Int. J. Anal. Appl. (2024), 22:184

International Journal of Analysis and Applications

A Comprehensive View of the Solvability and Stability of a Feedback Control Problem with a State-Dependent Delay Implicit Pantograph Equation

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Abstract. In this paper, we examine the existence of a unique solution of a feedback control problem with an implicit state-dependent pantograph equation. Additionally, the study implements the problem's Hyers-Ulam stability and the continuous dependence of the unique solution on the initial data and the parameters. Furthermore, we investigate this problem in the absence of feedback control. We also provide some examples to illustrate our results.

1. Introduction

The purpose of synthetic biology is to develop novel strategies based on design. A biological mechanism designed to control how other biological processes operate is called a controller. The outcomes of control theory, including strategies, can serve as the foundation for the construction of such controllers. The key to regulation, sensory adaptation, and long-term effects is integrated feedback control. When we talk about the disturbance functions, we mean the control variables. Dealing with issues involving control variables is crucial because unanticipated events frequently disrupt real-world ecosystems, which might alter biological traits. Due to unforeseen circumstances that disturb ecosystems in the real world, challenges involving feedback control are critical in a variety of areas; these problems are translated into mathematical models, see [16, 48, 61]. Unexpected factors that frequently disrupt biological systems may alter biological qualities in the real world. This emphasizes the importance of managing restrictions or control variables since they may alter biological traits like survival rates, see [13, 14, 18, 46]. Ecology is interested in the topic of whether an ecosystem can withstand those unpredictable, disturbing events that occasionally occur, see [16, 17, 41, 50].

Received: Aug. 22, 2024.

2020 *Mathematics Subject Classification.* 34A12, 47H09, 47H10, 34C08, 34K43.

Key words and phrases. feedback control; pantograph equation; state-dependent; existence of solution; Hyres-Ulam stability; continuous dependence; Schauder's fixed point theorem.

https://doi.org/10.28924/2291-8639-22-2024-184
ISSN: 2291-8639

Many researchers devoted their studies to this type of problem, the authors in [17] investigated a necessary condition for the existence of a positive periodic solution of the model of feedback control on chemostat models. A positive periodic solution with feedback control of a nonlinear neutral delay population problem has been reported in [44]. The asymptotic stability and solvability of a family of nonlinear functional-integral equations with feedback control were investigated by the author in [43]. The authors in [20] looked at a constrained problem with a quadratic functional integro-differential constraint and an arbitrary (fractional) order quadratic functional integro-differential equation.

Pantograph differential equation is A particular kind of delay differential equation that was first created by examining an electric locomotive [31, 45]. Ockendon and Taylor's research, which looked at the electric locomotive's catenary system, is where the name "pantograph" first appeared. They aimed to develop an equation for examining the pantograph head movement on an electric locomotive that runs on an overhead trolley wire [45]. The behavior of the pantograph differential equation is significant across various fields of research and have several applications in different areas; for example in the current collection system [45], the cell development model [32, 57], the ruin problem in risk theory [29], the quantum theory [51], the fusion of light in spiral galaxies [5], and some industrial applications. The pantograph equations have been studied and used by many researchers in a variety of mathematical and scientific fields, including number theory, probability, electrodynamics, and medicine, as demonstrated by the references in [45,58] and other works. In [21], the authors studied the solvability and the Hyers-Ulam stability of a non-local fractional orders pantograph equation with a feedback control. The authors in [34] introduced an efficient transferred Legendre pseudo-spectral method for finding the solution of pantograph delay differential equations.

Differential and integral equations with diverging arguments frequently depend only on time; see for example, [9, 38]. On the other side, the case when deviating arguments depend on time t and the state variable ζ is crucial from a theoretical and practical perspective. This type of equation is called self-reference or state-dependent equation. These formulas are extensively employed in nonlinear analysis and have wide applicability across several domains, especially in issues pertaining to memories of the past, for example, in hereditary phenomena, see [42, 52, 56]. These kinds of equations have been the focus of multiple published studies; Eder [19] presented a categorization of the solutions for self-reference differential equations in one of the earliest studies on this kind of equation. Fe'ckan [27] introduced a generalization of Eder's result by examining a self-referential functional differential equation. Buicá [11] investigated the data dependence theorems, existence, and uniqueness of the solution for a self-referencee initial value problem. Buicá's results was extended, and her assumptions are relaxed by the results presented by El-Sayed and Ebead in [22]. The authors of [25] examined the m-point boundary value problem of a self-reference differential equation and proved that the solution of this problem is unique. Please refer to [6, 23, 27, 28, 33, 39, 59] for further studies on this kind of equation.

P. Andrzej in [7], studied the initial value problem of a self-reference functional differential equation

$$\frac{d\zeta(t)}{dt} = F\Big(t, \zeta(t), \zeta(\zeta(t))\Big),\ t \in (0,T],\ \zeta(0) = \zeta_0.$$

In [37], Lauran used the technique of nonexpansive operators from [10] to study iterative and non iterative first order differential equations of the form

$$\frac{d\zeta(t)}{dt} = F(t, \zeta(t), \zeta(\lambda t))$$

and

$$\frac{d\zeta(t)}{dt} = F(t, \zeta(t), \zeta(\zeta(t)))$$

respectively, with initial condition $\zeta(t_0) = \zeta_0$. In [24], The authors generalized the results in [7] and [37], they studied the existence of positive nondecreasing solutions and data dependence of the initial value problem of the self-refereed differential equation with two state-delay functions

$$\frac{d\zeta(t)}{dt} = F\Big(t, \zeta\Big(g_1(t,\zeta(t))\Big), \zeta\Big(g_2(t,\zeta(t))\Big)\Big),\ a.e.\ t \in (0,T],\ \zeta(0) = \zeta_0 \geq 0$$

under suitable assumptions for the functions g_1 and g_2.
Motivated by the aforementioned, our aim in this work is to study the existence of the unique solution $\zeta \in C(\mathbb{k}),\ \mathbb{k} = [0,T]$ for the implicit state-dependent pantograph problem

$$\frac{d\zeta}{dt} = F_1(t, \zeta(t), \lambda_1 \zeta(\gamma_1 \frac{d\mathfrak{I}}{dt})),\ \zeta(0) = \zeta_0,\ a.e.\ t \in \mathbb{k} \tag{1.1}$$

with the feedback control

$$\frac{d\mathfrak{I}}{dt} = F_2(t, \mathfrak{I}(t), \lambda_2 \mathfrak{I}(\gamma_2 \frac{d\zeta}{dt})),\ \mathfrak{I}(0) = \mathfrak{I}_0,\ a.e.\ t \in \mathbb{k}, \tag{1.2}$$

where $\gamma_i,\ \lambda_i \in (0,1), i = 1,2$. Moreover, we investigate the stability of the problem through Hyres-Ulam stability and the stability of the solution through the continuous dependence of the solution on the initial data and the parameters. We also analyzed this problem in the absence of feedback control.

