Int. J. Anal. Appl. (2024), 22:193

International Journal of Analysis and Applications

On Intuitionistic Fuzzy Soft Pretopology

A. A. Azzam[1,2,*], Radwan Abu-Gdairi[3], M. Aldawood[1], Abdulaziz M. Alotaibi[1]

[1]*Mathematics Department, Faculty of Science and Humanities, Prince Sattam Bin Abdulaziz University, Alkharj 11942, Saudi Arabia*

[2]*Mathematics Department, Faculty of science, New Valley University, Elkharga 72511, Egypt*

[3]*Mathematics Department, Faculty of Science, Zarqa University, Zarqa 13132, Jordan*

[*]*Corresponding author:* aa.azzam@psau.edu.sa

Abstract. This paper defines the following terms: intuitionistic fuzzy soft (*IFS*) pretopological space, *IFS* interior function, *IFS* pre-open set, *IFS* pre-closed set, trace of a *IFS* pretopology, *IFS* separation axioms, *IFS* subspace, *IFS* compactness, *IFS* connectedness, and some of their properties. In addition, the degree of soft non-vacuity, the soft α-cut, and the *IFS* preneighbourhood system at a soft point are defined. *IFS* preneighbourhoods produce *IFS* pretopologies.

Numerous scientific disciplines work with ambiguous data that traditional mathematics may not be able to adequately describe. To address these issues, theories such as the soft set (S_s) theory [1, 3, 22] and fuzzy set (F_s) theory [28] have been created. One of the best techniques for handling multi-attribute decision-making situations is F_s theory. Numerous fields deal with ambiguous data, including the social sciences, medicine, and physics, so F_s [15, 28], intuitionistic fuzzy sets (IF_{ss}) [6-8], Pythagorean fuzzy sets (PYF_{ss}) and picture fuzzy sets (PF_{ss}) [13, 24], and other mathematical techniques are required to deal with the challenges of locating trustworthy data that is adequate and accurate given the ambiguity and imprecision of social economics. Mathematical techniques like IF_s and PF_{ss} [13, 24] are essential. Two mathematical tools exist for handling uncertainties: S_s theory, which was proposed by Molodtsov [22] and F_s theory, which was created by Zadeh [28]. A few attributes and the idea of a fuzzy soft set (FS_s) were presented by Maji et al. [21]. It was Tany and Kandemir [27] who first defined fuzzy soft topology (FST). Originally, fuzziness conditions were used to apply fuzzy sets F_{ss} as described by Zadeh [28]. Every domain set element ν in F_s has a single index, which is the degree of membership $md(\nu)$, which ranges from 0 to 1. A

Received: Aug. 17, 2024.

2020 *Mathematics Subject Classification.* 54A05, A40.

Key words and phrases. *IFS* pretopological space; *IFS* interior function; *IFS* pre-open set; *IFS* preneighbourhood system; *IFS* compactness; *IFS* connectedness.

https://doi.org/10.28924/2291-8639-22-2024-193
ISSN: 2291-8639

non-membership degree for the (F_s) is simply equal to $1 - md(\nu)$. and IF_s were created in order to take into consideration membership degree uncertainty, as noted in [9]. There are two indices used: the membership degree $(md(\nu))$ and the non-membership degree $(nd(\nu))$. Over the past few decades, a large number of researchers have investigated the IF_s, and it has been effectively used in numerous other fields, such as medical diagnostics and decision making [6, 9]. In [28], Zadeh's F_s was generalized, defining the IF_s with the two indexes $md(\nu)$ and $nd(\nu)$, and imposing the constraint that $0 \leq md(\nu) + nd(\nu) \leq 1$. The IF_s and interval valued intuitionistic fuzzy sets $(IVIf_{ss})$, which are widely utilized in various applications, including group decision making, have been studied by a large number of academics, including [2, 7, 8, 14, 18]. Pretopology was first proposed by M. Brissaud [11]. R. Badard first proposed the idea of a fuzzy pretopology (FPT)in [10]. Fuzzy soft pretopological spaces $(FSPT_{ss})$ proposed by Khedr et. al [16].

The new definitions based on the pretopological notion of intuitionistic fuzzy soft pretopological spaces $(IFSPT_{ss})$ that are presented in this study make a contribution. It is a generalization of the $FSPT_{ss}$ notion.

There are four main reasons to explore these models:

First, we'll introduce the innovative idea of $IFSPT$. Second, investigate the notions of the IFS neighbourhood system at a soft point.

Third, investigate the extent of soft no-vacuity and soft α-cut. Finally, $IFSPT_s$ is formed by IFS neighborhoods, as demonstrated in Section 6. The structure of this document is as follows. The concepts FS and PT_{ss}, this study is discussed in Section 2. Sect. 3, discuss some of topological structures on $IFST$. In Section 4, we examine the benefits of $FSPT$. Section 5 defines and investigates the $IFSPT$, IFS interior function, IFS pre-open set, IFS pre-closed set, the trace of a $FSPT$, and some of its properties. Section 6 introduce idea of intuitionistic fuzzy soft pretopology generated by fuzzy Soft preneighbourhoods. .

1. Preliminaries

In this part, we will go over the essential definitions and findings of fuzzy soft and pretopological spaces that we will need for this study.

Definition 1.1. [22] *A soft set, or parameterized family of subsets of the universe* Γ*, is a pair* (M, Π)*; denoted by* M_Π*, over the initial universe* Γ *and the set of parameters* Π*. i.e* $M_\Pi = \{M(\delta) : \delta \in \Pi,\ M : \Pi \to P(\Gamma)\}$*. For any* $\delta \in \Pi$*, if* $M(\delta) = \varphi$ *(or, alternatively,* $M(\delta) = \Gamma$*), then* (M, Π) *is referred to as a null (or, alternatively, an absolute) soft set, and will be represented by* $\tilde{\varphi}$ *(or, alternatively,* $\tilde{\Gamma}$*). The family of all soft sets is now denoted by* $S(\Gamma)_\Pi$*.*

Definition 1.2. [26] *A soft topology on* Γ *is defined as a collection* $\tau \subseteq S(\Gamma)_\Pi$ *that contains* $\tilde{\Gamma}, \tilde{\varphi}$ *and is closed under arbitrary soft union and finite soft intersection. The triplet* (Γ, τ, Π) *is known as a STS over* Γ*.*

Definition 1.3. [23] *If there is precisely one* $\delta \in \Pi$ *such that* $M(\delta) = \{\gamma\}$ *for some* $\gamma \in \Gamma$ *and* $M(\acute{\delta}) = \varphi$*, for every* $\acute{\delta} \in \Pi - \{\delta\}$*, then* M_Π *over* Γ *is referred to as a soft point. The symbol for it will be* γ_δ*.*

Definition 1.4. [23] *A soft point γ_δ is considered to belongs to a soft set G_Π if $\delta \in \Pi$ and $M(\delta) = \{\gamma\} \tilde{\sqsubseteq} G(\delta)$. We write $\gamma_\delta \tilde{G_\Pi}$.*

Definition 1.5. [28] *The mapping $\Phi_\Pi : \Gamma \to [0,1]$ defines a fuzzy set Π in any arbitrary set Γ. Here, $\Phi_\Pi(\gamma)$ indicates the $md(\gamma)$ in Π. In other words, the set of ordered pairs $\Pi = \{(\gamma, \Phi_\Pi(\gamma)) : \gamma \in \Gamma\}$ can represent a fuzzy set Π in Γ. $[0,1]^\Gamma$ represents the family of all F_{ss} in Γ.*

Definition 1.6. [19] *$[0,1]^\Gamma$ indicates the set of all F_{ss} of Γ, $\Pi \sqsubseteq A$, and Γ as the initial universe set and A as a set of parameters. If M is a mapping provided by $M : \Pi \to [0,1]^\Gamma$ such that $M(\delta) = 0_\Gamma$ if $\delta \notin \Pi$ and $M(\delta) \neq 0_\Gamma$ if $\delta \in \Pi$, then the pair (M, Π), represented by M_Π, is termed a FS_s over Γ. Therefore, the set of ordered pairs $M_\Pi = \{(\delta, M_\Pi) : \delta \in \Pi, M_\Pi \tilde{\in} [0,1]^\Gamma\}$ may be used to represent a FS_s M_Π over Γ. Stated differently, the family of parameterized fuzzy subsets of the set Γ is the fuzzy soft set.*
$FS(\Gamma)_A$ is the set of all fuzzy soft sets over an initial universe Γ and a collection of parameters A.

