\langle x, |T|^2x\rangle|^2\right)^r \\
=\ &\left(\frac{1}{2}|\varrho\langle x, (T + |T|^2)x\rangle|^2 + \frac{1}{2}|\varrho\langle x, (T - |T|^2)x\rangle|^2\right)^r \\
\leq\ &\frac{1}{2}|\varrho\langle x, (T + |T|^2)x\rangle|^{2r} + \frac{1}{2}|\varrho\langle x, (T - |T|^2)x\rangle|^{2r} \\
&-\frac{1}{2}\left|\frac{1}{2}|\varrho\langle x, (T + |T|^2)x\rangle|^2 - \frac{1}{2}|\varrho\langle x, (T - |T|^2)x\rangle|^2\right|^r \\
&-\frac{1}{2}\left|\frac{1}{2}|\varrho\langle x, (T - |T|^2)x\rangle|^2 - \frac{1}{2}|\varrho\langle x, (T + |T|^2)x\rangle|^2\right|^r (\textit{using Lemma } 2.2) \\
=\ &\frac{1}{2}|\varrho\langle x, (T + |T|^2)x\rangle|^{2r} + \frac{1}{2}|\varrho\langle x, (T - |T|^2)x\rangle|^{2r}
\end{aligned}$$

$$
\begin{aligned}
& -\frac{1}{2^r}\left||\varrho\langle x,(T+|T|^2)x\rangle|^2-|\varrho\langle x,(T-|T|^2)x\rangle|^2\right|^r \\
= \;& \frac{1}{2}|\varrho\langle x,(T+|T|^2)x\rangle|^{2r}+\frac{1}{2}|\varrho\langle x,(T-|T|^2)x\rangle|^{2r}-\frac{1}{2^r}2^{2r}\left|Re(\varrho\langle x,Tx\rangle\overline{\varrho\langle x,|T|^2x\rangle})\right|^r \\
\leq \;& \frac{1}{2}|\varrho\langle x,(T+|T|^2)x\rangle|^{2r}+\frac{1}{2}|\varrho\langle x,(T-|T|^2)x\rangle|^{2r}-2^r\varrho\langle Tx,Tx\rangle^r\left|Re(\varrho\langle x,Tx\rangle)\right|^r \\
\leq \;& \frac{1}{2}|\varrho\langle x,(T+|T|^2)x\rangle|^{2r}+\frac{1}{2}|\varrho\langle x,(T-|T|^2)x\rangle|^{2r}-2^r\|T\|^{2r}\varrho\langle x,x\rangle^r\left|Re(\varrho\langle x,Tx\rangle)\right|^r \\
\leq \;& \frac{1}{2}w_{\mathcal{A}}^{2r}(T+|T|^2)+\frac{1}{2}w_{\mathcal{A}}^{2r}(T-|T|^2)-2^r\|T\|^{2r}\inf_{\varrho\langle x,x\rangle=1}|Re(\varrho\langle x,Tx\rangle)|^r.
\end{aligned}
$$

By taking the supremum over all $x\in E$ and $\varrho\in S(\mathcal{A})$ with $\varrho\langle x,x\rangle=1$, we get

$$
dw_{\mathcal{A}}^{2r}(T)\leq\frac{1}{2}w_{\mathcal{A}}^{2r}(T+|T|^2)+\frac{1}{2}w_{\mathcal{A}}^{2r}(T-|T|^2)-2^r\|T\|^{2r}\inf_{\varrho\langle x,x\rangle=1,\varrho\in S(\mathcal{A})}|Re(\varrho\langle x,Tx\rangle)|^r.
$$

□

Corollary 2.4. *For any Let $T\in\mathcal{L}(E)$ and $r=1$ we have*

$$
dw_{\mathcal{A}}^{2}(T)\leq\frac{1}{2}w_{\mathcal{A}}^{2}(T+|T|^2)+\frac{1}{2}w_{\mathcal{A}}^{2}(T-|T|^2)-2\|T\|^{2}\inf_{\varrho\langle x,x\rangle=1,\varrho\in S(\mathcal{A})}|Re(\varrho\langle x,Tx\rangle)|. \tag{2.21}
$$

