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Applications of Strongly Deferred Weighted Convergence in the Environment of Uncertainty

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Abstract. In the context of uncertainty theory, we present strongly deferred weighted convergence of complex uncertain sequences. Also, we introduce strongly deferred weighted convergence of complex uncertain sequences in all five aspects of uncertainty, that is through almost surely, mean, measure, distribution and uniformly almost surely. Further, with the aid of interesting examples and diagram we investigate some interrelationships among these complex uncertain sequences.

1. Introduction and preliminaries

In real life human being usually come across situations when decisions are made in a state of indeterminacy. There are two ways to handle indeterminacy which were suggested by Liu [10]. One is the most familiar theory known as probability theory, which is unavoidable when the frequencies are very close to the distribution function. But when no sample is accessible to approximate a probability distribution, in that case probability theory never permits precise results.

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For example, we have no way to acquire the strength of the bridge in use. Thus, it is hard to obtain enough sample data for some events. In this case, we have to rely on the domain experts to give belief degree that each event would happen while making decisions. To overcome such kind of circumstances, Liu [8] investigated the hypothesis of uncertainty theory. Probability theory is based on the frequency of random events whereas the uncertainty theory is based on the belief degree. The uncertainty theory is used to handle randomness and fuzziness. In 2009, Liu [11] refined this theory which is based on an uncertain measure that satisfies monotonicity, normality, countable sub-additivity, self-duality and product measure properties. Moreover, Liu applied this theory of uncertainty on sequence and established the property of convergence of uncertain measure by presenting convergence in mean, in distribution, in measure and in almost surely. Peng [20] presented the conception of the complex uncertain variables which was further studied by Chen et al. [2]. Datta and Tripathy [3] extended this study by introducing double sequences of complex uncertain variables.

Since the convergence of sequences plays a vital role in mathematics. There are also many convergence notions in uncertainty theory i.e., almost surely convergence, convergence in mean, convergence in measure and convergence in distribution. Thereafter, You [27] carried out an important development by reporting a new type of convergence known as convergence uniformly almost surely. Complex uncertain sequence by means of statistical convergence was studied by Tripathy and Nath [26]. In uncertainty theory, a lot of development has been made by many researchers from different fields of mathematics like finance [1] , set theory [7] , risk and stability analysis [12] etc. For a detailed study, one may refer ([5], [6], [18], [19], [21], [22], [24], [25]).

Definition 1.1. *(Liu [9]) Suppose Ω be a non-empty set and $\mathcal{A}$ be a $\sigma-$algebra on Ω. A set function $\boldsymbol{\mu} : \mathcal{A} \rightarrow [0, 1]$ is called uncertain measure if it holds the following criteria:*

(i) $\boldsymbol{\mu}(\Omega) = 1$,

(ii) $\boldsymbol{\mu}\{\Lambda\} + \boldsymbol{\mu}\{\Lambda^c\} = 1$, for any event $\Lambda \in \mathcal{A}$,

(iii) For every countable sequence of $\{\Lambda_m\} \in \mathcal{A}$, we have

$$\boldsymbol{\mu}\left\{\bigcup_{m=1}^{\infty} \Lambda_m\right\} \leq \sum_{m=1}^{\infty} \boldsymbol{\mu}\{\Lambda_m\}.$$

The triplet $(\Omega, \mathcal{A}, \boldsymbol{\mu})$ is called an uncertainty space.

For obtaining uncertainty measure of compound event, product criteria of uncertain measure was presented by Liu [11] as

(iv) Suppose $(\Omega_m, \mathcal{A}_m, \boldsymbol{\mu}_m)$ be an uncertainty space for $m \in \mathbb{N}$. The product uncertain measure $\boldsymbol{\mu}$ is an uncertain measure that satisfies

$$\boldsymbol{\mu}\left\{\prod_{m=1}^{\infty} \Lambda_m\right\} = \bigwedge_{m=1}^{\infty} \boldsymbol{\mu}_m\{\Lambda_m\},$$

where Λ_m are events which are arbitrarily taken from $\mathcal{A}_m$.

Definition 1.2. *(Liu [13]) An uncertain variable ς is a measurable function from $(\Omega, \mathcal{A}, \boldsymbol{\mu}) \rightarrow \mathbb{R}$ (set of real numbers) i.e., for any Borel set B of real numbers, the set*

$$\{v \in \Omega : \varsigma(v) \in B\}$$

is an event.

Definition 1.3. *(Liu [13]) The uncertainty distribution Φ of an uncertain variable ς is defined as*

$$\Phi(u) \quad = \quad \boldsymbol{\mu}\{\varsigma \leq u\}, \ \text{for all } u \in \mathbb{R}.$$

Definition 1.4. *(Liu [13]) The expected value of E of an uncertain variable ς is given by*

$$E[\varsigma] \quad = \quad \int_0^{+\infty} \boldsymbol{\mu}\{\varsigma \geq y\}dy - \int_{-\infty}^{0} \boldsymbol{\mu}\{\varsigma \leq y\}dy$$

provided that at least one of two integrals is finite.
Now, let us discuss some concept of convergence for uncertain sequences which were introduced by Chen et al. [2].

Definition 1.5. *(Chen et al. [2]) Let (z_m) be complex uncertain sequence. Then (z_m) is almost surely convergent to z if $\exists$ an event Λ whose measure is 1 with*

$$\lim_{m\rightarrow\infty} \|z_m(v) - z(v)\| \quad = \quad 0,$$

for every $v \in \Lambda$.

Definition 1.6. *(Chen et al. [2]) Let (z_m) be a complex uncertain sequence, then (z_m) is convergent in measure to z if*

$$\lim_{m\rightarrow\infty} \boldsymbol{\mu}\{\|z_m - z\| \geq \epsilon\} \quad = \quad 0,$$

$\forall \epsilon > 0$.