1.1. **Structure of the paper.** This article is organized as follows: Section 1 introduces the basic background material about feedback control or constraint problems, pantograph differential equation and self-reference (state-dependence) equations and the importance of dealing with these kinds of problems; moreover, we outline some results and previous works to clarify our motivation and innovation. Section 2 states and demonstrates suitable assumptions and existence results for the unique solution of problem (1.1) with the feedback control (1.2) through Banach's fixed point theorem. In Section 3, we study the Hyres-Ulam stability of the problem additionally, we proved the continuous dependence of the solution ζ on ζ_0, $\mathfrak{I}_0$, and the parameters $\lambda,\ \gamma$. In Section 4, we study a special case of our problem in the absence of the control variable; we introduce some results for the existence and stability of the problem. In Section 5, we provide some examples to illustrate our results. Finally, a conclusion section will be presented.

2. Existence of solution

Consider the following assumptions:

(i) $F_i : \mathbb{k}\times R\times R \to \mathbb{k}$ are continuous and there exists a positive constant K_i such that

$$|F_i(t,\zeta,u) - F_i(s,\Im,v)| \leq K_i(|t-s| + |\zeta - \Im| + |u-v|).$$

From this assumption, we get

$$|F_i(t,\zeta,u)| - |F_i(t,0,0)| \leq |F_i(t,\zeta,u) - F_i(t,0,0)| \leq K_i(|\zeta| + |u|),$$

then

$$|F_i(t,\zeta,u)| \leq K_i(|\zeta| + |u|) + |F_i(t,0,0)|.$$

(ii) $\sup_{t\in\mathbb{k}} |F_i(t,0,0)| = B_i$.

(iii) There exists a real positive root of the algebraic equation

$$\lambda K\gamma r^2 + (KT-1)r + ((1+\lambda)KA + B) = 0,$$

where $A = \max\{\zeta_0, \Im_0\}$, $K = \max\{K_i\}$, $\gamma = \max\{\gamma_i\}$, $B = \max\{B_i\}$, $\lambda = \max\{\lambda_i\}$, $i = 1,\ 2$.

2.1. **Formulation of problem.** Let $\frac{d\zeta}{dt} = u$ and $\frac{d\Im}{dt} = v$, then

$$\zeta(t) = \zeta_0 + \int_0^t u(s)ds \tag{2.1}$$

and

$$\zeta(\gamma_1 v(t)) = \zeta_0 + \int_0^{\gamma_1 v(t)} u(s)ds.$$

Also

$$\Im(t) = \Im_0 + \int_0^t v(s)ds \tag{2.2}$$

and

$$\Im(\gamma_2 u(t)) = \Im_0 + \int_0^{\gamma_2 u(t)} v(s)ds,$$

and the problem (1.1)-(1.2) will be given by the coupled system

$$u(t) = F_1(t,\zeta_0 + \int_0^t u(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 v(t)} u(s)ds)), \tag{2.3}$$

$$v(t) = F_2(t,\Im_0 + \int_0^t v(s)ds, \lambda_2(\Im_0 + \int_0^{\gamma_2 u(t)} v(s)ds)). \tag{2.4}$$

Let $C(\mathbb{k})$ be the class of continuous functions define on $\mathbb{k}$. Let X be the Banach space of all ordered pairs $(\zeta,\Im)$ with the norm

$$\|(u,v)\|_X = \max\{\|u\|_C, \|v\|_C\},$$

where

$$\|u\|_C = \sup_{t\in[0,T]} |u(t)|.$$

Define the operator F associated with (3.1)-(3.2) by

$$F(u,v) = (F_1u, F_2v).$$

Where

$$F_1u = F_1(t, \zeta_0 + \int_0^t u(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 v(t)} u(s)ds)),$$

$$F_2v = F_2(t, \mathfrak{I}_0 + \int_0^t v(s)ds, \lambda_2(\mathfrak{I}_0 + \int_0^{\gamma_2 u(t)} v(s)ds)).$$

Let $Q_r = \{(u,v) \in X : \|u\| \le r,\ \|v\| \le r\}$.

Theorem 2.1. *Suppose the assumptions (i)–(ii) be hold. If* $(KT + 2\lambda Kr\gamma) < 1$, *then Problem (1.1)-(1.2) has a unique solution* $\zeta \in C(\mathbb{k})$.

Proof. For $(u,v) \in Q_r$, we have

$$\begin{aligned}
|F_1u(t)| &= |F_1(t, \zeta_0 + \int_0^t u(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 v(t)} u(s)ds))| \\
&\le K_1|\zeta_0 + \int_0^t u(s)ds| + \lambda_1 K_1|\zeta_0 + \int_0^{\gamma_1 v(t)} u(s)ds| + |F_1(t,0,0)| \\
&\le K_1|\zeta_0| + K_1\int_0^t |u(s)|ds + \lambda_1 K_1|\zeta_0| + \lambda_1 K_1 \int_0^{\gamma_1 v(t)} |u(s)|ds + B_1 \\
&\le (1+\lambda_1)K_1|\zeta_0| + K_1\|u\|T + \lambda_1 K_1\|u\|\gamma_1\|v\| + B_1 \\
&\le (1+\lambda)KA + KTr + \lambda K r^2\gamma + B = r.
\end{aligned}$$

This proves that $F_1 : Q_r \to Q_r$. Similarly,

$$\begin{aligned}
|F_2v(t)| &= |F_2(t, \mathfrak{I}_0 + \int_0^t v(s)ds, \lambda_2(\mathfrak{I}_0 + \int_0^{\gamma_2 u(t)} v(s)ds))| \\
&\le (1+\lambda)KA + KTr + \lambda K r^2\gamma + B = r.
\end{aligned}$$

Then $F_2 : Q_r \to Q_r$, and we deduce that

$$F(u,v) = (F_1u, F_2v) : Q_r \to Q_r.$$

Now, let $(u,v), (\bar{u},\bar{v}) \in Q_r$, then

$$\begin{aligned}
&|F_1u(t) - F_1\bar{u}(t)| \\
= \ & |F_1(t, \zeta_0 + \int_0^t u(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 v(t)} u(s)ds)) \\
- \ & F_1(t, \zeta_0 + \int_0^t \bar{u}(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 \bar{v}(t)} \bar{u}(s)ds))|
\end{aligned}$$