Example 2.2. *If we consider $\mathcal{A}=\mathbb{C}$ and $T=\begin{pmatrix}0&2\\0&0\end{pmatrix}$, then it follows from Corollary 2.4 that $dw_{\mathcal{A}}^2(T)\leq 5$, whereas Theorem 2.10 gives $dw_{\mathcal{A}}(T)\leq 5.999$. This shows that the upper bound of $dw_{\mathcal{A}}(.)$ obtained in Corollary 2.4 is better than that obtained in Theorem 2.10.*

In the next theorem, we obtain upper bounds for the Davis-Wielandt radius of $T\in\mathcal{L}(E)$.

First, we need the following lemma.

Lemma 2.3. *([20, page.12]). Let $x,y,z\in E$ with $\varrho\langle z,z\rangle=1$, then*

$$
|\varrho\langle x,z\rangle\varrho\langle z,y\rangle|\leqslant\frac{1}{2}(\varrho\langle x,x\rangle^{\frac{1}{2}}\varrho\langle y,y\rangle^{\frac{1}{2}}+|\varrho\langle x,y\rangle|).
$$

Theorem 2.17. *Let $T\in\mathcal{L}(E)$. Then*

$$
dw_{\mathcal{A}}^{2}(T)\leq\frac{1}{4}\|TT^*+T^*T\|+\frac{1}{2}w_{\mathcal{A}}(T^2)+\|T\|^4. \tag{2.22}
$$

Proof. Let $x\in E$ and $\varrho\in S(\mathcal{A})$ with $\varrho\langle x,x\rangle=1$. Then by using Lemma 2.3, we get

$$
\begin{aligned}
|\varrho\langle x,Tx\rangle|^2+|\varrho\langle x,|T|^2x\rangle|^2 &= |\varrho\langle T^*x,x\rangle\varrho\langle x,Tx\rangle|+\varrho\langle |T|^2x,x\rangle\varrho\langle x,|T|^2x\rangle \\
&\leq \frac{1}{2}(\varrho\langle T^*x,T^*x\rangle^{\frac{1}{2}}\varrho\langle Tx,Tx\rangle^{\frac{1}{2}}+|\varrho\langle T^*x,Tx\rangle|)+\varrho\langle x,|T|^2x\rangle^2 \\
&\leq \frac{1}{4}(\varrho\langle T^*x,T^*x\rangle+\varrho\langle Tx,Tx\rangle)+\frac{1}{2}|\varrho\langle x,T^2x\rangle|+\varrho\langle x,|T|^2x\rangle^2 (\textit{by Lemma } 1.6) \\
&\leq \frac{1}{4}\varrho\langle x,(TT^*+T^*T)x\rangle+\frac{1}{2}|\varrho\langle x,T^2x\rangle|+\varrho\langle x,|T|^4x\rangle (\textit{by Lemma } 1.7) \\
&\leq \frac{1}{4}\|TT^*+T^*T\|+\frac{1}{2}w_{\mathcal{A}}(T^2)+\|T\|^4.
\end{aligned}
$$

Taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

$$dw^2_{\mathcal{A}}(T) \leq \frac{1}{4}\|TT^* + T^*T\| + \frac{1}{2}w_{\mathcal{A}}(T^2) + \|T\|^4.$$

□

Remark 2.7. *Let $\mathcal{A} = \mathbb{C}$. If we take $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ then from theorem 2.17, we get $dw^2_{\mathcal{A}}(T) \leq 5.457$, while the inequalities from theorem 2.17 (i) and theorem 2.17 (ii) in [10, Theorem 2.17] respectively give $dw^2(T) \leq 6$ and $dw^2(T) \leq 5.6$. Thus, the $dw(.)$ obtained in theorem 2.17 is better than the existing ones.*

The next lemma will be used in Theorem 2.18 to obtain a new upper bound for the Davis-Wielandt radius of operators on Hilbert C^*-modules.

