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Dynamics and Expressions of Solutions of Fourth-Order Rational Systems of Difference Equations

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Abstract. The purpose of this article is to determine the expressions of solutions for the following rational difference systems

$$\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\alpha + \Phi_{n-2} + \Psi_{n-3}}, \quad \Psi_{n+1} = \frac{\Phi_n \Psi_{n-2}}{\beta \pm \Phi_{n-3} \pm \Psi_{n-2}}, \quad n = 0, 1, 2, ...,$$

where α and β are arbitrary real numbers. Furthermore, the solution's qualitative behavior is explored, such as local and global stability, as well as the boundedness of the solutions. We will offer numerical examples to demonstrate our results.

1. Introduction

The study of qualitative analysis of rational difference equations and systems of difference equations has sparked interest in recent years. Difference equations are used as approximations for continuous problems and as models to describe a variety of life situations. As a result, a rising number of mathematicians are focusing on the qualitative analysis of difference equations. This is due to the fact that difference equations are employed to model a wide range of real-world events. In addition, as discrete analogs and numerical solutions of differential equations, various results in the theory of difference equations have been obtained.

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Extensive research has been carried out regarding the method of determining the general form of the solution for certain special cases of the problem. Many studies have been written about the systems and behavior of rational difference equations (for example see [1]- [39]).

Alayachi et al. [4] have got the form of the solutions of the rational difference system:

$$\varkappa_{n+1} = \frac{\varkappa_{n-3}\psi_{n-4}}{\psi_n(1+\varkappa_{n-1}\psi_{n-2}\varkappa_{n-3}\psi_{n-4})},$$
$$\psi_{n+1} = \frac{\psi_{n-3}\varkappa_{n-4}}{\varkappa_n(\pm 1 \pm \psi_{n-1}\varkappa_{n-2}\psi_{n-3}\varkappa_{n-4})}.$$

Asiri et al. [8] studied the form of the solutions and the periodicity of the following systems:

$$\varkappa_{n+1} = \frac{\psi_{n-2}}{1-\psi_{n-2}\varkappa_{n-1}\psi_n}, \quad \psi_{n+1} = \frac{\varkappa_{n-2}}{\pm 1 \pm \varkappa_{n-2}\psi_{n-1}\varkappa_n}.$$

Elabbasy et al. [12] have solved specific situations of the following system of difference equations:

$$\chi_{n+1} = \frac{a_1 + a_2\Upsilon_n}{a_3\omega_n + a_4\chi_{n-1}\omega_n},$$
$$\Upsilon_{n+1} = \frac{b_1\omega_{n-1} + b_2\omega_n}{b_3\chi_n\Upsilon_n + b_4\chi_n\Upsilon_{n-1}},$$
$$\omega_{n+1} = \frac{c_1\omega_{n-1} + c_2\omega_n}{c_3\chi_{n-1}\Upsilon_{n-1} + c_4\chi_{n-1}\Upsilon_n + c_5\chi_n\Upsilon_n}.$$

El-Dessoky et al. [15] obtained the form of the solutions of the following system of difference equations:

$$\chi_{n+1} = \frac{\chi_n\psi_{n-1}}{\pm\psi_n \pm \psi_{n-1}}, \quad \psi_{n+1} = \frac{\psi_n\chi_{n-1}}{\pm\chi_n \pm \chi_{n-1}}.$$

Elsayed et al. [23] deal with the form of the solutions of the system:

$$R_{n+1} = \frac{\alpha_1 T_{n-1}S_{n-1}}{R_{n-1}+S_{n-1}+T_{n-1}}, \qquad S_{n+1} = \frac{\alpha_2 T_{n-1}R_{n-1}}{R_{n-1}+S_{n-1}+T_{n-1}}, \qquad T_{n+1} = \frac{\alpha_3 R_{n-1}S_{n-1}}{R_{n-1}+S_{n-1}+T_{n-1}}.$$

Halim et al. [30] have obtained the formula for the general solution to the system:

$$\varkappa_{n+1} = \frac{\psi_{n-1}\varkappa_{n-2}}{\psi_n(\alpha+\beta\psi_{n-1}\varkappa_{n-2})}, \quad \psi_{n+1} = \frac{\varkappa_{n-1}\psi_{n-2}}{\varkappa_n(\alpha+\beta\varkappa_{n-1}\psi_{n-2})}.$$

Yalçinkaya and Cinar [40] studied the global asymptotic stability of the system of difference equations:

$$\omega_{n+1} = \frac{\tau_n + \omega_{n-1}}{\tau_n\omega_{n-1} + a}, \qquad \tau_{n+1} = \frac{\omega_n + \tau_{n-1}}{\omega_n\tau_{n-1} + a}.$$

The main objective of this article is to obtain the expressions of solutions for the following rational difference systems:

$$\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\alpha + \Phi_{n-2} + \Psi_{n-3}}, \quad \Psi_{n+1} = \frac{\Phi_n\Psi_{n-2}}{\beta \pm \Phi_{n-3} \pm \Psi_{n-2}}, \quad n = 0, 1, 2, ..., \tag{1.1}$$

where the initial conditions $\Phi_{-3}, \Phi_{-2}, \Phi_{-1}, \Phi_0, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$ and Ψ_0 are nonzero real numbers, α and β are arbitrary real numbers. Furthermore, we investigate the behavior of solutions, such as local and global stability, as well as the boundedness of the solutions.

2. Main Results

Assume I_Φ and I_Ψ are any intervals of real numbers and $f : I_\Phi^3 \times I_\Psi^3 \to I_\Phi$, $g : I_\Phi^3 \times I_\Psi^3 \to I_\Psi$ are continuously differentiable functions. Then for each initial condition $(\Phi_i, \Psi_i) \in I_\Phi \times I_\Psi$ for $i \in \{-3, -2, -1, 0\}$, the system of difference equations:

$$\Phi_{n+1} = f(\Phi_n, \Phi_{n-2}, \Phi_{n-3}, \Psi_n, \Psi_{n-2}, \Psi_{n-3}) \tag{2.1}$$
$$\Psi_{n+1} = g(\Phi_n, \Phi_{n-2}, \Phi_{n-3}, \Psi_n, \Psi_{n-2}, \Psi_{n-3}) \quad n = 0, 1, ...,$$

has a unique solution $\{\Phi_n, \Psi_n\}_{n=-3}^{\infty}$.

Definition 2.1. *A point $(\bar{\Phi}, \bar{\Psi})$ is said to be an equilibrium point of (2.1) if*

$$\bar{\Phi} = f(\bar{\Phi}, \bar{\Phi}, \bar{\Phi}, \bar{\Psi}, \bar{\Psi}, \bar{\Psi}),$$
$$\bar{\Psi} = g(\bar{\Phi}, \bar{\Phi}, \bar{\Phi}, \bar{\Psi}, \bar{\Psi}, \bar{\Psi}),$$

are satisfied.

Definition 2.2. *Assume that $(\bar{\Phi}, \bar{\Psi})$ is a fixed point of (2.1) .*

(i): $(\bar{\Phi}, \bar{\Psi})$ is said to be stable if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every initial condition $(\Phi_i, \Psi_i) \in I_\Phi \times I_\Psi$ for $i \in \{-3, -2, -1, 0\}$ if

$$\left\| \sum_{i=-3}^{0} (\Phi_i, \Psi_i) - (\bar{\Phi}, \bar{\Psi}) \right\| < \delta \Rightarrow \left\| (\Phi_n, \Psi_n) - (\bar{\Phi}, \bar{\Psi}) \right\| < \varepsilon \text{ for all } n > 0.$$

(ii): $(\bar{\Phi}, \bar{\Psi})$ is said to be unstable if it is not stable.

(iii): $(\bar{\Phi}, \bar{\Psi})$ is called asymptotically stable if there exists $\gamma > 0$ such that

$$\left\| \sum_{i=-3}^{0} (\Phi_i, \Psi_i) - (\bar{\Phi}, \bar{\Psi}) \right\| < \gamma, \ (\Phi_n, \Psi_n) \to (\bar{\Phi}, \bar{\Psi}) \text{ as } n \to \infty.$$

(iv): $(\bar{\Phi}, \bar{\Psi})$ is called global attractor if $(\Phi_n, \Psi_n) \to (\bar{\Phi}, \bar{\Psi})$ as $n \to \infty$.

(v): $(\bar{\Phi}, \bar{\Psi})$ is called globally asymptotically stable if it is a global attractor and stable.

Theorem 2.1. *Assume that $(\Phi_{n+1}, \Psi_{n+1}) = F(\Phi_n, \Psi_n), n = 0, 1, ...,$ is a system of difference equations where F is continuously differentiable on open neighborhood $H \subseteq \mathbb{R}^{n+1}$ and $(\bar{\Phi}, \bar{\Psi})$ is a fixed point of F then*
(1) *If all eigenvalues of the Jacobian matrix J_F at equilibrium point $(\bar{\Phi}, \bar{\Psi})$ lie inside the unit disk i.e. $|\lambda_i| < 1$ then $(\bar{\Phi}, \bar{\Psi})$ is locally asymptotically stable.*
(2) *If at least one eigenvalues at equilibrium point $(\bar{\Phi}, \bar{\Psi})$ outside the unit disk, then $(\bar{\Phi}, \bar{\Psi})$ is unstable.*

Theorem 2.2. *[14] Let $[a_1, a_2]$ and $[b_1, b_2]$ be an interval of real numbers Moreover, suppose that $f : [a_1, a_2] \times [b_1, b_2] \to [a_1, a_2]$ and $g : [a_1, a_2] \times [b_1, b_2] \to [b_1, b_2]$ are continuous functions. Let*

$$\Phi_{n+1} = f(\Phi_n, \Psi_n) \quad \textit{and} \quad \Psi_{n+1} = g\,(\Phi_n, \Psi_n),$$

and assume (m_1, m_2, M_1, M_2) *be a solution of the system*

$$m_1 = f(m_1, m_2), \qquad M_1 = f(M_1, M_2),$$
$$m_2 = g(m_1, m_2), \qquad M_2 = g(M_1, M_2).$$

where

$$m_i = \begin{cases} m & \text{if } f \text{ or } g \quad \text{nondecreasing in } \Phi \text{ or } \Psi, \\ M & \text{if } f \text{ or } g \quad \text{nonincreasing in } \Phi \text{ or } \Psi, \end{cases}$$

and

$$M_i = \begin{cases} M & \text{if } f \text{ or } g \quad \text{nondecreasing in } \Phi \text{ or } \Psi, \\ m & \text{if } f \text{ or } g \quad \text{nonincreasing in } \Phi \text{ or } \Psi. \end{cases}$$

Thus, $m_1 = M_1$ *and* $m_2 = M_2$*. Then, the system of difference equations has a unique equilibrium point and it is a global attractor.*

3. On the System: $\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\alpha+\Phi_{n-2}+\Psi_{n-3}}$, $\Psi_{n+1} = \frac{\Phi_n\Psi_{n-2}}{\beta+\Phi_{n-3}+\Psi_{n-2}}$

In this section, we investigate the behavior of the solutions of the following system of difference equations

$$\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\alpha + \Phi_{n-2} + \Psi_{n-3}}, \quad \Psi_{n+1} = \frac{\Phi_n\Psi_{n-2}}{\beta + \Phi_{n-3} + \Psi_{n-2}}, \quad n = 0, 1, 2, ..., \tag{3.1}$$

and we take a special case $\alpha = \beta = 0$, in (3.1) to obtain a specific expression of the solutions of the following system of difference equations

$$\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2} + \Psi_{n-3}}, \quad \Psi_{n+1} = \frac{\Phi_n\Psi_{n-2}}{\Phi_{n-3} + \Psi_{n-2}}, \quad n = 0, 1, 2, ..., \tag{3.2}$$

where the initial conditions $\Phi_{-3}, \Phi_{-2}, \Phi_{-1}, \Phi_0, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$ and Ψ_0 are nonzero real numbers.

