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Existence and Uniqueness Solutions of Multi-Term Delay Caputo Fractional Differential Equations

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Abstract. This study investigates a novel type of nonlocal boundary value problem with multipoint-integral boundaries and multi-term delay Caputo fractional differential equations ($\mathcal{FDE}$). The provided problem is turned into an analogous fixed-point problem using fixed-point ($\mathcal{FP}$) theory tools. Additionally, discussing about stability, in Ulam-Hyers-Rassias ($\mathcal{UHR}$), Ulam-Hyers ($\mathcal{UH}$), generalized Ulam-Hyers-Rassias ($\mathcal{GUHR}$) and generalized Ulam-Hyers ($\mathcal{GUH}$) stability, for finding the problem. Based on our obtained results we given some examples. As of our obtained results are very useful to multi-term caputo $\mathcal{FDE}$ related to hydrodynamics.

1. Introduction

Due to its various applications in the natural and social sciences, fractional calculus has been thoroughly studied during the last several decades [1,2]. Unlike the corresponding classical integer order operators, the fractional order operators are nonlocal, which means that mathematical models based on them provide more light on the features of the phenomenon being studied.

This includes examples on relaxation filtration processes [3], zooplanktonphytoplankton system [4], epithelial cells [5], fractional kinetics [6], neural networks [7], chaos blood flow in small-lumen arterial vessels [8], synchronization [9], etc. We recommend the reader to [10] for basic information

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on the topic, and a recent monograph [11] contains some recent papers on nonlocal nonlinear fractional order boundary value problem. [12–16] contains some intriguing findings on boundary-value problem involving inclusions, systems of such equations, and multi-term $\mathcal{FDE}$ and as well as equipped with various types of boundary conditions. Recent research on multi-term $\mathcal{FDE}$ include [17–20]. Research on $\mathcal{FDE}$ with delay has increased significantly [21–23]. Equations are important for understanding the history of a phenomenon [24]. Many academics are interested in examining the stability features of $\mathcal{FDE}$ (e.g., [25–31]).

This study explores a novel type of boundary value equation using multi-term caputo $\mathcal{FDE}$ and nonlocal multipoint-integral boundaries. We investigate the existence conditions for solutions as follows:

$$\begin{cases} \sum_{\check{i}=1}^{\check{x}} \nu_{\check{i}}^{\mathcal{R}}\mathcal{D}^{\varpi_{\check{i}}}\check{a}(\aleph) = \check{u}(\aleph,\check{a}(\aleph),\check{a}(\check{w}\aleph)) + \int_0^{\aleph} \frac{(\aleph-\check{z})^{\phi-1}\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))}{\Gamma(\phi)}d\check{z}, \\ \varpi_\tau \in (\check{x}-1,\check{x}), \tau \neq \check{i}, 0 < \varpi_{\check{i}}, \phi \leq 1, \aleph \in [0,1], \\ \check{a}(0) = \check{a}_0, \check{a}^{(\kappa)}(0) = 0, \kappa = 1,2,3,\cdots,\check{x}-2, \\ \check{a}(1) = \upsilon \int_0^1 \check{a}(\mathfrak{s})d\mathfrak{s} - \sum_{\varrho=1}^{\check{r}} \check{h}_\varrho \check{a}(\pi_\varrho), \check{a}_0, \upsilon, \check{h}_\varrho \in \mathbb{R}, \pi_\varrho \in (0,1), \end{cases} \tag{1.1}$$

where ${}^{\mathcal{R}}\mathcal{D}^{\varpi_{\check{i}}}$ be the Caputo fractional derivative of order $\varpi_{\check{i}}, \check{w} \in (0,1), \nu_{\check{i}} \in \mathbb{R}$(denotes set of all real number), $\check{i} \in \{1,2,\cdots,\check{x}\}$, and $\check{u}\colon [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous.

This study aims to find a more generic solution by using the fractional integral operator of order ϖ_τ instead of ϖ_1, in [32] (further, more in Lemma 2.3). In [32] the authors used the multipoint boundary conditions from that we examine multipoint-integral boundary conditions. All of this conditions characterize the boundary response, eliminating multipoint locations at $\pi_\varrho, \varrho = 1,\cdots\check{r}$. They provide insight into fractional boundary-value problem for multi-term caputo $\mathcal{FDE}$. As previously mentioned, differential equations with multiple fractional-order differential operators are a fascinating field of study. Examples of these equations include the Bagley-Torvik equation [33], which models the motion of a rigid plate submerged in a Newtonian fluid, and the fractional calculus Basset equation [34], which expands on the Basset equation. It is essential to note that the outcomes related to exclusively integral boundary conditions are a particular case (see the conclusions section for an explanation). Using the instruments of fixed-point theory is one of the helpful methods to derive the existence theory for initial and boundary-value problem. Using this approach, we formulate the existence theory for the given equation (1.1).