Definition 1.7. [19] *A FS_s M_Π is a FS subset of a FS_s G_B over a common universe Γ if $\Pi \sqsubseteq B$ and $M(\delta) \leq G(\delta)$, for every $\delta \in \Pi$.*

Definition 1.8. [25,27] *A family of FS_{ss} over (Γ, A) is a FST τ on (Γ, A) that satisfies the following properties:*
i- $\tilde{\Gamma}, \tilde{\varphi} \in \tau$
ii- If $M_\Pi, G_B \in \tau$ then $M_\Pi \tilde{\sqcap} G_B \in \tau$
iii- If $(M_\Pi)_i \in \tau$, for every $i \in I$ then $\tilde{\sqcup}_{i \in I} \in \tau$.
The triple (Γ, A, τ) is said to be a FSTS. In (Γ, A, τ), each member of τ is referred to as a FS open set. FS closed is the complement of a FS open set.

Definition 1.9. [6] *Let Γ be the universe setting, then the set*
$\Pi = \{(\gamma, \Phi_\Pi(\gamma), \mathcal{N}_\Pi(\gamma)) : \gamma \in \Gamma\}$ is called IF_s of Γ, $\Phi_\Pi : \Gamma \to [0,1]$ and $\mathcal{N}_\Pi : \Gamma \to [0,1]$ are called degree of positive-membership of δ ($md(\delta)$) in Γ and negative-membership degree of δ ($nd(\delta)$) in Γ successively with the condition $0 \leq \Phi_\Pi(\delta) + \mathcal{N}_\Pi(\delta) \leq 1$, $\forall \gamma \in \Gamma$.

Definition 1.10. [6] *Let Π=$\{(\gamma, \Phi_\Pi(\gamma), \mathcal{N}_\Pi(\gamma)) : \gamma \in \Gamma\}$ and*
$C = \{(\gamma, \Phi_C(\gamma), \mathcal{N}_C(\gamma)) : \gamma \in \Gamma\}$ be IF_{ss} IF_s of Γ.
i- $\Pi \sqsubseteq C$ iff $\Phi_\Pi(\gamma) \leq \Phi_C(\gamma)$ and $\mathcal{N}_\Pi(\gamma) \geq \mathcal{N}_C(\gamma)$ for all $\gamma \in \Gamma$,
ii- $\Pi \sqcap C = \{(\gamma, min\{\Phi_\Pi(\gamma), \Phi_C(\gamma)\}, max\{\mathcal{N}_\Pi(\gamma), \mathcal{N}_C(\gamma)\}) : \gamma \in \Gamma\}$,
ii- $\Pi \sqcup C = \{(\gamma, max\{\Phi_\Pi(\gamma), \Phi_C(\gamma)\}, min\{\mathcal{N}_\Pi(\gamma), \mathcal{N}_C(\gamma)\}) : \gamma \in \Gamma\}$.

Definition 1.11. [6] *$\tilde{0}$ denotes a IF_s Π over the set Γ, which is defined as $\Pi = \{(\gamma, 0, 1) : \gamma \in \Gamma\}$. This is known as a IF null set.*

Definition 1.12. [6] *$\tilde{1}$ denotes a IF_s Π over the set Γ, which is defined as $\Pi = \{(\gamma, 1, 0) : \gamma \in \Gamma\}$. This is known as a IF absolute set.*

Definition 1.13. [20] *Let A be the parameter set and Γ be the starting universe set. The set of all IF subsets of Γ is represented by the symbol IF^Γ. Let $\Pi \sqsubseteq A$. An IFS_s over Γ is a pair (M, Π), where M is a mapping*

defined by $M : \Pi \to IF^{\Gamma}$.
Essentially, $M(\delta)$ *is a* IFS_s *of* Γ *for each* $\delta \in \Pi$. *This is known as the IF value set of parameter* δ. *It is evident that* $M(\delta)$ *may be expressed as a* IFS_s *such that* $M(\delta) = \{(\gamma, \Phi_{\Pi}(\gamma), \mathcal{N}_{\Pi}(\gamma)) : \gamma \in \Gamma\}$.
Referred to as $IFS(\Gamma_A)$, *the set of all* S_{ss} *over* Γ *with parameters from* A *is known as a IFS class.*

2. Some topological structures on *IFST*

This section presents the ideas of subspace of *IFSTS* and provides some examples.

Definition 2.1. [17] *If the conditions that follow are met,* Λ *is said to be a IFST on* Γ, *where* $\Lambda \sqsubseteq IFS(\Gamma_A)$.
i-$\tilde{\varphi}$, $\tilde{\Gamma}$ *belong to* Λ.
ii-The union of any number of IFS_{ss} *in* Λ *belongs to* Λ.
iii-The intersection of any two IFS_{ss} *in* Λ *belongs to* Λ.

The binary (Γ_A, Λ) is called a *STS* over Γ, and Λ is called a *IFST* on Γ. It is argued that the members of Λ are *IFS* open sets in Γ.
A IFS_s (M, A) over Γ is said to be a *IFS* closed set in Γ, If the complement $(M, A)^c \in \Lambda$ then, IFS_s (M, A) over Γ is called a *IFS* closed set in Γ.

Definition 2.2. *Let* $\Gamma = \{\gamma_1, \gamma_2\}$, $A = \{\delta_1, \delta_2\}$ *and* $\Lambda = \{\tilde{\varphi}, \tilde{\Gamma}, (M_1, A), (M_2, A), (M_3, A)\}$ *where* $(M_1, A), (M_2, A), (M_3, A)$ *are* IFS_{ss} *over* Γ, *defined as follows*
$M_1(\delta_1) = \{(\gamma_1, 0.2, 0.3), (\gamma_2, 0.1, 0.3)\}, M_1(\delta_2) = \{(\gamma_1, 0.2, 0.2), (\gamma_2, 0.1, 0.1)\},$
$M_2(\delta_1) = \{(\gamma_1, 0.8, 0.3), (\gamma_2, 0.1, 0.2)\}, M_2(\delta_2) = \{(\gamma_1, 0.8, 0.1), (\gamma_2, 0.1, 0.1)\},$
$M_3(\delta_1) = \{(\gamma_1, 0, 0.3), (\gamma_2, 0.5, 0.3)\}, M_3(\delta_2) = \{(\gamma_1, 0, 0.3), (\gamma_2, 0.5, 0.3)\}.$
Then Λ *defines a IFST over* Γ, (Γ_A, Λ) *is a IFSTS on* Γ.

Definition 2.3. *Let* (Γ_A, Λ_1), (Γ_A, Λ_2) *be* $IFST_{ss}$.
i- If $\Lambda_2 \sqsupseteq \Lambda_1$, *then* Λ_2 *is IFS finer than* Λ_1.
ii- If $\Lambda_2 \sqsupseteq \Lambda_1$, *then* Λ_2 *is IFS strictly finer than* Λ_1.
iii- If either $\Lambda_2 \sqsupseteq \Lambda_1$ *or* $\Lambda_2 \sqsubseteq \Lambda_1$, *then* Λ_2 *is comparable with* Λ_1.

Definition 2.4. *Let* Γ *be the universal set and* A *the parameter set.*
i-Let Λ *be the family of all defined* IFS_{ss} *on* Γ. *Next,* (Γ_A, Λ) *is designated a IFS discrete space over* Γ, *and* Λ *is named the IFS discrete topology on* Γ.
ii-Λ=$\{\tilde{\varphi}, \tilde{\gamma}\}$ *is the IFS indiscrete topology on* Γ, *and* (Γ_A, Λ) *is dubbed a IFS indiscrete space over* Γ.

Definition 2.5. *Let* Ψ *be a non-empty subset of* Γ *and* (Γ_A, Λ) *be* $IFST_s$ *over* Γ. *Then,* (Ψ_A, Λ_Ψ) *is called a IFS subspace of* (Γ_A, Λ), *where* $(M^{\Psi}, A) = \tilde{\Psi}_A \tilde{\sqcap} (M, A)$. *This is called the IFST on* Ψ, *and* $(\Psi_A, A) : (M, A) \in \Lambda\}$. *The fact that* Λ_Ψ *is in reality a IFST on* Ψ *may be easily verified.*

Example 2.1. *A IFS discrete topological space is any IFS subspace of another IFS discrete topological space. Moreover, a IFS indiscrete topological space is any IFS subspace of a IFS indiscrete topological space.*

3. Fuzzy soft pretopology

These terms are defined as follows in this section: pretopology, fuzzy pretopology, fuzzy soft pretopology, fuzzy pretopological space, fuzzy soft pretopology, fuzzy soft pre-open set, fuzzy soft pre-closed set, fuzzy soft pretopology trace, and some of its aspects are studied.

