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External Direct Product JU-Algebras

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Abstract. Lingcong and Endam introduced the idea of the direct product of a finite family of B-algebras. In this paper, we introduce the concept of the direct product of an infinite family of JU-algebras, which we call the external direct product. This is a general concept of the direct product in the sense of Lingcong and Endam. The result is based on different types of subsets of JU-algebras. These include JU-subalgebras, JU-ideals, *p*-ideals, *t*-ideals, strong JU-ideals, JU-filters, comparative JU-filters, and implicative JU-filters. Also, we introduce the concept of the weak direct product of JU-algebras. Finally, we provide several fundamental theorems of (anti-)JU-homomorphisms in view of the external direct product JU-algebras.

1. Introduction and Preliminaries

Imai and Iséki introduced two classes of abstract algebras called BCK-algebras and BCI-algebras, which are the two important classes of logical algebras and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras [14, 15]. After that, many authors have explained what BCK-algebras and BCI-algebras numbers are and how they work in different kinds of logical algebras. Their unique properties

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have also been talked about in different articles in the article [4]. In 2002, Neggers and Kim [20] constructed a new algebraic structure. They took some properties from BCI and BCK-algebras and called them B-algebra. Furthermore, Kim and Kim [17] introduced a new notion, called a BG-algebra, which is a generalization of B-algebra. They obtained several isomorphism theorems of BG-algebras and related properties. The notion of KU-algebras was introduced by Prabpayak and Leerawat [21], one of those logical algebras.

The concept of JU-algebras, as a generalization of KU-algebras, was introduced and analyzed in [2, 4]. It have been examined by several researchers, for example, the concept of pseudo-valuations on JU-algebras and have investigated the relationship between pseudo-valuations and ideals of JU-algebras by Ali et al. in 2019 [2]. In 2020, Ansari et al. [4] introduced the concept of p-closure for any nonempty subset J of a JU-algebra $X = (X; *, 0)$ and have investigated some related properties showing that $J_{pc} := \{x \in X \mid x * j \in J \text{ for some } j \in J\}$ is the least closed p-ideal containing J for any JU-ideal J of X. In the same year, Romano [22] introduced and analyzed the concepts of a few new types of JU-ideals of JU-algebras such as closed ideals, ag-ideals, t-ideals, $(\star)$-ideals and associative ideals. In 2021, Ansari [3] related JU-algebras (JU-ideals) with roughness through definitions, examples and results based on lower and upper approximations. In 2022, Romano [23] introduced and analyzed the concept of JU-filters of JU-algebras. Additionally, the concepts of implicative and comparative JU-filters in this class of algebras are introduced and discussed.

The concept of the direct product [24] was first defined in the group and obtained some properties. For example, a direct product of the group is also a group and a direct product of the abelian group is also an abelian group. Then, direct product groups are applied to other algebraic structures. In 2016, Lingcong and Endam [18] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms and obtained related properties, one of which is a direct product of two B-algebras, which is also a B-algebra. Then, they extended the concept of the direct product of B-algebra to the finite family B-algebra, and some of the related properties were investigated. Also, they introduced two canonical mappings of the direct product of B-algebras and we obtained some of their properties [19]. In the same year, Endam and Teves [13] defined the direct product of BF-algebras, 0-commutative BF-algebras, and BF-homomorphism and obtained related properties. In 2018, Abebe [1] introduced the concept of the finite direct product of BRK-algebras and proved that the finite direct product of BRK-algebras is a BRK-algebra. In 2019, Widianto et al. [25] defined the direct product of BG-algebras, 0-commutative BG-algebras, and BG-homomorphism, including related properties of BG-algebras. In 2020, Setiani et al. [24] defined the direct product of BP-algebras, which is equivalent to B-algebras. They obtained the relevant property of the direct product of BP-algebras and then defined the direct product of BP-algebras as applied to finite sets of BP-algebras, finite family 0-commutative BP-algebras, and finite family BP-homomorphisms. In 2021, Kavitha and Gowri [16] defined the direct product of GK-algebra. They derived the finite form of the direct product of GK-algebra and function as well. They investigated and applied the concept of the direct product of GK-algebra in GK-function and

GK-kernel and obtained interesting results. In 2002-2023, Chanmanee et al. conducted research on direct products in various algebras, which can be studied in detail in [6–12].

In this paper, we introduce the concept of the direct product of an infinite family of JU-algebras, which we call the external direct product, which is a general concept of the direct product in the sense of Lingcong and Endam [18], and find the result of the external direct product of special subsets of JU-algebras: JU-subalgebras, JU-ideals, p-ideals, t-ideals, strong JU-ideals, JU-filters, comparative JU-filters, implicative JU-filters. Moreover, we introduce the concept of the weak direct product of JU-algebras. Finally, we discuss several (anti-)JU-homomorphism theorems in view of the external direct product JU-algebras.

We first begin with the definition and example of JU-algebras, which was defined by Ansari et al. [4] in 2020, along with other relevant definitions for the studies in this paper.