$$
\begin{aligned}
&\leq K_1 \int_0^t |u(s)-\bar{u}(s)|ds + \lambda_1 K_1 |\int_0^{\gamma_1 v(t)} u(s)ds - \int_0^{\gamma_1 \bar{v}(t)} \bar{u}(s)ds))| \\
&\leq K_1 \int_0^t |u(s)-\bar{u}(s)|ds + \lambda_1 K_1 |\int_0^{\gamma_1 v(t)} u(s)ds - \int_0^{\gamma_1 \bar{v}(t)} u(s)ds + \int_0^{\gamma_1 \bar{v}(t)} u(s)ds \\
&- \int_0^{\gamma_1 \bar{v}(t)} \bar{u}(s)ds))| \\
&\leq K_1 \int_0^t |u(s)-\bar{u}(s)|ds + \lambda_1 K_1 |\int_0^{\gamma_1 \bar{v}(t)} u(s)ds - \int_0^{\gamma_1 v(t)} u(s)ds| \\
&+ \lambda_1 K_1 |\int_0^{\gamma_1 \bar{v}(t)} u(s)ds - \int_0^{\gamma_1 \bar{v}(t)} \bar{u}(s)ds))| \\
&\leq K_1 \int_0^t |u(s)-\bar{u}(s)|ds + \lambda_1 K_1 |\int_{\gamma_1 \bar{v}(t)}^{\gamma_1 v(t)} u(s)ds| + \lambda_1 K_1 \int_0^{\gamma_1 \bar{v}(t)} |u(s)-\bar{u}(s)|ds \\
&\leq K_1 T \|u-\bar{u}\| + \lambda_1 K_1 r \gamma_1 \|v-\bar{v}\| + \lambda_1 K_1 \|u-\bar{u}\| \int_0^{\gamma_1 \bar{u}(t)} ds \\
&\leq KT\|u-\bar{u}\|ds + \lambda_1 K r\gamma \|v-\bar{v}\| + \lambda_1 K r \gamma \|u-\bar{u}\| \\
&\leq (KT+\lambda K r\gamma)\|u-\bar{u}\| + \lambda K r\gamma \|v-\bar{v}\|.
\end{aligned}
$$

Then

$$\|F_1 u - F_1 \bar{u}\| \leq (KT+\lambda K r\gamma)\|u-\bar{u}\| + \lambda K r\gamma\|v-\bar{v}\|.$$

Similarly,

$$\|F_2 v - F_2 \bar{v}\| \leq (KT+\lambda K r\gamma)\|v-\bar{v}\| + \lambda K r\gamma\|u-\bar{u}\|.$$

Hence

$$
\begin{aligned}
\|F(u,v)-F(\bar{u},\bar{v})\|_X &= \|(F_1 u, F_2 v) - (F_1 \bar{u} - F_2 \bar{v})\|_X \\
&= \|(F_1 u - F_1 \bar{u}, F_2 v - F_2 \bar{v})\|_X = \max\{\|F_1 u - F_1 \bar{u}\|_C, \|F_2 v - F_2 \bar{v}\|_C\} \\
&\leq \max\{(KT+\lambda K r\gamma)\|u-\bar{u}\| + \lambda K r\gamma\|v-\bar{v}\|, \\
&\quad (KT+\lambda K r\gamma)\|u-\bar{u}\| + \lambda K r\gamma\|v-\bar{v}\|\} \\
&\leq (KT+2\lambda K r\gamma)\max\{\|u-\bar{u}\|, \|v-\bar{v}\|\} \\
&\leq (KT+2\lambda K r\gamma)\|(u,v)-(\bar{u},\bar{v})\|.
\end{aligned}
$$

Since $(KT+2\lambda K r\gamma) < 1$, then F is a contraction mapping and by the Banach fixed point Theorem [36], Problem (3.1)-(3.2) has a unique solution. Consequently, the feedback control problem (1.1)-(1.2) has a unique solution $\zeta \in C(\mathbb{k})$. □

3. Stability analysis

Stability analysis is a rich and versatile field with deep theoretical foundations and wide-ranging applications in engineering, economics, biology, physics, and other fields. It is a typical subject in the mathematical sciences [8,60]. An equation or problem can be used to simulate a physical process if a minor change to it yields a comparable small change in the result. The equation or problem is considered to be stable when this happens. In 1998, the Hyers-Ulam stability of a differential equation was initially investigated by Alsina and Ger [4]. Additionally, Jung [35] investigated the Hyers-Ulam stability of the first order differential equation $\phi(t)\mathfrak{I}'(t) = \mathfrak{I}(t)$ in 2004. Several authors looked into the Hyers-Ulam stability of second and third order differential equations between 2010 and 2015 (see [3,30,55]). Recently, Hyers-Ulam stability of numerous kinds of differential equations has been applied; they include integro-differential equations studied by O. Tunç et al. [53,54], as well as hypergeometric and Laguerre differential equations studied by Abdollahpour et al. [1,2]. Another important concept in stability theory is continuous dependence [47], which deals with how solutions behave in various mathematical situations. It guarantees that minor modifications to a problem's initial conditions or parameters produce equally minor modifications to the problem's solution. Many authors have investigated the continuous dependence of the solutions of their problems, (see [11,12,20,22,24,49]), which is crucial for using mathematical models to describe real-world situations. Integrating continuous dependency and Hyers-Ulam stability is necessary to guarantee the reliability of these models. Hyers-Ulam stability evaluates the problem's resistance to shocks, whereas continuous dependency looks at how even slight changes in parameters impact the problem's unique solution.

3.1. **Hyres-Ulam stability.**

Definition 3.1. *Let the solution $\zeta \in C(\mathbb{k})$ of (1.1)-(1.2) be exists, then Problem (1.1)-(1.2) is Hyers-Ulam stable if $\forall\ \epsilon > 0\ \exists\ \delta(\epsilon) > 0$ such that for any δ-approximate solution $\zeta_s \in C(\mathbb{k})$ of (1.1)-(1.2) satisfies*

$$\max\{|\frac{d\zeta_s}{dt} - F_1(t,\zeta_s(t),\lambda_1\zeta_s(\gamma_1(\frac{d\mathfrak{I}_s}{dt})))|,\ |\frac{d\mathfrak{I}_s}{dt} - F_2(t,\mathfrak{I}_s(t),\lambda_2\mathfrak{I}_s(\gamma_2(\frac{d\zeta_s}{dt})))|\} < \delta,$$

implies

$$||\zeta - \zeta_s||_X < \epsilon.$$

Theorem 3.1. *If the assumptions of Theorem 2.1 are met, then Problem (1.1)-(1.2) is Hyers-Ulam stable.*

Proof. Let

$$\max\{|\frac{d\zeta_s}{dt} - F_1(t,\zeta_s(t),\lambda_1\zeta_s(\gamma_1(\frac{d\mathfrak{I}_s}{dt})))|,\ |\frac{d\mathfrak{I}_s}{dt} - F_2(t,\mathfrak{I}_s(t),\lambda_2(\mathfrak{I}_s(\gamma_2(\frac{d\zeta_s}{dt})))|\} < \delta,$$

then

$$|\frac{d\zeta_s}{dt} - F_1(t,\zeta_s(t),\lambda_1\zeta_s(\gamma_1(\frac{d\mathfrak{I}_s}{dt})))| < \delta,$$

$$-\delta < \frac{d\zeta_s}{dt} - F_1(t,\zeta_s(t),\lambda_1\zeta_s(\gamma_1(\frac{d\mathfrak{I}_s}{dt}))) < \delta.$$