3.1. **Local Stability of Equilibrium Point.**

In this subsection, we study the stability of critical point $O = (0, 0)$ of the system (3.1).

Theorem 3.1. *The equilibrium point O is locally asymptotically stable.*

Proof. To investigate the stability of the critical point O, we assume

$$x_n = \Phi_{n-3}, y_n = \Phi_{n-2}, z_n = \Phi_{n-1},$$
$$u_n = \Psi_{n-3}, v_n = \Psi_{n-2}, w_n = \Psi_{n-1},$$

then the system (3.1) can be written as

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \\ \Phi_{n+1} \\ u_{n+1} \\ v_{n+1} \\ w_{n+1} \\ \Psi_{n+1} \end{pmatrix} = \begin{pmatrix} y_n \\ z_n \\ \Phi_n \\ \frac{y_n\Psi_n}{\alpha+y_n+u_n} \\ v_n \\ w_n \\ \Psi_n \\ \frac{\Phi_n v_n}{\beta+x_n+v_n} \end{pmatrix}. \tag{3.3}$$

The Jacobian matrix of (3.3) is given by

$$J_F = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & \frac{\alpha\Psi_n+u_n\Psi_n}{(\alpha+y_n+u_n)^2} & 0 & 0 & \frac{-y_n\Psi_n}{(\alpha+y_n+u_n)^2} & 0 & 0 & \frac{y_n}{\alpha+y_n+u_n} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \frac{-\Phi_n v_n}{(\beta+x_n+v_n)^2} & 0 & 0 & \frac{v_n}{\beta+x_n+v_n} & 0 & \frac{\beta\Phi_n+x_n\Phi_n}{(\beta+x_n+v_n)^2} & 0 & 0 \end{pmatrix}.$$

If we evaluate the Jacobian matrix about the equilibrium point, we get all eigenvalues $|\lambda_i| = 0, i = 1, 2, ..., 8$. So all eigenvalues are inside the unit disk. Therefore Theorem 2.1 ensures that the origin is locally asymptotically stable. □

3.2. **Global Attractor.** In this subsection, we will prove that the critical point is the global attractor.

Theorem 3.2. *The equilibrium point O of the system (3.1) is the global attractor.*

Proof. To prove this, we will assume that

$$f(p,q) = \frac{pq}{\alpha+p+q} \quad \text{and} \quad g(p,q) = \frac{pq}{\beta+p+q},$$

from here,

$$\frac{\partial f}{\partial p} = \frac{\alpha q + q^2}{(\alpha+p+q)^2}, \qquad \frac{\partial f}{\partial q} = \frac{\alpha p + p^2}{(\alpha+p+q)^2},$$
$$\frac{\partial g}{\partial p} = \frac{\beta q + q^2}{(\beta+p+q)^2}, \qquad \frac{\partial g}{\partial q} = \frac{\beta q + q^2}{(\beta+p+q)^2}.$$

We observe that $f(p,q)$ and $g(p,q)$ are nondecreasing in p and q. Let (m_1, m_2, M_1, M_2) be a solution of system (3.1) such that

$$m_1 = f(m_1, m_2), \quad M_1 = f(M_1, M_2),$$
$$m_2 = g(m_1, m_2), \quad M_2 = g(M_1, M_2).$$

So we have

$$m_1 = \frac{m_1 m_2}{\alpha + m_1 + m_2} \quad \Rightarrow \quad \alpha m_1 + m_1^2 + m_1 m_2 = m_1 m_2 \tag{3.4}$$

$$M_1 = \frac{M_1 M_2}{\alpha + M_1 + M_2} \quad \Rightarrow \alpha M_1 + M_1^2 + M_2 M_2 = M_1 M_2 \tag{3.5}$$

$$m_2 = \frac{m_1 m_2}{\beta + m_1 + m_2} \quad \Rightarrow \quad \beta m_2 + m_1 m_2 + m_2^2 = m_1 m_2 \tag{3.6}$$

$$M_2 = \frac{M_1 M_2}{\beta + M_1 + M_2} \quad \Rightarrow \beta M_1 + M_2 M_2 + M_2^2 = M_1 M_2 \tag{3.7}$$

By subtracting (3.4) from (3.5), we get

$$\alpha(m_1 - M_1) + (m_1^2 - M_1^2) = 0$$

$$\alpha(m_1 - M_1) + (m_1 - M_1)(m_1 + M_1) = 0$$

$$(m_1 - M_1)\left[\alpha + m_1 + M_1\right] = 0,$$

since $\alpha + m_1 + M_1 \neq 0$, hence

$$m_1 - M_1 = 0 \quad \Rightarrow \quad m_1 = M_1.$$

Similarly, we get $m_2 = M_2$. Then, from Theorem 2.2 there exists a unique equilibrium point of system (3.1) which is a global attractor. □

Theorem 3.3. *The equilibrium point O of the system (3.1) is globally asymptotically stable.*

Proof. We have from Theorem 3.1 that the equilibrium point O is locally stable. In addition, from Theorem 3.2 the equilibrium point O is globally attractor. Then from definition 2.2 the equilibrium point O is globally asymptotically stable. □

3.3. **On Solution of System (3.2).** In this subsection, we obtain the solution of system (3.2).

Theorem 3.4. *Assume $\{\Phi_n, \Psi_n\}_{n=-3}^{\infty}$ are a solution of system (3.2). Then for n=0,1,2,...,*

$$\Phi_{6n-3} = \frac{\eta^n \lambda^n \sigma^n \zeta \mu^n \tau^n \kappa^n}{\prod_{i=0}^{n-1} \left((2i+1)\eta + \tau\right)\left(\lambda + (2i)\,\mu\right)\left((2i)\lambda + \kappa\right)\left(\sigma + (2i+1)\tau\right)\left((2i+1)\sigma + \delta\right)\left(\zeta + (2i)\kappa\right)},$$

$$\Phi_{6n-2} = \frac{\eta^n \lambda^n \sigma^{n+1} \mu^n \tau^n \kappa^n}{\prod_{i=0}^{n-1} \left((2i)\eta + \tau\right)\left(\lambda + (2i+1)\,\mu\right)\left((2i+1)\lambda + \kappa\right)\left(\sigma + (2i)\tau\right)\left((2i+2)\sigma + \delta\right)\left(\zeta + (2i+1)\kappa\right)},$$

$$\Phi_{6n-1} = \frac{\eta^n \lambda^{n+1} \sigma^n \mu^n \tau^n \kappa^n}{\prod_{i=0}^{n-1} ((2i+1)\eta + \tau)(\lambda + (2i)\mu)((2i+2)\lambda + \kappa)(\sigma + (2i+1)\tau)((2i+1)\sigma + \delta)(\zeta + (2i+2)\kappa)},$$

$$\Phi_{6n} = \frac{\eta^{n+1} \lambda^n \sigma^n \mu^n \tau^n \kappa^n}{\prod_{i=0}^{n-1} ((2i+2)\eta + \tau)(\lambda + (2i+1)\mu)((2i+1)\lambda + \kappa)(\sigma + (2i+2)\tau)((2i+2)\sigma + \delta)(\zeta + (2i+1)\kappa)},$$

$$\Phi_{6n+1} = \frac{\eta^n \lambda^n \sigma^{n+1} \mu^{n+1} \tau^n \kappa^n}{\left[\begin{array}{c} (\sigma + \delta) \prod_{i=0}^{n-1} ((2i+1)\eta + \tau)(\lambda + (2i+2)\mu)((2i+2)\lambda + \kappa) \\ (\sigma + (2i+1)\tau)((2i+3)\sigma + \delta)(\zeta + (2i+2)\kappa) \end{array}\right]},$$

$$\Phi_{6n+2} = \frac{\eta^{n+1} \lambda^{n+1} \sigma^n \mu^n \tau^n \kappa^{n+1}}{\left[\begin{array}{c} (\lambda + \kappa)(\zeta + \kappa) \prod_{i=0}^{n-1} ((2i+2)\eta + \tau)(\lambda + (2i+1)\mu)((2i+3)\lambda + \kappa) \\ (\sigma + (2i+2)\tau)((2i+2)\sigma + \delta)(\zeta + (2i+3)\kappa) \end{array}\right]},$$

$$\Psi_{6n-3} = \frac{\eta^n \lambda^n \sigma^n \mu^n \tau^n \kappa^n \delta}{\prod_{i=0}^{n-1} ((2i)\eta + \tau)(\lambda + (2i+1)\mu)((2i+1)\lambda + \kappa)(\sigma + (2i)\tau)((2i)\sigma + \delta)(\zeta + (2i+1)\kappa)},$$

$$\Psi_{6n-2} = \frac{\eta^n \lambda^n \sigma^n \mu^n \tau^n \kappa^{n+1}}{\prod_{i=0}^{n-1} ((2i+1)\eta + \tau)(\lambda + (2i)\mu)((2i)\lambda + \kappa)(\sigma + (2i+1)\tau)((2i+1)\sigma + \delta)(\zeta + (2i+2)\kappa)},$$

$$\Psi_{6n-1} = \frac{\eta^n \lambda^n \sigma^n \mu^n \tau^{n+1} \kappa^n}{\prod_{i=0}^{n-1} ((2i)\eta + \tau)(\lambda + (2i+1)\mu)((2i+1)\lambda + \kappa)(\sigma + (2i+2)\tau)((2i+2)\sigma + \delta)(\zeta + (2i+1)\kappa)},$$

$$\Psi_{6n} = \frac{\eta^n \lambda^n \sigma^n \mu^{n+1} \tau^n \kappa^n}{\prod_{i=0}^{n-1} ((2i+1)\eta + \tau)(\lambda + (2i+2)\mu)((2i+2)\lambda + \kappa)(\sigma + (2i+1)\tau)((2i+1)\sigma + \delta)(\zeta + (2i+2)\kappa)},$$

$$\Psi_{6n+1} = \frac{\eta^{n+1} \lambda^n \sigma^n \mu^n \tau^n \kappa^{n+1}}{\left[\begin{array}{c} (\zeta + \kappa) \prod_{i=0}^{n-1} ((2i+2)\eta + \tau)(\lambda + (2i+1)\mu)((2i+1)\lambda + \kappa) \\ (\sigma + (2i+2)\tau)((2i+2)\sigma + \delta)(\zeta + (2i+3)\kappa) \end{array}\right]},$$