The structure of this paper is as follows. Section 2 comprises the introductory information pertaining to our research. Section 3 presents the key findings, which are based on the Banach contraction mapping principle and the nonlinear alternative of Leray-Schauder theorem ($\mathcal{LST}$). In Section 4, the stability of solutions to the equation (1.1) is examined, and in Section 5, examples are produced to show the findings gained. In Section 6, we conclude with a few noteworthy points.

2. Preliminaries

This section contains preliminary material relevant to our core study. We can start this subject with some fundamental notions in fractional calculus.

Definition 2.1. *[10] Assume that the Riemann-Liouville fractional integral $\mathcal{I}^{\phi}_{\check{h}}$ of order $\phi \in \mathbb{R}(\phi > 0)$ and let υ be a locally integrable real-valued function on $-\infty \leq \check{h} < \aleph < \check{f} \leq +\infty$ such that:*

$$\mathcal{I}^{\phi}_{\check{h}}\upsilon(\aleph) = (\upsilon * \mathcal{K}_{\phi})(\aleph) = \int_{\check{h}}^{\aleph} \frac{\upsilon(\rho)}{\Gamma(\phi)(\aleph-\rho)^{1-\phi}} d\rho,$$

where $\mathcal{K}_{\phi} = \frac{\aleph^{\phi-1}}{\Gamma(\phi)}$ and Euler gamma function is Γ.

Definition 2.2. *[10] Let the map $\upsilon\colon [\check{h},\infty) \to \mathbb{R}$ be $(\check{x}-1)$-times the continuous differentiable function and the Caputo derivative of fractional order ϕ such that:*

$$^{\mathcal{R}}\mathcal{D}^{\phi}\upsilon(\aleph) = \int_{\check{h}}^{\aleph} \frac{\upsilon^{(\check{x})}(\rho)}{\Gamma(\check{x}-\phi)(\aleph-\rho)^{1+\phi-\check{x}}} d\rho, \check{x}-1 < \phi \leq \check{x}, \check{x} = [\phi]+1,$$

where $[\phi]$ be intger part of the real number ϕ.

Lemma 2.1. *[10] For $\phi, \varpi \in (0,1]$. Then we have*

$$\mathcal{I}^{\phi}_{\check{h}}(\mathfrak{s}-\check{z})^{\varpi-1} = \frac{\Gamma(\varpi)}{\Gamma(\varpi+\phi)}(\mathfrak{s}-\check{z})^{\phi+\varpi-1}$$

$$\mathcal{D}^{\phi}_{\check{h}}(\mathfrak{s}-\check{z})^{\varpi-1} = \frac{\Gamma(\varpi)}{\Gamma(\varpi-\phi)}(\mathfrak{s}-\check{z})^{\phi-\varpi-1}.$$

Lemma 2.2. *[10] Let $\check{x}-1 < \phi < \check{x}$, the fractional differential equation's general solution $^{\mathcal{R}}\mathcal{D}^{\phi}\upsilon(\aleph) = 0, \aleph \in [\check{h}, \check{f}]$, such that*

$$\upsilon(\aleph) = \mathcal{R}_0 + \mathcal{R}_1(\aleph-\check{h}) + \mathcal{R}_2(\aleph-\check{h})^2 + \cdots + \mathcal{R}_{\check{x}-1}(\aleph-\check{h})^{\check{x}-1},$$

where $\mathcal{R}_{\check{i}} \in \mathbb{R}, \check{i} = 0,1,\cdots,\check{x}-1$. Furthermore,

$$\mathcal{I}^{\phi\mathcal{R}}\mathcal{D}^{\phi}\upsilon(\aleph) = \upsilon(\aleph) + \sum_{\check{i}=0}^{\check{x}-1}\mathcal{R}_{\check{i}}(\aleph-\check{h})^{\check{i}}.$$

In the upcoming analysis, the lemma that follows which relates to the linear form of the equation (1.1) will be crucial. Because of this conclusion, we may solve the nonlinear problem (1.1) and analyze the existence and uniqueness of its solutions by first turning it into an analogous fixed-point problem.

Lemma 2.3. *Assume $\check{y} \in \Lambda([0,1],\mathbb{R})$. Then, the equation is provided by*

$$\sum_{\check{i}=1}^{\check{x}} \nu_{\check{i}}\, {}^{\mathcal{R}}\mathcal{D}^{\varpi_{\check{i}}}\check{a}(\aleph) = \check{y}(\aleph) + \int_0^{\aleph}\frac{(\aleph-\check{z})^{\phi-1}\check{y}(\check{z})}{\Gamma(\phi)}d\check{z},$$
$$\varpi_{\tau} \in (\check{x}-1,\check{x}], 0 < \varpi_{\check{i}}, \phi \leq 1, \check{i} \in \{1,2,\cdots,\check{x}\}, \check{i} \neq \tau, \nu_{\check{i}} \in \mathbb{R}, \aleph \in [0,1],$$

