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The Existence of Weak Relative Pseudo-Complements in Almost Distributive Lattices

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Abstract. In this paper, we introduce weak relative annulets in almost distributive lattices and study their algebraic properties. Also, we introduce weak relative pseudo-complements and correlate with relative complements and pseudo-complements in an almost distributive lattice.

1. Introduction

In the realm of lattice theory, distributive lattices stands as a fundamental and captivating concept and holds a special place due to their elegant and far-reaching properties. At its core, a distributive lattice [11] is a lattice that satisfies the distributive property, a property that governs how the lattice operations of meet (infimum) and join (supremum) interact with each other. This property has a profound impact on the structure and behavior of elements within the lattice, leading to fascinating implications in various mathematical disciplines, including algebra, logic, and computer science.

The concept of annihilator ideals [2] emerges as a powerful and intriguing notion, shedding light on the interplay between elements within these lattice structures. Relative annihilator (weak-relative annihilators) [5, 7] ideals represent a specialized subset of annihilator ideals and these ideals capture a unique property: they are annihilated not just by individual elements of the lattice

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but by a specific subset of elements, forming a bridge between two distinct subsets and uncovering hidden connections.

The study of complements in distributive lattices [11] delves into the concept of an element having a unique counterpart that, when combined, yields the greatest or smallest element in the lattice. However, in certain situations, the presence of a true complement for every element is not guaranteed. This limitation leads to explore the concept of pseudo-complements [4] (weak-relative pseudo-complements [6]) and relative-complements [3], which extends the idea of complementarity in distributive lattices to encompass cases where true complements may not always exist. Traditional complements [3,4,6,10] in distributive lattices ([8,13,14] in almost distributive lattices) imbue each element with a unique counterpart that, when combined, results in the lattice's maximum or minimum element. However, in the realm of almost distributive lattices, strict distributivity may not always hold, challenging the existence of traditional complements. This motivates the study of weak relative pseudo-complements, which provides an extended framework to explore complementarity in almost distributive lattices.

2. Preliminaries

In this section, we recollect a few necessary definitions, examples and results which are thoroughly used in the sequel.

Definition 2.1. *[12] An algebra $(L, \vee, \wedge, 0)$ is called an Almost Distributive Lattice (ADL), if it satisfies the consecutive identities;*

(1) $0 \wedge a = 0$
(2) $a \vee 0 = a$
(3) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(4) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$
(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$
(6) $(a \vee b) \wedge b = b$

for all $a, b, c \in L$. -

Let us consider L means that an almost distributive lattice in the entire paper. Given $a, b \in L$, we say that $a \leq b$ if $a \wedge b = a$ (or $a \vee b = b$). It is easy to verify that L becomes a partial order set with respect to $\leq$. An element m in L is called a maximal if $m \wedge x = x$, for all $x \in L$.

Example 2.1. *[12] Let L be a set with three elements (say $0, a$ and b) such that*

$\wedge$	0	a	b
0	0	0	0
a	0	a	b
b	0	a	b

$\vee$	0	a	b
0	0	a	b
a	a	a	b
b	b	b	b

Then L is an almost distributive lattice in which every non-zero element is maximal. This ADL is called as a discrete ADL.

Definition 2.2. *[13] Given an element $x \in L$, an element $x^* \in L$ is said to be pseudo-complement of x if it satisfies the following;*

(i) $x \wedge x^* = 0$
(ii) *for any $y \in L, x \wedge y = 0$ implies $x^* \wedge y = y$.*

The authors [13] obtained a bijective relation between the class of maximal elements and the set of pseudo-complements, in an almost distributive lattice (provided maximal elements should exist). Also, they proved the class $\{a^* \mid a \in L\}$ forms a Boolean algebra [11], when L is an almost distributive lattice with maximal elements.

Definition 2.3. *[8] Given elements $a, b \in L$, an element $x \in L$ is called a relative pseudo-complement of a with respect to b if $\{y \in L \mid a \wedge y \in (b]\} = (x]$, where $(b]$ and $(x]$ are the ideals generated by x, b respectively.*

The authors [8] examined that the existence of x in Definition 2.3., need not be unique. If x is unique, then L becomes a lattice, and moreover L is relatively pseudo-complemented [4], since distributive lattice has a unique pseudo-complementation. Also, they have obtained an induced surjective undifferentiated relation between the set of maximal elements and the class of relative pseudo-complementation on an almost distributive lattice (provided with maximal elements) (provided there is a relative pseudo-complementation).

3. Weak relative annulets in ADLs

In this section, we define weak-relative annulets (which is a generalization of relative annulets [14]) in an almost distributive lattice and obtain several algebraic properties (whose proofs are straightforward, using algebraic properties of almost distributive lattices) on them.