Definition 1.1. *[4] An algebra $X = (X; *, 0)$ of type $(2, 0)$ with a single binary operation $*$ is said to be a JU-algebra satisfying the following axioms:*

$$(\forall x, y, z \in X)((y * z) * ((z * x) * (y * x)) = 0), \tag{JU-1}$$

$$(\forall x \in X)(0 * x = x), \tag{JU-2}$$

$$(\forall x, y \in X)(x * y = 0, y * x = 0 \Rightarrow x = y). \tag{JU-3}$$

Example 1.1. *Let $X = \{0, 1, 2, 3, 4, 5, 6, 7\}$ be a set with the Cayley table as follows:*

$*$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	0	0	1	1	4	5	6	7
2	0	0	0	1	4	5	6	7
3	0	0	0	0	4	5	6	7
4	0	0	0	0	0	5	6	7
5	0	0	0	0	0	0	6	7
6	0	0	0	0	0	0	0	7
7	0	0	0	0	0	0	0	0

Then $X = (X; *, 0)$ is a JU-algebra.

Definition 1.2. *[4, 22, 23] A nonempty subset S of a JU-algebra $X = (X; *, 0)$ is called*

(i) *a JU-subalgebra of X if it satisfies the following condition:*

$$(\forall x, y \in S)(x * y \in S), \tag{1.1}$$

(ii) *a JU-ideal of X if it satisfies the following conditions:*

$$\text{the constant } 0 \text{ of } X \text{ is in } S, \tag{1.2}$$

$$(\forall x, y \in X)(x * y \in S, x \in S \Rightarrow y \in S), \tag{1.3}$$

(*iii*) *a p-ideal of X if it satisfies the condition* (1.2) *and the following condition:*

$$(\forall x, y, z \in X)((z * x) * (z * y) \in S, x \in S \Rightarrow y \in S), \tag{1.4}$$

(*iv*) *a t-ideal of X if it satisfies the condition* (1.2) *and the following condition:*

$$(\forall x, y, z \in X)((x * y) * z \in S, y \in S \Rightarrow (x * z) \in S), \tag{1.5}$$

(*v*) *a strong JU-ideal of X if it is a JU-ideal of X which satisfies the following condition:*

$$(\forall x, y \in X)(x * y \in S, y \in S \Rightarrow x \in S), \tag{1.6}$$

(*vi*) *a JU-filter of X if it satisfies the conditions* (1.2) *and* (1.6),

(*vii*) *a comparative JU-filter of X if it satisfies the condition* (1.2) *and the following condition:*

$$(\forall x, y, z \in X)((y * z) * (z * x) \in S, x \in S \Rightarrow y \in S), \tag{1.7}$$

(*viii*) *an implicative JU-filter of X if it satisfies the condition* (1.2) *and the following condition:*

$$(\forall x, y, z \in X)(x * (y * z) \in S, x * y \in S \Rightarrow x * z \in S). \tag{1.8}$$

The concept of JU-homomorphisms was also introduced by Ansari et al. [4].

Let $A = (A; *_A, 0_A)$ and $B = (B; *_B, 0_B)$ be JU-algebras. A map $\varphi : A \to B$ is called a *JU-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(x) *_B \varphi(y))$$

and an *anti-JU-homomorphism* if

$$(\forall x, y \in A)(\varphi(x *_A y) = \varphi(y) *_B \varphi(x)).$$

The *kernel* of φ, denoted by $\ker\varphi$, is defined to be $\{x \in A \mid \varphi(x) = 0_B\}$. The $\ker\varphi$ is a JU-ideal of A, and $\ker\varphi = \{0_A\}$ if and only if φ is injective. An (anti-)JU-homomorphism φ is called an (anti-)JU-monomorphism, (anti-)JU-epimorphism, or (anti-)JU-isomorphism if φ is injective, surjective, or bijective, respectively.

In a JU-algebra $X = (X; *, 0)$, the following assertions are valid (see [4]).

$$(\forall x \in X)(x * x = 0), \tag{1.9}$$

$$(\forall x, y, z \in X)(z * x = 0, y * z = 0 \Rightarrow y * x = 0), \tag{1.10}$$

$$(\forall x, y, z \in X)(y * x = 0 \Rightarrow (x * z) * (y * z) = 0), \tag{1.11}$$

$$(\forall x, y, z \in X)(y * x = 0 \Rightarrow (z * y) * (z * x) = 0), \tag{1.12}$$

$$(\forall x, y \in X)((y * z) * ((z * x) * (y * x)) = 0), \tag{1.13}$$

$$(\forall x, y \in X)(y * ((y * x) * x) = 0), \tag{1.14}$$

$$(\forall x, y, z \in X)(z * (y * x) = y * (z * x)), \tag{1.15}$$

$$(\forall u, x, y, z \in X)((y * x) * 0 = (y * 0) * (x * 0)). \tag{1.16}$$

According to [4], the binary relation $\leq$ on X is defined as follows:

$$(\forall x, y \in X)(x \leq y \Leftrightarrow y * x = 0).$$

2. External Direct Product of JU-algebras

Lingcong and Endam [18] discussed the notion of the direct product of B-algebras, 0-commutative B-algebras, and B-homomorphisms and obtained related properties, one of which is a direct product of two B-algebras, which is also a B-algebra. Then, they extended the concept of the direct product of B-algebra to finite family B-algebra, and some of the related properties were investigated as follows:

Definition 2.1. *[18] Let $(X_i; *_i)$ be an algebra for each $i \in \{1, 2, ..., k\}$. Define the direct product of algebras $X_1, X_2, ..., X_k$ to be the structure $(\prod_{i=1}^{k} X_i; \otimes)$, where*

$$\prod_{i=1}^{k} X_i = X_1 \times X_2 \times ... \times X_k = \{(x_1, x_2, ..., x_k) \mid x_i \in X_i \ \forall i = 1, 2, ..., k\}$$

and whose operation $\otimes$ is given by

$$(x_1, x_2, ..., x_k) \otimes (y_1, y_2, ..., y_k) = (x_1 *_1 y_1, x_2 *_2 y_2, ..., x_k *_k y_k)$$

for all $(x_1, x_2, ..., x_k), (y_1, y_2, ..., y_k) \in \prod_{i=1}^{k} X_i$.