Definition 1.7. *(Chen et al. [2]) Let (z_m) be a complex uncertain sequence, then (z_m) is convergent in mean to z if*

$$\lim_{m\rightarrow\infty} \mathbf{E}[\|z_m - z\|] \quad = \quad 0.$$

Definition 1.8. *(Chen et al. [2]) Let (z_m) be complex uncertain sequence. If $\Phi, \Phi_1, \Phi_2, \cdots$ are uncertainty distributions of uncertain variables $z, z_1, z_2, \cdots$ respectively, then (z_m) is convergent in distribution to z if*

$$\lim_{m\rightarrow\infty} \Phi_m(c) \quad = \quad \Phi(c),$$

$\forall c \in \mathbb{R}$, at which $\Phi(c)$ is continuous.

Definition 1.9. *(Chen et al. [2]) Let (z_m) be complex uncertain sequence. Then (z_m) is uniformly almost surely convergent to z if $\exists$ $(E^{'}(m))$ with $\boldsymbol{\mu}(E^{'}(m)) \rightarrow 0$ such that (z_m) is uniformly convergent to z in $\Omega \setminus E^{'}(m)$, for any fixed natural number m, where $(E^{'}(m))$ represents a sequence of events.*

Remark (Chen et al. [2]) Suppose $z_m = \vartheta_m + i\eta_m$, for $m = 1, 2, 3\cdots$ and $z = \vartheta + i\eta$, we have

$$||z_m - z|| = \sqrt{(\vartheta_m - \vartheta)^2 + (\eta_m - \eta)^2}$$

is just an uncertain variable.

Recently, Nath and Tripathy [16] introduced statistical convergence of complex uncertain sequence via Orlicz function. Almost convergence of complex uncertain triple sequences, was presented by Das et al. [4]. Strongly almost convergence in sequences of complex uncertain variables is presented by Nath et al. [17]. Motivated by these works, we introduced the notion of strongly deferred weighted convergence in concern with mean, measure, distribution, almost surely and uniform almost surely of complex uncertain sequences. Moreover, we have discussed some interrelationships between these uncertain sequences.

A sequence (ς_m) is said to be Cesaro summable to a number l if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} \varsigma_m = l.$$

Similarly, a sequence (ς_m) is said to be strongly $p-$Cesaro summable to l if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{m=1}^{n} |\varsigma_m - l|^p = 0,$$

where $p \in \mathbb{R}^+$. For more details on Cesaro summability and strongly Cesaro summabilty one may consult ([14] and [15]).

Definition 1.10. *Assume that (a_n) and (b_n) be sequences of non-negative integers satisfying the following criteria:*

(i) $a_n < b_n$, ($\forall n \in \mathbb{N}$) and

(ii) $\lim_{n\to\infty} b_n = \infty$.

Now, suppose (t_m) and (l_m) are two sequences of non-negative real numbers s.t.

$$T_n = \sum_{m=a_{n+1}}^{b_n} t_m$$

and

$$L_n = \sum_{m=a_{n+1}}^{b_n} l_m.$$

The convolution of the above sequences is defined as:

$$R_n = (T * L)_n = \sum_{m=a_{n+1}}^{b_n} t_{b_n - m} l_m \varsigma_m.$$

Then, the deferred weighted mean is defined as

$$\nu_n = \frac{1}{R_n} \sum_{m=a_{n+1}}^{b_n} t_{b_n - m} l_m \varsigma_m.$$

To know more about deferred weighted mean (see [23]).

Definition 1.11. *A sequence (ς_m) is said to be strongly weighted convergent to a number l if*

$$\lim_{n\to\infty}\frac{1}{R_n}\sum_{m=1}^{n}|t_{b_n-m}l_m\varsigma_m - l| = 0.$$

Throughout the paper, we assume (z_m) be complex uncertain sequence.

2. Main Results

Definition 2.1. *A sequence (z_m) is called strongly deferred weighted convergent in concern with almost surely to z if $\exists\ \Lambda$ with uncertain measure 1 s.t., for every $\upsilon \in \Lambda$,*

$$\lim_{n\to\infty}\frac{1}{R_n}\sum_{m=a_{n+1}}^{b_n}||t_{b_n-m}l_m z_m(\upsilon) - z(\upsilon)|| = 0.$$

Definition 2.2. *A sequence (z_m) is called strongly deferred weighted convergent in measure to z if $\forall\ \epsilon > 0$*

$$\lim_{n\to\infty}\frac{1}{R_n}\sum_{m=a_{n+1}}^{b_n}\mu\Big\{||t_{b_n-m}l_m z_m(\upsilon) - z(\upsilon)|| \geq \epsilon\Big\} = 0.$$

Definition 2.3. *A sequence (z_m) is called strongly deferred weighted convergent in mean to z if*

$$\lim_{n\to\infty}\frac{1}{R_n}\sum_{m=a_{n+1}}^{b_n}E\Big[||t_{b_n-m}l_m z_m(\upsilon) - z(\upsilon)||\Big] = 0.$$

Definition 2.4. *Suppose Φ and Φ_m be complex uncertain distribution of complex uncertain variable z and z_m respectively. Then (z_m) is called strongly deferred weighted convergent in distribution to z if*

$$\lim_{n\to\infty}\frac{1}{R_n}\sum_{m=a_{n+1}}^{b_n}||t_{b_n-m}l_m\Phi_m(c) - \Phi(c)|| = 0.$$

$\forall\ c$ at which $\Phi(c)$ is continuous.

Definition 2.5. *A sequence (z_m) is called strongly deferred weighted convergent in concern with uniformly almost surely to z if $\forall\ \epsilon > 0$, $\exists\ E_k'$ with $\mu\{E_k'\} \to 0$ s.t., (z_m) is strongly deferred weighted uniformly convergent in $\zeta - E_k'$ for any fixed $k \in \mathbb{N}$*

The following example shows the existence of a strongly deferred weighted convergent with respect to almost surely in a given uncertain space.