Definition 3.1. *Given $a, b \in L$. An element $x \in L$ is said to be a weak relative annulet of a with respect to b, if $a \wedge x = b$. Let us denote the set of weak relative annulets of a with respect to b as $wra < a, b >$.*

Propositiom 3.1. *For any $a \in L$,*

(1) $wra < a, 0 >= (a)^*$, *where* $(a)^* = \{x \in L \mid a \wedge x = 0\}$.
(2) $wra < 0, a >= L$
(3) $wra < m, a >= \{a\}$, *where m is a maximal element in L.*
(4) $wra < d, 0 >= \{0\}$, *where d is a dense element in L.*
(5) *If $a \neq 0$, then $wra < 0, a >= \emptyset$.*
(6) $wra < a, a >= [a)$, *where* $[a) = \{a \vee x \mid x \in L\}$.

Proposition 3.2. *For any $a, b \in L$,*

(1) $a \wedge b = b \iff b \in wra < a, b > \iff a \vee b = b$.
(2) $a \in wra < a, b > \iff a = b$.
(3) $0 \in wra < a, b > \iff b = 0$.
(4) $a \leq b \implies a \in wra < b, a >$ *and* $b \in wra < a, a >$.

(5) $a \leq b \implies wra < b, 0 > \subseteq wra < a, 0 >$.
(6) $wra < a, 0 > \cap wra < b, 0 >= wra < a \vee b, 0 >$.
(7) $wra < a, 0 > \cup wra < b, 0 > \subseteq wra < a \wedge b, 0 >$.
(8) $b \in wra < a \vee b, b >$ *and* $a \in wra < b \vee a, a >$.
(9) $b \wedge a \in wra < b, b \wedge a >$ *and* $a \wedge b \in wra < a, a \wedge b >$.

Propositiom 3.3. *For any* $a, b, x, y \in L$,

(1) $x \in (a] \iff x \in wra < a, x >$.
(2) $wra < x, a > \cap wra < y, a >\subseteq wra < x \vee y, a >$.
(3) $y \in wrc < x, a > \implies y \wedge a \in wrc < x, a >$.
(4) $wra < x \wedge y, a >= wra < y \wedge x, a >$.
(5) $wra < x \vee y, a >= wra < y \vee x, a >$.
(6) $wra < x, a > \cap wra < x, b >\neq \emptyset \implies a = b$.
(7) *If* $x \leq y$, *then* $wra < y, a >\subseteq wra < x, x \wedge a >$.
(8) $a \vee x \in wra < a, a >$, *for all* $x \in L$.
(9) $x \in wra < a, b > \implies x \wedge b = b$.

Propositiom 3.4. *The following are equivalent in* L,

(1) L *is a lattice.*
(2) $a \vee x \in wra < a, a >$, *for all* $x \in L$.
(3) $x \vee a \in wra < a, a >$, *for all* $x \in L$.

Propositiom 3.5. *Let* $a, b, x, y \in L$. *Then*

(1) *If* $x \in wra < a, b >, y \in wra < b, a >$, *then* $y \in wra < x, b \wedge a >$ *and* $x \in wra < y, a \wedge b >$.
(2) *If* $x \in wra < a \wedge b, a \wedge c >$, *then* $a \wedge x \in wra < b, a \wedge c >$.
(3) *If* $x \in wra < b \wedge a, c \wedge a >$, *then* $x \wedge a \in wra < b, c \wedge a >$.

Propositiom 3.6. *Let* $a, b, c \in L$. *Then*

(1) $wra < a, c > \subseteq wra < a \wedge c, c >$.
(2) $wra < a, c > \cap wra < b, c > \subseteq wra < a \vee b, c >$.
(3) $wra < a, c > \cap wra < b, c > \subseteq wra < a \wedge b, c >$.
(4) $wra < b, a \wedge c > \subseteq wra < a \wedge b, a \wedge c >$.
(5) $wra < b, c \wedge a > \subseteq wra < b \wedge a, c \wedge a >$.

4. Weak relative pseudo-complements in ADLs

In this section, we define and examine weak relative pseudo-complementation in almost distributive lattices. We extract pseudo-complementation from weak relative pseudo-complementation in an almost distributive lattice.

Definition 4.1. *Given any two elements* $a, b \in L$, *an element* x *of* L *is said to be a weak relative pseudo-complement of* a *with respect to* b, *if it satisfies the following;*

(i) $a \wedge x = b$

(ii) *for any* $y \in L, a \wedge y = b$ *implies* $x \wedge y = y$.

In another words, a maximal element in the set $wra < a, b >$ (provided $wra < a, b > \neq \emptyset$) is called a weak relative pseudo-complement of a with respect to b.

Let us denote the set of weak relative pseudo-complements of a with respect to b in L as $wrp((a, b)$.