Now, we extend the concept of the direct product to an infinite family of JU-algebras and provide some of its properties.

Definition 2.2. *Let X_i be a nonempty set for each $i \in I$. Define the external direct product of sets X_i for all $i \in I$ to be the set $\prod_{i \in I} X_i$, where*

$$\prod_{i \in I} X_i = \{f : I \to \bigcup_{i \in I} X_i \mid f(i) \in X_i \ \forall i \in I\}.$$

For convenience, we define an element of $\prod_{i \in I} X_i$ with a function $(x_i)_{i \in I} : I \to \bigcup_{i \in I} X_i$, where $i \mapsto x_i \in X_i$ for all $i \in I$.

Remark 2.1. *Let X_i be a nonempty set and S_i a subset of X_i for all $i \in I$. Then $\prod_{i \in I} S_i$ is a nonempty subset of the external direct product $\prod_{i \in I} X_i$ if and only if S_i is a nonempty subset of X_i for all $i \in I$.*

Definition 2.3. *Let $X_i = (X_i; *_i)$ be an algebra for all $i \in I$. Define the binary operation $\otimes$ on the external direct product $\prod_{i \in I} X_i = (\prod_{i \in I} X_i; \otimes)$ as follows:*

$$(\forall (x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} X_i)((x_i)_{i \in I} \otimes (y_i)_{i \in I} = (x_i *_i y_i)_{i \in I}). \tag{2.1}$$

We shall show that $\otimes$ is a binary operation on $\prod_{i \in I} X_i$. Let $(x_i)_{i \in I}, (y_i)_{i \in I} \in \prod_{i \in I} X_i$. Since $*_i$ is a binary operation on X_i for all $i \in I$, we have $x_i *_i y_i \in X_i$ for all $i \in I$. Then $(x_i *_i y_i)_{i \in I} \in \prod_{i \in I} X_i$ such that

$$(x_i)_{i \in I} \otimes (y_i)_{i \in I} = (x_i *_i y_i)_{i \in I}.$$

Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (x'_i)_{i\in I}, (y'_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} = (y_i)_{i\in I}$ and $(x'_i)_{i\in I} = (y'_i)_{i\in I}$. We shall show that $(x_i)_{i\in I} \otimes (x'_i)_{i\in I} = (y_i)_{i\in I} \otimes (y'_i)_{i\in I}$. Then

$$x_i = y_i \text{ for all } i \in I \text{ and } x'_i = y'_i \text{ for all } i \in I.$$

Since $*_i$ is a binary operation on X_i for all $i \in I$, we have $x_i *_i x'_i = y_i *_i y'_i$ for all $i \in I$. Thus

$$\begin{aligned}(x_i)_{i\in I} \otimes (x'_i)_{i\in I} &= (x_i *_i x'_i)_{i\in I}\\ &= (y_i *_i y'_i)_{i\in I}\\ &= (y_i)_{i\in I} \otimes (y'_i)_{i\in I}.\end{aligned}$$

Hence, $\otimes$ is a binary operation on $\prod_{i\in I} X_i$.

Let $X_i = (X_i; *_i, 0_i)$ be a JU-algebra for all $i \in I$. For $i \in I$, let $x_i \in X_i$. We define the function $f_{x_i} : I \to \bigcup_{i\in I} X_i$ as follows:

$$(\forall j \in I)\left(f_{x_i}(j) = \begin{cases} x_i & \text{if } j = i\\ 0_j & \text{otherwise}\end{cases}\right). \tag{2.2}$$

Then $f_{x_i} \in \prod_{i\in I} X_i$.

Lemma 2.1. *Let $X_i = (X_i; *_i, 0_i)$ be a JU-algebra for all $i \in I$. For $i \in I$, let $x_i, y_i \in X_i$. Then $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i}$.*

Proof. Now,

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i\\ 0_j *_j 0_j & \text{otherwise}\end{cases}\right).$$

By (1.9), we have

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i\\ 0_j & \text{otherwise}\end{cases}\right).$$

By (2.2), we have $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i}$. □

The following theorem shows that the direct product of JU-algebras in terms of an infinite family of JU-algebras is also.

Theorem 2.1. *$X_i = (X_i; *_i, 0_i)$ is a JU-algebra for all $i \in I$ if and only if $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a JU-algebra, where the binary operation $\otimes$ is defined in Definition 2.3.*

Proof. Assume that $X_i = (X_i; *_i, 0_i)$ is a JU-algebra for all $i \in I$.