Example 1: Let us consider infinite uncertainty space $\Omega = \{\upsilon_1, \upsilon_2, \cdots\}$ with measurable function as follows

$$\mu\{\Lambda\} = \begin{cases} \sup_{\upsilon_{m\in\Lambda}}\frac{1}{(m^2+1)}, & \text{if } \sup_{\upsilon_{m\in\Lambda}}\frac{1}{(m^2+1)} < \frac{1}{2}; \\ 0, & \text{otherwise.}\end{cases}$$

Now, let us define complex uncertain variable by

$$z_m(\upsilon) = \begin{cases} (m^2+1)i, & \text{if } \upsilon = \upsilon_m; \\ 0, & \text{otherwise.}\end{cases}$$

Consider $t_{b_n-m} = 1$, $l_m = 1$, $a_n = 2n$, $b_n = 4n$, we get

$$\begin{aligned} \|t_{b_n-m}l_m z_m - z\| &= \int_0^{+\infty} \mu\{\|t_{b_n-m}l_m z_m - z\| \geq \epsilon\}d\epsilon - \int_{-\infty}^{0} \mu\{\|t_{b_n-m}l_m z_m - z\| \geq \epsilon\}d\epsilon \\ &= \int_0^{\frac{1}{2}} \mu\{v_m\}d\epsilon = \frac{1}{2(m^2+1)} \end{aligned}$$

therefore,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{R_n} \sum_{m=2n+1}^{4n} \mu(\|t_{b_n-m}l_m z_m - z\|) &= \lim_{n\to\infty} \frac{1}{2n} \sum_{m=2n+1}^{4n} \left\{\frac{1}{2(m^2+1)}\right\} \\ &= 0. \end{aligned}$$

Thus, (z_m) is strongly deferred weighted convergent to z in almost surely.

Theorem 2.1. *If a sequence (z_m) is strongly deferred weighted convergent in mean to z, then it is strongly deferred weighted convergent in measure to z.*

Proof. Let the sequence (z_m) be strongly deferred weighted convergent to z in mean, then from Markov's inequality, for any $\epsilon > 0$, we get

$$\mu\{\|z_m - z\| \geq \epsilon\} \leq \frac{E[\|z_m - z\|]}{\epsilon}$$

or

$$\frac{1}{R_n} \sum_{m=a_n+1}^{b_n} \mu\{\|t_{b_n-m}l_m z_m - z\| \geq \epsilon\} \leq \frac{1}{R_n} \sum_{m=a_n+1}^{b_n} \frac{E[\|t_{b_n-m}l_m z_m - z\|]}{\epsilon} \to 0$$

as $n \to \infty$. Thus, (z_m) is strongly deferred weighted convergent to z in measure. □

To show converse of above result is not true we present an example below.

Example 2: Consider the uncertainty space $(\Omega, \mathcal{A}, \mu)$ with $\Omega = \{v_1, v_2, \cdots\}$ having uncertain measurable function as follows

$$\mu\{\Lambda\} = \begin{cases} \sup_{v_{m\in\Lambda}} \frac{1}{(m+1)}, & \text{if } \sup_{v_{m\in\Lambda}} \frac{1}{(m+1)} < \frac{1}{2}; \\ 1 - \sup_{v_{m\in\Lambda^c}} \frac{1}{(m+1)}, & \text{if } \sup_{v_{m\in\Lambda^c}} \frac{1}{(m+1)} < \frac{1}{2}; \\ \frac{1}{2}, & \text{otherwise.} \end{cases}$$

Now, let us define complex uncertain variable by

$$z_m(v) = \begin{cases} (m+1)i, & \text{if } v = v_m; \\ 0, & \text{otherwise.} \end{cases}$$

for $m = 1, 2, \cdots$ and $z = 0$. Then, for any $\epsilon > 0$, $t_{b_n-m} = 1$, $l_m = 1$, $a_n = 2n$, $b_n = 4n$ and $m \geq 2$, we have

$$\frac{1}{R_n} \sum_{m=a_n+1}^{b_n} \mu\{v : \|t_{b_n-m}l_m z_m - z\| \geq \epsilon\} = \frac{1}{2n} \sum_{m=2n+1}^{4n} \frac{1}{m+1} \to 0 \text{ as } n \to \infty.$$

Thus, (z_m) is strongly deferred weighted convergent to z in measure. Now, for each $m \geq 2$, and $t_{b_n-m} = l_m = 1$, we have the uncertain distribution of $\|t_{b_n-m}l_m z_m - z\| = \|z_m - z\|$ as

$$\Phi_m(u) = \begin{cases} 0, & \text{if } u < 0; \\ 1 - \frac{1}{(m+1)}, & \text{if } 0 \leq u < (m+1); \\ 1, & \text{if } u \geq (m+1). \end{cases}$$

So, we have

$$E[\|t_{b_n-m}l_m z_m - z\| - 1] = \Big[\int_0^{m^2+1} 1 - \Big(1 - \frac{1}{m^2+1}\Big)du - 1\Big] = 0.$$

Thus, (z_m) is not strongly deferred weighted convergent in mean to z.

Theorem 2.2. *Suppose (z_m) with real (ϑ_m) and imaginary part (η_m) respectively are strongly deferred weighted convergent in measure to ϑ and η respectively iff (z_m) is also strongly deferred weighted convergent in measure to $z = \vartheta + i\eta$.*

Proof. As (ϑ_m) and (η_m) are strongly deferred weighted convergent in measure to ϑ and η respectively. Then for every $\epsilon > 0$, we have

$$\lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{bn} \mu\{|t_{b_n-m}l_m\vartheta_m - \vartheta| \geq \frac{\epsilon}{\sqrt{2}}\} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{bn} \mu\Big\{|t_{b_n-m}l_m\eta_m - \eta| \geq \frac{\epsilon}{\sqrt{2}}\Big\} = 0.$$

Also,

$$\|t_{b_n-m}l_m z_m - z\| = \sqrt{|t_{b_n-m}l_m\vartheta_m - \vartheta|^2 + |t_{b_n-m}l_m\eta_m - \eta|^2},$$

we have

$$\Big\{\|t_{b_n-m}l_m z_m - z\| \geq \epsilon\Big\} \subset \Big\{|t_{b_n-m}l_m\vartheta_m - \vartheta| \geq \frac{\epsilon}{\sqrt{2}}\Big\} \cup \Big\{|t_{b_n-m}l_m\eta_m - \eta| \geq \frac{\epsilon}{\sqrt{2}}\Big\}.$$