$$\Psi_{6n+2} = \frac{\eta^n \lambda^n \sigma^{n+1} \mu^{n+1} \tau^{n+1} \kappa^n}{\left[\begin{array}{c} (\sigma + \tau)(\sigma + \delta) \prod_{i=0}^{n-1} ((2i+1)\eta + \tau)(\lambda + (2i+2)\mu)((2i+2)\lambda + \kappa) \\ (\sigma + (2i+3)\tau)((2i+3)\sigma + \delta)(\zeta + (2i+2)\kappa) \end{array}\right]},$$

where $\Phi_{-3} = \zeta, \Phi_{-2} = \sigma, \Phi_{-1} = \lambda, \Phi_0 = \eta, \Psi_{-3} = \delta, \Psi_{-2} = \kappa, \Psi_{-1} = \tau$ *and* $\Psi_0 = \mu$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and our assumption holds for $n-1$, that is;

$$\Phi_{6n-9} = \frac{\eta^{n-1} \lambda^{n-1} \sigma^{n-1} \zeta \mu^{n-1} \tau^{n-1} \kappa^{n-1}}{\prod_{i=0}^{n-2} ((2i+1)\eta + \tau)(\lambda + (2i)\mu)((2i)\lambda + \kappa)(\sigma + (2i+1)\tau)((2i+1)\sigma + \delta)(\zeta + (2i)\kappa)},$$

$$\Phi_{6n-8}=\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}\left((2i)\eta+\tau\right)\left(\lambda+(2i+1)\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma+(2i)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta+(2i+1)\kappa\right)},$$

$$\Phi_{6n-7}=\frac{\eta^{n-1}\lambda^{n}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left(\lambda+(2i)\mu\right)\left((2i+2)\lambda+\kappa\right)\left(\sigma+(2i+1)\tau\right)\left((2i+1)\sigma+\delta\right)\left(\zeta+(2i+2)\kappa\right)},$$

$$\Phi_{6n-6}=\frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}\left((2i+2)\eta+\tau\right)\left(\lambda+(2i+1)\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma+(2i+2)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta+(2i+1)\kappa\right)},$$

$$\Phi_{6n-5}=\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\delta)\prod_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left(\lambda+(2i+2)\mu\right)\left((2i+2)\lambda+\kappa\right)\\ \left(\sigma+(2i+1)\tau\right)\left((2i+3)\sigma+\delta\right)\left(\zeta+(2i+2)\kappa\right)\end{array}\right]},$$

$$\Phi_{6n-4}=\frac{\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\left[\begin{array}{c}(\lambda+\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}\left((2i+2)\eta+\tau\right)\left(\lambda+(2i+1)\mu\right)\left((2i+3)\lambda+\kappa\right)\\ \left(\sigma+(2i+2)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta+(2i+3)\kappa\right)\end{array}\right]},$$

$$\Psi_{6n-9}=\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}\delta}{\prod_{i=0}^{n-2}\left((2i)\eta+\tau\right)\left(\lambda+(2i+1)\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma+(2i)\tau\right)\left((2i)\sigma+\delta\right)\left(\zeta+(2i+1)\kappa\right)},$$

$$\Psi_{6n-8}=\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\prod_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left(\lambda+(2i)\mu\right)\left((2i)\lambda+\kappa\right)\left(\sigma+(2i+1)\tau\right)\left((2i+1)\sigma+\delta\right)\left(\zeta+(2i+2)\kappa\right)},$$

$$\Psi_{6n-7}=\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n}\kappa^{n-1}}{\prod_{i=0}^{n-2}\left((2i)\eta+\tau\right)\left(\lambda+(2i+1)\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma+(2i+2)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta+(2i+1)\kappa\right)},$$

$$\Psi_{6n-6}=\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left(\lambda+(2i+2)\mu\right)\left((2i+2)\lambda+\kappa\right)\left(\sigma+(2i+1)\tau\right)\left((2i+1)\sigma+\delta\right)\left(\zeta+(2i+2)\kappa\right)},$$

$$\Psi_{6n-5}=\frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\left[\begin{array}{c}(\zeta+\kappa)\prod_{i=0}^{n-2}\left((2i+2)\eta+\tau\right)\left(\lambda+(2i+1)\mu\right)\left((2i+1)\lambda+\kappa\right)\\ \left(\sigma+(2i+2)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta+(2i+3)\kappa\right)\end{array}\right]},$$

$$\Psi_{6n-4}=\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma+\delta)\prod_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left(\lambda+(2i+2)\mu\right)\left((2i+2)\lambda+\kappa\right)\\ \left(\sigma+(2i+3)\tau\right)\left((2i+3)\sigma+\delta\right)\left(\zeta+(2i+2)\kappa\right)\end{array}\right]}.$$

Now, from the system (3.2) it follows that

$$\Phi_{6n-3}=\frac{\Phi_{6n-6}\Psi_{6n-4}}{\Phi_{6n-6}+\Psi_{6n-7}}$$

$$=\frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+1)\kappa)}\right]\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma+\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)}\right]}{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+1)\kappa)}\right]+\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n}\kappa^{n-1}}{\prod_{i=0}^{n-2}((2i)\eta+\tau)(\lambda+(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+1)\kappa)}\right]}$$

$$=\frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma+\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)}\right]}{\left[\frac{\eta}{\prod_{i=0}^{n-2}((2i+2)\eta+\tau)}\right]+\left[\frac{\tau}{\prod_{i=0}^{n-2}((2i)\eta+\tau)}\right]}$$

$$=\frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma+\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)}\right]}{\eta+\left[\frac{\tau\prod_{i=0}^{n-2}((2i+2)\eta+\tau)}{\prod_{i=0}^{n-2}((2i)\eta+\tau)}\right]}$$

$$=\frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma+\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)}\right]}{\eta+((2n-2)\eta+\tau)}$$

$$=\frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\tau)(\sigma+\delta)((2n-1)\eta+\tau)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)\\(\sigma+(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta+(2i+2)\kappa)\end{array}\right]}.$$

Consequently, we obtain

$$\Phi_{6n-3}=\frac{\eta^{n}\lambda^{n}\sigma^{n}\zeta\mu^{n}\tau^{n}\kappa^{n}}{\prod_{i=0}^{n-1}((2i+1)\eta+\tau)(\lambda+(2i)\mu)((2i)\lambda+\kappa)(\sigma+(2i+1)\tau)((2i+1)\sigma+\delta)(\zeta+(2i)\kappa)}.$$

In a similar manner, we have from the system (3.2) that

$$\Psi_{6n-3}=\frac{\Phi_{6n-4}\Psi_{6n-6}}{\Phi_{6n-7}+\Psi_{6n-6}}$$

$$=\frac{\left[\frac{\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{(\lambda+\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda+\kappa)(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)}\right]\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+1)\tau)((2i+1)\sigma+\delta)(\zeta+(2i+2)\kappa)}\right]}{\left[\frac{\eta^{n-1}\lambda^{n}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+1)\tau)((2i+1)\sigma+\delta)(\zeta+(2i+2)\kappa)}\right]+\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda+(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma+(2i+1)\tau)((2i+1)\sigma+\delta)(\zeta+(2i+2)\kappa)}\right]}$$

$$=\frac{\left[\frac{\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n}}{(\lambda+\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda+\kappa)(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)}\right]}{\left[\frac{\lambda}{\prod_{i=0}^{n-2}(\lambda+(2i)\mu)}\right]+\left[\frac{\mu}{\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)}\right]}$$

$$=\frac{\left[\frac{\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n}}{(\lambda+\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda+\kappa)(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)}\right]}{\left[\frac{\lambda\prod_{i=0}^{n-2}(\lambda+(2i+2)\mu)}{\prod_{i=0}^{n-2}(\lambda+(2i)\mu)}\right]+\mu}$$

$$=\frac{\left[\frac{\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n}}{(\lambda+\kappa)(\zeta+\kappa)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda+\kappa)(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)}\right]}{(\lambda+(2n-2)\mu)+\mu}$$

$$=\frac{\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n}}{\left[\begin{array}{c}(\lambda+\kappa)(\zeta+\kappa)(\lambda+(2n-1)\mu)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda+(2i+1)\mu)((2i+3)\lambda+\kappa)\\(\sigma+(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta+(2i+3)\kappa)\end{array}\right]}.$$

Hence, we get

$$\Psi_{6n-3} = \frac{\eta^n\lambda^n\sigma^n\mu^n\tau^n\kappa^n\delta}{\prod_{i=0}^{n-1}\left((2i)\eta+\tau\right)\left(\lambda+(2i+1)\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma+(2i)\tau\right)\left((2i)\sigma+\delta\right)\left(\zeta+(2i+1)\kappa\right)}.$$

Other expressions can be investigated in the same way. □

3.4. **Boundedness of The Solution.** In this subsection, we demonstrate that the positive solutions of system (3.2) are bounded.

Lemma 3.1. *Every positive solution of system (3.2) is bounded and converges to zero.*

Proof. System (3.2) shows that

$$\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2}+\Psi_{n-3}} \le \frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2}} \le \Psi_n,$$
$$\Psi_{n+1} = \frac{\Phi_n\Psi_{n-2}}{\Phi_{n-3}+\Psi_{n-2}} \le \frac{\Phi_n\Psi_{n-2}}{\Psi_{n-2}} \le \Phi_n.$$

We conclude that

$$\Phi_{n+1} \le \Psi_n \qquad \text{and} \qquad \Psi_{n+1} \le \Phi_n.$$

If we set $n = n+1, n+2, ...,$ we get

$$\Phi_{n+2} \le \Psi_{n+1} \le \Phi_n \qquad \text{and} \qquad \Psi_{n+2} \le \Phi_{n+1} \le \Psi_n,$$
$$\Phi_{n+3} \le \Psi_{n+2} \le \Phi_{n+1} \qquad \text{and} \qquad \Psi_{n+3} \le \Phi_{n+2} \le \Psi_{n+1} \quad , ...,$$

and so on. This implies that the subsequences $\{\Phi_{6n-3}\}_{n=0}^{\infty}, \{\Phi_{6n-2}\}_{n=0}^{\infty}, \{\Phi_{6n-1}\}_{n=0}^{\infty}, \{\Phi_{6n}\}_{n=0}^{\infty}$, $\{\Phi_{6n+1}\}_{n=0}^{\infty}, \{\Phi_{6n+2}\}_{n=0}^{\infty}$ and $\{\Psi_{6n-3}\}_{n=0}^{\infty}, \{\Psi_{6n-2}\}_{n=0}^{\infty}, \{\Psi_{6n-1}\}_{n=0}^{\infty}, \{\Psi_{6n}\}_{n=0}^{\infty}, \{\Psi_{6n+1}\}_{n=0}^{\infty}, \{\Psi_{6n+2}\}_{n=0}^{\infty}$ are decreasing and so are bounded from above by

$$\Phi_{\max} = \max\{\Phi_{-3}, \Phi_{-2}, \Phi_{-1}, \Phi_0\},$$

and

$$\Psi_{\max} = \max\{\Psi_{-3}, \Psi_{-2}, \Psi_{-1}, \Psi_0\}.$$

□

In the following cases, we will obtain the solution expression when we take $\alpha = \beta = 0$, in (1.1).