Definition 3.1. [10] $b : 2^{\Gamma} \to 2^{\Gamma}$ *is a function that describes a PT on* Γ *and satisfies the following conditions:*
i-$b(\varphi) = \varphi$;
ii-For each $\Pi \in I^{\Gamma}$, $b(\Pi) \sqsupseteq \Pi$.
A PT_s *is the pair* (Γ, b).

We now give the following definition of the term "*SPT*":

Definition 3.2. [16] *A function* $b : S(\Gamma)_A \to S(\Gamma)_A$ *that satisfies the following conditions is a SPT on* (Γ, A).
i-$b(\tilde{\varphi}) = \tilde{\varphi}$;
ii-$b(M_{\Pi}) \sqsupseteq M_{\Pi}$, *for every* $M_{\Pi} \tilde{\in} MS(\Gamma)_A$.
A SPT_s *is the triple* (Γ, A, b).

Definition 3.3. [11] *The function* $b : I^{\Gamma} \to I^{\Gamma}$ *can be used to describe a FPT on* Γ *if it fulfills:*
i-$b(\varphi) = \varphi$;
ii-For each $\Pi \in I^{\Gamma}$, $b(\Pi) \sqsupseteq \Pi$.
A FPT_s *is the pair* (Γ, b).
Let (Γ, b) *be a* FPT_s*, and let us consider the following properties:*
iii-If $b(\Pi) \sqsubseteq b(C)$ *for any* $\Pi, C \in I^{\Gamma}$ *such that* $\Pi \sqsubseteq C$*, then* (Γ, b) *is of type I.*
iv-If $b(\Pi \sqcup C) = b(\Pi) \sqcup b(C)$ *holds true for each* $\Pi, C \in I^{\Gamma}$*, then* (Γ, b) *is considered to be of type D.*
v-If $b^2(\Pi) = b(b(\Pi)) = b(\Pi)$ *for every* $\Pi \in I^{\Gamma}$*, then* (Γ, b) *is of type S.*
It should be noted that any FPT_s *of type* D, S *and* $(iv) \Rightarrow (iii)$ *are* FPT_s*, where b is Kuratowski closure.*

Definition 3.4. [11] *Assume that* (Γ, b) *is a* FPT_s*, the fuzzy interior function* $\hbar_b : I^{\Gamma} \to I^{\Gamma}$ *is defined by* $\hbar_b(\Pi) = (b(\Pi^c))^c$. *The following conditions are satisfied by the fuzzy interior function* $\hbar_b$:
i-$b(\varphi) = \varphi$;
ii-For each $\Pi \in I^{\Gamma}$, $\hbar_b(\Pi) \sqsubseteq \Pi$;
iii-For every $\Pi, C \in I^{\Gamma}$ *if* $\Pi \sqsubseteq C$*, then* $\hbar_b(\Pi) \sqsubseteq \hbar_b(C)$;
iv-For every $\Pi, C \in I^{\Gamma}$, $\hbar_b(\Pi \sqcap C) = \hbar_b(\Pi) \sqcap \hbar_b(C)$;
v-For every $\Pi \in I^{\Gamma}$, $\hbar_b^2(\Pi) = \hbar_b(\Pi)$.

Definition 3.5. [11] *Let* (Γ, b) *be a* FPT_s *and* Π *be a fuzzy subset of* Γ. *The trace of b on* Π*, denoted by* b_{Π}*, is defined by:* $b_{\Pi}(C) = b(C) \sqcap \Pi$*, for every fuzzy subset C of* Π.

Definition 3.6. [16] *A FSPT on* (Γ, A) *is a function*
$b : FS(\Gamma)_A \to FS(\Gamma)_A$ *which satisfies:*
i-$b(\tilde{\varphi}) = \tilde{\varphi}$;

ii-$b(M_\Pi) \sqsupseteq M_\Pi$*, for every* $M_\Pi \tilde{\in} FS(\Gamma_A)$*.*
The triple (Γ, A, b) *is then said to be a* $FSPT_s$*.*
A $FSPT_s$ (Γ, A, b) *is said to be of:*
iii-Type I: if for every $M_\Pi, G_B \tilde{\in} FS(\Gamma_A)$ *such that* $M_\Pi \tilde{\sqsubseteq} G_B$*, we have* $b(M_\Pi) \tilde{\sqsubseteq} b(G_B)$*;*
iv-Type D: if for every $M_\Pi, G_B \tilde{\in} FS(\Gamma)_A$*, we have* $b(M_\Pi \tilde{\sqcup} G_B) = b(M_\Pi) \tilde{\sqcup} b(G_B)$*;*
v-Type S: if for every $M_\Pi \tilde{\in} FS(\Gamma_A)$*, we have* $b^2(M_\Pi) = b(b(M_\Pi)) = b(M_\Pi)$*.*
We say that b is of type $\sigma\eta$ *if b is of type* σ *and* η*, where* $\sigma, \eta \in \{I, D, S\}$*.*
One may notice that $(iv) \Rightarrow (iii)$ *and any* $FSPT_s$ *of type DS is a* $FSPT_s$ *where b is Kuratowski closure.*

Proposition 3.1. [16] *Let* b, a *be two FSPT on* (Γ, A)*. Then*
i- The composition function of b, a *denoted by* $b \circ a$ *is a FSPT on* (Γ, A)*.*
ii-If b, a *are type I (or D), then the composition function* $b \circ a$ *of b and a is of type I (or D).*

Proof. i-$(b \circ a)(\tilde{\varphi}) = b(a(\tilde{\varphi})) = \tilde{\varphi}$. Also, $(b \circ a)(M_\Pi) = b(a(M_\Pi))$. Put $a(M_\Pi) = G_\Pi$, then $(b \circ a)(M_\Pi) = b(G_\Pi) \tilde{\sqsupseteq} G_\Pi = a(M_\Pi) \tilde{\sqsupseteq} M_\Pi$. Therefore $(b \circ a)(M_\Pi) \tilde{\sqsupseteq} M_\Pi$.
ii-* Let b, a of type I and $M_\Pi \tilde{\sqsubseteq} G_\Pi$, then $a(M_\Pi) \tilde{\sqsubseteq} a(G_\Pi)$ and $b(a(M_\Pi)) \tilde{\sqsubseteq} b(a(G_\Pi))$.
Therefore $(b \circ a)(M_\Pi) = b(a(M_\Pi)) \tilde{\sqsubseteq} b(a(G_\Pi)) = (b \circ a)(G_\Pi)$. This shoes that $b \circ a$ is of type I.
** Let b, a of type D, then $b(M_\Pi \tilde{\sqcup} G_\Pi) = b(M_\Pi) \tilde{\sqcup} b(G_\Pi)$ and $a(M_\Pi \tilde{\sqcup} G_\Pi) = a(M_\Pi) \tilde{\sqcup} a(G_\Pi)$. Now, $(b \circ a)(M_\Pi \tilde{\sqcup} G_\Pi) = b(a(M_\Pi \tilde{\sqcup} G_\Pi)) = b(a(M_\Pi) \tilde{\sqcup} a(G_\Pi)) = b(a(M_\Pi)) \tilde{\sqcup} b(a(G_\Pi)) = (b \circ a)(M_\Pi) \tilde{\sqcup} b(a)(G_\Pi)$.
Therefore, $(b \circ a)$ is a type D. □

4. Intuitionistic fuzzy soft pretopology

The *IFSPT*, *IFS* interior function, *IFS* pre-open set, *IFS* pre-closed set, the trace of a *FSPT* and some of its attributes are defined and studied in this section.