Using sub-additivity axiom of uncertain measure we get,

$$\begin{aligned} \lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{b_n} \mu\Big\{\|t_{b_n-m}l_m z_m - z\| \geq \epsilon\Big\} &\leq \lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{b_n} \mu\Big\{|t_{b_n-m}l_m\vartheta_m - \vartheta| \geq \frac{\epsilon}{\sqrt{2}}\Big\} \\ &+ \lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{b_n} \mu\Big\{|t_{b_n-m}l_m\eta_m - \eta| \geq \frac{\epsilon}{\sqrt{2}}\Big\}. \end{aligned}$$

So,

$$\lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{b_n} \mu\Big\{\|t_{b_n-m}l_m z_m - z\| \geq \epsilon\Big\} = 0.$$

That is (z_m) is strongly deferred weighted convergent to z in measure. Conversely, suppose (z_m) is strongly deferred weighted convergent to z in measure. Then for any $\delta > 0$, we have

$$\begin{aligned}
\lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{b_n} \mu\Big\{\|t_{b_n-m} l_m z_m - z\| \geq \delta\Big\} &= 0 \\
\Rightarrow \lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{b_n} \mu\Big\{|t_{b_n-m} l_m(\vartheta_m + i\eta_m) - (\vartheta + i\eta)| \geq \delta\Big\} &= 0 \\
\Rightarrow \lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{b_n} \mu\Big\{|(t_{b_n-m} l_m \vartheta_m - \vartheta) + i(t_{b_n-m} l_m \eta_m - \eta)| \geq \delta\Big\} &= 0.
\end{aligned}$$

Then, $\exists$ a positive number with r' with $0 < r' < \frac{\delta}{2}$ such that,

$$\lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{b_n} \mu\Big\{|t_{b_n-m} l_m \vartheta_m - \vartheta| \geq r'\Big\} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{b_n} \mu\Big\{|t_{b_n-m} l_m \eta_m - \eta| \geq r'\Big\} = 0.$$

Hence, the real part (ϑ_m) and imaginary part (η_m) of complex uncertain sequence (z_m) are strongly deferred weighted convergent in measure to ϑ and η respectively. □

Theorem 2.3. *Consider $z = \vartheta + i\eta$ be a complex normal uncertain variable. If real part (ϑ_m) and imaginary part (η_m) of a complex uncertain sequence (z_m), for $m = 1, 2, \cdots$. are strongly deferred weighted convergent in measure to ϑ and η, respectively, then (z_m) is deferred weighted convergent in distribution to $z = \vartheta + i\eta$. But converse is not true in general.*

Proof. For complex uncertainty distribution Φ, consider $c = e + id$ be a point of continuity. Also, for any $\gamma > e$ and $\rho > d$,

$$\begin{aligned}
\{t_{b_n-m} l_m \vartheta_m \leq e, t_{b_n-m} l_m \eta_m \leq d\} &= \{t_{b_n-m} l_m \vartheta_m \leq e, t_{b_n-m} l_m \eta_m \leq d, \vartheta \leq \gamma, \eta \leq \rho\} \\
&\quad \cup \{t_{b_n-m} l_m \vartheta_m \leq e, t_{b_n-m} l_m \eta_m \leq d, \vartheta > \gamma, \eta > \rho\} \\
&\quad \cup \{t_{b_n-m} l_m \vartheta_m \leq e,\ t_{b_n-m} l_m \eta_m \leq d, \vartheta \leq \gamma, \eta > \rho\} \\
&\quad \cup \{t_{b_n-m} l_m \vartheta_m \leq e,\ t_{b_n-m} l_m \eta_m \leq d, \vartheta > \gamma, \eta \leq \rho\} \\
&\quad \subset \{\vartheta \leq \gamma, \eta \leq \rho\} \cup \{|t_{b_n-m} l_m \vartheta_m - \vartheta| \geq \gamma - e\} \\
&\quad \cup \{|t_{b_n-m} l_m \eta_m - \eta| \geq \rho - d\}.
\end{aligned}$$

Using subadditivity property, we have

$$\begin{aligned}
\Phi_m(c) &= \Phi_m(e + id) \\
&\leq \Phi(\gamma + i\rho) + \mu\{|t_{b_n-m} l_m \vartheta_m - \vartheta| \geq \gamma - e\} + \mu\{|t_{b_n-m} l_m \eta_m - \eta| \geq \rho - d\}.
\end{aligned}$$

Since, both (ϑ_m) and (η_m) are strongly deferred weighted convergent in measure to ϑ and η respectively, we have

$$\lim_{n\to\infty}\frac{1}{R_n}\sum_{m=a_n+1}^{b_n}\mu\Big\{|t_{b_n-m}l_m\vartheta_m-\vartheta|\geq\gamma-e\Big\} = 0$$

and

$$\lim_{n\to\infty}\frac{1}{R_n}\sum_{m=a_n+1}^{b_n}\mu\Big\{|t_{b_n-m}l_m\eta_m-\eta|\geq\rho-d\Big\} = 0$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{R_n}\sum_{m=a_n+1}^{b_n}t_{b_n-m}l_m\Phi_m(c) &\leq \Phi(\gamma+i\rho)\\ &+ \lim_{n\to\infty}\frac{1}{R_n}\sum_{m=a_n+1}^{b_n}\mu\Big\{|t_{b_n-m}l_m\vartheta_m-\vartheta|\geq\gamma-e\Big\}\\ &+ \lim_{n\to\infty}\frac{1}{R_n}\sum_{m=a_n+1}^{b_n}\mu\Big\{|t_{b_n-m}l_m\eta_m-\eta|\geq\rho-d\Big\}.\end{aligned}$$

For any $\gamma>e$, $\rho>d$. Suppose $\gamma+i\rho\ \to e+id$, we get

$$\limsup_{n\to\infty}\left\{\frac{1}{R_n}\sum_{m=a_n+1}^{b_n}t_{b_n-m}l_m\Phi_m(c)\right\}\leq\Phi(\gamma+i\rho)=\Phi(c). \tag{2.1}$$