4. On the System: $\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2}+\Psi_{n-3}}, \Psi_{n+1} = \frac{\Phi_n\Psi_{n-2}}{\Phi_{n-3}-\Psi_{n-2}}$

In this section, we will get a specific expression of the solutions of the following system of difference equations

$$\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2}+\Psi_{n-3}}, \ \ \Psi_{n+1} = \frac{\Phi_n\Psi_{n-2}}{\Phi_{n-3}-\Psi_{n-2}}, \ n = 0, 1, 2, ..., \tag{4.1}$$

where the initial conditions $\Phi_{-3}, \Phi_{-2}, \Phi_{-1}, \Phi_0, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$ and Ψ_0 are nonzero real numbers.

Theorem 4.1. *Assume $\{\Phi_n, \Psi_n\}_{n=-3}^{\infty}$ are a solution of system (4.1). Then for n=0,1,2,...,*

$$\Phi_{6n-3} = \frac{\eta^n\lambda^n\sigma^n\zeta\mu^n\tau^n\kappa^n}{\prod_{i=0}^{n-1}\left((2i+1)\eta+\tau\right)\left(\lambda-(2i)\,\mu\right)\left((2i)\lambda+\kappa\right)\left(\sigma-(2i+1)\tau\right)\left((2i+1)\sigma+\delta\right)\left(\zeta-(2i)\kappa\right)},$$

$$\Phi_{6n-2} = \frac{\eta^n\lambda^n\sigma^{n+1}\mu^n\tau^n\kappa^n}{\prod_{i=0}^{n-1}\left((2i)\eta+\tau\right)\left(\lambda-(2i+1)\,\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma-(2i)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta-(2i+1)\kappa\right)},$$

$$\Phi_{6n-1} = \frac{\eta^n\lambda^{n+1}\sigma^n\mu^n\tau^n\kappa^n}{\prod_{i=0}^{n-1}\left((2i+1)\eta+\tau\right)\left(\lambda-(2i)\,\mu\right)\left((2i+2)\lambda+\kappa\right)\left(\sigma-(2i+1)\tau\right)\left((2i+1)\sigma+\delta\right)\left(\zeta-(2i+2)\kappa\right)},$$

$$\Phi_{6n} = \frac{\eta^{n+1}\lambda^n\sigma^n\mu^n\tau^n\kappa^n}{\prod_{i=0}^{n-1}\left((2i+2)\eta+\tau\right)\left(\lambda-(2i+1)\,\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma-(2i+2)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta-(2i+1)\kappa\right)},$$

$$\Phi_{6n+1} = \frac{\eta^n\lambda^n\sigma^{n+1}\mu^{n+1}\tau^n\kappa^n}{\left[\begin{array}{c}(\sigma+\delta)\prod_{i=0}^{n-1}\left((2i+1)\eta+\tau\right)\left(\lambda-(2i+2)\,\mu\right)\left((2i+2)\lambda+\kappa\right)\\ \left(\sigma-(2i+1)\tau\right)\left((2i+3)\sigma+\delta\right)\left(\zeta-(2i+2)\kappa\right)\end{array}\right]},$$

$$\Phi_{6n+2} = \frac{\eta^{n+1}\lambda^{n+1}\sigma^n\mu^n\tau^n\kappa^{n+1}}{\left[\begin{array}{c}(\lambda+\kappa)(\zeta-\kappa)\prod_{i=0}^{n-1}\left((2i+2)\eta+\tau\right)\left(\lambda-(2i+1)\,\mu\right)\left((2i+3)\lambda+\kappa\right)\\ \left(\sigma-(2i+2)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta-(2i+3)\kappa\right)\end{array}\right]},$$

$$\Psi_{6n-3} = \frac{\eta^n\lambda^n\sigma^n\mu^n\tau^n\kappa^n\delta}{\prod_{i=0}^{n-1}\left((2i)\eta+\tau\right)\left(\lambda-(2i+1)\,\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma-(2i)\tau\right)\left((2i)\sigma+\delta\right)\left(\zeta-(2i+1)\kappa\right)},$$

$$\Psi_{6n-2} = \frac{\eta^n\lambda^n\sigma^n\mu^n\tau^n\kappa^{n+1}}{\prod_{i=0}^{n-1}\left((2i+1)\eta+\tau\right)\left(\lambda-(2i)\,\mu\right)\left((2i)\lambda+\kappa\right)\left(\sigma-(2i+1)\tau\right)\left((2i+1)\sigma+\delta\right)\left(\zeta-(2i+2)\kappa\right)},$$

$$\Psi_{6n-1} = \frac{\eta^n\lambda^n\sigma^n\mu^n\tau^{n+1}\kappa^n}{\prod_{i=0}^{n-1}\left((2i)\eta+\tau\right)\left(\lambda-(2i+1)\,\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma-(2i+2)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta-(2i+1)\kappa\right)},$$

$$\Psi_{6n} = \frac{\eta^n\lambda^n\sigma^n\mu^{n+1}\tau^n\kappa^n}{\prod_{i=0}^{n-1}\left((2i+1)\eta+\tau\right)\left(\lambda-(2i+2)\,\mu\right)\left((2i+2)\lambda+\kappa\right)\left(\sigma-(2i+1)\tau\right)\left((2i+1)\sigma+\delta\right)\left(\zeta-(2i+2)\kappa\right)},$$

$$\Psi_{6n+1} = \frac{\eta^{n+1}\lambda^n\sigma^n\mu^n\tau^n\kappa^{n+1}}{\left[\begin{array}{c}(\zeta-\kappa)\prod_{i=0}^{n-1}\left((2i+2)\eta+\tau\right)\left(\lambda-(2i+1)\,\mu\right)\left((2i+1)\lambda+\kappa\right)\\ \left(\sigma-(2i+2)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta-(2i+3)\kappa\right)\end{array}\right]},$$

$$\Psi_{6n+2} = \frac{\eta^n\lambda^n\sigma^{n+1}\mu^{n+1}\tau^{n+1}\kappa^n}{\left[\begin{array}{c}(\sigma-\tau)(\sigma+\delta)\prod_{i=0}^{n-1}\left((2i+1)\eta+\tau\right)\left(\lambda-(2i+2)\,\mu\right)\left((2i+2)\lambda+\kappa\right)\\ \left(\sigma-(2i+3)\tau\right)\left((2i+3)\sigma+\delta\right)\left(\zeta-(2i+2)\kappa\right)\end{array}\right]},$$

where $\Phi_{-3} = \zeta, \Phi_{-2} = \sigma, \Phi_{-1} = \lambda, \Phi_0 = \eta, \Psi_{-3} = \delta, \Psi_{-2} = \kappa, \Psi_{-1} = \tau$ and $\Psi_0 = \mu$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and our assumption holds for $n-1$, that is;

$$\Phi_{6n-9} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\zeta\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left(\lambda-(2i)\mu\right)\left((2i)\lambda+\kappa\right)\left(\sigma-(2i+1)\tau\right)\left((2i+1)\sigma+\delta\right)\left(\zeta-(2i)\kappa\right)},$$

$$\Phi_{6n-8} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}\left((2i)\eta+\tau\right)\left(\lambda-(2i+1)\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma-(2i)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta-(2i+1)\kappa\right)},$$

$$\Phi_{6n-7} = \frac{\eta^{n-1}\lambda^{n}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left(\lambda-(2i)\mu\right)\left((2i+2)\lambda+\kappa\right)\left(\sigma-(2i+1)\tau\right)\left((2i+1)\sigma+\delta\right)\left(\zeta-(2i+2)\kappa\right)},$$

$$\Phi_{6n-6} = \frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}\left((2i+2)\eta+\tau\right)\left(\lambda-(2i+1)\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma-(2i+2)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta-(2i+1)\kappa\right)},$$

$$\Phi_{6n-5} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma+\delta)\prod_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left(\lambda-(2i+2)\mu\right)\left((2i+2)\lambda+\kappa\right)\\ \left(\sigma-(2i+1)\tau\right)\left((2i+3)\sigma+\delta\right)\left(\zeta-(2i+2)\kappa\right)\end{array}\right]},$$

$$\Phi_{6n-4} = \frac{\eta^{n}\lambda^{n}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\left[\begin{array}{c}(\lambda+\kappa)(\zeta-\kappa)\prod_{i=0}^{n-2}\left((2i+2)\eta+\tau\right)\left(\lambda-(2i+1)\mu\right)\left((2i+3)\lambda+\kappa\right)\\ \left(\sigma-(2i+2)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta-(2i+3)\kappa\right)\end{array}\right]},$$

$$\Psi_{6n-9} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}\delta}{\prod_{i=0}^{n-2}\left((2i)\eta+\tau\right)\left(\lambda-(2i+1)\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma-(2i)\tau\right)\left((2i)\sigma+\delta\right)\left(\zeta-(2i+1)\kappa\right)},$$

$$\Psi_{6n-8} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\prod_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left(\lambda-(2i)\mu\right)\left((2i)\lambda+\kappa\right)\left(\sigma-(2i+1)\tau\right)\left((2i+1)\sigma+\delta\right)\left(\zeta-(2i+2)\kappa\right)},$$

$$\Psi_{6n-7} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n}\kappa^{n-1}}{\prod_{i=0}^{n-2}\left((2i)\eta+\tau\right)\left(\lambda-(2i+1)\mu\right)\left((2i+1)\lambda+\kappa\right)\left(\sigma-(2i+2)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta-(2i+1)\kappa\right)},$$

$$\Psi_{6n-6} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left(\lambda-(2i+2)\mu\right)\left((2i+2)\lambda+\kappa\right)\left(\sigma-(2i+1)\tau\right)\left((2i+1)\sigma+\delta\right)\left(\zeta-(2i+2)\kappa\right)},$$

$$\Psi_{6n-5} = \frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\left[\begin{array}{c}(\zeta-\kappa)\prod_{i=0}^{n-2}\left((2i+2)\eta+\tau\right)\left(\lambda-(2i+1)\mu\right)\left((2i+1)\lambda+\kappa\right)\\ \left(\sigma-(2i+2)\tau\right)\left((2i+2)\sigma+\delta\right)\left(\zeta-(2i+3)\kappa\right)\end{array}\right]},$$

$$\Psi_{6n-4} = \frac{\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma-\tau)(\sigma+\delta)\prod_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left(\lambda-(2i+2)\mu\right)\left((2i+2)\lambda+\kappa\right)\\ \left(\sigma-(2i+3)\tau\right)\left((2i+3)\sigma+\delta\right)\left(\zeta-(2i+2)\kappa\right)\end{array}\right]}.$$