Similarly,

$$|\frac{d\mathfrak{I}_s}{dt} - F_2(t, \mathfrak{I}_s(t), \lambda_2 \mathfrak{I}_s(\gamma_2(\frac{d\zeta_s}{dt})))| < \delta,$$

$$-\delta < \frac{d\mathfrak{I}_s}{dt} - F_2(t, \mathfrak{I}_s(t), \lambda_2 \mathfrak{I}_s(\gamma_2(\frac{d\zeta_s}{dt}))) < \delta.$$

Let $\frac{d\zeta_s}{dt} = u_s$ and $\frac{d\mathfrak{I}_s}{dt} = v_s$, then

$$\zeta_s(t) = \zeta_0 + \int_0^t u_s(s)ds \;\Rightarrow\; \zeta_s(\gamma_1 v_s(t)) = \zeta_0 + \int_0^{\gamma_1 v_s(t)} u_s(s)ds,$$

$$\mathfrak{I}_s(t) = \mathfrak{I}_0 + \int_0^t u_s(s)ds \;\Rightarrow\; \mathfrak{I}_s(\gamma_2 u_s(t)) = \mathfrak{I}_0 + \int_0^{\gamma_2 u_s(t)} v_s(s)ds.$$

Hence

$$-\delta < u_s(t) - F_1(t, \zeta_0 + \int_0^t u_s(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 v_s(t)} u_s(s)ds)) < \delta$$

$$-\delta < v_s(t) - F_2(t, \mathfrak{I}_0 + \int_0^t v_s(s)ds, \lambda_2(\mathfrak{I}_0 + \int_0^{\gamma_2 u_s(t)} v_s(s)ds)) < \delta,$$

and

$$\begin{aligned} |\zeta(t) - \zeta_s(t)| &= |\zeta_0 + \int_0^t u(s)ds - \zeta_0 - \int_0^t u_s(s)ds| \\ &\leq \int_0^t |u(s) - u_s(s)|ds \leq \|u - u_s\|\, T, \end{aligned}$$

then

$$\|\zeta - \zeta_s\| \leq \|u - u_s\|\, T.$$

Similarly,

$$\|\mathfrak{I} - \mathfrak{I}_s\| \leq \|v - v_s\|\, T.$$

But

$$\begin{aligned} & |u(t) - u_s(t)| \\ = \; & |F_1(t, \zeta_0 + \int_0^t u(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 v(t)} u(s)ds) - u_s(t))| \\ = \; & |F_1(t, \zeta_0 + \int_0^t u(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 v(t)} u(s)ds)) - F_1(t, \zeta_0 + \int_0^t u_s(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 v_s(t)} u_s(s)ds)) \\ + \; & F_1(t, \zeta_0 + \int_0^t u_s(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 v_s(t)} u_s(s)ds)) - u_s(t)| \\ \leq \; & |F_1(t, \zeta_0 + \int_0^t u_s(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 v_s(t)} u_s(s)ds)) - u_s(t)| \\ + \; & |F_1(t, \zeta_0 + \int_0^t u(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 v(t)} u(s)ds)) - F_1(t, \zeta_0 + \int_0^t u_s(s)ds, \lambda_1(\zeta_0 + \int_0^{\gamma_1 v_s(t)} u_s(s)ds))| \\ \leq \; & \delta + K_1 \int_0^t |u(s) - u_s(s)|ds + \lambda_1 K_1 |\int_0^{\gamma_1 v(t)} u(s)ds - \int_0^{\gamma_1 v_s(t)} u_s(s)ds| \end{aligned}$$

$$\begin{aligned}
&\leq \delta + K_1T\|u-u_s\| + \lambda_1K_1|\int_0^{\gamma_1 v(t)} u(s)ds - \int_0^{\gamma_1 v_s(t)} u(s)ds + \int_0^{\gamma_1 v_s(t)} u(s)ds - \int_0^{\gamma_1 v_s(t)} u_s(s)ds| \\
&\leq \delta + K_1T\|u-u_s\| + \lambda_1K_1|\int_0^{\gamma_1 v(t)} u(s)ds - \int_0^{\gamma_1 v_s(t)} u(s)ds| \\
&+ \lambda_1K_1|\int_0^{\gamma_1 v_s(t)} u(s)ds - \int_0^{\gamma_1 v_s(t)} u_s(s)ds| \\
&\leq \delta + K_1T\|u-u_s\| + \lambda_1K_1|\int_{\gamma_1 v_s(t)}^{\gamma_1 v(t)} u(s)ds| + \lambda_1K_1\int_0^{\gamma_1 v_s(t)} |u(s)-u_s(s)|ds \\
&\leq \delta + K_1T\|u-u_s\| + \lambda_1K_1\|u\|\gamma_1\|v-v_s\| + \lambda_1K_1\|u-u_s\|\int_0^{\gamma_1 u_s(t)} ds \\
&\leq \delta + K_1T\|u-u_s\| + \lambda_1K_1r\gamma_1\|v-v_s\| + \lambda_1K_1r\gamma_1\|u-u_s\| \\
&\leq \delta + KT\|u-u_s\| + \lambda Kr\gamma\|v-v_s\| + \lambda Kr\gamma\|u-u_s\| \\
&\leq \delta + (KT+\lambda Kr\gamma)\|u-u_s\| + \lambda Kr\gamma\|v-v_s\|,
\end{aligned}$$

and

$$(1-(KT+\lambda Kr\gamma))\|u-u_s\| \leq \delta + \lambda Kr\gamma\|v-v_s\|,$$

hence

$$\|u-u_s\| \leq \frac{\delta}{1-(KT+\lambda Kr\gamma)} + \frac{\lambda Kr\gamma}{1-(KT+\lambda Kr\gamma)}\|v-v_s\|.$$

Similarly,

$$\|v-v_s\| \leq \frac{\delta}{1-(KT+\lambda Kr\gamma)} + \frac{\lambda Kr\gamma}{1-(KT+\lambda Kr\gamma)}\|u-u_s\|.$$

Then

$$\begin{aligned}
\|(u,v)-(u_s,v_s)\|_X &= \|((u-u_s),(v-v_s))\|_X = \max\{\|(u-u_s)\|_C, \|(v-v_s)\|_C\} \\
&\leq \frac{\delta}{1-(KT+\lambda Kr\gamma)} + \max\{\frac{Kr\gamma}{1-(KT+Kr\gamma)}\|v-v_s\|, \frac{Kr\gamma}{1-(KT+Kr\gamma)}\|u-u_s\|\} \\
&\leq \frac{\delta}{1-(KT+\lambda Kr\gamma)} + \frac{\lambda Kr\gamma}{1-(KT+\lambda Kr\gamma)}\max\{\|v-v_s\|, \|u-u_s\|\} \\
&\leq \frac{\delta}{1-(KT+\lambda Kr\gamma)} + \frac{\lambda Kr\gamma}{1-(KT+\lambda Kr\gamma)}(\|((u-u_s),(v-v_s))\|_X) \\
&\leq \frac{\delta}{1-(KT+\lambda Kr\gamma)} + \frac{\lambda Kr\gamma}{1-(KT+\lambda Kr\gamma)}\|(u,v)-(u_s,v_s)\|_X,
\end{aligned}$$

and

$$(1-\frac{\lambda Kr\gamma}{1-(KT+\lambda Kr\gamma)})\|(u,v)-(u_s,v_s)\| \leq \frac{\delta}{1-(KT+\lambda Kr\gamma)},$$

then

$$\|(u,v)-(u_s,v_s)\| \leq \frac{\delta}{1-(KT+2\lambda Kr\gamma)} = \epsilon_1.$$