Also, for $u<e,\ y<d$, we have

$$\begin{aligned}\{t_{b_n-m}l_m\vartheta_m\leq u,t_{b_n-m}l_m\eta\leq y\} &= \{t_{b_n-m}l_m\vartheta_m\leq e,t_{b_n-m}l_m\vartheta_m\leq d,\vartheta\leq u,\eta\leq y\}\\ &\cup\{t_{b_n-m}l_m\vartheta_m\leq e,t_{b_n-m}l_m\eta_m>d,\vartheta\leq u,\eta\leq y\}\\ &\cup\{t_{b_n-m}l_m\vartheta_m\leq e,t_{b_n-m}l_m\eta_m\leq d,\vartheta\leq u,\eta>\rho\}\\ &\cup\{t_{b_n-m}l_m\vartheta_m\leq e,t_{b_n-m}l_m\eta_m\leq d,\vartheta>u,\eta\leq y\}\\ &\subset\{|t_{b_n-m}l_m\vartheta_m\leq e,t_{b_n-m}l_m\eta_m\leq d\}\\ &\cup\{|t_{b_n-m}l_m\vartheta_m-\vartheta|\geq e-u\}\cup\{|t_{b_n-m}l_m\eta_m-\eta|\\ &\geq d-y\},\end{aligned}$$

this means that

$$\Phi(u+iy) \leq \Phi_m(e+id)+\mu\{|t_{b_n-m}l_m\vartheta_m-\vartheta|\geq e-u\}+\mu\{|t_{b_n-m}l_m\eta_m-\eta|\geq d-y\}.$$

As, for preassigned $\epsilon>0$, we have

$$\lim_{n\to\infty}\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\mu\{|t_{b_n-m}l_m\vartheta_m-\vartheta|\geq\epsilon\} = 0$$

and

$$\lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{bn} \mu\Big\{|t_{b_n-m} l_m \eta_m - \eta| \geq \epsilon\Big\} \quad = \quad 0.$$

So, we get

$$\Phi(u+iy) \quad \leq \quad \lim_{n\to\infty} \inf \left\{ \frac{1}{R_n} \sum_{m=a_n+1}^{bn} t_{b_n-m} l_m \Phi_m(e+id) \right\},$$

for any $u < e$, $y < d$. When $u + iy \ \to \ e + id$, we have

$$\Phi(c) \leq \lim_{n\to\infty} \inf \left\{ \frac{1}{R_n} \sum_{m=a_n+1}^{bn} t_{b_n-m} l_m \Phi_m(c) \right\}. \tag{2.2}$$

Using equation (2.1) and (2.2) we get $\Phi_m(c)$ is strongly deferred weighted convergent in distribution to $\Phi(c)$ as $n \to \infty$. □

Converse of above result is not true. To show this we present an example below.
Example 3: Consider the uncertainty space $(\Omega, \mathcal{A}, \mu)$ to be $\{v_1, v_2, v_3\}$ with $\mu(v_1) = \frac{7}{10}$, $\mu(v_2) = \frac{1}{5}$, $\mu(v_3) = \frac{3}{10}$, $\mu(v_1, v_2) = \frac{7}{10}$, $\mu(v_1, v_3) = \frac{4}{5}$, and $\mu(v_2, v_3) = \frac{3}{10}$.
Define complex uncertain variables as

$$z_m(v) = \begin{cases} i, & \text{if } v = v_1; \\ -i, & \text{if } v = v_2; \\ 2i & \text{if } v = v_3. \end{cases}$$

Define $z_m = -z$ for $m = 1, 2, \cdots$. Then, z_m and z have the same distribution as

$$\Phi_m(c) = \begin{cases} 0, & \text{if } e < 0,\ -\infty < d < +\infty; \\ 0, & \text{if } e \geq 0,\ d < -1; \\ 1/5, & \text{if } e \geq 0,\ -1 \leq d < 1; \\ 7/10, & \text{if } e \geq 0,\ 1 \leq d < 2; \\ 1, & \text{if } e \geq 0, 1,\ d \geq 2. \end{cases}$$

So, (z_m) is strongly deferred weighted convergent to z in distribution. Now, for any $\epsilon > 0$, $t_{b_n-m} = l_m = 1$ and $a_n = 2n$, $b_n = 4n$ we have

$$\sum_{m=a_n+1}^{bn} \mu\{\|t_{b_n-m} l_m z_m - z\| \geq \epsilon\} = 1.$$

Thus, $\lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{bn} \mu\{\|t_{b_n-m} l_m z_m - z\| \geq \epsilon\}$

$$= \lim_{n\to\infty} \frac{1}{2n} \sum_{m=2n+1}^{4n} \mu\{v : \|t_{b_n-m} l_m z_m(v) - z(v)| \geq \epsilon\} = 1.$$

Hence, (z_m) is not strongly deferred weighted convergent to z in measure.

Corollary 2.1. *If (z_m) is strongly deferred weighted convergent in measure, then it is also almost $\lambda-$statistical convergent in distribution.*

Proof. On combining Theorem 2.7 and Theorem 2.8, we get the result as desired. □

Corollary 2.2. *If (z_m) is strongly deferred weighted convergent in mean then it is also strongly deferred weighted convergent in distribution.*

Proof. From Theorem 1 and corollary above, it can be easily proved. □

Remark 2: Strongly deferred weighted convergent with respect to almost surely does not imply strongly deferred weighted convergent in mean.

Example 4: Consider the uncertain space $(\Omega, \mathcal{A}, \boldsymbol{\mu})$ to be $\{v_1, v_2, \cdots\}$ with $\boldsymbol{\mu}(\Lambda) = 2\sum_{v_m \in \Lambda} \frac{1}{3^m}$.