Now, it follows from system (4.1) that

$$\Phi_{6n-3}=\frac{\Phi_{6n-6}\Psi_{6n-4}}{\Phi_{6n-6}+\Psi_{6n-7}}$$

$$=\frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda-(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma-(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta-(2i+1)\kappa)}\right]\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma-\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda-(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma-(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta-(2i+2)\kappa)}\right]}{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda-(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma-(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta-(2i+1)\kappa)}\right]+\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n}\kappa^{n-1}}{\prod_{i=0}^{n-2}((2i)\eta+\tau)(\lambda-(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma-(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta-(2i+1)\kappa)}\right]}$$

$$=\frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma-\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda-(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma-(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta-(2i+2)\kappa)}\right]}{\left[\frac{\eta}{\prod_{i=0}^{n-2}((2i+2)\eta+\tau)}\right]+\left[\frac{\tau}{\prod_{i=0}^{n-2}((2i)\eta+\tau)}\right]}$$

$$=\frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma-\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda-(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma-(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta-(2i+2)\kappa)}\right]}{\eta+\left[\frac{\tau\prod_{i=0}^{n-2}((2i+2)\eta+\tau)}{\prod_{i=0}^{n-2}((2i)\eta+\tau)}\right]}$$

$$=\frac{\left[\frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma-\tau)(\sigma+\delta)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda-(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma-(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta-(2i+2)\kappa)}\right]}{\eta+((2n-2)\eta+\tau)}$$

$$=\frac{\eta^{n}\lambda^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left[\begin{array}{c}(\sigma-\tau)(\sigma+\delta)((2n-1)\eta+\tau)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda-(2i+2)\mu)((2i+2)\lambda+\kappa)\\(\sigma-(2i+3)\tau)((2i+3)\sigma+\delta)(\zeta-(2i+2)\kappa)\end{array}\right]}.$$

Therefore, we get

$$\Phi_{6n-3}=\frac{\eta^{n}\lambda^{n}\sigma^{n}\zeta\mu^{n}\tau^{n}\kappa^{n}}{\prod_{i=0}^{n-1}((2i+1)\eta+\tau)(\lambda-(2i)\mu)((2i)\lambda+\kappa)(\sigma-(2i+1)\tau)((2i+1)\sigma+\delta)(\zeta-(2i)\kappa)}.$$

$$\Psi_{6n-3} = \frac{\Phi_{6n-4}\Psi_{6n-6}}{\Phi_{6n-7}-\Psi_{6n-6}}$$

$$= \frac{\left[\frac{\eta^n\lambda^n\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^n}{(\lambda+\kappa)(\zeta-\kappa)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda-(2i+1)\mu)((2i+3)\lambda+\kappa)(\sigma-(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta-(2i+3)\kappa)}\right]\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^n\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda-(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma-(2i+1)\tau)((2i+1)\sigma+\delta)(\zeta-(2i+2)\kappa)}\right]}{\left[\frac{\eta^{n-1}\lambda^n\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda-(2i)\mu)((2i+2)\lambda+\kappa)(\sigma-(2i+1)\tau)((2i+1)\sigma+\delta)(\zeta-(2i+2)\kappa)}\right]-\left[\frac{\eta^{n-1}\lambda^{n-1}\sigma^{n-1}\mu^n\tau^{n-1}\kappa^{n-1}}{\prod_{i=0}^{n-2}((2i+1)\eta+\tau)(\lambda-(2i+2)\mu)((2i+2)\lambda+\kappa)(\sigma-(2i+1)\tau)((2i+1)\sigma+\delta)(\zeta-(2i+2)\kappa)}\right]}$$

$$= \frac{\left[\frac{\eta^n\lambda^n\sigma^{n-1}\mu^n\tau^{n-1}\kappa^n}{(\lambda+\kappa)(\zeta-\kappa)\prod_{i=0}^{n-2}(\lambda-(2i+2)\mu)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda-(2i+1)\mu)((2i+3)\lambda+\kappa)(\sigma-(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta-(2i+3)\kappa)}\right]}{\left[\frac{\lambda}{\prod_{i=0}^{n-2}(\lambda-(2i)\mu)}\right]-\left[\frac{\mu}{\prod_{i=0}^{n-2}(\lambda-(2i+2)\mu)}\right]}$$

$$= \frac{\left[\frac{\eta^n\lambda^n\sigma^{n-1}\mu^n\tau^{n-1}\kappa^n}{(\lambda+\kappa)(\zeta-\kappa)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda-(2i+1)\mu)((2i+3)\lambda+\kappa)(\sigma-(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta-(2i+3)\kappa)}\right]}{\left[\frac{\lambda\prod_{i=0}^{n-2}(\lambda-(2i+2)\mu)}{\prod_{i=0}^{n-2}(\lambda-(2i)\mu)}\right]-\mu}$$

$$= \frac{\left[\frac{\eta^n\lambda^n\sigma^{n-1}\mu^n\tau^{n-1}\kappa^n}{(\lambda+\kappa)(\zeta-\kappa)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda-(2i+1)\mu)((2i+3)\lambda+\kappa)(\sigma-(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta-(2i+3)\kappa)}\right]}{(\lambda-(2n-2)\mu)-\mu}$$

$$= \frac{\eta^n\lambda^n\sigma^{n-1}\mu^n\tau^{n-1}\kappa^n}{\left[\begin{array}{c}(\lambda+\kappa)(\zeta-\kappa)(\lambda-(2n-1)\mu)\prod_{i=0}^{n-2}((2i+2)\eta+\tau)(\lambda-(2i+1)\mu)((2i+3)\lambda+\kappa)\\(\sigma-(2i+2)\tau)((2i+2)\sigma+\delta)(\zeta-(2i+3)\kappa)\end{array}\right]}.$$

Hence, we obtain

$$\Psi_{6n-3} = \frac{\eta^n\lambda^n\sigma^n\mu^n\tau^n\kappa^n\delta}{\prod_{i=0}^{n-1}((2i)\eta+\tau)(\lambda-(2i+1)\mu)((2i+1)\lambda+\kappa)(\sigma-(2i)\tau)((2i)\sigma+\delta)(\zeta-(2i+1)\kappa)}.$$

The following cases can be proved using a similar technique. □

5. On the System: $\Phi_{n+1}=\frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2}+\Psi_{n-3}}, \Psi_{n+1}=\frac{\Phi_n\Psi_{n-2}}{-\Phi_{n-3}+\Psi_{n-2}}$

In this section, we will obtain a specific expression of the solutions of the following system of difference equations

$$\Phi_{n+1}=\frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2}+\Psi_{n-3}},\ \ \Psi_{n+1}=\frac{\Phi_n\Psi_{n-2}}{-\Phi_{n-3}+\Psi_{n-2}},\ n=0,1,2,..., \tag{5.1}$$

where the initial conditions $\Phi_{-3},\Phi_{-2},\Phi_{-1},\Phi_0,\Psi_{-3},\Psi_{-2},\Psi_{-1}$ and Ψ_0 are nonzero real numbers.

Theorem 5.1. *Assume $\{\Phi_n,\Psi_n\}_{n=-3}^{\infty}$ are a solution of system (5.1). Then for n=0,1,2,...,*

$$\Phi_{6n-3}=\frac{\eta^n\sigma^n\mu^n\tau^n\kappa^n}{\zeta^{n-1}\left(\tau-\sigma\right)^n\prod\limits_{i=0}^{n-1}\left((2i+1)\eta+\tau\right)\left((2i)\lambda+\kappa\right)\left((2i+1)\sigma+\delta\right)},$$

$$\Phi_{6n-2}=\frac{\eta^n\lambda^n\sigma\mu^n\tau^n\kappa^n}{\left(\mu-\lambda\right)^n\left(\kappa-\zeta\right)^n\prod\limits_{i=0}^{n-1}\left((2i)\eta+\tau\right)\left((2i+1)\lambda+\kappa\right)\left((2i+2)\sigma+\delta\right)},$$

$$\Phi_{6n-1}=\frac{\eta^n\lambda\sigma^n\mu^n\tau^n\kappa^n}{\zeta^n\left(\tau-\sigma\right)^n\prod\limits_{i=0}^{n-1}\left((2i+1)\eta+\tau\right)\left((2i+2)\lambda+\kappa\right)\left((2i+1)\sigma+\delta\right)},$$

$$\Phi_{6n}=\frac{\eta^{n+1}\lambda^n\mu^n\tau^n\kappa^n}{\left(\mu-\lambda\right)^n\left(\kappa-\zeta\right)^n\prod\limits_{i=0}^{n-1}\left((2i+2)\eta+\tau\right)\left((2i+1)\lambda+\kappa\right)\left((2i+2)\sigma+\delta\right)},$$

$$\Phi_{6n+1}=\frac{\eta^n\sigma^{n+1}\mu^{n+1}\tau^n\kappa^n}{\left(\sigma+\delta\right)\zeta^n\left(\tau-\sigma\right)^n\prod\limits_{i=0}^{n-1}\left((2i+1)\eta+\tau\right)\left((2i+2)\lambda+\kappa\right)\left((2i+3)\sigma+\delta\right)},$$

$$\Phi_{6n+2}=\frac{\eta^{n+1}\lambda^{n+1}\mu^n\tau^n\kappa^{n+1}}{\left(\lambda+\kappa\right)\left(\mu-\lambda\right)^n\left(\kappa-\zeta\right)^{n+1}\prod\limits_{i=0}^{n-1}\left((2i+2)\eta+\tau\right)\left((2i+3)\lambda+\kappa\right)\left((2i+2)\sigma+\delta\right)},$$

$$\Psi_{6n-3}=\frac{\eta^n\lambda^n\mu^n\tau^n\kappa^n\delta}{\left(\mu-\lambda\right)^n\left(\kappa-\zeta\right)^n\prod\limits_{i=0}^{n-1}\left((2i)\eta+\tau\right)\left((2i+1)\lambda+\kappa\right)\left((2i)\sigma+\delta\right)},$$

$$\Psi_{6n-2}=\frac{\eta^n\sigma^n\mu^n\tau^n\kappa^{n+1}}{\zeta^n\left(\tau-\sigma\right)^n\prod\limits_{i=0}^{n-1}\left((2i+1)\eta+\tau\right)\left((2i)\lambda+\kappa\right)\left((2i+1)\sigma+\delta\right)},$$

$$\Psi_{6n-1}=\frac{\eta^n\lambda^n\mu^n\tau^{n+1}\kappa^n}{\left(\mu-\lambda\right)^n\left(\kappa-\zeta\right)^n\prod\limits_{i=0}^{n-1}\left((2i)\eta+\tau\right)\left((2i+1)\lambda+\kappa\right)\left((2i+2)\sigma+\delta\right)},$$