(JU-1) Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (JU-1), we have $(y_i *_i z_i) *_i ((z_i *_i x_i) *_i (y_i *_i x_i)) = 0_i$ for all $i \in I$. Thus

$$\begin{aligned}
&((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \otimes (((z_i)_{i\in I} \otimes (x_i)_{i\in I}) \otimes ((y_i)_{i\in I} \otimes (x_i)_{i\in I}))\\
&= (y_i *_i z_i)_{i\in I} \otimes ((z_i *_i x_i)_{i\in I} \otimes (y_i *_i x_i)_{i\in I})\\
&= (y_i *_i z_i)_{i\in I} \otimes ((z_i *_i x_i) *_i (y_i *_i x_i))_{i\in I}\\
&= ((y_i *_i z_i) *_i ((z_i *_i x_i) *_i (y_i *_i x_i)))_{i\in I}\\
&= (0_i)_{i\in I}.
\end{aligned}$$

(JU-2) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Since X_i satisfies (JU-2), we have $0_i *_i x_i = x_i$ for all $i \in I$. Thus

$$(0_i)_{i\in I} \otimes (x_i)_{i\in I} = (0_i *_i x_i)_{i\in I} = (x_i)_{i\in I}.$$

(JU-3) Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (0_i)_{i\in I}$ and $(y_i)_{i\in I} \otimes (x_i)_{i\in I} = (0_i)_{i\in I}$. Then $(x_i *_i y_i)_{i\in I} = (0_i)_{i\in I}$ and $(y_i *_i x_i)_{i\in I} = (0_i)_{i\in I}$, so $x_i *_i y_i = 0_i$ and $y_i *_i x_i = 0_i$ for all $i \in I$. Since X_i satisfies (JU-3), we have $x_i = y_i$ for all $i \in I$. Therefore, $(x_i)_{i\in I} = (y_i)_{i\in I}$.

Hence, $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a JU-algebra.

Conversely, assume that $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ is a JU-algebra, where the binary operation $\otimes$ is defined in Definition 2.3. Let $i \in I$.

(JU-1) Let $x_i, y_i, z_i \in X_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$, which is defined by (2.2). Since $\prod_{i\in I} X_i$ satisfies (JU-1), we have $(f_{y_i} \otimes f_{z_i}) \otimes ((f_{z_i} \otimes f_{x_i}) \otimes (f_{y_i} \otimes f_{x_i})) = (0_i)_{i\in I}$. Now,

$$(\forall j \in I)\left(((f_{y_i} \otimes f_{z_i}) \otimes ((f_{z_i} \otimes f_{x_i}) \otimes (f_{y_i} \otimes f_{x_i})))(j) = \begin{cases} (y_i *_i z_i) *_i ((z_i *_i x_i) *_i (y_i *_i x_i)) & \text{if } j = i \\ (0_j *_j 0_j) *_j ((0_j *_j 0_j) *_j (0_j *_j 0_j)) & \text{otherwise} \end{cases}\right),$$

this implies that $(y_i *_i z_i) *_i ((z_i *_i x_i) *_i (y_i *_i x_i)) = 0_i$.

(JU-2) Let $x_i \in X_i$. Then $f_{x_i} \in \prod_{i\in I} X_i$, which is defined by (2.2). Since $\prod_{i\in I} X_i$ satisfies (JU-2), we have $(0_i)_{i\in I} \otimes f_{x_i} = f_{x_i}$. Now,

$$(\forall j \in I)\left(((0_i)_{i\in I} \otimes f_{x_i})(j) = \begin{cases} 0_i *_i x_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

this implies that $0_i *_i x_i = x_i$.

(JU-3) Let $x_i, y_i \in X_i$ be such that $x_i *_i y_i = 0_i$ and $y_i *_i x_i = 0_i$ for all $i \in I$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (2.2). Now,

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

and

$$(\forall j \in I)\left((f_{y_i} \otimes f_{x_i})(j) = \begin{cases} y_i *_i x_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By assumption and (1.9), we have

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} 0_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases}\right),$$

and

$$(\forall j \in I)\left((f_{y_i} \otimes f_{x_i})(j) = \begin{cases} 0_i & \text{if } j = i \\ 0_j & \text{otherwise} \end{cases}\right).$$

Thus $f_{x_i} \otimes f_{y_i} = (0_i)_{i\in I}$ and $f_{y_i} \otimes f_{x_i} = (0_i)_{i\in I}$. Since $\prod_{i\in I} X_i$ satisfies (JU-3), we have $f_{x_i} = f_{y_i}$. Therefore, $x_i = y_i$.

Hence, $X_i = (X_i; *_i, 0_i)$ is a JU-algebra for all $i \in I$. □

We call the JU-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$ in Theorem 2.1 the external direct product JU-algebra induced by a JU-algebra $X_i = (X_i; *_i, 0_i)$ for all $i \in I$.

Next, we introduce the concept of the weak direct product of infinite family of JU-algebras and obtain some of its properties as follows:

Definition 2.4. *Let $X_i = (X_i; *_i, 0_i)$ be a JU-algebra for all $i \in I$. Define the weak direct product of a JU-algebra X_i for all $i \in I$ to be the structure $\prod^{\mathrm{w}}_{i\in I} X_i = (\prod^{\mathrm{w}}_{i\in I} X_i; \otimes)$, where*

$$\prod^{\mathrm{w}}_{i\in I} X_i = \{(x_i)_{i\in I} \in \prod_{i\in I} X_i \mid x_i \neq 0_i, \text{ where the number of such } i \text{ is finite}\}.$$

Then $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i \subseteq \prod_{i\in I} X_i$.