Define the complex uncertain variables by

$$z_m(v) = \begin{cases} i3^m, & \text{if } v = v_m; \text{ for } m \in \mathbb{N} \\ 0, & \text{otherwise,} \end{cases}$$

for $m = 1, 2, \cdots$ and $z = 0$. Then, (z_m) is strongly deferred weighted convergent in concern with almost surely to z. Also,

$$\Phi_m(u) = \begin{cases} 0, & \text{if } u < 0; \\ 1 - \frac{1}{3^m}, & \text{if } 0 \le u < 3^m; \\ 1, & \text{if } u \ge 3^m, \end{cases}$$

for $m = 1, 2, 3, \cdots$, $t_{b_n - m} = l_m = 1$, $a_n = 2n$, and $b_n = 4n$, we have

$$\begin{aligned} E[\|t_{b_n-m} l_m z_m - z\|] &= \int_0^{3^m} [1 - \left(1 - \frac{1}{3^m}\right)] du \\ &+ \int_{3^m}^{\infty} (1-1) du - \int_{-\infty}^{0} 0 du \\ &= 1. \end{aligned}$$

This implies, $\lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{b_n} E[\|t_{b_n-m} l_m z_m - z\|]$

$$= \lim_{n\to\infty} \frac{1}{2n} \sum_{m=2n+1}^{4n} E[\|t_{b_n-m} l_m z_m - z\|] = 1.$$

So, (z_m) does not strongly deferred weighted convergent to z in mean.

Remark 3: Strongly deferred weighted convergent with respect to almost surely does not imply strongly deferred weighted convergent in measure.

Example 5: Consider the uncertainty space $(\Omega, \mathcal{A}, \boldsymbol{\mu})$ to be $\{v_1, v_2, v_3, v_4\}$. Define

$$\boldsymbol{\mu}\{\boldsymbol{\Lambda}\} = \begin{cases} 0, & \text{if } \Lambda = \phi; \\ 1, & \text{if } \Lambda = \Omega; \\ 0.6, & \text{if } v_1 \in \Lambda; \\ 0.4, & \text{if } v_1 \notin \Lambda. \end{cases}$$

Define the sequence of complex uncertain variables by

$$z_m(v) = \begin{cases} i, & \text{if } v = v_1; \\ 2i, & \text{if } v = v_2; \\ 3i, & \text{if } v = v_3; \\ 4i, & \text{if } v = v_4; \\ 0, & \text{otherwise,} \end{cases}$$

for $m = 1, 2, 3, \cdots$, and $z = 0$. Then, (z_m) is strongly deferred weighted convergent to z with respect to almost surely. Also, for $\epsilon > 0$, $t_{b_n - m} = l_m = 1$, $a_n = 2n$, $b_n = 4n$ we have

$$\mu\{\|t_{b_n-m} l_m z_m - z\| \geq \epsilon\} = \mu\{v : \|t_{b_n-m} l_m z_m(v) - z(v)\| \geq \epsilon\} = 1.$$

That is,

$$\lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{bn} \mu\{\|t_{b_n-m} l_m z_m - z\| \geq \epsilon\} = 1.$$

Thus, (z_m) does not strongly deferred weighted in measure.

Remark 4: Strongly deferred weighted convergent in measure does not imply strongly deferred weighted convergent with respect to almost surely.

Example 6: Suppose the uncertainty space $(\Omega, \mathcal{A}, \mu)$ to be $[0, 1]$ with Borel algebra and Lebesgue measure. For any positive integer m $\exists$ a integer s such that, $m = 2^s + Q$. Now, define a complex uncertain variable by

$$z_m(v) = \begin{cases} i, & \text{if } \frac{Q}{2^s} < v \leq \frac{Q+1}{2^s}; \\ 0, & \text{otherwise,} \end{cases}$$

for $m = 1, 2, 3, \cdots$, $Q \in \mathbb{Z}$ (set of integers) and $z = 0$. Now, for $\epsilon > 0$, $t_{b_n - m} = l_m = 1$, $a_n = 2n$, $b_n = 4n$ and $m \geq 2$, we get

$$\lim_{n\to\infty} \tfrac{1}{R_n} \sum_{m=a_n+1}^{bn} \mu\{\|t_{b_n-m} l_m z_m - z\| \geq \epsilon\}$$

$$\begin{aligned} &= \lim_{n\to\infty} \frac{1}{2n} \sum_{m=2n+1}^{4n} \mu\{v : \|t_{b_n-m} l_m z_m(v) - z(v)\| \geq \epsilon\} \\ &= \lim_{n\to\infty} \frac{1}{2n} \times \frac{1}{2^s} = 0. \end{aligned}$$

Thus, (z_m) is strongly deferred weighted convergent to z in measure. Further, we have

$$\lim_{n\to\infty} \frac{1}{R_n} \sum_{m=a_n+1}^{bn} E[\|t_{b_n-m} l_m z_m - z\|] = 0.$$

Therefore, (z_m) is also strongly weighted convergent to z in mean. However, for any $v \in [0, 1]$, $\exists$ an infinite number of closed intervals which are of the form $[\frac{Q}{2^s}, \frac{Q+1}{2^s}]$ containing v. Thus, $z_m(v)$ does not strongly deferred weighted convergent with respect to almost surely to z.

Proposition 2.1. *The sequence (z_m) is strongly deferred weighted convergent in concern with almost surely to z if and only if*

$$\mu\Bigg(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m(\upsilon)-z(\upsilon)\|\geq\epsilon\Bigg) \;=\; 0.$$

Proof. By the definition of strongly deferred weighted convergence with respect to almost surely, $\exists$ an event with $\mu(\Lambda)=1$ such that,

$$\lim_{n\to\infty}\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m(\upsilon)-z(\upsilon)\|=0,$$

then for any $\epsilon>0$ and $\upsilon\ \in\Lambda$, we have

$$\mu\Bigg(\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m(\upsilon)-z(\upsilon)\|<\epsilon\Bigg) \;=\; 1.$$

Applying criteria (ii) of uncertain measure, we get

$$\mu\Bigg(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\frac{1}{R_n}\Bigg\{\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m(\upsilon)-z(\upsilon)\|\geq\epsilon\Bigg\}\Bigg) \;=\; 0.$$

Conversely, suppose for $\epsilon>0$, and for $\upsilon\in\Lambda$, we have

$$\mu\Bigg(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\frac{1}{R_n}\Bigg\{\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m(\upsilon)-z(\upsilon)\|\geq\epsilon\Bigg\}\Bigg) \;=\; 0.$$