Definition 4.1. *An IFSPT on* (Γ, A) *is a function* $b : IFS(\Gamma_A) \to IFS(\Gamma_A)$ *which satisfies the conditions:*
i-$b(\tilde{\varphi}) = \tilde{\varphi}$*;*
ii-$b(M_\Pi) \sqsupseteq M_\Pi$*, for each* $M_\Pi \tilde{\in} IFS(\Gamma_A)$*.*
The triple (Γ, A, b) *is then said to be an* $IFSPT_s$*.*
One classifies a $IFSPT_s$ (Γ, A, b) *as follows:*
iii-Type I: We have $b(M_\Pi) \tilde{\sqsubseteq} b(G_B)$ *if, for every* $M_\Pi, G_B \tilde{\in} IFS(\Gamma_A)$ *such that* $M_\Pi \tilde{\sqsubseteq} G_B$*;*
iv-Type D: if for every $M_\Pi, G_B \tilde{\in} IFS(\Gamma_A)$*, we have* $b(M_\Pi \tilde{\sqcup} G_B) = b(M_\Pi) \tilde{\sqcup} b(G_B)$*;*
v-Type S: if for every $M_\Pi \tilde{\in} IFS(\Gamma_A)$*, we have* $b^2(M_\Pi) = b(b(M_\Pi)) = b(M_\Pi)$*.*
We say that b is of type $\sigma\eta$ *if b is of type* ξ *and* ζ*, where* $\xi, \zeta \in \{I, D, S\}$*.*
One may notice that $(iv) \Rightarrow (iii)$ *and any* $FSPT_s$ *of type DS is a* $FSPT_s$ *where b is Kuratowski closure.*

Proposition 4.1. *Let* b, a *be two IFSPT on* (Γ, A)*. Then*
i- The composition function of b, a *denoted by* $b \circ a$ *is a IFSPT on* (Γ, A)*.*
ii-If b, a *are type I (or D), then the composition function* $b \circ a$ *of b and a is of type I (or D).*

Proof. i-$(b \circ a)(\tilde{\varphi}) = b(a(\tilde{\varphi})) = \tilde{\varphi}$. Also, $(b \circ a)(M_\Pi) = b(a(M_\Pi))$. Put $a(M_\Pi) = G_\Pi$, then $(b \circ a)(M_\Pi) = b(G_\Pi) \tilde{\sqsupseteq} G_\Pi = a(M_\Pi) \tilde{\sqsupseteq} M_\Pi$. Therefore $(b \circ a)(M_\Pi) \tilde{\sqsupseteq} M_\Pi$.

ii-* Let b, a of type I and $M_\Pi \tilde{\sqsubseteq} G_\Pi$, then $a(M_\Pi)\tilde{)}\sqsubseteq a(G_\Pi)$ and $b(a(M_\Pi))\tilde{)}\sqsubseteq b(a(G_\Pi))$.
Therefore $(b \circ a)(M_\Pi) = b(a(M_\Pi))\tilde{\sqsubseteq} b(a(G_\Pi)) = (b \circ a)(G_\Pi)$. This shoes that $b \circ a$ is of type I.
** Let b, a of type D, then $b(M_\Pi \tilde{\sqcup} G_\Pi) = b(M_\Pi)\tilde{\sqcup} b(G_\Pi)$ and $a(M_\Pi \tilde{\sqcup} G_\Pi) = a(M_\Pi)\tilde{\sqcup} a(G_\Pi)$. Now, $(b \circ a)(M_\Pi \tilde{\sqcup} G_\Pi) = b(a(M_\Pi \tilde{\sqcup} G_\Pi)) = b(a(M_\Pi)\tilde{\sqcup} a(G_\Pi)) = b(a(M_\Pi))\tilde{\sqcup} b(a(G_\Pi)) = (b \circ a)(M_\Pi)\tilde{\sqcup} b(a)(G_\Pi)$.
Therefore, $(b \circ a)$ is a type D. □

Definition 4.2. *i-Given two IFSPT on (Γ, A), let b_1, b_2. For every $M_\Pi \tilde{\in} IFS(\Gamma_A)$, b_1 is considered coarser than b_2. This can be expressed as $b_1 \tilde{\leq} b_2$ if $b_1(M_\Pi)\tilde{\sqsubseteq} b_2(M_\Pi)$.*
ii- Let $\{b_\ell, \ell \in L\}$ be a family of IFSPT on (Γ, A), where L is an indexed set. We define $sup b_\ell = \tilde{\vee}_{\ell \in L}$, the least upper bound IFSPT on (Γ, A) and $inf b_\ell = \tilde{\wedge}_{\ell \in L}$, the greatest lower bound IFSPT on (Γ, A), as follows:
$(\tilde{\vee}_{\ell \in L} b_\ell)(M_\Pi) = \tilde{\sqcup}_{\ell \in L} b_\ell(M_\Pi)$ *and* $(\tilde{\wedge}_{\ell \in L} b_\ell)(M_\Pi) = \tilde{\sqcap}_{\ell \in L} b_\ell(M_\Pi)$.

Proposition 4.2. *Let $\{b_\ell, \ell \in L\}$ be a family of IFSPT on (Γ, A). Then,*
i-For each $\ell \in L$, $\tilde{\vee}_{\ell \in L} b_\ell$ and $\tilde{\wedge}_{\ell \in L} b_\ell$ are of type I if b_ℓ is of type I.
ii-If b_ℓ is of type D, then for every $\ell \in L$, $\tilde{\vee}_{\ell \in L} b_\ell$ likewise belongs to type D.
iii-For every $\ell \in L$, $\tilde{\wedge}_{\ell \in L} b_\ell$ is also of type S if b_ℓ is of type S.

Proof. Let $G_B, M_\Pi \tilde{\in} IFS(\Gamma_A)$.
i(1)-Let b_ℓ be of type I and $M_\Pi \tilde{\sqsubseteq} G_B$, then $b_\ell(M_\Pi)\tilde{\sqsubseteq} b_\ell(G_B)$.
$(\tilde{\vee}_{\ell \in L} b_\ell)(M_\Pi) = \tilde{\sqcup}_{\ell \in L} b_\ell(M_\Pi)\tilde{\sqsubseteq}\tilde{\sqcup}_{\ell \in L} b_\ell(G_B) = (\tilde{\vee}_{\ell \in L} b_\ell)(G_B)$. This indicates that
$(\tilde{\vee}_{\ell \in L} b_\ell)(M_\Pi)\tilde{\sqsubseteq}(\tilde{\vee}_{\ell \in L} b_\ell)(G_B)$ and for that reason $\tilde{\vee}_{\ell \in L} b_\ell$ is of type I.
i(2)-$(\tilde{\wedge}_{\ell \in L} b_\ell)(M_\Pi) = \tilde{\sqcap}_{\ell \in L} b_\ell(M_\Pi)\tilde{\sqsubseteq}\tilde{\sqcap}_{\ell \in L} b_\ell(G_B) = (\tilde{\wedge}_{\ell \in L} b_\ell)(G_B)$. This indicates that
$(\tilde{\wedge}_{\ell \in L} b_\ell)(M_\Pi)\tilde{\sqsubseteq}(\tilde{\wedge}_{\ell \in L} b_\ell)(G_B)$ and for that reason $\tilde{\wedge}_{\ell \in L} b_\ell$ is of type I.
ii-Let b_ℓ be of type D, then $b_\ell(M_\Pi \tilde{\sqcup} G_B) = b_\ell(M_\Pi)\tilde{\sqcup} b_\ell(G_B)$. At hence,
$(\tilde{\vee}_{\ell \in L} b_\ell)(M_\Pi \tilde{\sqcup} G_B) = \tilde{\sqcup} b_\ell(M_\Pi \tilde{\sqcup} G_B) = \tilde{\sqcup}(b_\ell(M_\Pi)\tilde{\sqcup}(b_\ell(G_B)) = (\tilde{\sqcup} b_\ell(M_\Pi))\tilde{\sqcup}(\tilde{\sqcup} b_\ell(G_B))$
$= ((\tilde{\vee}_{\ell \in L} b_\ell)(M_\Pi))\tilde{\sqcup}((\tilde{\vee}_{\ell \in L} b_\ell)(G_B))$. This suggests that the type of $\tilde{\vee}_{\ell \in L} b_\ell$ is D.
iii-Let b_ℓ be of type S, then $b_\ell^2(M_\Pi) = b_\ell(M_\Pi)$. Hence,
$(\tilde{\wedge}_{\ell \in L} b_\ell)^2(M_\Pi) = (\tilde{\wedge}_{\ell \in L} b_\ell)((\tilde{\wedge}_{\jmath \in L} b_\jmath)(M_\Pi)) = \tilde{\sqcap}_{\ell \in L} b_\ell(\tilde{\sqcap}_{\jmath \in L} b_\jmath(M_\Pi)) =$
$(\tilde{\sqcap}_{\ell \in L} b_\ell^2(M_\Pi))\tilde{\sqcap}(\tilde{\sqcap}_{\jmath \neq \ell} b_\ell(b_\jmath(M_\Pi))) = (\tilde{\sqcap}_{\ell \in L} b_\ell(M_\Pi))\tilde{\sqcap}(\tilde{\sqcap}_{\jmath \neq \ell} b_\ell(b_\jmath(M_\Pi))$
$\tilde{\sqsubseteq}(\tilde{\sqcap}_{\ell \in L} b_\ell(M_\Pi)) = (\tilde{\wedge}_{\ell \in L} b_\ell)(M_\Pi)$. Then, $(\tilde{\wedge}_{\ell \in L} b_\ell^2)(M_\Pi)\tilde{\sqsubseteq}(\tilde{\wedge}_{\ell \in L} b_\ell)(M_\Pi)$.
On the other hand since $b_\ell(M_\Pi)\tilde{\sqsupseteq} M_\Pi$, we have $(\tilde{\wedge}_{\ell \in L} b_\ell)(M_\Pi)\tilde{\sqsupseteq} M_\Pi$, which leads to
$(\tilde{\wedge}_{\ell \in L} b_\ell^2)(M_\Pi)\tilde{\sqsupseteq}(\tilde{\wedge}_{\ell \in L} b_\ell)(M_\Pi)$. therefore, we have $(\tilde{\wedge}_{\ell \in L} b_\ell^2)(M_\Pi) = (\tilde{\wedge}_{\ell \in L} b_\ell)(M_\Pi)$ and thus, $\tilde{\wedge}_{\ell \in L} b_\ell$ is type S. □