Theorem 2.2. *Let $X_i = (X_i; *_i, 0_i)$ be a JU-algebra for all $i \in I$. Then $\prod^{\mathrm{w}}_{i\in I} X_i$ is a JU-subalgebra of the external direct product JU-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. We see that $(0_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$, where $I_1 = \{i \in I \mid x_i \neq 0_i\}$ and $I_2 = \{i \in I \mid y_i \neq 0_i\}$ are finite. Then $|I_1 \cup I_2|$ is finite. Thus

$$(\forall j \in I)\left(((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) = \begin{cases} x_j *_j 0_j & \text{if } j \in I_1 - I_2 \\ x_j *_j y_j & \text{if } j \in I_1 \cap I_2 \\ 0_j *_j y_j & \text{if } j \in I_2 - I_1 \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right).$$

By (JU-2) and (1.9), we have

$$(\forall j \in I)\left(((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) = \begin{cases} x_j *_j 0_j & \text{if } j \in I_1 - I_2 \\ x_j *_j y_j & \text{if } j \in I_1 \cap I_2 \\ y_j & \text{if } j \in I_2 - I_1 \\ 0_j & \text{otherwise} \end{cases}\right).$$

This implies that the number of such $((x_i)_{i\in I} \otimes (y_i)_{i\in I})(j) \neq 0_j$ is not more than $|I_1 \cup I_2|$, that is, it is finite. Thus $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod^{\mathrm{w}}_{i\in I} X_i$. Hence, $\prod^{\mathrm{w}}_{i\in I} X_i$ is a JU-subalgebra of $\prod_{i\in I} X_i$. □

Theorem 2.3. *Let $X_i = (X_i; *_i, 0_i)$ be a JU-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a JU-subalgebra of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a JU-subalgebra of the external direct product JU-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a JU-subalgebra of X_i for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 2.1, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $x_i, y_i \in S_i$ for all $i \in I$. By (1.1), we have $x_i *_i y_i \in S_i$ for all $i \in I$ and so $(x_i)_{i\in I} \otimes (y_i)_{i\in I} = (x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a JU-subalgebra of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a JU-subalgebra of $\prod_{i\in I} X_i$. Since $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 2.1, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By (1.1) and Lemma 2.1, we have $f_{x_i *_i y_i} = f_{x_i} \otimes f_{y_i} \in \prod_{i\in I} S_i$. By (2.2), we have $x_i *_i y_i \in S_i$. Hence, S_i is a JU-subalgebra of X_i for all $i \in I$. □

Theorem 2.4. *Let $X_i = (X_i; *_i, 0_i)$ be a JU-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a JU-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a JU-ideal of the external direct product JU-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a JU-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i y_i \in S_i$ and $x_i \in S_i$, it follows from (1.3) that $y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a JU-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a JU-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $x_i *_i y_i \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.1, we have $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (1.3), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (2.2), we have $y_i \in S_i$. Hence, S_i is a JU-ideal of X_i for all $i \in I$. □

Theorem 2.5. *Let $X_i = (X_i; *_i, 0_i)$ be a JU-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a p-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a p-ideal of the external direct product JU-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a p-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((z_i)_{i\in I} \otimes (x_i)_{i\in I}) \otimes ((z_i)_{i\in I} \otimes (y_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $((z_i *_i x_i) *_i (z_i *_i y_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $(z_i *_i x_i) *_i (z_i *_i y_i) \in S_i$ and $x_i \in S_i$, it follows from (1.4) that $y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a p-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a p-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $(z_i *_i x_i) *_i (z_i *_i y_i) \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{(z_i *_i x_i) *_i (z_i *_i y_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.1, we have $(f_{z_i} \otimes f_{x_i}) \otimes (f_{z_i} \otimes f_{y_i}) = f_{(z_i *_i x_i) *_i (z_i *_i y_i)} \in \prod_{i\in I} S_i$. By (1.4), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (2.2), we have $y_i \in S_i$. Hence, S_i is a p-ideal of X_i for all $i \in I$. □

Theorem 2.6. *Let $X_i = (X_i; *_i, 0_i)$ be a JU-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a t-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a t-ideal of the external direct product JU-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a t-ideal of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((x_i)_{i\in I} \otimes (y_i)_{i\in I}) \otimes (z_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $((x_i *_i y_i) *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $(x_i *_i y_i) *_i z_i \in S_i$ and $y_i \in S_i$, it follows from (1.5) that $x_i *_i z_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \otimes (z_i)_{i\in I} = (x_i *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a t-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a t-ideal of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $(x_i *_i y_i) *_i z_i \in S_i$ and $y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{(x_i *_i y_i) *_i z_i} \in \prod_{i\in I} S_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.1, we have $(f_{x_i} \otimes f_{y_i}) \otimes f_{z_i} = f_{(x_i *_i y_i) *_i z_i} \in \prod_{i\in I} S_i$. By (1.5) and Lemma 2.1, we have $f_{x_i *_i z_i} = f_{x_i} \otimes f_{z_i} \in \prod_{i\in I} S_i$. By (2.2), we have $x_i *_i z_i \in S_i$. Hence, S_i is a t-ideal of X_i for all $i \in I$. □

Theorem 2.7. *Let $X_i = (X_i; *_i, 0_i)$ be a JU-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a strong JU-ideal of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a strong JU-ideal of the external direct product JU-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. By Theorem 2.4, we are left to prove that X_i satisfies (1.6) for all $i \in I$ if and only if $\prod_{i\in I} X_i$ satisfies (1.6).