Now,

$$\begin{aligned}\|(\zeta,\Im)-(\zeta_s,\Im_s)\|_X &= \|((\zeta-\zeta_s),(\Im-\Im_s))\|_X=\max\{\|(\zeta-\zeta_s)\|_C,\|(\Im-\Im_s)\|_C\\ &\le T\max\{\|(u-u_s)\|_C,\|(v-v_s)\|_C\}\le T\|((u-u_s),(v-v_s))\|\\ &\le T\|(u,v)-(u_s,v_s)\|\le\epsilon_1 T=\epsilon.\end{aligned}$$

Hence

$$\|(\zeta,\Im)-(\zeta_s,\Im_s)\|_X\le\epsilon.$$

Then we deduce that

$$\|\zeta-\zeta_s\|\le\epsilon.$$

□

3.2. **Continuous dependence.**

Definition 3.2. *The solution* $(u,v)\in Q_r$ *of (3.1)-(3.2) depends continuously on* $\zeta_0,\ \Im_0,\ \gamma,\ \lambda$ *if* $\forall\ \epsilon>0\ \exists\ \delta>0$ *such that*

$$\max\{|\zeta_0-\zeta_0^*|,\ |\Im_0-\Im_0^*|,\ |\gamma-\gamma^*|,\ |\lambda-\lambda^*|\}<\delta\ \Rightarrow\ \|(u,v)-(u^*,v^*)\|_X<\epsilon,$$

where

$$u^*(t)=F_1(t,\zeta_0^*+\int_0^t u^*(s)ds,\lambda_1^*(\zeta_0^*+\int_0^{\gamma_1^*v^*(t)}u^*(s)ds)). \tag{3.1}$$

$$v^*(t)=F_2(t,\Im_0^*+\int_0^t v^*(s)ds,\lambda_2^*(\Im_0^*+\int_0^{\gamma_2^*u^*(t)}v^*(s)ds)), \tag{3.2}$$

Theorem 3.2. *Let the assumptions of Theorem (2.1) be satisfied, then the solution* (u,v) *depends continuously on the parameters* $\zeta_0,\ \Im_0,\gamma,\ \lambda$.

Proof. Let $\delta>0$ be given such that

$$\max\{|\zeta_0-\zeta_0^*|,\ |\Im_0-\Im_0^*|,\ |\gamma-\gamma^*|,\ |\lambda-\lambda^*|\}<\delta.$$

Then

$$\begin{aligned}&|u(t)-u^*(t)|\\ =\ &|F_1(t,\zeta_0+\int_0^t u(s)ds,\lambda_1(\zeta_0+\int_0^{\gamma_1v(t)}u(s)ds))-F_1(t,\zeta_0^*+\int_0^t u^*(s)ds,\lambda_1^*(\zeta_0^*+\int_0^{\gamma_1^*v^*(t)}u^*(s)ds))|\\ \le\ &K_1|\zeta_0-\zeta_0^*|+K_1\int_0^t|u(s)-u^*(s)|ds+K_1|\lambda_1\int_0^{\gamma_1v(t)}u(s)ds-\lambda_1^*\int_0^{\gamma_1^*v^*(t)}u^*(s)ds|\\ \le\ &K_1\delta+K_1T\|u-u^*\|+K_1|\lambda_1\int_0^{\gamma_1v(t)}u(s)ds-\lambda_1^*\int_0^{\gamma_1^*v^*(t)}u^*(s)ds|\\ \le\ &K_1\delta+K_1T\|u-u^*\|+K_1|\lambda_1\int_0^{\gamma_1v(t)}u(s)ds-\lambda_1\int_0^{\gamma_1^*v^*(t)}u^*(s)ds\end{aligned}$$

$$
\begin{aligned}
&+ \quad \lambda_1 \int_0^{\gamma_1^* v(t)} u^*(s)ds) - \lambda_1^* \int_0^{\gamma_1^* v^*(t)} u^*(s)ds| \\
&\leq \quad K_1\delta + K_1T\|u-u^*\| + K_1\lambda_1| \int_0^{\gamma_1 v(t)} u(s)ds - \int_0^{\gamma_1^* v^*(t)} u^*(s)ds| + K_1|\lambda_1 - \lambda_1^*| \int_0^{\gamma_1^* v^*(t)} u^*(s)ds \\
&\leq \quad K_1\delta + K_1T\|u-u^*\| + K_1\lambda_1| \int_0^{\gamma_1 v(t)} u(s)ds - \int_0^{\gamma_1 v(t)} u^*(s)ds + \int_0^{\gamma_1 v(t)} u^*(s)ds - \int_0^{\gamma_1^* v^*(t)} u^*(s)ds| \\
&+ \quad K_1\delta\|u^*\| \int_0^{\gamma_1^* v(t)} ds \\
&\leq \quad K_1\delta + K_1T\|u-u^*\| + K_1\lambda_1 \int_0^{\gamma_1 v(t)} |u(s)-u^*(s)|ds + K_1\lambda_1| \int_{\gamma_1^* v^*(t)}^{\gamma_1 v(t)} u^*(s)ds| + K_1\delta\|u^*\|\gamma_1^*\|v\| \\
&\leq \quad K_1\delta + K_1T\|u-u^*\| + K_1\lambda_1\gamma_1 r\|u-u^*\| + K_1\lambda_1 r| \int_{\gamma_1^* v^*(t)}^{\gamma_1 v(t)} ds| + K_1\delta r^2\gamma_1^* \\
&\leq \quad K_1\delta + K_1T\|u-u^*\| + K_1\lambda_1\gamma_1 r\|u-u^*\| + K_1\lambda_1 r|\gamma_1 v(t) - \gamma_1^* v^*(t)| + K_1 r^2\gamma_1^*\delta \\
&\leq \quad K_1\delta + K_1T\|u-u^*\| + K_1\lambda_1\gamma_1 r\|u-u^*\| + K_1\lambda_1 r|\gamma_1 v(t) - \gamma_1 v^*(t) + \gamma_1 v^*(t) - \gamma_1^* v^*(t)| + K_1 r^2\gamma_1^*\delta \\
&\leq \quad K_1\delta + K_1T\|u-u^*\| + K_1\lambda_1\gamma_1 r\|u-u^*\| + K_1\lambda_1 r\gamma_1|v(t) - v^*(t)| + K_1\lambda_1 r^2|\gamma_1^* - \gamma_1| + K_1 r^2\gamma_1^*\delta \\
&\leq \quad K_1\delta + K_1T\|u-u^*\| + K_1\lambda_1\gamma_1 r\|u-u^*\| + K_1\lambda_1 r\gamma_1\|v-v^*\| + K_1\lambda_1 r^2\delta + K_1 r^2\gamma_1^*\delta \\
&\leq \quad (1+\lambda r^2 + \gamma^* r^2)K\delta + (KT + \lambda K r\gamma)\|u-u^*\| + K\lambda r\gamma\|v-v^*\|,
\end{aligned}
$$