Definition 4.3. *Let b be an IFSPT, we define the IFS interior function*
$I_b : IFS(\Gamma_A) \to IFS(\Gamma_A)$ by: $I_b(M_\Pi) = (b(M_\Pi^c))^c$.
The following characteristics of the function I_b are met:
1-$I_b(\tilde{\varphi}) = \tilde{\varphi}$;
2-For each $M_\Pi \tilde{\in} IFS(\Gamma_A)$, we have $I_b(M_\Pi)\tilde{\sqsubseteq} M_\Pi$;
3-For each $M_\Pi, G_B \tilde{\in} IFS(\Gamma_A)$ such that $M_\Pi \tilde{\sqsubseteq} G_B$, we have $I_b(M_\Pi)\tilde{\sqsubseteq} I_b(G_B)$;

4-For each $M_\Pi, G_B \tilde{\in} IFS(\Gamma_A)$*, we have* $I_b((M_\Pi \tilde{\sqcap} G_B) = I_b(M_\Pi)\tilde{\sqcap} I_b(G_B)$*;*
5-For each $M_\Pi \tilde{\in} IFS(\Gamma_A)$*, we have* $I_b^2(M_\Pi) = I_b(M_\Pi)$*.*

Definition 4.4. *Let b be an IFSPT and* $M_\Pi \tilde{\in} IFS(\Gamma_A)$*. Then* M_Π *is said to be a IFSP-open (resp. IFSP-closed) set in* (Γ, A, b) *if* $I_b(M_\Pi) = M_\Pi$ *(resp.* $b(M_\Pi) = M_\Pi$*).*
It is obvious that M_Π *is an IFSP-closed if* M_Π^c *is IFSP-open.*

Proposition 4.3. *Let* b_1, b_2 *be two IFSPT on* (Γ, A)*. If* $b_1 \tilde{\leq} b_2$ *and* M_Π *is an IFSP-open (resp. IFSP-closed) with respect to* b_2*, then* M_Π *is IFSP-open (resp. IFSP-closed) with respect to* b_1*.*

Proof. First, let M_Π be *IFSP*-open with respect to b_2 and $b_1 \tilde{\leq} b_2$, then $b_1(M_\Pi)\tilde{\sqsubseteq} b_2(M_\Pi)$ with implies that $I_{b_1}(M_\Pi)\tilde{\sqsupseteq} I_{b_2}(M_\Pi)$. Since M_Π is *IFSP*-open with respect to b_2, then $I_b(M_\Pi) = M_\Pi$ and therefore $I_{b_1}(M_\Pi)\tilde{\sqsupseteq} M_\Pi$. Since $I_{b_1}(M_\Pi)\tilde{\sqsubseteq} M_\Pi$, then $I_{b_1}(M_\Pi) = M_\Pi\ M_\Pi$ and so M_Π is *IFSP*-open with respect to b_1.
Now, let M_Π be *IFSP*-closed with respect to b_2 and $b_1 \tilde{\leq} b_2$, then $b_1(M_\Pi)\tilde{\sqsubseteq} b_2(M_\Pi)$. Since M_Π is *IFSP*-closed with respect to b_2, then $b_2(M_\Pi) = M_\Pi$ and therefore $b_1(M_\Pi)\tilde{\sqsubseteq} M_\Pi$. Since, $b_1(M_\Pi)\tilde{\sqsupseteq} M_\Pi$, then $b_1(M_\Pi) = M_\Pi$ and so, M_Π is *IFSP*-closed with respect to b_1. □

Proposition 4.4. *Let b be IFSPT of type D, then the finite union of IFSP-closed sets in* (Γ, A, b) *is IFSP-closed.*

Proof. Let b be a type D and M_Π, G_B *IFSP*-closed sets in (Γ, A, b), then $b(M_\Pi) = M_\Pi$ and $b(G_B) = G_B$. Also, $b(M_\Pi \tilde{\sqcup} G_B) = b(M_\Pi)\tilde{\sqcup} b(G_B) = M_\Pi \tilde{\sqcup} G_B$. This shows that $M_\Pi \tilde{\sqcup} G_B$ is *IFSP*-closed. Therefore, the finite union of *IFSP*-closed sets in (Γ, A, b) is *IFSP*-closed. □

Definition 4.5. *Let b be a IFSPT with* $M_\Pi \tilde{\in} IFS(\Gamma_A)$*. The trace of b on* M_Π *is defined as* $b_{M_\Pi}(G_B) = b(G_B)\tilde{\sqcap} M_\Pi$*, for any IFS subset* G_B *of* M_Π*.*

Proposition 4.5. *The trace* b_{M_Π} *represents an IFSPT. Furthermore, if b is of type I (resp. D, IS), so is* b_{M_Π} *(resp. D, IS).*

Proof. First, we show that b_{M_Π} is a *IFSPT*.
(1) $b_{M_\Pi}(\tilde{\phi}) = b(\tilde{\phi})\tilde{\sqcap} M_\Pi = \tilde{\phi} \sqcap M_\Pi = \tilde{\phi}$.
(2) $b_{M_\Pi}(G_B) = b(G_B)\tilde{\sqcap} M_\Pi \tilde{\sqsupseteq} G_B \tilde{\sqcap} M_\Pi = G_B$. Therefore, $b_{M_\Pi}(G_B)\tilde{\sqsupseteq} G_B$.
b_{M_Π} is a *IFSPT* according to (1) and (2).
(i) Assume b is of type I and $G_B, H_C \tilde{\sqsubseteq} M_\Pi$ such that $G_B \tilde{\sqsubseteq} H_C$. Then $b(G_B)\tilde{\sqsubseteq} b(H_C)$. Now, $b_{M_\Pi}(G_B) = b(G_B)\tilde{\sqcap} M_\Pi \tilde{\sqsubseteq} b(H_C)\tilde{\sqcap} M_\Pi = b_{M_\Pi}(H_C)$.
This show that $b_{M_\Pi}(G_B)\tilde{\sqsubseteq} b_{M_\Pi}(H_C)$. Since $H_C \tilde{\sqsubseteq} M_\Pi$ implies $b_{M_\Pi}(G_B)\tilde{\sqsubseteq} b_{M_\Pi}(H_C)$, then b_{M_Π} of type I.
(ii) Assume b is of type D, then $b(G_B \tilde{\sqcup} H_C) = b(G_B)\tilde{\sqcup} b(H_C)$. Now, $b_{M_\Pi}(G_B \tilde{\sqcup} H_C) = b(G_B \tilde{\sqcup} H_C)\tilde{\sqcap} M_\Pi = (b(G_B)\tilde{\sqcup} b(H_C))\tilde{\sqcap} M_\Pi = (b(G_B \tilde{\sqcup} M_\Pi)\tilde{\sqcup}(b(H_C \tilde{\sqcup} M_\Pi) = b_{M_\Pi}(G_B)\tilde{\sqcup} b_{M_\Pi}(H_C)$. Therefore, b_{M_Π} of type D.
(iii) Assume b is of type IS, then we have (with $G_B \tilde{\sqsubseteq} M_\Pi$)
$b^2_{M_\Pi}(G_B) = b_{M_\Pi}(b_{M_\Pi}(G_B)) = b(b(G_B)\tilde{\sqcap} M_\Pi)\tilde{\sqcap} M_\Pi$, but $b(G_B)\tilde{\sqcap} M_\Pi \tilde{\sqsubseteq} b(G_B)$ which implies that $b(b(G_B)\tilde{\sqcap} M_\Pi)\tilde{\sqsubseteq} b^2(G_B) = b(G_B)$.