Assume that S_i satisfies (1.6) for all $i \in I$. Since S_i is a nonempty subset of X_i for all $i \in I$ and by Remark 2.1, we have $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$. be such that $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i y_i \in S_i$ and $y_i \in S_i$, it follows from (1.6) that $x_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a strong JU-ideal of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ satisfies (1.6). Since $\prod_{i\in I} S_i$ is a nonempty subset of $\prod_{i\in I} X_i$ and by Remark 2.1, we have S_i is a nonempty subset of X_i for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $x_i *_i y_i \in S_i$ and $y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.1, we have $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (1.6), we have $f_{x_i} \in \prod_{i\in I} S_i$. By (2.2), we have $x_i \in S_i$. Hence, S_i is a strong JU-ideal of X_i for all $i \in I$. □

Theorem 2.8. *Let $X_i = (X_i; *_i, 0_i)$ be a JU-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a JU-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a JU-filter of the external direct product JU-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a JU-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I} \in \prod_{i\in I} S_i$. be such that $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod_{i\in I} S_i$ and $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i y_i \in S_i$ and $y_i \in S_i$, it follows from (1.6) that $x_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a JU-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a JU-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i \in X_i$ be such that $x_i *_i y_i \in S_i$ and $y_i \in S_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$ and

$f_{x_i *_i y_i} \in \prod_{i\in I} S_i$ and $f_{y_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.1, we have $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (1.6), we have $f_{x_i} \in \prod_{i\in I} S_i$. By (2.2), we have $x_i \in S_i$. Hence, S_i is a JU-filter of X_i for all $i \in I$. □

Theorem 2.9. *Let $X_i = (X_i; *_i, 0_i)$ be a JU-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is a comparative JU-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is a comparative JU-filter of the external direct product JU-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is a comparative JU-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \otimes ((z_i)_{i\in I} \otimes (x_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $((y_i *_i z_i) *_i (z_i *_i x_i))_{i\in I} \in \prod_{i\in I} S_i$. Thus $(y_i *_i z_i) *_i (z_i *_i x_i) \in S_i$ and $x_i \in S_i$, it follows from (1.7) that $y_i \in S_i$ for all $i \in I$. Thus $(y_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is a comparative JU-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is a comparative JU-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $(y_i *_i z_i) *_i (z_i *_i x_i) \in S_i$ and $x_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{(y_i *_i z_i) *_i (z_i *_i x_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.1, we have $(f_{y_i} \otimes f_{z_i}) \otimes (f_{z_i} \otimes f_{x_i}) = f_{(y_i *_i z_i) *_i (z_i *_i x_i)} \in \prod_{i\in I} S_i$. By (1.7), we have $f_{y_i} \in \prod_{i\in I} S_i$. By (2.2), we have $y_i \in S_i$. Hence, S_i is a comparative JU-filter of X_i for all $i \in I$. □

Theorem 2.10. *Let $X_i = (X_i; *_i, 0_i)$ be a JU-algebra and S_i a subset of X_i for all $i \in I$. Then S_i is an implicative JU-filter of X_i for all $i \in I$ if and only if $\prod_{i\in I} S_i$ is an implicative JU-filter of the external direct product JU-algebra $\prod_{i\in I} X_i = (\prod_{i\in I} X_i; \otimes, (0_i)_{i\in I})$.*

Proof. Assume that S_i is an implicative JU-filter of X_i for all $i \in I$. Then $0_i \in S_i$ for all $i \in I$, so $(0_i)_{i\in I} \in \prod_{i\in I} S_i \neq \emptyset$. Let $(x_i)_{i\in I}, (y_i)_{i\in I}, (z_i)_{i\in I} \in \prod_{i\in I} X_i$ be such that $(x_i)_{i\in I} \otimes ((y_i)_{i\in I} \otimes (z_i)_{i\in I}) \in \prod_{i\in I} S_i$ and $(x_i)_{i\in I} \otimes (y_i)_{i\in I} \in \prod_{i\in I} S_i$. Then $(x_i *_i (y_i *_i z_i))_{i\in I} \in \prod_{i\in I} S_i$ and $(x_i *_i y_i)_{i\in I} \in \prod_{i\in I} S_i$. Thus $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i *_i y_i \in S_i$, it follows from (1.8) that $x_i *_i z_i \in S_i$ for all $i \in I$. Thus $(x_i)_{i\in I} \otimes (z_i)_{i\in I} = (x_i *_i z_i)_{i\in I} \in \prod_{i\in I} S_i$. Hence, $\prod_{i\in I} S_i$ is an implicative JU-filter of $\prod_{i\in I} X_i$.