and

$$(1-(KT+\lambda K r\gamma))\|u-u^*\| \leq (1+\lambda r^2+\gamma^* r^2)K\delta + \lambda K r\gamma\|v-v^*\|,$$

then

$$\|u-u^*\| \leq \frac{(1+\lambda r^2+\gamma^* r^2)K}{1-(KT+\lambda K r\gamma)}\delta + \frac{\lambda K r\gamma}{1-(KT+\lambda K r\gamma)}\|v-v^*\|.$$

Similarly,

$$\|v-v^*\| \leq \frac{(1+\lambda r^2+\gamma^* r^2)K}{1-(KT+\lambda K r\gamma)}\delta + \frac{\lambda K r\gamma}{1-(KT+\lambda K r\gamma)}\|u-u^*\|.$$

Then

$$
\begin{aligned}
\|(u,v)-(u^*,v^*)\|_X &= \|((u-u^*),(v-v^*))\|_X = \max\{\|(u-u^*)\|_C, \|(v-v^*)\|_C\} \\
&\leq \max\{\frac{(1+\lambda r^2+\gamma^* r^2)K}{1-(KT+\lambda K r\gamma)}\delta + \frac{\lambda K r\gamma}{1-(KT+\lambda K r\gamma)}\|v-v^*\|, \\
&\quad \frac{(1+\lambda r^2+\gamma^* r^2)K}{1-(KT+\lambda K r\gamma)}\delta + \frac{\lambda K r\gamma}{1-(KT+\lambda K r\gamma)}\|u-u^*\|\} \\
&\leq \frac{(1+\lambda r^2+\gamma^* r^2)K}{1-(KT+K r\gamma)}\delta + \max\{\frac{K r\gamma}{1-(KT+K r\gamma)}\|v-v^*\|, \\
&\quad \frac{K r\gamma}{(1-(KT+K r\gamma)}\|u-u^*\|\} \\
&\leq \frac{(1+\lambda r^2+\gamma^* r^2)K}{1-(KT+\lambda K r\gamma)}\delta + \frac{\lambda K r\gamma}{1-(KT+\lambda K r\gamma)} \max\{\|v-v^*\|, \|u-u^*\|\}
\end{aligned}
$$

$$\begin{aligned} &\leq \frac{(1+\lambda r^2+\gamma^* r^2)K}{1-(KT+\lambda Kr\gamma)}\delta + \frac{\lambda Kr\gamma}{1-(KT+\lambda Kr\gamma)}\|((u-u^*),(v-v^*))\|_X \\ &\leq \frac{(1+\lambda r^2+\gamma^* r^2)K}{1-(KT+\lambda Kr\gamma)}\delta + \frac{\lambda Kr\gamma}{1-(KT+\lambda Kr\gamma)}\|((u,v)-(u^*,v^*))\|_X \end{aligned}$$

and

$$(1-\frac{\lambda Kr\gamma}{1-(KT+\lambda Kr\gamma)})\|(u,v)-(u^*,v^*)\| \leq \frac{2K+Kr^2}{1-(KT+\lambda Kr\gamma)}\delta,$$

then

$$\|(u,v)-(u^*,v^*)\| \leq \frac{2K+Kr^2}{1-(KT+2\lambda Kr\gamma)}\delta = \epsilon.$$

This prove that the solution $(u,v) \in X$ depends continuously on the parameters $\zeta_0,\ \Im_0, \gamma,\ \lambda$. □

Definition 3.3. *The solution $(\zeta,\Im) \in X$ of (1.1)-(1.2) depends continuously on $u,\ v$ if $\forall\ \epsilon > 0\ \exists\ \delta > 0$ such that*

$$\max\{|u-u^*|,\ |v-v^*|\} < \delta \ \Rightarrow\ \|(\zeta,\Im)-(\zeta^*,\Im^*)\| < \epsilon,$$

where

$$\zeta^*(t) = \zeta_0 + \int_0^t u^*(s)ds. \tag{3.3}$$

$$\Im^*(t) = \Im_0 + \int_0^t v^*(s)ds. \tag{3.4}$$

Theorem 3.3. *Let the assumptions of Theorem 3.2 be satisfied, then the solution $(\zeta,\Im) \in X$ depends continuously on $u,\ v$.*

Proof. Let $(\zeta^*,\Im^*)$ be the solution of (3.3)-(3.4), then

$$\begin{aligned} |\zeta(t)-\zeta^*(t)| &\leq |\int_0^t u(s)ds - \int_0^t u^*(s)ds| \\ &\leq \int_0^t |u(s)-u^*(s)|ds \leq \|u-u^*\|T \leq \delta T, \end{aligned}$$

then

$$\|\zeta-\zeta^*\| \leq \epsilon.$$

Similarly,

$$\begin{aligned} |\Im(t)-\Im^*(t)| &\leq |\int_0^t v(s)ds - \int_0^t v^*(s)ds| \\ &\leq \int_0^t |v(s)-v^*(s)|ds \leq \|v-v^*\|T \leq \delta T, \end{aligned}$$

then

$$\|\Im-\Im^*\| \leq \epsilon.$$

Now,

$$\|(\zeta,\Im)-(\zeta^*,\Im^*)\|_X = \|((\zeta-\zeta^*),(\Im-\Im^*))\|_X = \max\{\|\zeta-\zeta^*\|_C, \|\Im-\Im^*\|_C\} \leq \epsilon.$$

Then

$$\|(\zeta, \mathfrak{I}) - (\zeta^*, \mathfrak{I}^*)\|_X \leq \epsilon.$$

□

Corollary 3.1. *Let the assumptions of Theorem 3.3 be satisfied, then the solution $(\zeta, \mathfrak{I}) \in X$ depends continuously on $\zeta_0, \mathfrak{I}_0$ and the parameters $\lambda,\ \gamma$.*

4. General discussion

In the absence of the feedback control; as a special case of our work; we can study the next problem

$$\frac{d\zeta}{dt} = F(t, \zeta(t), \lambda\zeta(\gamma\frac{d\zeta}{dt})),\ \zeta(0) = \zeta_0,\ a.e.\ t \in \mathbb{k}. \tag{4.1}$$

This problem can be solved under the following assumptions

(i) $F : \mathbb{k} \times R \times R \to \mathbb{k}$ is continuous and there exists a positive constant L such that

$$|F(t, \zeta, u) - F(s, \mathfrak{I}, v)| \leq L(|t - s| + |\zeta - \mathfrak{I}| + |u - v|).$$

From this assumption, we get

$$|F(t, \zeta, u)| \leq L(|\zeta| + |u|) + |F(t, 0, 0)|.$$

(ii) $\sup_{t\in\mathbb{k}} |F(t, 0, 0)| = N$.