So, $b^2_{M_\Pi}(G_B) = b(b(G_B)\tilde{\sqcap}M_\Pi)\tilde{\sqcap}M_\Pi\tilde{\sqsubseteq}b(G_B)\tilde{\sqcap}M_\Pi = b_{M_\Pi}(G_B)$.
But $b_{M_\Pi}(b_{M_\Pi}(G_B))\tilde{\sqsupseteq}b_{M_\Pi}(G_B)$, and these two inclusions that $b^2_{M_\Pi}(G_B) = b_{M_\Pi}(G_B)$, so b_{M_Π} of a type *IS*. □

5. Intuitionistic fuzzy soft pretopology generated by fuzzy soft preneighbourhoods

The ideas of the intuitionistic fuzzy soft preneighbourhood system at a soft point, the degree of soft non-vacuity, soft α-cut, and how to construct intuitionistic fuzzy soft pretopologies by intuitionistic fuzzy soft preneighbourhoods are introduced and studied in this part.

Definition 5.1. *The family of all intuitionistic fuzzy soft subsets Λ that fulfill $\Phi_\Lambda(\gamma_\delta) = 1$ is the intuitionistic fuzzy soft preneighbourhood system at a soft point γ_δ in (Γ, A), denoted by $\zeta(\gamma_\delta)$.*

Definition 5.2. *A function $\Psi : IFS(\Gamma_A) \to [0,1]$ is referred to as a soft non-vacuity degree (it associates to every intuitionistic fuzzy soft subset a number which represents the fact that it is more or less empty intuitionistic fuzzy soft set) if it fulfills:*
(i) $\Psi(\tilde{\phi}) = 0$,
(ii) $\Psi(G_B) = sup_{\gamma_\delta\tilde{\in}\Gamma_A}\Phi_{G_B}(\gamma_\delta)$,
(iii) $G_B\tilde{\sqsubseteq}H_C \Rightarrow \Psi(G_B) \leq \Psi(H_C)$.

In particular $\Psi(G_B) = 1$ if $\exists$ a soft point γ_δ that $\Phi_{G_B}(\gamma_\delta) = 1$.

To construct the adherence, we follow the same procedure as in the classical case; however, we employ the degree of soft non-vacuity to characterize how the intersection $\Lambda\tilde{\sqcap}G_B$, with $\Lambda\tilde{\in}\zeta(\gamma_\delta)$, is essentially empty.

Proposition 5.1. *The function $b : IFS(\Gamma_A) \to IFS(\Gamma_A)$ build by*
$\Phi_b(G_B) = inf_{\Lambda\tilde{\in}\zeta(\gamma_\delta)}\Psi(\Lambda\tilde{\sqcap}G_B)$ is an intuitionistic fuzzy soft adherence (intuitionistic fuzzy soft pretopology) of type I.

Proof. From $\Phi_{b(\tilde{\phi})}(\gamma_\delta) = inf_{\Lambda\tilde{\in}\zeta(\gamma_\delta)}\Psi(\Lambda\tilde{\sqcap}\tilde{\phi}) = inf_{\Lambda\tilde{\in}\zeta(\gamma_\delta)}\Psi(\tilde{\phi}) = 0$, then $\Phi_{b(\tilde{\phi})}(\gamma_\delta) = 0$ for each $\gamma_\delta\tilde{\in}\Gamma_A$. At hence, $b(\tilde{\phi}) = \tilde{\phi}$.
Since foe each $\gamma_\delta\tilde{\in}\Gamma_A$ and for each $\Lambda\tilde{\in}\zeta(\gamma_\delta)$, we have $\Phi_\Lambda(\gamma_\delta) = 1$, then
$\Phi_{b(G_B)}(\gamma_\delta) = inf_{\Lambda\tilde{\in}\zeta(\gamma_\delta)}\Psi(\Lambda\tilde{\sqcap}G_B) \geq inf_{\Lambda\tilde{\in}\zeta(\gamma_\delta)}\Psi(G_B) = \Psi(G_B) = sup_{\gamma_\delta\tilde{\in}\zeta(\gamma_\delta)}\Phi_{G_B}(\gamma_\delta) \geq \Phi_{G_B}(\gamma_\delta)$.
This shows that $\Phi_{b(G_B)}(\gamma_\delta) \geq \Phi_{G_B}(\gamma_\delta)$ and therefore $b(G_B)\tilde{\sqsupseteq}G_B$.
Given arbitrary intuitionistic fuzzy soft subsets G_B, H_C, let $G_B\tilde{\sqsubseteq}H_C$. Then for each $\Lambda\tilde{\in}\zeta(\gamma_\delta)$, we have
$G_B\tilde{\sqcap}\Lambda\tilde{\sqsubseteq}H_C\tilde{\sqcap}\Lambda$ it suggests that $\Psi(G_B\tilde{\sqcap}\Lambda) \leq \Psi(H_C\tilde{\sqcap}\Lambda)$.
Therefore $inf_{\Lambda\tilde{\in}\zeta(\gamma_\delta)}\Psi(G_B\tilde{\sqcap}\Lambda) \leq inf_{\Lambda\tilde{\in}\zeta(\gamma_\delta)}\Psi(H_C\tilde{\sqcap}\Lambda)$, and so $\Phi_{b(G_B)}(\gamma_\delta) \leq \Phi_{b(H_C)}(\gamma_\delta)$.
Thus $b(G_B)\tilde{\sqsubseteq}b(H_C)$. Based on the aforementioned, we deduce that b represents intuitionistic fuzzy soft adherence (also known as intuitionistic fuzzy soft pretopology) of type *I*. □

Definition 5.3. *Let $\alpha \in [0,1]$ and let G_B be an intuitionistic fuzzy soft subset of (Γ, A). The soft subset $\widetilde{(G_B)}_\alpha$ of (Γ, A) is the soft α-cut of G_B, and it is defined as follows:*
$\widetilde{(G_B)}_\alpha = \{\gamma_\delta \tilde{\in} (\Gamma, A) : \Phi_{G_B}(\gamma_\delta) \geq \alpha\}$. We now present a second method for constructing an intuitionistic fuzzy soft adherence, and we will show subsequently that these two approaches can produce the same result. Let us suppose that with every $\eta \in [0,1]$ is associated a classical type-I soft pretopology $\widetilde{(b)}_\eta$ and that if $\eta_1 \geq \eta_2$ we have $\widetilde{(b)}_{\eta_1} \tilde{\leq} \widetilde{(b)}_{\eta_2}$.

Proposition 5.2. *$\eta_1 \geq \eta_2$ can be satisfied by a family of soft adherences of type I, denoted by $\widetilde{(b)}_\eta$ implies $\widetilde{(b)}_{\eta_1} \tilde{\leq} \widetilde{(b)}_{\eta_2}$.*
Next, the two equivalent methods define the function $\acute{b} : IFS(\Gamma_A) \to IFS(\Gamma_A)$
(i) $\Phi_{\acute{b}}(\gamma_\delta) = sup\{\eta : \gamma_\delta \tilde{\in} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)\}$
(ii) $\widetilde{(\acute{b}(G_B))}_\alpha = \tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)$ that is a type-I intuitionistic fuzzy soft pretopology.

Proof. First we show that if $\eta_1 \geq \eta_2$ then $\widetilde{(b)}_{\eta_1}(\widetilde{(G_B)}_{\eta_1}) \tilde{\sqsubseteq} \widetilde{(b)}_{\eta_2}(\widetilde{(G_B)}_{\eta_2})$
Since $\widetilde{(G_B)}_{\eta_1} \tilde{\sqsubseteq} \widetilde{(G_B)}_{\eta_2}$ and $\widetilde{(b)}_\eta$ of type I, then $\widetilde{(b)}_{\eta_1}(\widetilde{(G_B)}_{\eta_1}) \tilde{\sqsubseteq} \widetilde{(b)}_{\eta_1}(\widetilde{(G_B)}_{\eta_2})$. Also, if $\widetilde{(b)}_{\eta_1} \tilde{\leq} \widetilde{(b)}_{\eta_2}$, then $\widetilde{(b)}_{\eta_1}(\widetilde{(G_B)}_{\eta_2}) \tilde{\sqsubseteq} \widetilde{(b)}_{\eta_2}(\widetilde{(G_B)}_{\eta_2})$.
Therefore, we have $\widetilde{(b)}_{\eta_1}(\widetilde{(G_B)}_{\eta_1}) \tilde{\sqsubseteq} \widetilde{(b)}_{\eta_2}(\widetilde{(G_B)}_{\eta_2})$.
However, it is well known that $\widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta) \mid \eta \in [o,1]$ often has no membership function attached to it, and this is true for each α. If it isn't, we have
$\tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta) = \widetilde{(b)}_\alpha(\widetilde{(G_B)}_\alpha)$.
However, the intuitionistic fuzzy soft pretopology $\acute{b}$ associated with
$\widetilde{(\acute{b}(G_B))}_\alpha = \tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)$ which will satisfies $\acute{b}(\tilde{\phi}) = \tilde{\phi}$ and $\acute{b}(G_B) \tilde{\sqsupseteq} G_B$ for every η because $\widetilde{(\acute{b}(\tilde{\phi}))}_\alpha = \tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(\tilde{\phi})}_\eta) = \tilde{\phi}$, then $\acute{b}(\tilde{\phi}) = \tilde{\phi}$.