Conversely, assume that $\prod_{i\in I} S_i$ is an implicative JU-filter of $\prod_{i\in I} X_i$. Then $(0_i)_{i\in I} \in \prod_{i\in I} S_i$, so $0_i \in S_i \neq \emptyset$ for all $i \in I$. Let $i \in I$ and let $x_i, y_i, z_i \in X_i$ be such that $x_i *_i (y_i *_i z_i) \in S_i$ and $x_i *_i y_i \in S_i$. Then $f_{x_i}, f_{y_i}, f_{z_i} \in \prod_{i\in I} X_i$ and $f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i *_i y_i} \in \prod_{i\in I} S_i$, which are defined by (2.2). By Lemma 2.1, we have $f_{x_i} \otimes (f_{y_i} \otimes f_{z_i}) = f_{x_i *_i (y_i *_i z_i)} \in \prod_{i\in I} S_i$ and $f_{x_i} \otimes f_{y_i} = f_{x_i *_i y_i} \in \prod_{i\in I} S_i$. By (1.8) and Lemma 2.1, we have $f_{x_i *_i z_i} = f_{x_i} \otimes f_{z_i} \in \prod_{i\in I} S_i$. By (2.2), we have $x_i *_i z_i \in S_i$. Hence, S_i is an implicative JU-filter of X_i for all $i \in I$. □

Moreover, we discuss several homomorphism theorems in view of the external direct product of JU-algebras.

Definition 2.5. *[7] Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Define the function $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ given by*

$$(\forall (x_i)_{i\in I} \in \prod_{i\in I} X_i)(\psi(x_i)_{i\in I} = (\psi_i(x_i))_{i\in I}). \tag{2.3}$$

Then $\psi : \prod_{i\in I} X_i \to \prod_{i\in I} S_i$ is a function (see [7]).

Theorem 2.11. *[7] Let $X_i = (X_i; *_i)$ and $S_i = (S_i; \circ_i)$ be algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$.*

- (i) *ψ_i is injective for all $i \in I$ if and only if ψ is injective, which is defined in Definition 2.5,*
- (ii) *ψ_i is surjective for all $i \in I$ if and only if ψ is surjective,*
- (iii) *ψ_i is bijective for all $i \in I$ if and only if ψ is bijective.*

Theorem 2.12. *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be JU-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

- (i) *ψ_i is a JU-homomorphism for all $i \in I$ if and only if ψ is a JU-homomorphism, which is defined in Definition 2.5,*
- (ii) *ψ_i is a JU-monomorphism for all $i \in I$ if and only if ψ is a JU-monomorphism,*
- (iii) *ψ_i is a JU-epimorphism for all $i \in I$ if and only if ψ is a JU-epimorphism,*
- (iv) *ψ_i is a JU-isomorphism for all $i \in I$ if and only if ψ is a JU-isomorphism,*
- (v) $\ker \psi = \prod_{i\in I} \ker \psi_i$ *and* $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$.

Proof. (i) Assume that ψ_i is a JU-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}
\psi((x_i)_{i\in I} \otimes (x'_i)_{i\in I}) &= \psi(x_i *_i x'_i)_{i\in I} \\
&= (\psi_i(x_i *_i x'_i))_{i\in I} \\
&= (\psi_i(x_i) *_i \psi_i(x'_i))_{i\in I} \\
&= (\psi_i(x_i))_{i\in I} \otimes (\psi_i(x'_i))_{i\in I} \\
&= \psi(x_i)_{i\in I} \otimes \psi(x'_i)_{i\in I}.
\end{aligned}$$

Hence, ψ is a JU-homomorphism.

Conversely, assume that ψ is a JU-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which is defined by (2.2). Since ψ is a JU-homomorphism, we have $\psi(f_{x_i} \otimes f_{y_i}) = \psi(f_{x_i}) \otimes \psi(f_{y_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{x_i} \otimes f_{y_i})(j) = \begin{cases} \psi_i(x_i *_i y_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise} \end{cases}\right). \tag{2.4}$$

Since

$$(\forall j \in I)\left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right)$$

and

$$(\forall j \in I)\left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{x_i}) \otimes \psi(f_{y_i}))(j) = \begin{cases} \psi_i(x_i) \circ_i \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases}\right). \tag{2.5}$$

By (2.4) and (2.5), we have $\psi_i(x_i *_i y_i) = \psi_i(x_i) \circ_i \psi_i(y_i)$. Hence, ψ_i is a JU-homomorphism for all $i \in I$.

(ii) It is straightforward from (i) and Theorem 2.11 (i).

(iii) It is straightforward from (i) and Theorem 2.11 (ii).

(iv) It is straightforward from (i) and Theorem 2.11 (iii).

(v) Let $(x_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned}
(x_i)_{i\in I} \in \ker\psi &\Leftrightarrow \psi(x_i)_{i\in I} = (1_i)_{i\in I} \\
&\Leftrightarrow (\psi_i(x_i))_{i\in I} = (1_i)_{i\in I} \\
&\Leftrightarrow \psi_i(x_i) = 1_i \ \forall i \in I \\
&\Leftrightarrow x_i \in \ker\psi_i \ \forall i \in I \\
&\Leftrightarrow (x_i)_{i\in I} \in \prod_{i\in I} \ker\psi_i.
\end{aligned}$$

Hence, $\ker\psi = \prod_{i\in I} \ker\psi_i$. Now,

$$\begin{aligned}
(y_i)_{i\in I} \in \psi(\prod_{i\in I} X_i) &\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = \psi(x_i)_{i\in I} \\
&\Leftrightarrow \exists (x_i)_{i\in I} \in \prod_{i\in I} X_i \text{ s.t. } (y_i)_{i\in I} = (\psi_i(x_i))_{i\in I} \\
&\Leftrightarrow \exists x_i \in X_i \text{ s.t. } y_i = \psi_i(x_i) \in \psi(X_i) \ \forall i \in I \\
&\Leftrightarrow (y_i)_{i\in I} \in \prod_{i\in I} \psi_i(X_i).
\end{aligned}$$

Hence, $\psi(\prod_{i\in I} X_i) = \prod_{i\in I} \psi_i(X_i)$. □

Finally, we discuss several anti-JU-homomorphism theorems in view of the external direct product of JU-algebras.