Lemma 2.4. *Let $x, y, z \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle z, z\rangle = 1$, $\varrho\langle ., .\rangle$ is a semi-inner product. Then*

$$|\varrho\langle x, z\rangle\varrho\langle z, y\rangle|^r \leqslant \frac{1+\alpha}{2}\varrho\langle x, x\rangle^{\frac{r}{2}}\varrho\langle y, y\rangle^{\frac{r}{2}} + \frac{1-\alpha}{2}|\varrho\langle x, y\rangle|^r,$$

for every $\alpha \in [0, 1]$ and $r \geqslant 1$.

Proof. By Lemma 2.3 and the convexity of t^r, $r \geqslant 1$, we have

$$\begin{aligned}
|\varrho\langle x, z\rangle\varrho\langle z, y\rangle|^r &\leqslant \left(\frac{\varrho\langle x, x\rangle^{\frac{1}{2}}\varrho\langle y, y\rangle^{\frac{1}{2}}}{2} + \frac{|\varrho\langle x, y\rangle|}{2}\right)^r \\
&\leqslant \frac{\varrho\langle x, x\rangle^{\frac{r}{2}}\varrho\langle y, y\rangle^{\frac{r}{2}}}{2} + \frac{|\varrho\langle x, y\rangle|^r}{2} \\
&= \frac{\varrho\langle x, x\rangle^{\frac{r}{2}}\varrho\langle y, y\rangle^{\frac{r}{2}}}{2} + \frac{1}{2}(\alpha|\varrho\langle x, y\rangle|^r + (1-\alpha)|\varrho\langle x, y\rangle|^r) \\
&\leqslant \frac{1+\alpha}{2}\varrho\langle x, x\rangle^{\frac{r}{2}}\varrho\langle y, y\rangle^{\frac{r}{2}} + \frac{1-\alpha}{2}|\varrho\langle x, y\rangle|^r.
\end{aligned}$$

□

Theorem 2.18. *Let $T \in \mathcal{L}(E)$. Then*

$$dw^{2r}_{\mathcal{A}}(T) \leqslant 2^{r-1}\left(\frac{1+\alpha}{4}\||T^*|^{2r} + |T|^{2r}\| + \frac{1-\alpha}{2}w^r_{\mathcal{A}}(T^2) + \frac{1+\alpha}{2}\|T\|^{4r} + \frac{1-\alpha}{2}w^r_{\mathcal{A}}(|T|^4)\right).$$

for every $\alpha \in [0, 1]$ and $r \geqslant 1$.

Proof. For every $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$. Then by using Lemma 2.4, we have