Also, we have $\widetilde{(\acute{b}(G_B))}_\alpha = \tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)$, since $\widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta \tilde{\sqsupseteq} \widetilde{(G_B)}_\eta)$,
then $\widetilde{(\acute{b}(G_B))}_\eta \tilde{\sqsupseteq} \tilde{\sqcap}_{\eta<\alpha} \widetilde{(G_B)}_\eta$. Thus $\acute{b}(G_B) \tilde{\sqsupseteq} G_B$.
We now demonstrate that $\acute{b}$ is of a type I, it is a consequence of the fact that the $\widetilde{(b)}_\eta$ are of type I. We have $G_B \tilde{\sqsubseteq} H_C$ so, $\widetilde{(G_B)}_\eta \tilde{\sqsubseteq} \widetilde{(H_C)}_\eta$, for every η. Since $\widetilde{(b)}_\eta$ are of type I, then $\widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta) \tilde{\sqsubseteq} \widetilde{(b)}_\eta(\widetilde{(H_C)}_\eta)$, which implies that $\tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta) \tilde{\sqsubseteq} \tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(H_C)}_\eta)$. Therefor, $\widetilde{(\acute{b}(G_B))}_\alpha \tilde{\sqsubseteq} \widetilde{(\acute{b}(H_C))}_\eta$ and thus $\acute{b}(G_B) \tilde{\sqsubseteq} \acute{b}(H_C)$.
This demonstrates the type I of $\acute{b}$.
We now demonstrate the equivalentity of the two approaches to creating $\acute{b}$.
We not that $\acute{\acute{b}}$ is a soft adherence built with $\Phi_{\acute{b}(G_B)}(\gamma_\delta) = sup\{\eta : \gamma_\delta \tilde{\in} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)\}$. Let γ_δ be a soft point of $\widetilde{(\acute{\acute{b}}(G_B))}_\alpha$, then $\gamma_\delta \tilde{\in} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)$, for every $\eta < \alpha$.
(Given that $[0,1]$ is hole-free), therefore $\gamma_\delta \tilde{\in} \tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)$ and so $\gamma_\delta \tilde{\in} (\acute{b}\widetilde{(G_B)}_\eta)$ $(*)$
Let $\gamma_\delta \tilde{\in} (\acute{b}\widetilde{(G_B)}_\eta)$, then $\gamma_\delta \tilde{\in} \tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)$. Therefore, $\gamma_\delta \tilde{\in} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)$ for each $\eta < \alpha$ and $sup\{\eta : \gamma_\delta \tilde{\in} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)\} \geq \alpha$. At hence, $\gamma_\delta \tilde{\in} \widetilde{(\acute{\acute{b}}(G_B))}_\alpha$ $(**)$
Based on $*$ and $(**)$, both approaches are equivalent. □

Proposition 5.3. *Let A be a set of parameters and Γ be the universe set, γ_δ be a soft point in Γ_A, $\zeta(\gamma_\delta)$ be the family of all intuitionistic fuzzy soft preneighbourhoods of γ_δ and $\widetilde{(\zeta(\gamma_\delta))}_\alpha$ is soft η-cuts. So, we can*

build a soft pretopology $\widetilde{(b)}_\eta$ *by*
$\widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta) = \{\gamma_\delta : \forall \widetilde{(\Lambda)}_\eta \tilde{\in} \widetilde{(\zeta(\gamma_\delta))}_\eta, \widetilde{(\Lambda)}_\eta \tilde{\sqcap} \widetilde{(G_B)}_\eta \neq \tilde{\phi}\}$ *on* $SS(\Gamma_A)$.
We now consider the intuitionistic fuzzy soft adheerences:
(i) $b(G_B)$ *defined by* $\Phi_b(G_B) = inf_{\Lambda \tilde{\in} \zeta(\gamma_\delta)} \Psi(\Lambda \tilde{\sqcap} G_B)$
(ii) $\acute{b}(G_B)$ *defined by* $\widetilde{(\acute{b}(G_B))}_\alpha = \tilde{\sqcap}_{\delta<\alpha} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)$, *for every* $\alpha \in [0,1]$.
Then the two intuitionistic fuzzy soft adherences are similar.

Proof. If for α we have $\gamma_\delta \tilde{\in} \widetilde{(b)}_\alpha(\widetilde{(G_B)}_\alpha)$, then $\widetilde{(\Lambda)}_\alpha \tilde{\sqcap} \widetilde{(G_B)}_\alpha \neq \tilde{\phi}$, for each $\Lambda \tilde{\in} \zeta(\gamma_\delta)$.
Therefore, $\Psi(\Lambda \tilde{\sqcap} G_B) \geq \alpha$ and so $sup_{z_\delta \tilde{\in} \Gamma_A} \Phi_{\Lambda \tilde{\sqcap} G_B}(z_\delta) \geq \alpha$, for $\Lambda \tilde{\in} \zeta(\gamma_\delta)$.
Thus $inf_{\Lambda \tilde{\in} \zeta(\gamma_\delta)} sup_{z_\delta \tilde{\in} \Gamma_A} \Phi_{\Lambda \tilde{\sqcap} G_B}(z_\delta) = \xi_1 \geq \alpha$.
Let's think about $sup\{\eta : \gamma_\delta \tilde{\in} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)\} = \xi_2$ for the same soft point γ_δ, we have $\xi_1 \geq \xi_2$.
Assume, for now, that $\xi_1 > \xi_2$. This suggests that $\alpha_\circ$ exists such that $\xi_1 > \alpha_\circ > \xi_2$, then $\alpha_\circ > \xi_2$ and for that reason $\gamma_\delta \tilde{\notin} \widetilde{(b)}_{\alpha_\circ}(\widetilde{(G_B)}_{\alpha_\circ})$, then there are $\Lambda \tilde{\in} \zeta(\gamma_\delta)$ such that $\widetilde{(\Lambda)}_{\alpha_\circ} \tilde{\sqcap} \widetilde{(G_B)}_{\alpha_\circ} = \tilde{\phi}$. This suggests that $\Psi(\Lambda \tilde{\sqcap} G_B) \leq \alpha_\circ$ and for that reason
$sup_{z_\delta \tilde{\in} \Gamma_A} \Phi_{\Lambda \tilde{\sqcap} G_B}(z_\delta) \leq \alpha_\circ$, then $inf_{\Lambda \tilde{\in} \zeta(\gamma_\delta)} sup_{z_\delta \tilde{\in} \Gamma_A} \Phi_{\Lambda \tilde{\sqcap} G_B}(z_\delta) = \xi_1 \leq \alpha_\circ$. This suggests a contradication: $\xi_1 \leq \alpha_\circ$. $\xi_1 = \xi_2$ as a result. □

Proposition 5.4. *For any* $\eta \in [0,1]$, *a type-I soft adherence* $\widetilde{(b)}_\eta$ *has the following property:* $\eta_1 \geq \eta_2$ *indicates that* $\widetilde{(b)}_{\eta_1} \tilde{\leq} \eta_2$ *where* A *is the set of parameters and* Γ *is the universe set.*
We construct the intuitionistic fuzzy soft type-I adherence b using:
$\widetilde{(b(G_B))}_\alpha) = \tilde{\sqcap}_{\delta<\alpha} \widetilde{(b)}_\eta \widetilde{(G_B)}_\eta)$. *If for every* η, $\widetilde{(b)}_\eta$ *is of type D then b is of type D. If for each* η, $\widetilde{(b)}_\eta$ *is of type S then b is of type S.*