Theorem 2.13. *Let $X_i = (X_i; *_i, 0_i)$ and $S_i = (S_i; \circ_i, 1_i)$ be JU-algebras and $\psi_i : X_i \to S_i$ be a function for all $i \in I$. Then*

(i) *ψ_i is an anti-JU-homomorphism for all $i \in I$ if and only if ψ is an anti-JU-homomorphism which is defined in Definition 2.5,*

(ii) *ψ_i is an anti-JU-monomorphism for all $i \in I$ if and only if ψ is an anti-JU-monomorphism,*

(iii) *ψ_i is an anti-JU-epimorphism for all $i \in I$ if and only if ψ is an anti-JU-epimorphism,* (*iv*) *ψ_i is an anti-JU-isomorphism for all $i \in I$ if and only if ψ is an anti-JU-isomorphism.*

Proof. (*i*) Assume that ψ_i is an anti-JU-homomorphism for all $i \in I$. Let $(x_i)_{i\in I}, (x'_i)_{i\in I} \in \prod_{i\in I} X_i$. Then

$$\begin{aligned} \psi((x_i)_{i\in I} \otimes (x'_i)_{i\in I}) &= \psi(x_i *_i x'_i)_{i\in I} \\ &= (\psi_i(x_i *_i x'_i))_{i\in I} \\ &= (\psi_i(x'_i) *_i \psi_i(x_i))_{i\in I} \\ &= (\psi_i(x'_i))_{i\in I} \otimes (\psi_i(x_i))_{i\in I} \\ &= \psi(x'_i)_{i\in I} \otimes \psi(x_i)_{i\in I}. \end{aligned}$$

Hence, ψ is an anti-JU-homomorphism.

Conversely, assume that ψ is an anti-JU-homomorphism. Let $i \in I$. Let $x_i, y_i \in X_i$. Then $f_{x_i}, f_{y_i} \in \prod_{i\in I} X_i$, which are defined by (2.2). Since ψ is an anti-JU-homomorphism, we have $\psi(f_{x_i} \otimes f_{y_i}) = \psi(f_{y_i}) \otimes \psi(f_{x_i})$. Since

$$(\forall j \in I)\left((f_{x_i} \otimes f_{y_i})(j) = \begin{cases} x_i *_i y_i & \text{if } j = i \\ 0_j *_j 0_j & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left(\psi(f_{x_i} \otimes f_{y_i})(j) = \begin{cases} \psi_i(x_i *_i y_i) & \text{if } j = i \\ \psi_j(0_j *_j 0_j) & \text{otherwise} \end{cases}\right). \tag{2.6}$$

Since

$$(\forall j \in I)\left(\psi(f_{y_i})(j) = \begin{cases} \psi_i(y_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right)$$

and

$$(\forall j \in I)\left(\psi(f_{x_i})(j) = \begin{cases} \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) & \text{otherwise} \end{cases}\right),$$

we have

$$(\forall j \in I)\left((\psi(f_{y_i}) \otimes \psi(f_{x_i}))(j) = \begin{cases} \psi_i(y_i) \circ_i \psi_i(x_i) & \text{if } j = i \\ \psi_j(0_j) \circ_j \psi_j(0_j) & \text{otherwise} \end{cases}\right). \tag{2.7}$$

By (2.6) and (2.7), we have $\psi_i(x_i *_i y_i) = \psi_i(y_i) \circ_i \psi_i(x_i)$. Hence, ψ_i is an anti-JU-homomorphism for all $i \in I$.

(*ii*) It is straightforward from (*i*) and Theorem 2.11 (*i*).

(*iii*) It is straightforward from (*i*) and Theorem 2.11 (*ii*).

(*iv*) It is straightforward from (*i*) and Theorem 2.11 (*iii*). □

3. Conclusions and Future Work

In this paper, we have introduced the concept of the direct product of an infinite family of JU algebras, which we call the external direct product, which is a general concept of the direct product in the sense of Lingcong and Endam [18]. We proved that the external direct product of JU-algebras is also a JU-algebra. Also, we have introduced the concept of the weak direct product of JU-algebras. We proved that the weak direct product of JU-algebras is a JU-subalgebra and the external direct product of JU-subalgebras (resp., JU-ideals, p-ideals, t-ideals, strong JU-ideals, JU-filters, comparative JU-filters, implicative JU-filters) is also a JU-subalgebra (resp., JU-ideal, p-ideal, t-ideal, strong JU-ideal, JU-filter, comparative JU-filter, implicative JU-filter) of the external direct product JU-algebras. Finally, we have provided several fundamental theorems of (anti-)JU-homomorphisms in view of the external direct product JU-algebras.

Based on the notion of the external direct product in JU-algebras presented in this article, this concept can be extended to explore the external direct product in other algebraic systems. Future research will focus on investigating external and weak direct products, particularly within the framework of GE-algebras, as introduced by Bandaru et al. [5].

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