$$\check{a}(0) = \check{a}_0, \check{a}^{(\kappa)}(0) = 0, \kappa = 1,2,3,\cdots,\check{x}-2, \check{a}(1) = \check{b}\int_0^1 \check{a}(\mathfrak{s})d\mathfrak{s} - \sum_{\varrho=1}^{\check{r}}\check{h}_{\varrho}\check{a}(\pi_{\varrho}), \tag{2.1}$$

is given by

$$\begin{aligned}
\breve{a}(\aleph) =&\breve{a}_0 + \frac{\aleph^{\check{x}-1}}{\gamma_1}\Bigg[\breve{a}_0\Bigg(\breve{b} - 1 - \sum_{\varrho=1}^{\check{r}} \check{h}_\varrho\Bigg) + \frac{\breve{b}}{\nu_\tau \Gamma(\varpi_\tau)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \check{z})^{\varpi_\tau - 1} \check{y}(\check{z}) d\check{z} d\mathfrak{s} \\
&+ \frac{\breve{b}}{\nu_\tau \Gamma(\phi + \varpi_\tau)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \check{z})^{\phi + \varpi_\tau - 1} \check{y}(\check{z}) d\check{z} d\mathfrak{s} \\
&- \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{\breve{b}}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \breve{a}(\check{z}) d\check{z} d\mathfrak{s} \\
&- \sum_{\varrho=1}^{\check{r}} \frac{\check{h}_\varrho}{\nu_\tau \Gamma(\varpi_\tau)} \int_0^{\pi_\varrho} (\pi_\varrho - \check{z})^{\varpi_\tau - 1} \check{y}(\check{z}) d\check{z} \\
&- \sum_{\varrho=1}^{\check{r}} \frac{\check{h}_\varrho}{\nu_\tau \Gamma(\phi + \varpi_\tau)} \int_0^{\pi_\varrho} (\pi_\varrho - \check{z})^{\phi + \varpi_\tau - 1} \check{y}(\check{z}) d\check{z} \\
&+ \sum_{\varrho=1}^{\check{r}} \check{h}_\varrho \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^{\pi_\varrho} (\pi_\varrho - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \breve{a}(\check{z}) d\check{z} \\
&- \frac{1}{\nu_\tau \Gamma(\varpi_\tau)} \int_0^1 (1 - \check{z})^{\varpi_\tau - 1} \check{y}(\check{z}) d\check{z} \\
&- \frac{1}{\nu_\tau \Gamma(\phi + \varpi_\tau)} \int_0^1 (1 - \check{z})^{\phi + \varpi_\tau - 1} \check{y}(\check{z}) d\check{z} \\
&+ \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^1 (1 - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \breve{a}(\check{z}) d\check{z}\Bigg] \\
&+ \frac{1}{\nu_\tau} \int_0^{\aleph} \frac{(\aleph - \check{z})^{\varpi_\tau - 1}}{\Gamma(\varpi_\tau)} \check{y}(\check{z}) d\check{z} \\
&+ \frac{1}{\nu_\tau} \int_0^{\aleph} \frac{(\aleph - \check{z})^{\phi + \varpi_\tau - 1} \check{y}(\check{z})}{\Gamma(\phi + \varpi_\tau)} d\check{z} \\
&- \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^{\aleph} (\aleph - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \breve{a}(\check{z}) d\check{z}, \qquad (2.2)
\end{aligned}$$

where

$$\gamma_1 = 1 - \frac{\breve{b}}{\check{x}} + \sum_{\varrho=1}^{\check{r}} \check{h}_\varrho \pi_1^{\check{x}-1}. \qquad (2.3)$$

Proof. Both sides, we take fractional integral operator of order ϖ_τ in (2.1), we get

$$\begin{aligned}
\breve{a}(\aleph) =&\mathcal{E}_1 + \mathcal{E}_2\aleph + \cdots + \mathcal{E}_{\check{x}}\aleph^{\check{x}-1} + \frac{1}{\nu_\tau} \int_0^{\aleph} \frac{(\aleph - \check{z})^{\varpi_\tau - 1}}{\Gamma(\varpi_\tau)} \check{y}(\check{z}) d\check{z} \\
&+ \frac{1}{\nu_\tau} \int_0^{\aleph} \frac{(\aleph - \check{z})^{\phi + \varpi_\tau - 1} \check{y}(\check{z})}{\Gamma(\phi + \varpi_\tau)} d\check{z} - \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^{\aleph} (\aleph - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \breve{a}(\check{z}) d\check{z}, \qquad (2.4)
\end{aligned}$$

where $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_{\check{x}}$ are undefined constants. Using (2.4) and the condition $\check{a}(0) = \check{a}_0$ and $\check{a}^{(\kappa)}(0) = 0, \kappa = 1, 2, 3, \cdots, \check{x} - 2$, Our findings indicate that $\mathcal{E}_1 = \check{a}_0$ and $\mathcal{E}_{\check{i}} = 0$ for $\check{i} = 2, \cdots, (\check{x} - 1)$. Thus,