(iii) There exists a real positive root of the algebraic equation

$$\lambda L\gamma r^2 + (LT - 1)r + (L|\zeta_0| + \lambda L|\zeta_0| + N) = 0.$$

The formulation of this problem can be obtained if we put $\frac{d\zeta}{dt} = \mathfrak{I}$, then

$$\zeta(t) = \zeta_0 + \int_0^t \mathfrak{I}(s)ds, \tag{4.2}$$

and

$$\zeta(\gamma\mathfrak{I}(t)) = \zeta_0 + \int_0^{\gamma\mathfrak{I}(t)} \mathfrak{I}(s)ds,$$

and the problem (4.1), will be given by

$$\mathfrak{I}(t) = F(t, \zeta_0 + \int_0^t \mathfrak{I}(s)ds, \lambda(\zeta_0 + \int_0^{\gamma\mathfrak{I}(t)} \mathfrak{I}(s)ds)). \tag{4.3}$$

Let $C(\mathbb{k})$ be the class of continuous functions define on $\mathbb{k}$. Define the operator G associated with (4.3) by

$$G\mathfrak{I} = F(t, \zeta_0 + \int_0^t \mathfrak{I}(s)ds, \lambda(\zeta_0 + \int_0^{\gamma\mathfrak{I}(t)} \mathfrak{I}(s)ds)).$$

Let $Q_r = \{\mathfrak{I} \in C : \|\mathfrak{I}\| \leq r\}$.

Theorem 4.1. *Suppose the assumptions (i)–(iii) be hold. If $(LT + 2\lambda Lr\gamma) < 1$, then the functional integral equation (4.3) has a unique solution $\mathfrak{I} \in C(\mathbb{k})$.*

Proof. For $\mathfrak{I} \in Q_r$. By the same way, we can show that the functional integral equation (4.3) has a unique solution $\mathfrak{I} \in C(\mathbb{k})$. Consequently, Problem 4.1 has a unique solution $\zeta \in C(\mathbb{k})$. □

The next theorems can be also proved for Problem 4.1.

Theorem 4.2. *If the assumptions of Theorem 4.1 are met, then (4.1) is Hyers-Ulam stable.*

Theorem 4.3. *Let the assumptions of Theorem 4.1 be satisfied, then the solution $\mathfrak{I} \in C(\mathbb{k})$ depends continuously on ζ_0 and the parameters λ, γ.*

Theorem 4.4. *Let the assumptions of Theorem 4.1 be satisfied, then the solution $\zeta \in C(\mathbb{k})$ depends continuously on $\mathfrak{I}$.*

Corollary 4.1. *Let the assumptions of Theorem 4.1 and Theorem 4.4 be satisfied, then the solution $\zeta \in C(\mathbb{k})$ depends continuously on ζ_0 and the parameters λ, γ.*

5. Examples

Example 1. Consider the problem

$$\frac{d\zeta}{dt} = \frac{1}{4}ln(1+t) + \frac{1}{5}\zeta(t) + \frac{1}{3}\zeta(0.9\frac{d\mathfrak{I}}{dt}),\ t \in (0, \frac{1}{2}],\ \zeta(0) = 0 \tag{5.1}$$

$$\frac{d\mathfrak{I}}{dt} = \frac{1}{7}\sqrt{t^2+2} + \frac{1}{10}\mathfrak{I}(t) + \frac{1}{2}\mathfrak{I}(0.8\frac{d\zeta}{dt}),\ t \in (0, \frac{1}{2}],\ \mathfrak{I}(0) = 0.1. \tag{5.2}$$

Set

$$F_1(t,\zeta,\zeta_1) = \frac{1}{4}ln(1+t) + \frac{1}{5}\zeta(t) + \frac{1}{3}\zeta(0.9\frac{d\mathfrak{I}}{dt}),$$
$$F_2(t,\mathfrak{I},\mathfrak{I}_1) = \frac{1}{7}\sqrt{t^2+2} + \frac{1}{10}\mathfrak{I}(t) + \frac{1}{2}u(0.8\frac{d\zeta}{dt}),$$

thus

$$\begin{aligned}|F_1(t,\mathfrak{I},\mathfrak{I}_1) - F_1(s,\bar{\mathfrak{I}},\bar{\mathfrak{I}}_1)| &\leq \frac{1}{4}|ln(1+t) - ln(1+s)| + \frac{1}{5}|\zeta - \bar{\zeta}| + \frac{1}{3}|\zeta_1 - \bar{\zeta}_1| \\ &\leq \frac{1}{3}(|t-s| + |\zeta - \bar{\zeta}| + |\zeta_1 - \bar{\zeta}_1|).\end{aligned}$$

$$\begin{aligned}|F_2(t,\zeta,\zeta_1) - F_2(s,\bar{\zeta},\bar{\zeta}_1)| &\leq \frac{1}{7}|\sqrt{t^2+2} - \sqrt{s^2+2}| + \frac{1}{10}|\mathfrak{I} - \bar{\mathfrak{I}}| + \frac{1}{2}|\mathfrak{I}_1 - \bar{\mathfrak{I}}_1| \\ &\leq \frac{1}{2}(|t-s| + |\mathfrak{I} - \bar{\mathfrak{I}}| + |\mathfrak{I}_1 - \bar{\mathfrak{I}}_1|).\end{aligned}$$

Where $\gamma = \max\{\gamma_1, \gamma_2\} = 0.9$, $K = \max\{\frac{1}{3}, \frac{1}{2}\} = \frac{1}{2}$, $\lambda = \max\{\lambda_1, \lambda_2\} = \frac{1}{2}$, $A = \max\{0, 0.1\} = 0.1$ and we have

$$F_1(t,0,0) = \frac{1}{4}ln(1+t),\ F_2(t,0,0) = \frac{1}{7}\sqrt{t^2+2}.$$

Since $B = \max\{0.044, 0.21\} = 0.21$. Then, we get $r = 0.4373$ and $KT + 2\lambda K r\gamma = 0.4468 < 1$. It is clear that all assumptions of Theorem 2.1 are satisfied. Hence there exist a unique solution

$\zeta \in C[0, \frac{1}{2}]$ of Problem (5.1)-(5.2).