$$\Psi_{6n}=\frac{\eta^n\sigma^n\mu^{n+1}\tau^n\kappa^n}{\zeta^n\left(\tau-\sigma\right)^n\prod\limits_{i=0}^{n-1}\left((2i+1)\eta+\tau\right)\left((2i+2)\lambda+\kappa\right)\left((2i+1)\sigma+\delta\right)},$$

$$\Psi_{6n+1} = \frac{\eta^{n+1}\lambda^{n}\mu^{n}\tau^{n}\kappa^{n+1}}{(\mu-\lambda)^{n}(\kappa-\zeta)^{n+1}\prod_{i=0}^{n-1}((2i+2)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)},$$

$$\Psi_{6n+2} = \frac{\eta^{n}\sigma^{n+1}\mu^{n+1}\tau^{n+1}\kappa^{n}}{(\sigma+\delta)\zeta^{n}(\tau-\sigma)^{n+1}\prod_{i=0}^{n-1}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)},$$

where $\Phi_{-3} = \zeta, \Phi_{-2} = \sigma, \Phi_{-1} = \lambda, \Phi_0 = \eta, \Psi_{-3} = \delta, \Psi_{-2} = \kappa, \Psi_{-1} = \tau$ *and* $\Psi_0 = \mu$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and our assumption holds for $n-1$, that is;

$$\Phi_{6n-9} = \frac{\eta^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\zeta^{n-2}(\tau-\sigma)^{n-1}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i)\lambda+\kappa)((2i+1)\sigma+\delta)},$$

$$\Phi_{6n-8} = \frac{\eta^{n-1}\lambda^{n-1}\sigma\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{(\mu-\lambda)^{n-1}(\kappa-\zeta)^{n-1}\prod_{i=0}^{n-2}((2i)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)},$$

$$\Phi_{6n-7} = \frac{\eta^{n-1}\lambda\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\zeta^{n-1}(\tau-\sigma)^{n-1}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+1)\sigma+\delta)},$$

$$\Phi_{6n-6} = \frac{\eta^{n}\lambda^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{(\mu-\lambda)^{n-1}(\kappa-\zeta)^{n-1}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)},$$

$$\Phi_{6n-5} = \frac{\eta^{n-1}\sigma^{n}\mu^{n}\tau^{n-1}\kappa^{n-1}}{(\sigma+\delta)\zeta^{n-1}(\tau-\sigma)^{n-1}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)},$$

$$\Phi_{6n-4} = \frac{\eta^{n}\lambda^{n}\mu^{n-1}\tau^{n-1}\kappa^{n}}{(\lambda+\kappa)(\mu-\lambda)^{n-1}(\kappa-\zeta)^{n}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)((2i+3)\lambda+\kappa)((2i+2)\sigma+\delta)},$$

$$\Psi_{6n-9} = \frac{\eta^{n-1}\lambda^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}\delta}{(\mu-\lambda)^{n-1}(\kappa-\zeta)^{n-1}\prod_{i=0}^{n-2}((2i)\eta+\tau)((2i+1)\lambda+\kappa)((2i)\sigma+\delta)},$$

$$\Psi_{6n-8} = \frac{\eta^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\zeta^{n-1}(\tau-\sigma)^{n-1}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i)\lambda+\kappa)((2i+1)\sigma+\delta)},$$

$$\Psi_{6n-7} = \frac{\eta^{n-1}\lambda^{n-1}\mu^{n-1}\tau^{n}\kappa^{n-1}}{(\mu-\lambda)^{n-1}(\kappa-\zeta)^{n-1}\prod_{i=0}^{n-2}((2i)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)},$$

$$\Psi_{6n-6} = \frac{\eta^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\zeta^{n-1}(\tau-\sigma)^{n-1}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+1)\sigma+\delta)},$$

$$\Psi_{6n-5}=\frac{\eta^{n}\lambda^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{(\mu-\lambda)^{n-1}(\kappa-\zeta)^{n}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)},$$

$$\Psi_{6n-4}=\frac{\eta^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma+\delta)\zeta^{n-1}(\tau-\sigma)^{n}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)}.$$

Now, we find from system (5.1) that

$$\begin{aligned}\Phi_{6n-3}&=\frac{\Phi_{6n-6}\Psi_{6n-4}}{\Phi_{6n-6}+\Psi_{6n-7}}\\
&=\frac{\left[\frac{\eta^{n}\lambda^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{(\mu-\lambda)^{n-1}(\kappa-\zeta)^{n-1}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)}\right]\left[\frac{\eta^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma+\delta)\zeta^{n-1}(\tau-\sigma)^{n}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)}\right]}{\left[\frac{\eta^{n}\lambda^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{(\mu-\lambda)^{n-1}(\kappa-\zeta)^{n-1}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)}\right]+\left[\frac{\eta^{n-1}\lambda^{n-1}\mu^{n-1}\tau^{n}\kappa^{n-1}}{(\mu-\lambda)^{n-1}(\kappa-\zeta)^{n-1}\prod_{i=0}^{n-2}((2i)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)}\right]}\\
&=\frac{\left[\frac{\eta^{n}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma+\delta)\zeta^{n-1}(\tau-\sigma)^{n}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)}\right]}{\left[\frac{\eta}{\prod_{i=0}^{n-2}((2i+2)\eta+\tau)}\right]+\left[\frac{\tau}{\prod_{i=0}^{n-2}((2i)\eta+\tau)}\right]}\\
&=\frac{\left[\frac{\eta^{n}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma+\delta)\zeta^{n-1}(\tau-\sigma)^{n}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)}\right]}{\eta+\left[\frac{\tau\prod_{i=0}^{n-2}((2i+2)\eta+\tau)}{\prod_{i=0}^{n-2}((2i)\eta+\tau)}\right]}\\
&=\frac{\left[\frac{\eta^{n}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{(\sigma+\delta)\zeta^{n-1}(\tau-\sigma)^{n}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)}\right]}{\eta+((2n-2)\eta+\tau)}\end{aligned}$$

$$= \frac{\eta^n\sigma^n\mu^n\tau^n\kappa^{n-1}}{(\sigma+\delta)\,\zeta^{n-1}\,(\tau-\sigma)^n\,((2n-1)\eta+\tau)\prod\limits_{i=0}^{n-2}((2i+1)\eta+\tau)\,((2i+2)\lambda+\kappa)\,((2i+3)\sigma+\delta)}.$$

Thus, we obtain

$$\Phi_{6n-3} = \frac{\eta^n\sigma^n\mu^n\tau^n\kappa^n}{\zeta^{n-1}\,(\tau-\sigma)^n\prod\limits_{i=0}^{n-1}((2i+1)\eta+\tau)\,((2i)\lambda+\kappa)\,((2i+1)\sigma+\delta)}.$$

Similarly, by using the same method, we can investigate the relations

$$\Psi_{6n-3} = \frac{\Phi_{6n-4}\Psi_{6n-6}}{-\Phi_{6n-7}+\Psi_{6n-6}}$$

$$= \frac{\left[\frac{\eta^n\lambda^n\mu^{n-1}\tau^{n-1}\kappa^n}{(\lambda+\kappa)(\mu-\lambda)^{n-1}(\kappa-\zeta)^n\prod\limits_{i=0}^{n-2}((2i+2)\eta+\tau)((2i+3)\lambda+\kappa)((2i+2)\sigma+\delta)}\right]\left[\frac{\eta^{n-1}\sigma^{n-1}\mu^n\tau^{n-1}\kappa^{n-1}}{\zeta^{n-1}(\tau-\sigma)^{n-1}\prod\limits_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+1)\sigma+\delta)}\right]}{-\left[\frac{\eta^{n-1}\lambda\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\zeta^{n-1}(\tau-\sigma)^{n-1}\prod\limits_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+1)\sigma+\delta)}\right]+\left[\frac{\eta^{n-1}\sigma^{n-1}\mu^n\tau^{n-1}\kappa^{n-1}}{\zeta^{n-1}(\tau-\sigma)^{n-1}\prod\limits_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+1)\sigma+\delta)}\right]}$$

$$= \frac{\left[\frac{\eta^n\lambda^n\mu^n\tau^{n-1}\kappa^n}{(\lambda+\kappa)(\mu-\lambda)^{n-1}(\kappa-\zeta)^n\prod\limits_{i=0}^{n-2}((2i+2)\eta+\tau)((2i+3)\lambda+\kappa)((2i+2)\sigma+\delta)}\right]}{(-\lambda+\mu)}$$

$$= \frac{\eta^n\lambda^n\mu^n\tau^{n-1}\kappa^n}{(\lambda+\kappa)\,(\mu-\lambda)^n\,(\kappa-\zeta)^n\prod\limits_{i=0}^{n-2}((2i+2)\eta+\tau)\,((2i+3)\lambda+\kappa)\,((2i+2)\sigma+\delta)}.$$

Then

$$\Psi_{6n-3} = \frac{\eta^n\lambda^n\mu^n\tau^n\kappa^n\delta}{(\mu-\lambda)^n\,(\kappa-\zeta)^n\prod\limits_{i=0}^{n-1}((2i)\eta+\tau)\,((2i+1)\lambda+\kappa)\,((2i)\sigma+\delta)}.$$

Other relations can be proven in the same way. □

6. On the System: $\Phi_{n+1}=\frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2}+\Psi_{n-3}}$, $\Psi_{n+1}=\frac{\Phi_n\Psi_{n-2}}{-\Phi_{n-3}-\Psi_{n-2}}$

In this section, we will obtain a specific expression of the solutions of the following system of difference equations

$$\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2}+\Psi_{n-3}},\quad \Psi_{n+1} = \frac{\Phi_n\Psi_{n-2}}{-\Phi_{n-3}-\Psi_{n-2}},\ n=0,1,2,..., \tag{6.1}$$

where the initial conditions $\Phi_{-3}, \Phi_{-2}, \Phi_{-1}, \Phi_0, \Psi_{-3}, \Psi_{-2}, \Psi_{-1}$ and Ψ_0 are nonzero real numbers.