$$\begin{aligned}
\check{a}(\aleph) =& \check{a}_0 + \mathcal{E}_{\check{x}} \aleph^{\check{x}-1} + \frac{1}{\nu_\tau} \int_0^{\aleph} \frac{(\aleph - \check{z})^{\varpi_\tau - 1}}{\Gamma(\varpi_\tau)} \check{y}(\check{z}) d\check{z} \\
&+ \frac{1}{\nu_\tau} \int_0^{\aleph} \frac{(\aleph - \check{z})^{\phi + \varpi_\tau - 1} \check{y}(\check{z})}{\Gamma(\phi + \varpi_\tau)} d\check{z} \\
&- \sum_{\check{i}=1, \check{i} \neq \tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^{\aleph} (\aleph - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \check{a}(\check{z}) d\check{z}.
\end{aligned} \tag{2.5}$$

Applying (2.5) with the condition $\check{a}(1) = \check{b} \int_0^1 \check{a}(\mathfrak{s}) d\mathfrak{s} - \sum_{\varrho=1}^{\check{r}} \check{h}_\varrho \check{a}(\pi_\varrho)$, we find that

$$\begin{aligned}
\mathcal{E}_{\check{x}} =& \frac{1}{\gamma_1} \Bigg[\check{a}_0 \Bigg(\check{b} - 1 - \sum_{\varrho=1}^{\check{r}} \check{h}_\varrho \Bigg) + \frac{\check{b}}{\nu_\tau \Gamma(\varpi_\tau)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \check{z})^{\varpi_\tau - 1} \check{y}(\check{z}) d\check{z} d\mathfrak{s} \\
&+ \frac{\check{b}}{\nu_\tau \Gamma(\phi + \varpi_\tau)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \check{z})^{\phi + \varpi_\tau - 1} \check{y}(\check{z}) d\check{z} d\mathfrak{s} \\
&- \sum_{\check{i}=1, \check{i} \neq \tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{\check{b}}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \check{a}(\check{z}) d\check{z} d\mathfrak{s} \\
&- \sum_{\varrho=1}^{\check{r}} \frac{\check{h}_\varrho}{\nu_\tau \Gamma(\varpi_\tau)} \int_0^{\pi_\varrho} (\pi_\varrho - \check{z})^{\varpi_\tau - 1} \check{y}(\check{z}) d\check{z} \\
&- \sum_{\varrho=1}^{\check{r}} \frac{\check{h}_\varrho}{\nu_\tau \Gamma(\phi + \varpi_\tau)} \int_0^{\pi_\varrho} (\pi_\varrho - \check{z})^{\phi + \varpi_\tau - 1} \check{y}(\check{z}) d\check{z} \\
&+ \sum_{\varrho=1}^{\check{r}} \check{h}_\varrho \sum_{\check{i}=1, \check{i} \neq \tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^{\pi_\varrho} (\pi_\varrho - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \check{a}(\check{z}) d\check{z} \\
&- \frac{1}{\nu_\tau \Gamma(\varpi_\tau)} \int_0^1 (1 - \check{z})^{\varpi_\tau - 1} \check{y}(\check{z}) d\check{z} \\
&- \frac{1}{\nu_\tau \Gamma(\phi + \varpi_\tau)} \int_0^1 (1 - \check{z})^{\phi + \varpi_\tau - 1} \check{y}(\check{z}) d\check{z} \\
&+ \sum_{\check{i}=1, \check{i} \neq \tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^1 (1 - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \check{a}(\check{z}) d\check{z} \Bigg],
\end{aligned}$$

where γ_1 is defined in (2.3). Substitute $\mathcal{E}_{\check{x}}$ in (2.5) which yields (2.2). This concludes the proof. □

Theorem 2.1. *[35] Let $\mathcal{R}$ be a non-void convex subset of $\mathcal{U}$ and $\mathcal{U}$ be a Banach space. Let $\check{T} \colon \overline{\mathcal{N}} \to \mathcal{R}$ be a continuous and compact operator and $\mathcal{N}$ be a non-void open subset of $\mathcal{R}$ at $0 \in \mathcal{N}$. Then, $\check{T}$ has $\mathcal{FP}$ or we can find that $\check{a} \in \partial \mathcal{N}$ such that $\check{a} = \lambda_* \check{T}(\check{a})$ at $\lambda_* \in (0, 1)$.*