By self duality axiom for any $\epsilon>0$ and $\upsilon\in\Lambda$, we have

$$\mu\Bigg(\bigcup_{m=1}^{\infty}\bigcap_{n=m}^{\infty}\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m(\upsilon)-z(\upsilon)\|<\epsilon\Bigg) \;=\; 1.$$

That is, for any $\epsilon>0$, $\exists$ an event Λ with uncertain measure 1, we get

$$\lim_{n\to\infty}\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m(\upsilon)-z(\upsilon)\|=0.$$

□

Proposition 2.2. *The sequence (z_m) is strongly deferred weighted convergent to z in concern with uniformly almost surely if and only if*

$$\mu\Bigg(\bigcup_{n=m}^{\infty}\Bigg\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m-z\|\geq\epsilon\Bigg\}\Bigg) \;=\; 0.$$

Proof. Assume (z_m) be strongly deferred weighted convergent in concern with uniformly almost surely to z, then for any $\xi>0$ $\exists$ S such that $\mu(S)<\xi$ and (z_m) is strongly deferred weighted uniformly converges to z on $\Omega\setminus S$. So, for any $\epsilon>0$, $\exists$ $m>0$ such that

$$\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m(\upsilon)-z(\upsilon)\|<\epsilon$$

for $n \geq m$ and $\upsilon \in \Omega \setminus S$, that is

$$\bigcup_{n=m}^{\infty}\left\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_mz_m(\upsilon)-z(\upsilon)\|\geq\epsilon\right\} \subset S.$$

Now, by using subadditivity criteria, we have

$$\mu\left(\bigcup_{n=m}^{\infty}\left\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_mz_m(\upsilon)-z(\upsilon)\|\geq\epsilon\right\}\right) \leq \mu\{S\}<\xi.$$

Then,

$$\lim_{n\to\infty}\mu\left(\bigcup_{n=m}^{\infty}\left\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_mz_m(\upsilon)-z(\upsilon)\|\geq\epsilon\right\}\right) = 0.$$

Conversely, let

$$\lim_{n\to\infty}\mu\left(\bigcup_{n=m}^{\infty}\left\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_mz_m(\upsilon)-z(\upsilon)\|\geq\epsilon\right\}\right) = 0.$$

So, for preassigned $\delta>0$ and $m\geq 1$, $\exists\, m_k$ such that

$$\mu\left(\bigcup_{n=m_k}^{\infty}\left\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_mz_m(\upsilon)-z(\upsilon)\|\geq\frac{1}{m}\right\}\right) < \frac{\delta}{2^m}.$$

Suppose

$$S=\bigcup_{m=1}^{\infty}\bigcup_{n=m_k}^{\infty}\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_mz_{m(\upsilon)}-z(\upsilon)\|\geq\frac{1}{m}\Big\},$$

then

$$\mu\{S\}\leq\mu\left(\bigcup_{n=m}^{\infty}\left\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_mz_m(\upsilon)-z(\upsilon)\|\geq\frac{1}{m}\right\}\right) \leq \delta,$$

but

$$\sup_{\upsilon\in\Omega-S}\|t_{b_n-m}l_mz_m(\upsilon)-z(\upsilon)\|<\frac{1}{m}$$

for any $m=1,2,3\cdots$. □

Theorem 2.4. *If (z_m) is strongly deferred weighted convergent to z with respect to uniformly almost surely then (z_m) is strongly deferred weighted convergent to z with respect to almost surely.*

Proof. Taking proposition 2.12 in consideration, if (z_m) is strongly deferred weighted convergent in concern with uniformly almost surely to z then we have

$$\lim_{n\to\infty}\mu\left(\bigcup_{n=m}^{\infty}\left\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_mz_m-z\|\geq\epsilon\right\}\right) = 0.$$

Since,

$$\mu\left(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\left\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_mz_m-z\|\geq\epsilon\right\}\right)$$

$$\leq \quad \mu\Bigg(\bigcup_{n=m}^{\infty}\Bigg\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m - z\| \geq \epsilon\Bigg\}\Bigg).$$

As $n \to \infty$ on both side, we get

$$\mu\Bigg(\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty}\Bigg\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m - z\| \geq \epsilon\Bigg\}\Bigg) \quad = \quad 0.$$

By proposition 2.11, (z_m) is strongly deferred weighted convergent to z. □

Theorem 2.5. *If (z_m) is strongly deferred weighted convergent to z with respect to uniformly almost surely. Then (z_m) is strongly deferred weighted convergent to z in measure.*

Proof. If (z_m) is strongly deferred weighted convergent to z with respect to uniformly almost surely to z, then from proposition 2.12, we have

$$\lim_{n\to\infty}\mu\Bigg(\bigcup_{n=m}^{\infty}\Bigg\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m - z\| \geq \epsilon\Bigg\}\Bigg) \quad = \quad 0$$

and

$$\mu\Bigg\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m - z\| \geq \epsilon\Bigg\} \quad \leq \quad \mu\Bigg(\bigcup_{n=m}^{\infty}\Bigg\{\frac{1}{R_n}\sum_{m=a_n+1}^{bn}\|t_{b_n-m}l_m z_m - z\| \geq \epsilon\Bigg\}\Bigg)$$

as $n \to \infty$, we get (z_m) is strongly deferred convergent in measure to z. □

The interrelation among almost surely, mean, measure, distribution and uniformly almost surely is represented in Figure 1.

Diagram representing above convergence relations :

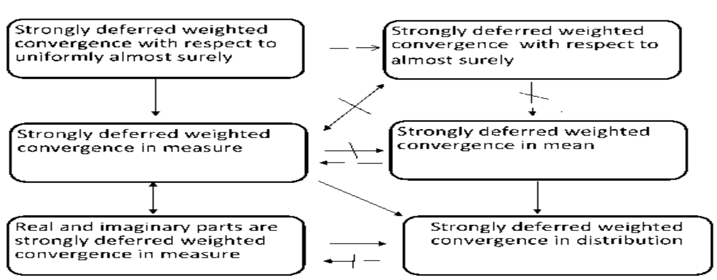

Figure 1. → means implies, ↛ means does not implies, ↮ does not imply each other.