Theorem 6.1. *Assume $\{\Phi_n, \Psi_n\}_{n=-3}^{\infty}$ are a solution of system (6.1). Then for n=0,1,2,...,*

$$\Phi_{6n-3} = \frac{(-1)^n \eta^n \sigma^n \mu^n \tau^n \kappa^n}{\zeta^{n-1} (\sigma+\tau)^n \prod_{i=0}^{n-1} ((2i+1)\eta+\tau)((2i)\lambda+\kappa)((2i+1)\sigma+\delta)},$$

$$\Phi_{6n-2} = \frac{\eta^n \lambda^n \sigma \mu^n \tau^n \kappa^n}{(\lambda+\mu)^n (\zeta+\kappa)^n \prod_{i=0}^{n-1} ((2i)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)},$$

$$\Phi_{6n-1} = \frac{(-1)^n \eta^n \lambda \sigma^n \mu^n \tau^n \kappa^n}{\zeta^n (\sigma+\tau)^n \prod_{i=0}^{n-1} ((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+1)\sigma+\delta)},$$

$$\Phi_{6n} = \frac{\eta^{n+1} \lambda^n \mu^n \tau^n \kappa^n}{(\lambda+\mu)^n (\zeta+\kappa)^n \prod_{i=0}^{n-1} ((2i+2)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)},$$

$$\Phi_{6n+1} = \frac{(-1)^n \eta^n \sigma^{n+1} \mu^{n+1} \tau^n \kappa^n}{(\sigma+\delta) \zeta^n (\sigma+\tau)^n \prod_{i=0}^{n-1} ((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)},$$

$$\Phi_{6n+2} = \frac{-\eta^{n+1} \lambda^{n+1} \mu^n \tau^n \kappa^{n+1}}{(\lambda+\kappa)(\lambda+\mu)^n (\zeta+\kappa)^{n+1} \prod_{i=0}^{n-1} ((2i+2)\eta+\tau)((2i+3)\lambda+\kappa)((2i+2)\sigma+\delta)},$$

$$\Psi_{6n-3} = \frac{\eta^n \lambda^n \mu^n \tau^n \kappa^n \delta}{(\lambda+\mu)^n (\zeta+\kappa)^n \prod_{i=0}^{n-1} ((2i)\eta+\tau)((2i+1)\lambda+\kappa)((2i)\sigma+\delta)},$$

$$\Psi_{6n-2} = \frac{(-1)^n \eta^n \sigma^n \mu^n \tau^n \kappa^{n+1}}{\zeta^n (\sigma+\tau)^n \prod_{i=0}^{n-1} ((2i+1)\eta+\tau)((2i)\lambda+\kappa)((2i+1)\sigma+\delta)},$$

$$\Psi_{6n-1} = \frac{\eta^n \lambda^n \mu^n \tau^{n+1} \kappa^n}{(\lambda+\mu)^n (\zeta+\kappa)^n \prod_{i=0}^{n-1} ((2i)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)},$$

$$\Psi_{6n} = \frac{(-1)^n \eta^n \sigma^n \mu^{n+1} \tau^n \kappa^n}{\zeta^n (\sigma+\tau)^n \prod_{i=0}^{n-1} ((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+1)\sigma+\delta)},$$

$$\Psi_{6n+1} = \frac{-\eta^{n+1} \lambda^n \mu^n \tau^n \kappa^{n+1}}{(\lambda+\mu)^n (\zeta+\kappa)^{n+1} \prod_{i=0}^{n-1} ((2i+2)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)},$$

$$\Psi_{6n+2} = \frac{(-1)^{n+1} \eta^n \sigma^{n+1} \mu^{n+1} \tau^{n+1} \kappa^n}{(\sigma+\delta) \zeta^n (\sigma+\tau)^{n+1} \prod_{i=0}^{n-1} ((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)},$$

where $\Phi_{-3} = \zeta, \Phi_{-2} = \sigma, \Phi_{-1} = \lambda, \Phi_0 = \eta, \Psi_{-3} = \delta, \Psi_{-2} = \kappa, \Psi_{-1} = \tau$ and $\Psi_0 = \mu$.

Proof. For $n = 0$ the result holds. Now suppose that $n > 0$ and our assumption holds for $n-1$, that is;

$$\Phi_{6n-9} = \frac{(-1)^{n-1}\eta^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\zeta^{n}\left(\sigma+\tau\right)^{n-1}\prod\limits_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left((2i)\lambda+\kappa\right)\left((2i+1)\sigma+\delta\right)},$$

$$\Phi_{6n-8} = \frac{\eta^{n-1}\lambda^{n-1}\sigma\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left(\lambda+\mu\right)^{n-1}\left(\zeta+\kappa\right)^{n-1}\prod\limits_{i=0}^{n-2}\left((2i)\eta+\tau\right)\left((2i+1)\lambda+\kappa\right)\left((2i+2)\sigma+\delta\right)},$$

$$\Phi_{6n-7} = \frac{(-1)^{n-1}\eta^{n-1}\lambda\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\zeta^{n-1}\left(\sigma+\tau\right)^{n-1}\prod\limits_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left((2i+2)\lambda+\kappa\right)\left((2i+1)\sigma+\delta\right)},$$

$$\Phi_{6n-6} = \frac{\eta^{n}\lambda^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\left(\lambda+\mu\right)^{n-1}\left(\zeta+\kappa\right)^{n-1}\prod\limits_{i=0}^{n-2}\left((2i+2)\eta+\tau\right)\left((2i+1)\lambda+\kappa\right)\left((2i+2)\sigma+\delta\right)},$$

$$\Phi_{6n-5} = \frac{(-1)^{n-1}\eta^{n-1}\sigma^{n}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\left(\sigma+\delta\right)\zeta^{n-1}\left(\sigma+\tau\right)^{n-1}\prod\limits_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left((2i+2)\lambda+\kappa\right)\left((2i+3)\sigma+\delta\right)},$$

$$\Phi_{6n-4} = \frac{-\eta^{n}\lambda^{n}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\left(\lambda+\kappa\right)\left(\lambda+\mu\right)^{n-1}\left(\zeta+\kappa\right)^{n}\prod\limits_{i=0}^{n-2}\left((2i+2)\eta+\tau\right)\left((2i+3)\lambda+\kappa\right)\left((2i+2)\sigma+\delta\right)},$$

$$\Psi_{6n-9} = \frac{\eta^{n-1}\lambda^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}\delta}{\left(\lambda+\mu\right)^{n-1}\left(\zeta+\kappa\right)^{n-1}\prod\limits_{i=0}^{n-2}\left((2i)\eta+\tau\right)\left((2i+1)\lambda+\kappa\right)\left((2i)\sigma+\delta\right)},$$

$$\Psi_{6n-8} = \frac{(-1)^{n-1}\eta^{n-1}\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\zeta^{n-1}\left(\sigma+\tau\right)^{n-1}\prod\limits_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left((2i)\lambda+\kappa\right)\left((2i+1)\sigma+\delta\right)},$$

$$\Psi_{6n-7} = \frac{\eta^{n-1}\lambda^{n-1}\mu^{n-1}\tau^{n}\kappa^{n-1}}{\left(\lambda+\mu\right)^{n-1}\left(\zeta+\kappa\right)^{n-1}\prod\limits_{i=0}^{n-2}\left((2i)\eta+\tau\right)\left((2i+1)\lambda+\kappa\right)\left((2i+2)\sigma+\delta\right)},$$

$$\Psi_{6n-6} = \frac{(-1)^{n-1}\eta^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\zeta^{n-1}\left(\sigma+\tau\right)^{n-1}\prod\limits_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left((2i+2)\lambda+\kappa\right)\left((2i+1)\sigma+\delta\right)},$$

$$\Psi_{6n-5} = \frac{-\eta^{n}\lambda^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n}}{\left(\lambda+\mu\right)^{n-1}\left(\zeta+\kappa\right)^{n}\prod\limits_{i=0}^{n-2}\left((2i+2)\eta+\tau\right)\left((2i+1)\lambda+\kappa\right)\left((2i+2)\sigma+\delta\right)},$$

$$\Psi_{6n-4} = \frac{(-1)^{n}\eta^{n-1}\sigma^{n}\mu^{n}\tau^{n}\kappa^{n-1}}{\left(\sigma+\delta\right)\zeta^{n-1}\left(\sigma+\tau\right)^{n}\prod\limits_{i=0}^{n-2}\left((2i+1)\eta+\tau\right)\left((2i+2)\lambda+\kappa\right)\left((2i+3)\sigma+\delta\right)}.$$

Now, we prove that the results hold for n. From system (6.1), it follows that

$$\Phi_{6n-3} = \frac{\Phi_{6n-6}\Psi_{6n-4}}{\Phi_{6n-6}+\Psi_{6n-7}}$$

$$= \frac{\left[\frac{\eta^n\lambda^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{(\lambda+\mu)^{n-1}(\zeta+\kappa)^{n-1}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)}\right]\left[\frac{(-1)^n\eta^{n-1}\sigma^n\mu^n\tau^n\kappa^{n-1}}{(\sigma+\delta)\zeta^{n-1}(\sigma+\tau)^n\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)}\right]}{\left[\frac{\eta^n\lambda^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{(\lambda+\mu)^{n-1}(\zeta+\kappa)^{n-1}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)}\right]+\left[\frac{\eta^{n-1}\lambda^{n-1}\mu^{n-1}\tau^n\kappa^{n-1}}{(\lambda+\mu)^{n-1}(\zeta+\kappa)^{n-1}\prod_{i=0}^{n-2}((2i)\eta+\tau)((2i+1)\lambda+\kappa)((2i+2)\sigma+\delta)}\right]}$$

$$= \frac{\left[\frac{(-1)^n\eta^n\sigma^n\mu^n\tau^n\kappa^{n-1}}{(\sigma+\delta)\zeta^{n-1}(\sigma+\tau)^n\prod_{i=0}^{n-2}((2i+2)\eta+\tau)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)}\right]}{\left[\frac{\eta}{\prod_{i=0}^{n-2}((2i+2)\eta+\tau)}\right]+\left[\frac{\tau}{\prod_{i=0}^{n-2}((2i)\eta+\tau)}\right]}$$

$$= \frac{\left[\frac{(-1)^n\eta^n\sigma^n\mu^n\tau^n\kappa^{n-1}}{(\sigma+\delta)\zeta^{n-1}(\sigma+\tau)^n\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)}\right]}{\eta+\left[\frac{\tau\prod_{i=0}^{n-2}((2i+2)\eta+\tau)}{\prod_{i=0}^{n-2}((2i)\eta+\tau)}\right]}$$

$$= \frac{\left[\frac{(-1)^n\eta^n\sigma^n\mu^n\tau^n\kappa^{n-1}}{(\sigma+\delta)\zeta^{n-1}(\sigma+\tau)^n\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+3)\sigma+\delta)}\right]}{\eta+((2n-2)\eta+\tau)}$$

$$= \frac{(-1)^n\eta^n\sigma^n\mu^n\tau^n\kappa^{n-1}}{(\sigma+\delta)\,\zeta^{n-1}\,(\sigma+\tau)^n\,((2n-1)\eta+\tau)\prod_{i=0}^{n-2}((2i+1)\eta+\tau)\,((2i+2)\lambda+\kappa)\,((2i+3)\sigma+\delta)}.$$

So, we have

$$\Phi_{6n-3} = \frac{(-1)^n\eta^n\sigma^n\mu^n\tau^n\kappa^n}{\zeta^{n-1}\,(\sigma+\tau)^n\prod_{i=0}^{n-1}((2i+1)\eta+\tau)\,((2i)\lambda+\kappa)\,((2i+1)\sigma+\delta)}.$$