$$\begin{aligned}
|\varrho\langle x, Tx\rangle|^{2r} &= |\varrho\langle T^*x, x\rangle\varrho\langle x, Tx\rangle|^r \\
&\leq \frac{1+\alpha}{2}\varrho\langle T^*x, T^*x\rangle^{\frac{r}{2}}\varrho\langle Tx, Tx\rangle^{\frac{r}{2}} + \frac{1-\alpha}{2}|\varrho\langle T^*x, Tx\rangle|^r \\
&\leq \frac{1+\alpha}{2}\varrho\langle x, |T^*|^{2r}x\rangle^{\frac{1}{2}}\varrho\langle x, |T|^{2r}x\rangle^{\frac{1}{2}} + \frac{1-\alpha}{2}|\varrho\langle x, T^2x\rangle|^r (\text{by Lemma } 1.7) \\
&\leq \frac{1+\alpha}{4}(\varrho\langle x, |T^*|^{2r}x\rangle + \varrho\langle x, |T|^{2r}x\rangle) + \frac{1-\alpha}{2}|\varrho\langle x, T^2x\rangle|^r (\text{by Lemma } 1.6)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1+\alpha}{4}\varrho\langle x, (|T^*|^{2r}+|T|^{2r})x\rangle + \frac{1-\alpha}{2}|\varrho\langle x, T^2x\rangle|^r \\
&\le \frac{1+\alpha}{4}\||T^*|^{2r}+|T|^{2r}\| + \frac{1-\alpha}{2}w^r_{\mathcal{A}}(T^2).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|\varrho\langle x, |T|^2x\rangle|^{2r} &= |\varrho\langle |T|^2x, x\rangle\varrho\langle x, |T|^2x\rangle|^r \\
&\le \frac{1+\alpha}{2}\varrho\langle |T|^2x, |T|^2x\rangle^{\frac{r}{2}}\varrho\langle |T|^2x, |T|^2x\rangle^{\frac{r}{2}} + \frac{1-\alpha}{2}|\varrho\langle |T|^2x, |T|^2x\rangle|^r \\
&= \frac{1+\alpha}{2}\varrho\langle x, |T|^4x\rangle^r + \frac{1-\alpha}{2}|\varrho\langle x, |T|^4x\rangle|^r \\
&\le \frac{1+\alpha}{2}\varrho\langle x, |T|^{4r}x\rangle + \frac{1-\alpha}{2}|\varrho\langle x, |T|^4x\rangle|^r \ (\textit{by Lemma } 1.7) \\
&\le \frac{1+\alpha}{2}\|T\|^{4r} + \frac{1-\alpha}{2}w^r_{\mathcal{A}}(|T|^4).
\end{aligned}
$$

By convexity of the function $f(t) = t^r$ on $[0, \infty)$, we have

$$
\begin{aligned}
\left(|\varrho\langle x, Tx\rangle|^2 + |\varrho\langle x, |T|^2x\rangle|^2\right)^r &\leqslant 2^{r-1}\left(|\varrho\langle x, Tx\rangle|^{2r} + |\varrho\langle x, T^*Tx\rangle|^{2r}\right) \\
&\leqslant 2^{r-1}\Bigg(\frac{1+\alpha}{4}\||T^*|^{2r}+|T|^{2r}\| + \frac{1-\alpha}{2}w^r_{\mathcal{A}}(T^2) \\
&\qquad + \frac{1+\alpha}{2}\|T\|^{4r} + \frac{1-\alpha}{2}w^r_{\mathcal{A}}(|T|^4)\Bigg).
\end{aligned}
$$

Now, taking the supremum over all $x \in E$ and $\varrho \in S(\mathcal{A})$ with $\varrho\langle x, x\rangle = 1$, we get