Example 2. Consider the problem

$$\frac{d\zeta}{dt} = \frac{1}{25}\frac{t^2}{(9-t)} + \frac{1}{5}\zeta(t) + \frac{1}{7}\zeta(0.7\frac{dy}{dt}),\ t \in (0,1],\ \zeta(0) = 0.1 \tag{5.3}$$

$$\frac{d\mathfrak{I}}{dt} = \frac{1}{8}(t^2-1) + \frac{1}{7}\mathfrak{I}(t) + \frac{1}{10}\mathfrak{I}(0.5\frac{d\zeta}{dt}),\ t \in (0,1],\ \mathfrak{I}(0) = 0.25. \tag{5.4}$$

Set

$$F_1(t,\zeta,\zeta_1) = \frac{1}{25}\frac{t^2}{(9-t)} + \frac{1}{7}\zeta(t) + \frac{1}{5}\zeta_1(0.7\frac{dy}{dt}).$$

$$F_2(t,\mathfrak{I},\mathfrak{I}_1) = \frac{1}{12}(t^2-1) + \frac{1}{8}\mathfrak{I}(t) + \frac{1}{10}\mathfrak{I}_1(0.5\frac{d\zeta}{dt}),$$

thus

$$\begin{aligned}
|F_1(t,\zeta,\zeta_1) - F_1(s,\bar{\zeta},\bar{\zeta}_1)| &\leq \frac{1}{25}|\frac{t^2}{9-t} - \frac{s^2}{9-s}| + \frac{1}{7}|\zeta-\bar{\zeta}| + \frac{1}{5}|\zeta_1-\bar{\zeta}_1| \\
&\leq \frac{1}{25}|\frac{t^2(9-s)-s^2(9-t)}{(9-t)(9-s)}| + \frac{1}{7}|\zeta-\bar{\zeta}| + \frac{1}{5}|\zeta_1-\bar{\zeta}_1| \\
&\leq \frac{18}{200}|t-s| + \frac{1}{200}|t-s| + \frac{1}{7}|\zeta-\bar{\zeta}| + \frac{1}{5}|\zeta_1-\bar{\zeta}_1| \\
&\leq \frac{1}{5}(|t-s| + |\zeta-\bar{\zeta}| + |\zeta_1-\bar{\zeta}_1|),
\end{aligned}$$

$$\begin{aligned}
|F_2(t,\mathfrak{I},\mathfrak{I}_1) - F_2(s,\bar{\mathfrak{I}},\bar{\mathfrak{I}}_1)| &\leq \frac{1}{12}|(t^2-1)-(s^2-1)| + \frac{1}{8}|\mathfrak{I}-\bar{\mathfrak{I}}| + \frac{1}{10}|\mathfrak{I}_1-\bar{\mathfrak{I}}_1| \\
&\leq \frac{1}{12}|t^2-s^2| + \frac{1}{8}|\mathfrak{I}-\bar{\mathfrak{I}}| + \frac{1}{10}|\mathfrak{I}_1-\bar{\mathfrak{I}}_1| \\
&\leq \frac{1}{6}(|t-s| + |\mathfrak{I}-\bar{\mathfrak{I}}| + |\mathfrak{I}_1-\bar{\mathfrak{I}}_1|).
\end{aligned}$$

Where $\gamma = \max\{\gamma_1,\gamma_2\} = 0.7$, $K = \max\{\frac{1}{5},\frac{1}{6}\} = \frac{1}{5}$, $\lambda = \max\{\lambda_1,\lambda_2\} = \frac{1}{7}$, $A = \max\{0.25, 0.1\} = 0.1$ and we have

$$F_1(t,0,0) = \frac{1}{25}\frac{t^2}{9-t},\ F_2(t,0,0) = \frac{1}{12}(t^2-1).$$

Since $B = \max\{0, 0.05\} = 0.05$. Then, we get $r = 0.0908$ and $KT + 2\lambda Kr\gamma = 0.2036 < 1$. It is clear that all assumptions of Theorem 2.1 are satisfied. Hence there exist unique solution $\zeta \in C[0,1]$ of Problem (5.3)-(5.4).

Example 3. Consider the problem

$$\frac{d\zeta}{dt} = \frac{1+2t}{20} + \frac{2}{3}\zeta(t) + \frac{1}{2}\zeta(0.6\frac{d\zeta}{dt}),\ \zeta(0) = \frac{1}{5},\ t \in (0,\frac{1}{3}]. \tag{5.5}$$

Set

$$F(t,\zeta,\bar{\zeta}) = \frac{1+2t}{20} + \frac{2}{3}\zeta(t) + \frac{1}{2}\zeta(0.6\zeta).$$

We can easily deduce that

$$\begin{aligned}|F(t_1,\zeta_1,\bar{\zeta}_1)-F(t_2,\zeta_2,\bar{\zeta}_2)| &\le \frac{1+2t_1}{20}-\frac{1+2t_2}{20}+\frac{2}{3}|\zeta_1-\zeta_2|+\frac{1}{2}(\bar{\zeta}_1-\bar{\zeta}_2)\\ &\le \frac{1}{10}|t_1-t_2|+\frac{2}{3}|\zeta_1-\zeta_2|+\frac{1}{2}|\bar{\zeta}_1-\bar{\zeta}_2|\\ &\le \frac{2}{3}(|t_1-t_2|+|\zeta_1-\zeta_2|+|\bar{\zeta}_1-\bar{\zeta}_2|).\end{aligned}$$

Where $\gamma = 0.6$, $L = \frac{2}{3}$, $\lambda = \frac{1}{2}$ and $N = \frac{1}{5}$, $\zeta_0 = \frac{1}{5}$ and we have $F(t,0,0) = \frac{1+2t}{20}$, since $N = \sup_{t\in[0,\frac{1}{3}]} |F(t,0,0)| = 0.08$. Then, we get $r = 0.5975$ and $LT + 2\lambda Lr\gamma = 0.4612 < 1$. It is clear that all assumptions of Theorem 4.1 are satisfied. Hence there exist a unique solution $\zeta \in C[0,\frac{1}{3}]$ of Problem (5.5).

6. Conclusion

Problems with differential equations involving control variables are very applicable due to their influence on a wide range of fields. When these equations also have deviating arguments depending on both the time t and the state variable ζ, the problem in this case possesses broad relevance in a variety of fields. This kind of delay introduces memory effects into the system, which implies intricate and rich dynamics. Many researchers devoted their works to such types of differential equations; they used a variety of analytical and numerical methodologies to study the existence and stability of the solution. Feedback control problems with pantograph equations have various applications in most fields, such as biology, ecology, physics, engineering, and others. In this manuscript, we are concerned with the study of the solvability of the state-dependent implicit pantograph problem (1.1) under the feedback control (1.2). We also introduced suitable assumptions and analyzed the uniqueness of the solution. We investigated the stability of the problem due to the concept of Hyres-Ulam stability. The continuous dependence of the solution of the problem on the initial data and some parameters has been proved. Furthermore, we presented a general discussion section for the problem of a state-dependent implicit pantograph equation without feedback control and introduced some theorems for the existence, uniqueness, and stability of the problem. Finally, we provided some examples to illustrate our results.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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