Propositiom 4.1. *$wrp(0, a) = \emptyset$, for all $a \in L \setminus \{0\}$.*

Proof. Let $x \in L$. Then $0 = 0 \wedge x \neq a$, for all $a \in L \setminus \{0\}$. □

Propositiom 4.2. *$wrp(d, 0) = \{0\}$, for all dense elements $d \in L$.*

Proof. Let d be a dense element in L. Then $d \wedge 0 = 0$. Let $y \in L$ such that $d \wedge y = 0$. Then $y = 0$ (since d is dense). Now, $d \wedge y = d \wedge 0 = 0 = y$. Therefore $0 \in wrp(d, 0)$. On the other hand, let $x \in L$ such that $x \in wrp(d, 0)$. Then $d \wedge x = 0$ which implies that $x = 0$ (since d is dense). Therefore $wrp(d, 0) = \{0\}$. □

Remark 4.1. *Every weak relative pseudo-complement of an element with respect to 0 may not be zero. For, consider the lattice $L = \{0, a, b, 1\}$, in which $\vee$ and $\wedge$ defined as follow;*

$\vee$	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	1
1	0	a	b	1

Then b is a weak relative pseudo-complement of a relative to 0, which is not equal to 0.

Propositiom 4.3. *If $x_1, x_2 \in wrp(a, b)$, then $x_1 \wedge x_2 = x_2$ and $x_2 \wedge x_1 = x_1$.*

Proof. Let $x_1, x_2 \in wrp(a, b)$. Then $a \wedge x_1 = b = a \wedge x_2$. Since x_1 is a weak relative pseudo-complement of a with respect to b and $a \wedge x_2 = b$, we get $x_1 \wedge x_2 = x_2$. Similarly we can prove $x_2 \wedge x_1 = x_1$. □

Propositiom 4.4. *If $a \in wrp(a, b)$, then $b = a$.*

Proof. Suppose that $a \in wrp(a, b)$. Then $b = a \wedge a = a$. □

Propositiom 4.5. *If $wrp(a, b) \neq \emptyset$, then $a \wedge b = b$.*

Proof. Let x be an element in $wrp(a, b)$. Then $a \wedge x = b$. Now, $a \wedge b = a \wedge a \wedge x = a \wedge x = b$. □

Propositiom 4.6. *If L is a lattice, then, for any $a, b \in L$, $wrp(a, b)$ has atmost one element.*

Proof. Let $a, b \in L$ such that $wrp(a, b) \neq \emptyset$. Let $x_1, x_2 \in wrp(a, b)$. Then $a \wedge x_1 = b = a \wedge x_2$. Since x_1 is a weak relative pseudo-complement of a with respect to b, and $a \wedge x_2 = b$, we get $x_1 \wedge x_2 = x_2$. Similarly we can prove $x_2 \wedge x_1 = x_1$. Therefore $x_1 \leq x_2$ and $x_2 \leq x_1$ (since L is a lattice) and hence $x_1 = x_2$. Thus $wrp(a, b)$ contains at most one element. □

Remark 4.2. *The converse of Proposition 4.6., need not be true. For, see Example 2.1., it is easy to observe that $wrp(x, y)$ has at most one element, for all $x, y \in L$ and L is not a lattice.*

Propositiom 4.7. *If $*$ is a pseudo-complementation on L, then $a^* \in wrp(a, 0)$, for all $a \in L$.*

Proof. Let $*$ be a pseudo complementation on L and $a \in L$. Then $a \wedge a^* = 0$ and, for $y \in L$, $a \wedge y = 0$ implies $a^* \wedge y = y$ (since $*$ is a pseudo-complementation on L). Therefore a^* is a weak relative pseudo-complementation of a with respect to 0. □

Propositiom 4.8. *If L has dense elements, then every weak relative pseudo-complement of an element $a \in L$ with respect to 0 is a pseudo-complement of a.*

Proof. Let x be a weak relative pseudo-complement of an element $a \in L$ with respect to 0. Then $a \wedge x = 0$. Since L has dense elements, it is possible to choose $y \in L$ such that $a \wedge y = 0$. As x is a weak relative complement of a with respect to 0, we get $x \wedge y = 0$. Therefore x is pseudo-complement of a with respect to 0. □

Remark 4.3. *Weak relative pseudo-complementation is a stronger property than the pseudo-complementation on an ADL. That is, if an ADL has dense elements, then only we can discuss pseudo-complementation. Where as the concept of weak relative pseudo-complementation on an ADL, we may not require dense elements.*

L with maximal elements is said to be a relatively complemented almost distributive lattice [12], if for any $a \in L$, there exists $x \in L$ such that $a \wedge x = 0$ and $a \vee x$ is maximal. In this case, we can say that x is a relative complement of a. Now, we have the following.

Proposition 4.9. *If x is a relative complement of a in L, then $x \in wrp(a, 0)$*

Proof. Let x be a relative complement of a. Then $a \wedge x = 0$ and $a \vee x$ maximal. Let $y \in L$ such that $a \wedge y = 0$. Then,

$$x \wedge y = 0 \vee (x \wedge y) = (a \wedge y) \vee (x \wedge y) = (a \vee x) \wedge y = y \text{ (since } a \vee x \text{ is maximal)}$$

Therefore x is a weak relative pseudo-complement a with respect to 0. □

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