3. Main results

Using the standard fixed-point theory tools, we prove the existence and uniqueness findings for the equation (1.1) in this section. First, let's convert the (1.1) problem into a $\mathcal{FP}$ problem:

$$\check{T}\check{a}(\aleph) = \check{a}(\aleph),$$

where $\check{T}\colon \Lambda([0,1],\mathbb{R}) \longmapsto \Lambda([0,1],\mathbb{R})$ is the fixed operator described as

$$\begin{aligned}
\check{T}\check{a}(\aleph) =& \check{a}_0 + \frac{\aleph^{\check{x}-1}}{\gamma_1}\Bigg[\check{a}_0\Bigg(\check{b} - 1 - \sum_{\varrho=1}^{\check{r}} \check{h}_\varrho\Bigg) \\
&+ \frac{\check{b}}{\nu_\tau \Gamma(\varpi_\tau)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \check{z})^{\varpi_\tau - 1} \check{u}(\check{z}, \check{y}(\check{z}), \check{y}(\check{w}\check{z})) d\check{z} d\mathfrak{s} \\
&+ \frac{\check{b}}{\nu_\tau \Gamma(\phi + \varpi_\tau)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \check{z})^{\phi + \varpi_\tau - 1} \check{u}(\check{z}, \check{y}(\check{z}), \check{y}(\check{w}\check{z})) d\check{z} d\mathfrak{s} \\
&- \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{\check{b}}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \check{a}(\check{z}) d\check{z} d\mathfrak{s} \\
&- \sum_{\varrho=1}^{\check{r}} \frac{\check{h}_\varrho}{\nu_\tau \Gamma(\varpi_\tau)} \int_0^{\pi_\varrho} (\pi_\varrho - \check{z})^{\varpi_\tau - 1} \check{u}(\check{z}, \check{y}(\check{z}), \check{y}(\check{w}\check{z})) d\check{z} \\
&- \sum_{\varrho=1}^{\check{r}} \frac{\check{h}_\varrho}{\nu_\tau \Gamma(\phi + \varpi_\tau)} \int_0^{\pi_\varrho} (\pi_\varrho - \check{z})^{\phi + \varpi_\tau - 1} \check{u}(\check{z}, \check{y}(\check{z}), \check{y}(\check{w}\check{z})) d\check{z} \\
&+ \sum_{\varrho=1}^{\check{r}} \check{h}_\varrho \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^{\pi_\varrho} (\pi_\varrho - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \check{a}(\check{z}) d\check{z} \\
&- \frac{1}{\nu_\tau \Gamma(\varpi_\tau)} \int_0^1 (1 - \check{z})^{\varpi_\tau - 1} \check{u}(\check{z}, \check{y}(\check{z}), \check{y}(\check{w}\check{z})) d\check{z} \\
&- \frac{1}{\nu_\tau \Gamma(\phi + \varpi_\tau)} \int_0^1 (1 - \check{z})^{\phi + \varpi_\tau - 1} \check{u}(\check{z}, \check{y}(\check{z}), \check{y}(\check{w}\check{z})) d\check{z} \\
&+ \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^1 (1 - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \check{a}(\check{z}) d\check{z}\Bigg] \\
&+ \frac{1}{\nu_\tau} \int_0^{\aleph} \frac{(\aleph - \check{z})^{\varpi_\tau - 1}}{\Gamma(\varpi_\tau)} \check{u}(\check{z}, \check{y}(\check{z}), \check{y}(\check{w}\check{z})) d\check{z} \\
&+ \frac{1}{\nu_\tau} \int_0^{\aleph} \frac{(\aleph - \check{z})^{\phi + \varpi_\tau - 1} \check{u}(\check{z}, \check{y}(\check{z}), \check{y}(\check{w}\check{z}))}{\Gamma(\phi + \varpi_\tau)} d\check{z} \\
&- \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{\nu_{\check{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}})} \int_0^{\aleph} (\aleph - \check{z})^{\varpi_\tau - \varpi_{\check{i}} - 1} \check{a}(\check{z}) d\check{z}.
\end{aligned} \tag{3.1}$$

$\Lambda([0,1],\mathbb{R}) = \{\mathfrak{f} | \mathfrak{f}\colon [0,1] \to \mathbb{R} \text{ is continuous}\}$ is a Banach space with the norm $\|\check{a}\| = \sup\{|\check{a}(\aleph)|, \aleph \in (0,1)\}$.

We require the following hypotheses in the sequel.

(A1) The map $\breve{u}\colon [0,1]\times\mathbb{R}\times\mathbb{R}\longmapsto\mathbb{R}$ is continuous.

(A2) For every $\aleph\in[0,1]$ and $\breve{a}_1,\breve{a}_2,\breve{y}_1,\breve{y}_2\in\mathbb{R}$, we can find $\mathcal{L}_1,\mathcal{L}_2>0$ such that

$$|\breve{u}(\aleph,\breve{a}_1,\breve{a}_2)-\breve{u}(\aleph,\breve{y}_1,\breve{y}_2)|\le\mathcal{L}_1\|\breve{a}_1-\breve{y}_1\|+\mathcal{L}_2\|\breve{a}_2-\breve{y}_2\|.$$

(A3) $|\breve{u}(\check{z},\breve{a}(\check{z}),\breve{a}(\breve{w}\check{z}))|\le|\psi_{\breve{u}}(\aleph)|$ for all $(\aleph,\breve{a}_1,\breve{a}_2)\in[0,1]\times\mathbb{R}\times\mathbb{R}$, where $\psi_{\breve{u}}\in\Lambda([0,1],\mathbb{R}^+)$.