Also, we can observe that from system (6.1)

$$\Psi_{6n-3} = \frac{\Phi_{6n-4}\Psi_{6n-6}}{-\Phi_{6n-7} - \Psi_{6n-6}}$$

$$= \frac{\left[\frac{-\eta^{n}\lambda^{n}\mu^{n-1}\tau^{n-1}\kappa^{n}}{(\lambda+\kappa)(\lambda+\mu)^{n-1}(\zeta+\kappa)^{n}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)((2i+3)\lambda+\kappa)((2i+2)\sigma+\delta)}\right]\left[\frac{(-1)^{n-1}\eta^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\zeta^{n-1}(\sigma+\tau)^{n-1}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+1)\sigma+\delta)}\right]}{-\left[\frac{(-1)^{n-1}\eta^{n-1}\lambda\sigma^{n-1}\mu^{n-1}\tau^{n-1}\kappa^{n-1}}{\zeta^{n-1}(\sigma+\tau)^{n-1}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+1)\sigma+\delta)}\right]-\left[\frac{(-1)^{n-1}\eta^{n-1}\sigma^{n-1}\mu^{n}\tau^{n-1}\kappa^{n-1}}{\zeta^{n-1}(\sigma+\tau)^{n-1}\prod_{i=0}^{n-2}((2i+1)\eta+\tau)((2i+2)\lambda+\kappa)((2i+1)\sigma+\delta)}\right]}$$

$$= \frac{\left[\frac{-\eta^{n}\lambda^{n}\mu^{n}\tau^{n-1}\kappa^{n}}{(\lambda+\kappa)(\lambda+\mu)^{n-1}(\zeta+\kappa)^{n}\prod_{i=0}^{n-2}((2i+2)\eta+\tau)((2i+3)\lambda+\kappa)((2i+2)\sigma+\delta)}\right]}{-\lambda-\mu}$$

$$= \frac{\eta^{n}\lambda^{n}\mu^{n}\tau^{n-1}\kappa^{n}}{(\lambda+\kappa)\left(\lambda+\mu\right)^{n}\left(\zeta+\kappa\right)^{n}\prod_{i=0}^{n-2}\left((2i+2)\eta+\tau\right)\left((2i+3)\lambda+\kappa\right)\left((2i+2)\sigma+\delta\right)}.$$

Hence, we obtain

$$\Psi_{6n-3} = \frac{\eta^{n}\lambda^{n}\mu^{n}\tau^{n}\kappa^{n}\delta}{\left(\lambda+\mu\right)^{n}\left(\zeta+\kappa\right)^{n}\prod_{i=0}^{n-1}\left((2i)\eta+\tau\right)\left((2i+1)\lambda+\kappa\right)\left((2i)\sigma+\delta\right)}.$$

Similarly, by using the same method, we can investigate other relations. □

7. Numerical Examples

To demonstrate our prior results, we present some numerical examples which represent different types of solutions of the system (1.1).

Example 7.1. *In numerical simulation, we assumed that for system (3.2) the initial conditions are* $\Phi_{-3} = 0.3, \Phi_{-2} = 1.5, \Phi_{-1} = 0.2, \Phi_0 = 0.6, \Psi_{-3} = 0.9, \Psi_{-2} = 0.7, \Psi_{-1} = 0.8$ *and* $\Psi_0 = 0.4$. *Then the dynamics of the solution appear in Figure 1.*

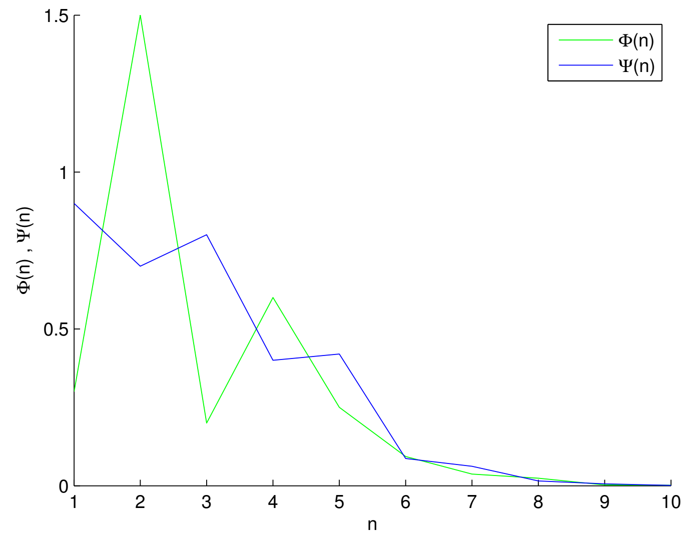

FIGURE 1. Plotting the solutions of $\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2}+\Psi_{n-3}}$, $\Psi_{n+1} = \frac{\Phi_n\Psi_{n-2}}{\Phi_{n-3}+\Psi_{n-2}}$.

From Figure 1, we conclude that the solutions are bounded and O is globally asymptotically stable and bounded. The results confirm Theorem 3.3 and Lemma 3.1.

Example 7.2. *In order to confirm the results of the system (5.1), Figure 2 shows the numerical simulations when the initial values are* $\Phi_{-3} = -0.15, \Phi_{-2} = 1.08, \Phi_{-1} = 2.3, \Phi_0 = 0.6, \Psi_{-3} = 1.9, \Psi_{-2} = 0.2, \Psi_{-1} = -0.12$ *and* $\Psi_0 = 3.3$.

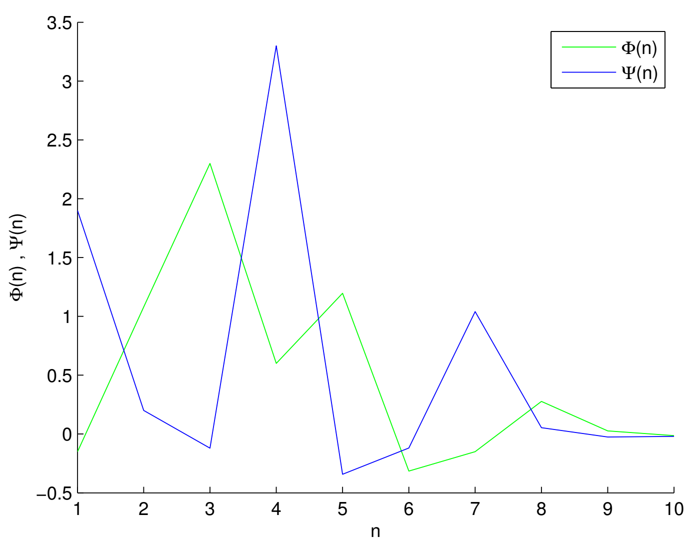

FIGURE 2. Plotting the solutions of $\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2}+\Psi_{n-3}}$, $\Psi_{n+1} = \frac{\Phi_n\Psi_{n-2}}{\Phi_{n-3}-\Psi_{n-2}}$.

Example 7.3. *Figures 3 depict the behavior of system (5.1), with initial conditions* $\Phi_{-3} = 0.3, \Phi_{-2} = 1.5, \Phi_{-1} = -0.9, \Phi_0 = 0.2, \Psi_{-3} = 1.2, \Psi_{-2} = 0.7, \Psi_{-1} = 0.3$ *and* $\Psi_0 = 2.4$. *We observe that the results are consistent with Theorem 6.1.*

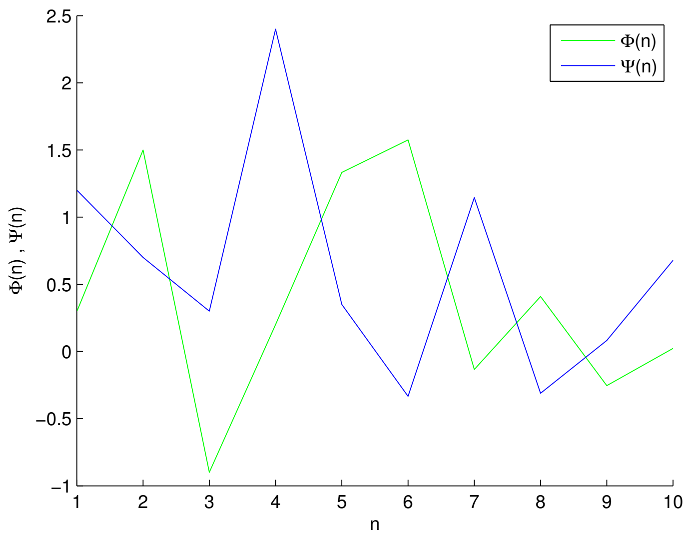

FIGURE 3. Plotting the solutions of $\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2}+\Psi_{n-3}}$, $\Psi_{n+1} = \frac{\Phi_n\Psi_{n-2}}{-\Phi_{n-3}+\Psi_{n-2}}$.

Example 7.4. *We consider numerical simulations to verify the results presented for system (6.1) the initial conditions are set as follows:* $\Phi_{-3} = 1.2, \Phi_{-2} = 2.5, \Phi_{-1} = 0.8, \Phi_0 = 3.5, \Psi_{-3} = -0.9, \Psi_{-2} = 1.7, \Psi_{-1} = -1.8$ *and* $\Psi_0 = 1.4$, *then the results which obtain in Figure 4 are confirmed with Theorem 6.1.*

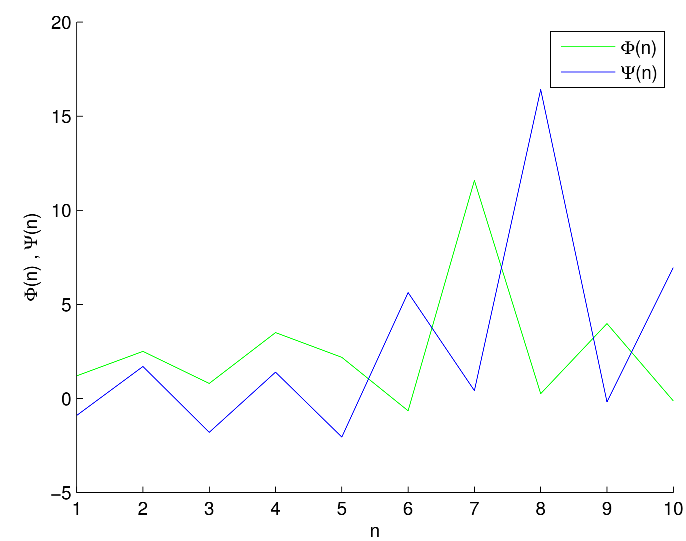

FIGURE 4. Plotting the solutions of $\Phi_{n+1} = \frac{\Phi_{n-2}\Psi_n}{\Phi_{n-2}+\Psi_{n-3}}$, $\Psi_{n+1} = \frac{\Phi_n\Psi_{n-2}}{-\Phi_{n-3}-\Psi_{n-2}}$.

8. Conclusions

Most research on nonlinear rational difference equations focuses on analyzing how solutions behave by presenting a general solution form. Researchers look at the stability properties of the equilibrium point because getting the solution formulations can be difficult.

In this article, we have found the expressions of solutions in some special cases as applications of rational difference systems of order four. In section 3, we have investigated the solution's qualitative behavior, such as local and global stability, as well as the boundedness of the solutions. Also, we have obtained the general form of the solution of system 3.2. In sections 4,5 and 6, we have obtained solutions for three special cases of the studied systems 4.1, 5.1 and 6.1. Finally, some illustrative examples are provided to support our theoretical discussion.

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