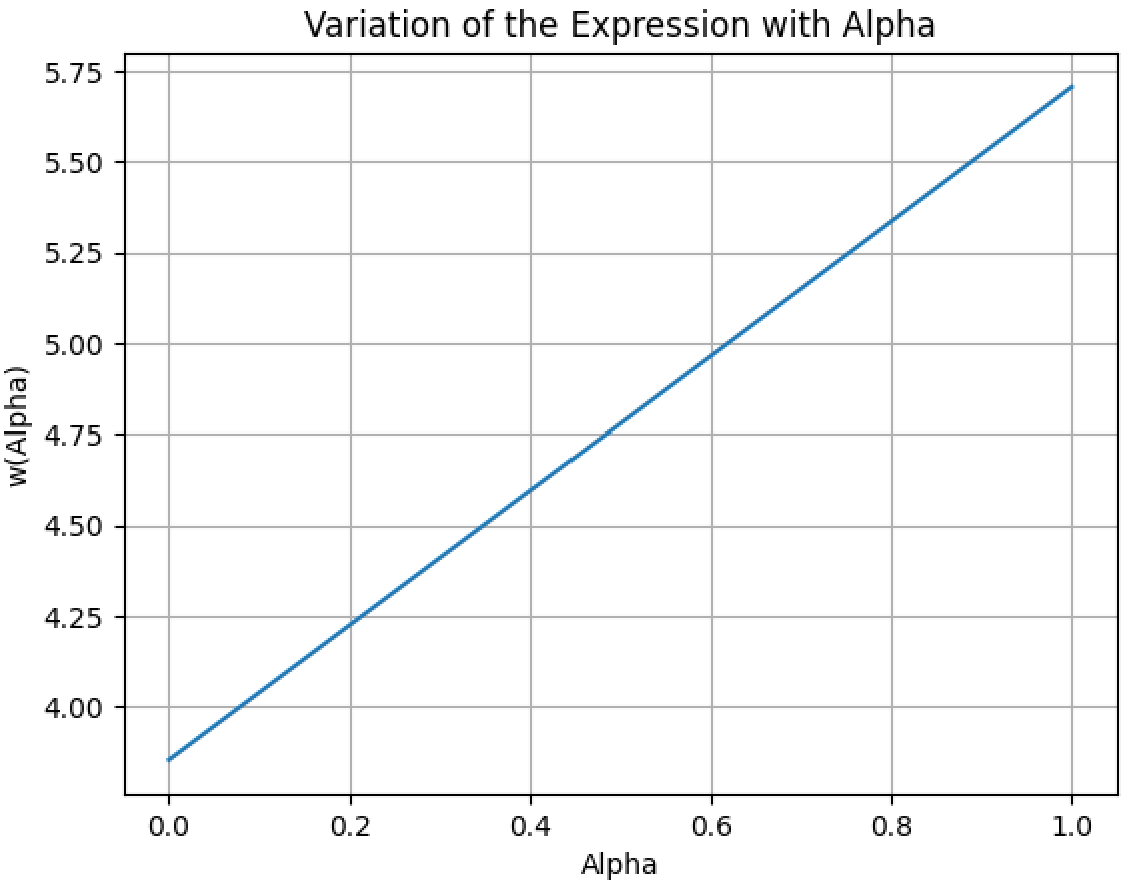
$$
dw^{2r}_{\mathcal{A}}(T) \leqslant 2^{r-1}\left(\frac{1+\alpha}{4}\||T^*|^{2r}+|T|^{2r}\| + \frac{1-\alpha}{2}w^r_{\mathcal{A}}(T^2) + \frac{1+\alpha}{2}\|T\|^{4r} + \frac{1-\alpha}{2}w^r_{\mathcal{A}}(|T|^4)\right).
$$

□

Corollary 2.5. *The particular case $r = 1$ produces the inequality*

$$
dw^2_{\mathcal{A}}(T) \leqslant \frac{1+\alpha}{4}\||T^*|^2+|T|^2\| + \frac{1-\alpha}{2}w_{\mathcal{A}}(T^2) + \frac{1+\alpha}{2}\|T\|^4 + \frac{1-\alpha}{2}w_{\mathcal{A}}(|T|^4).
$$

Remark 2.8. *If we consider $\mathcal{A} = \mathbb{C}$ and $T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$, then from Corollary 2.5, we get $dw^2_{\mathcal{A}}(T) \le 5.707$, whereas the inequalities in [12, Th. 2.1, Th. 2.7, Th. 2.13, Th. 2.14, Th. 2.16, Th. 2.17] respectively gives $dw^2(T) \le$ 6.283, 35.416, 6.828, 6.828, 6.325, 6.58. Thus, the bounds of $dw_{\mathcal{A}}(T)$ obtained in Theorem 2.18 with $r = 1$ is better than the existing ones. This graph shows how the expression $w(\alpha) = \frac{1+\alpha}{4}\||T^*|^2+|T|^2\| + \frac{1-\alpha}{2}w_{\mathcal{A}}(T^2) + \frac{1+\alpha}{2}\|T\|^4 + \frac{1-\alpha}{2}w_{\mathcal{A}}(|T|^4)$ varies as a function of α over the interval $[0, 1]$.*



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