(A4) $\zeta\in\Lambda([0,1],\mathbb{R}^+)$ be a non-decreasing function, we can find a positive constant $\breve{q}$ such that $\mathcal{I}^{\varpi_\tau}\zeta(\aleph)=\breve{q}\zeta(\aleph)$ and $\mathcal{I}^{\phi+\varpi_\tau}\zeta(\aleph)=(\breve{q}+1)\zeta(\aleph)$ $\forall\aleph\in[0,1]$.

In our first finding, we use the Banach fixed-point theorem to demonstrate the existence of a unique solution to the problem (1.1).

Theorem 3.1. *Let us assume (A1)-(A2) are fulfilled, then there is only one solution to the problem:* (1.1) *on* $[0,1]$*, given* $\theta<1$*, where*

$$\begin{aligned}\theta=&\frac{1}{|\gamma_1|}\frac{\mathcal{L}_3}{|\nu_\tau|\Gamma(\varpi_\tau+1)}\left(\frac{|\breve{b}|}{(\varpi_\tau+1)}+\sum_{\varrho=1}^{\breve{r}}|\breve{h}_\varrho|\pi_\varrho^{\varpi_\tau}+1+|\gamma_1|\aleph^{\varpi_\tau}\right)\\&+\frac{1}{|\gamma_1|}\frac{\mathcal{L}_3}{|\nu_\tau|\Gamma(\phi+\varpi_\tau+1)}\left(\frac{|\breve{b}|}{(\phi+\varpi_\tau+1)}+\sum_{\varrho=1}^{\breve{r}}|\breve{h}_\varrho|\pi_\varrho^{\phi+\varpi_\tau}+1+|\gamma_1|\aleph^{\phi+\varpi_\tau}\right)\\&+\sum_{\varrho=1}^{\breve{r}}\frac{|\breve{h}_\varrho|}{|\gamma_1|}\sum_{\breve{i}=1,\breve{i}\neq\tau}^{\breve{x}}\frac{|\nu_{\breve{i}}|}{|\nu_\tau|\Gamma(\varpi_\tau-\varpi_{\breve{i}}+1)}\pi_\varrho^{\varpi_\tau-\varpi_{\breve{i}}}\\&+\sum_{\breve{i}=1,\breve{i}\neq\tau}^{\breve{x}}\frac{|\nu_{\breve{i}}|}{|\nu_\tau|\Gamma(\varpi_\tau-\varpi_{\breve{i}}+1)}\left(\frac{1}{|\gamma_1|}+\frac{|\breve{b}|}{|\gamma_1|(\varpi_\tau-\varpi_{\breve{i}}+1)}+\aleph^{\varpi_\tau-\varpi_{\breve{i}}}\right)\end{aligned}\tag{3.2}$$

Proof. Define $\check{T}\colon\Lambda([0,1],\mathbb{R})\longmapsto\Lambda([0,1],\mathbb{R})$ is given by (3.1). Let $\breve{a}_1,\breve{a}_2\in\Lambda([0,1],\mathbb{R})$. Then,

$$\begin{aligned}&\|\check{T}\breve{a}_1(\aleph)-\check{T}\breve{a}_2(\aleph)\|\\&\le\sup_{\aleph\in[0,1]}\left\{\frac{|\aleph^{\breve{x}-1}|}{|\gamma_1|}\left[\frac{|\breve{b}|}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\varpi_\tau-1}|\breve{u}(\check{z},\breve{a}_1(\check{z}),\breve{a}_1(\phi\check{z}))-\breve{u}(\check{z},\breve{a}_2(\check{z}),\breve{a}_2(\phi\check{z}))|d\check{z}d\mathfrak{s}\right.\right.\\&+\frac{|\breve{b}|}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\phi+\varpi_\tau-1}|\breve{u}(\check{z},\breve{a}_1(\check{z}),\breve{a}_1(\phi\check{z}))-\breve{u}(\check{z},\breve{a}_2(\check{z}),\breve{a}_2(\phi\check{z}))|d\check{z}d\mathfrak{s}\\&+\sum_{\varrho=1}^{\breve{r}}\frac{|\breve{h}_\varrho|}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\check{z})^{\varpi_\tau-1}|\breve{u}(\check{z},\breve{a}_1(\check{z}),\breve{a}_1(\phi\check{z}))-\breve{u}(\check{z},\breve{a}_2(\check{z}),\breve{a}_2(\phi\check{z}))|d\check{z}\\&+\sum_{\varrho=1}^{\breve{r}}\frac{|\breve{h}_\varrho|}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\check{z})^{\phi+\varpi_\tau-1}|\breve{u}(\check{z},\breve{a}_1(\check{z}),\breve{a}_1(\phi\check{z}))-\breve{u}(\check{z},\breve{a}_2(\check{z}),\breve{a}_2(\phi\check{z}))|d\check{z}\\&+\frac{1}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^1(1-\check{z})^{\varpi_\tau-1}|\breve{u}(\check{z},\breve{a}_1(\check{z}),\breve{a}_1(\phi\check{z}))-\breve{u}(\check{z},\breve{a}_2(\check{z}),\breve{a}_2(\phi\check{z}))|d\check{z}\\&+\frac{1}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^1(1-\check{z})^{\phi+\varpi_\tau-1}|\breve{u}(\check{z},\breve{a}_1(\check{z}),\breve{a}_1(\phi\check{z}))-\breve{u}(\check{z},\breve{a}_2(\check{z}),\breve{a}_2(\phi\check{z}))|d\check{z}\end{aligned}$$