Proof. It has been demonstrated that (Γ, A, b) is an intuitionistic fuzzy soft pretopology of type-I.
(i) Let us suppose that $\widetilde{(b)}_\eta$ are of type D, then $\widetilde{(b(G_B \tilde{\sqcup} H_C))}_\alpha = \tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta \tilde{\sqcup} \widetilde{(H_C)}_\eta) =$
$(\tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(G_B)}_\eta)) \tilde{\sqcup} (\tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(H_C)}_\eta)) =$
$\widetilde{(b(G_B))}_\alpha \tilde{\sqcup} \widetilde{(b(H_C))}_\alpha$ which prove that b is of type D.
(ii) Let $\widetilde{(b)}_\eta$ of type S. Since $b(G_B) \tilde{\sqsupseteq} G_B$, then $b^2(G_B) \tilde{\sqsupseteq} b(G_B)$. However, given that $\widetilde{(b^2(G_B))}_\alpha =$ $\tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta(\widetilde{(b(G_B))}_\eta) = \tilde{\sqcap}_{\eta<\alpha} \widetilde{(b)}_\eta((\tilde{\sqcap}_{\Phi<\eta} \widetilde{(b)}_\Phi(\widetilde{(G_B)}_\Phi))$ and since
$\widetilde{(b)}_\eta(\tilde{\sqcap}_{\Phi<\alpha} \widetilde{(b)}_\Phi \widetilde{(G_B)}_\Phi) \tilde{\sqsubseteq} \tilde{\sqcap}_{\Phi<\alpha} (\widetilde{(b)})^2_\Phi(\widetilde{(G_B)}_\Phi) = \tilde{\sqcap}_{\Phi<\alpha} (\widetilde{(b)})_\Phi(\widetilde{(G_B)}_\Phi)$, we have
$\widetilde{(b)}_\eta(\tilde{\sqcap}_{\Phi<\eta} \widetilde{(b)}_\Phi(\widetilde{(G_B)}_\Phi)) \tilde{\sqsubseteq} \widetilde{(b(G_B))}_\eta)$ it suggests that
$\widetilde{(b^2(G_B))}_\alpha \tilde{\sqsubseteq} \tilde{\sqcap}_{\eta<\alpha} \widetilde{(b(G_B))}_\eta$. So $b^2(G_B) \tilde{\sqsubseteq} b(G_B)$. Therefore, we obtain $b^2(G_B) = b(G_B)$, where b is of type S. □

6. Conclusion

This paper defines a few terms related to intuitionistic fuzzy soft for pretopological space, including *IFS* interior function, *IFS* pre-open set, *IFS* pre-closed set, trace of a *IFS* pretopology, *IFS* separation axioms, *IFS* subspace, *IFS* compactness, and *IFS* connectedness. Furthermore, the *IFS* preneighbourhood system at a soft point, the soft α-cut, and the degree of soft non-vacuity are given, and *IFS* preneighbourhoods produced *IFS* pretopologies.

Acknowledgment: The authors extend their appreciation to Prince Sattam bin Abdulaziz University for funding this research work through the project number (PSAU/2024/01/29167).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] R.A. Abu-Gdairi, A. Azzam, I. Noaman, Nearly Soft β - Open Sets via Soft Ditopological Spaces, Eur. J. Pure Appl. Math. 15 (2022), 126–134. https://doi.org/10.29020/nybg.ejpam.v15i1.4249.

[2] J.C.R. Alcantud, A.Z. Khameneh, A. Kilicman, Aggregation of Infinite Chains of Intuitionistic Fuzzy Sets and Their Application to Choices with Temporal Intuitionistic Fuzzy Information, Inf. Sci. 514 (2020), 106–117. https://doi.org/10.1016/j.ins.2019.12.008.

[3] T.M. Al-shami, Z.A. Ameen, A.A. Azzam, M.E. El-Shafei, Soft Separation Axioms via Soft Topological Operators, AIMS Math. 7 (2022), 15107–15119. https://doi.org/10.3934/math.2022828.

[4] T.M. Al-shami, (2,1)-Fuzzy Sets: Properties, Weighted Aggregated Operators and Their Applications to Multi-Criteria Decision-Making Methods, Complex Intell. Syst. 9 (2023), 1687–1705. https://doi.org/10.1007/s40747-022-00878-4.

[5] T.M. Al-shami, H.Z. Ibrahim, A.A. Azzam, A.I. EL-Maghrabi, SR-Fuzzy Sets and Their Weighted Aggregated Operators in Application to Decision-Making, J. Function Spaces 2022 (2022), 3653225. https://doi.org/10.1155/2022/3653225.

[6] K.T. Atanassov, Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 20 (1986), 87–96. https://doi.org/10.1016/S0165-0114(86)80034-3.

[7] K.T. Atanassov, Operators over Interval Valued Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 64 (1994), 159–174. https://doi.org/10.1016/0165-0114(94)90331-X.

[8] K. Atanassov, G. Gargov, Interval Valued Intuitionistic Fuzzy Sets, Fuzzy Sets Syst. 31 (1989), 343–349. https://doi.org/10.1016/0165-0114(89)90205-4.

[9] K. Atanassov, Remark on Intuitionistic Fuzzy Numbers, Notes Intuitionistic Fuzzy Sets, 13 (2007), 29–32.

[10] A.A. Azzam, Asmaa.M. Nasr, H. ElGhawalby, R. Mareay, Application on Similarity Relation and Pretopology, Fractal Fract. 7 (2023), 168. https://doi.org/10.3390/fractalfract7020168.

[11] R. Badard, Fuzzy Pretopological Spaces and Their Representation, J. Math. Anal. Appl. 81 (1981), 378–390. https://doi.org/10.1016/0022-247X(81)90071-8.

[12] M. Brissaud, Les Espaces Prétopologiques, Compt. l'Acad. Sci. 280 (1975), 705–708.

[13] B.C. Cuong, V. Kreinovich, Picture Fuzzy Sets - A New Concept for Computational Intelligence Problems, in: 2013 Third World Congress on Information and Communication Technologies (WICT 2013), IEEE, Hanoi, Vietnam, 2013: pp. 1–6. https://doi.org/10.1109/WICT.2013.7113099.

[14] S.K. De, R. Biswas, A.R. Roy, An Application of Intuitionistic Fuzzy Sets in Medical Diagnosis, Fuzzy Sets Syst. 117 (2001), 209–213. https://doi.org/10.1016/S0165-0114(98)00235-8.

[15] W.L. Gau, D.J. Buehrer, Vague Sets, IEEE Transactions on Systems, Man Cybern. 23 (1993), 610–614. https://doi.org/10.1109/21.229476.

[16] F.H. Khedr, M.A. Abd-Allah, E.A. Abdelgaber, Fuzzy Soft Pretopological Spaces, Glob. J. Math. 13 (2019), 879–889.

[17] Z. Li, R. Cui, On the Topological Structure of Intuitionistic Fuzzy Soft Sets, Ann. Fuzzy Math. Inf. 5 (2013), 229–239.

[18] P.K. Maji, R. Biswas, A.R. Roy, Fuzzy Soft Sets, J. Fuzzy Math. 9 (2001), 589–602.

[19] P.K. Maji, R. Biswas, A.R. Roy, Fuzzy Soft Sets, J. Fuzzy Math. Appl. 59 (2010), 1425–1432.

[20] P.K. Maji, R. Biswas, A.R. Roy, Intuitionistic Fuzzy Soft Sets, J. Fuzzy Math. 9 (2001), 677–692.

[21] P.K. Maji, R. Biswas, A.R. Roy, Fuzzy Soft Sets, J. Fuzzy Math. Appl. 59 (2010), 1425–1432.

[22] D. Molodtsov, Soft Set Theory–First Results, Comp. Math. Appl. 37 (1999), 19–31. https://doi.org/10.1016/S0898-1221(99)00056-5.

[23] M. Yazar, G. Çuğdem, S. Bayramov, Fixed Point Theorems of Soft Contractive Mappings, Filomat 30 (2016), 269–279. https://doi.org/10.2298/FIL1602269Y.

[24] M. Olgun, M. Ünver, Ş. Yardımcı, Pythagorean Fuzzy Topological Spaces, Complex Intell. Syst. 5 (2019), 177–183. https://doi.org/10.1007/s40747-019-0095-2.

[25] S. Roy, T.K. Samanta, A Note on Fuzzy Soft Topological Spaces, Ann. Fuzzy Math. Inf. 3 (2012), 305–311.

[26] M. Shabir, M. Naz, On Soft Topological Spaces, Comp. Math. Appl. 61 (2011), 1786–1799. https://doi.org/10.1016/j.camwa.2011.02.006.

[27] B. Tanay, M.B. Kandemir, Topological Structure of Fuzzy Soft Sets, Comp. Math. Appl. 61 (2011), 2952–2957. https://doi.org/10.1016/j.camwa.2011.03.056.

[28] L.A. Zadeh, Fuzzy Sets, Inf. Control 8 (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.