3. Conclusion

Upon prior analysis, our interest is to investigate and discuss strongly deferred weighted convergence with respect to almost surely, strongly deferred weighted convergence in mean, strongly deferred weighted convergence in measure, strongly deferred weighted convergence in distribution and strongly deferred weighted convergence with respect to uniformly almost surely for complex uncertain sequences. Also, the interrelationship among these concepts are presented diagrammatically. In further studies, strongly lacunary statistical convergence by using double sequences can be studied for complex uncertain sequences.

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References

[1] X. Chen, American Option Pricing Formula for Uncertain Financial Market, Int. J. Oper. Res. 8 (2011), 27–32.
[2] X. Chen, Y. Ning, X. Wang, Convergence of Complex Uncertain Sequences, J. Intell. Fuzzy Syst. 30 (2016), 3357–3366. https://doi.org/10.3233/ifs-152083.
[3] D. Datta, B.C. Tripathy, Convergence of Complex Uncertain Double Sequences, New Math. Nat. Comp. 16 (2020), 447–459. https://doi.org/10.1142/s1793005720500271.
[4] B. Das, B.C. Tripathy, P. Debnath, J. Nath, B. Bhattacharya, Almost Convergence of Complex Uncertain Triple Sequences, Proc. Natl. Acad. Sci., India, Sect. A Phys. Sci. 91 (2020), 245–256. https://doi.org/10.1007/s40010-020-00721-w.
[5] B. Das, B. Tripathy Chandra, P. Debnath, B. Bhattacharya, Almost Convergence of Complex Uncertain Double Sequences, Filomat 35 (2021), 61–78. https://doi.org/10.2298/fil2101061d.
[6] H. Guo, C. Xu, A Necessary and Sufficient Condition of Convergence in Mean Square for Uncertain Sequences, Int. Interdiscip. J. Sch. Res. 16 (2013), 1091–1096.
[7] B. Liu, Uncertain Set Theory and Uncertain Inference Rule With Application to Uncertain Control, J. Uncertain Syst. 4 (2010), 83–98.
[8] B. Liu, Uncertainty Theory, Springer, Berlin, 2007. https://doi.org/10.1007/978-3-540-73165-8.
[9] B. Liu, Uncertainty Theory, in: Uncertainty Theory, Springer, Berlin, 2010: pp. 1–79. https://doi.org/10.1007/978-3-642-13959-8_1.
[10] B. Liu, Why is There a Need for Uncertainty Theory?, J. Uncertain Syst. 6 (2012), 3–10.
[11] B. Liu, Some Research Problems in Uncertain Theory, J. Uncertain Syst. 3 (2009), 3–10.
[12] B. Liu, Uncertain Risk Analysis and Uncertain Stability Analysis, J. Uncertain Syst. 4 (2010), 163–170.
[13] B. Liu, Uncertainty Theory: A Branch of Mathematics for Modeling Human Uncertainty, Springer, Berlin, 2010. https://doi.org/10.1007/978-3-642-13959-8.

[14] I.J. Maddox, Spaces of Strongly Summable Sequences, Q. J. Math. 18 (1967), 345–355. https://doi.org/10.1093/qmath/18.1.345.

[15] Mursaleen, O.H.H. Edely, Statistical convergence of double sequences, J. Math. Anal. Appl. 288 (2003), 223–231. https://doi.org/10.1016/j.jmaa.2003.08.004.

[16] P.K. Nath, B.C. Tripathy, Convergent Complex Uncertain Sequences Defined by Orlicz Function, Ann. Univ. Craiova - Math. Comp. Sci. Ser. 46 (2019), 139–149.

[17] J. Nath, B.C. Tripathy, P. Debnath, B. Bhattacharya, Strongly Almost Convergence in Sequences of Complex Uncertain Variables, Comm. Stat. - Theory Meth. 52 (2021), 714–729. https://doi.org/10.1080/03610926.2021.1921802.

[18] K. Ömer, H.K. Ünal, Lacunary Statistical Convergence of Complex Uncertain Sequence, Sigma J. Eng. Nat. Sci. 10 (2019), 277–286.

[19] K. Ömer, $S_\lambda(I)$-Convergence of Complex Uncertain Sequence, Mat. Stud. 51 (2019), 183–194.

[20] Z. Peng, Complex Uncertain Variables, Doctoral Dissertation, Tsinghua University, 2012.

[21] K. Raj, S. Sharma, M. Mursaleen, Almost λ-Statistical Convergence of Complex Uncertain Sequences, Int. J. Unc. Fuzz. Knowl. Based Syst. 30 (2022), 795–811. https://doi.org/10.1142/s0218488522500234.

[22] K. Saini, K. Raj, Applications of Statistical Convergence in Complex Uncertain Sequences via Deferred Riesz Mean, Int. J. Unc. Fuzz. Knowl. Based Syst. 29 (2021), 337–351. https://doi.org/10.1142/s021848852150015x.

[23] H. M. Srivastava, B. B. Jena and S. K. Paikray, Statistical Probability Convergence via the Deferred Nörlund Mean and Its Applications to Approximation Theorems, Rev. R. Acad. Cienc. Exactas Fís. Nat., Ser. A Mat., RACSAM 114 (2020), 144. https://doi.org/10.1007/s13398-020-00875-7.

[24] S. Saha, B.C. Tripathy, S. Roy, On Almost Convergent of Complex Uncertain Sequences, New Math. Nat. Comp. 16 (2020), 573–580. https://doi.org/10.1142/s1793005720500349.

[25] B.C. Tripathy, P.J. Dowari, Nörlund and Riesz Mean of Sequences of Complex Uncertain Variables, Filomat 32 (2018), 2875–2881. https://doi.org/10.2298/fil1808875t.

[26] B.C. Tripathy, P.K. Nath, Statistical Convergence of Complex Uncertain Sequences, New Math. Nat. Comp. 13 (2017), 359–374. https://doi.org/10.1142/s1793005717500090.

[27] C. You, On the Convergence of Uncertain Sequences, Math. Comp. Model. 49 (2009), 482–487. https://doi.org/10.1016/j.mcm.2008.07.007.