$$+\sum_{\varrho=1}^{\check{r}}|\check{h}_\varrho|\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_\tau|\Gamma(\varpi_\tau-\varpi_{\check{i}})}\int_0^{\pi_\varrho}(\pi_\varrho-\check{z})^{\varpi_\tau-\varpi_{\check{i}}-1}|\check{a}_1(\check{z})-\check{a}_2(\check{z})|d\check{z}$$

$$+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_\tau|\Gamma(\varpi_\tau-\varpi_{\check{i}})}\int_0^{1}(1-\check{z})^{\varpi_\tau-\varpi_{\check{i}}-1}|\check{a}_1(\check{z})-\check{a}_2(\check{z})|d\check{z}$$

$$+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}||\check{b}|}{|\nu_\tau|\Gamma(\varpi_\tau-\varpi_{\check{i}})}\int_0^{1}\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\varpi_\tau-\varpi_{\check{i}}-1}|\check{a}_1(\check{z})-\check{a}_2(\check{z})|d\check{z}d\mathfrak{s}\Bigg]$$

$$+\frac{1}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^{\aleph}(\aleph-\check{z})^{\varpi_\tau-1}|\check{u}(\check{z},\check{a}_1(\check{z}),\check{a}_1(\phi\check{z}))-\check{u}(\check{z},\check{a}_2(\check{z}),\check{a}_2(\phi\check{z}))|d\check{z}$$

$$+\frac{1}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^{\aleph}(\aleph-\check{z})^{\phi+\varpi_\tau-1}|\check{u}(\check{z},\check{a}_1(\check{z}),\check{a}_1(\phi\check{z}))-\check{u}(\check{z},\check{a}_2(\check{z}),\check{a}_2(\phi\check{z}))|d\check{z}$$

$$+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_\tau|\Gamma(\varpi_\tau-\varpi_{\check{i}})}\int_0^{\aleph}(\aleph-\check{z})^{\varpi_\tau-\varpi_{\check{i}}-1}|\check{a}_1(\check{z})-\check{a}_2(\check{z})|d\check{z}\Bigg\}$$

Using (A2) in the above inequality with $\mathcal{L}_3=\mathcal{L}_1+\mathcal{L}_2$, we obtain

$$\|\check{T}\check{a}_1(\aleph)-\check{T}\check{a}_2(\aleph)\|$$

$$\leq\|\check{a}_1-\check{a}_2\|\sup_{\aleph\in[0,1]}\Bigg\{\frac{1}{|\gamma_1|}\Bigg[\mathcal{L}_3\frac{|\check{b}|}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\varpi_\tau-1}d\check{z}d\mathfrak{s}$$

$$+\mathcal{L}_3\frac{|\check{b}|}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\phi+\varpi_\tau-1}d\check{z}d\mathfrak{s}$$

$$+\sum_{\varrho=1}^{\check{r}}\mathcal{L}_3\frac{|\check{h}_\varrho|}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\check{z})^{\varpi_\tau-1}d\check{z}$$

$$+\sum_{\varrho=1}^{\check{r}}\mathcal{L}_3\frac{|\check{h}_\varrho|}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\check{z})^{\phi+\varpi_\tau-1}d\check{z}$$

$$+\frac{\mathcal{L}_3}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^1(1-\check{z})^{\varpi_\tau-1}d\check{z}+\frac{\mathcal{L}_3}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^1(1-\check{z})^{\phi+\varpi_\tau-1}d\check{z}$$

$$+\sum_{\varrho=1}^{\check{r}}|\check{h}_\varrho|\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_\tau|\Gamma(\varpi_\tau-\varpi_{\check{i}})}\int_0^{\pi_\varrho}(\pi_\varrho-\check{z})^{\varpi_\tau-\varpi_{\check{i}}-1}d\check{z}$$

$$+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_\tau|\Gamma(\varpi_\tau-\varpi_{\check{i}})}\int_0^{1}(1-\check{z})^{\varpi_\tau-\varpi_{\check{i}}-1}d\check{z}$$

$$+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}||\check{b}|}{|\nu_\tau|\Gamma(\varpi_\tau-\varpi_{\check{i}})}\int_0^{1}\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\varpi_\tau-\varpi_{\check{i}}-1}d\check{z}d\mathfrak{s}\Bigg]$$

$$+\frac{\mathcal{L}_3}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^{\aleph}(\aleph-\check{z})^{\varpi_\tau-1}d\check{z}+\frac{\mathcal{L}_3}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^{\aleph}(\aleph-\check{z})^{\phi+\varpi_\tau-1}d\check{z}$$

$$+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_\tau|\Gamma(\varpi_\tau-\varpi_{\check{i}})}\int_0^{\aleph}(\aleph-\check{z})^{\varpi_\tau-\varpi_{\check{i}}-1}d\check{z}\Bigg\}$$

$$\leq\|\check{a}_1-\check{a}_2\|\Bigg[\frac{1}{|\gamma_1|}\frac{\mathcal{L}_3}{|\nu_\tau|\Gamma(\varpi_\tau+1)}\Bigg(\frac{|\check{b}|}{(\varpi_\tau+1)}+\sum_{\varrho=1}^{\check{r}}|\check{h}_\varrho|\pi_\varrho^{\varpi_\tau}+1+|\gamma_1|\Bigg)$$

$$+\frac{1}{|\gamma_1|}\frac{\mathcal{L}_3}{|\nu_\tau|\Gamma(\phi+\varpi_\tau+1)}\Bigg(\frac{|\check{b}|}{(\phi+\varpi_\tau+1)}+\sum_{\varrho=1}^{\check{r}}|\check{h}_\varrho|\pi_\varrho^{\phi+\varpi_\tau}+1+|\gamma_1|\Bigg)$$

$$+\sum_{\varrho=1}^{\check{r}}\frac{|\check{h}_\varrho|}{|\gamma_1|}\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_\tau|\Gamma(\varpi_\tau-\varpi_{\check{i}}+1)}\pi_\varrho^{\varpi_\tau-\varpi_{\check{i}}}$$

$$+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_\tau|\Gamma(\varpi_\tau-\varpi_{\check{i}}+1)}\Bigg(\frac{1}{|\gamma_1|}+\frac{|\check{b}|}{|\gamma_1|(\varpi_\tau-\varpi_{\check{i}}+1)}+1\Bigg)\Bigg]$$

$$=\theta\|\check{a}_1-\check{a}_2\|,$$

where (3.2) provides the value of θ. We prove $\check{T}$ is a contraction given the specified condition, i.e., $\theta<1$. Thus, by the Banach contraction mapping principle's end, the equation (1.1) on $[0,1]$ has a unique solution. □

The following theorem is based on the nonlinear alternative of $\mathcal{LST}$ [35].

Theorem 3.2. *Let us assume (A1) and (A3) hold, then we can find at least one solution to* (1.1) *on* $[0,1]$.

Proof. Let $\mathfrak{r}_0\in\mathbb{R}^+$. Consider a set $\mathcal{A}=\{\check{a}\in\Lambda([0,1],\mathbb{R})\colon\|\check{a}\|\leq\mathfrak{r}_0\}$. There will be several stages to finish the proof.

(1) We state that $\check{T}$ is uniformly bounded. Let $\check{a}\in\Lambda([0,1],\mathbb{R})$. Defined by

$$\|\check{T}\check{a}\|\leq|\check{a}_0|+\frac{1}{|\gamma_1|}\Bigg[\Bigg|\check{a}_0\Bigg(\check{b}-1-\sum_{\varrho=1}^{\check{r}}\check{h}_\varrho\Bigg)\Bigg|$$

$$+\frac{|\check{b}|}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\varpi_\tau-1}|\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}d\mathfrak{s}$$

$$+\frac{|\check{b}|}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\phi+\varpi_\tau-1}|\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}d\mathfrak{s}$$

$$+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_\tau|}\frac{|\check{b}|}{\Gamma(\varpi_\tau-\varpi_{\check{i}})}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\varpi_\tau-\varpi_{\check{i}}-1}|\check{a}(\check{z})|d\check{z}d\mathfrak{s}$$

$$+\sum_{\varrho=1}^{\check{r}}\frac{|\check{h}_\varrho|}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\check{z})^{\varpi_\tau-1}|\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}$$

$$+\sum_{\varrho=1}^{\check{r}}\frac{|\check{h}_\varrho|}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\check{z})^{\phi+\varpi_\tau-1}|\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}$$

$$+\sum_{\varrho=1}^{\check{r}}|\check{h}_{\varrho}|\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_{\tau}|}\frac{1}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}})}\int_{0}^{\pi_{\varrho}}(\pi_{\varrho}-\check{z})^{\varpi_{\tau}-\varpi_{\check{i}}-1}|\check{a}(\check{z})|d\check{z}$$

$$+\frac{1}{|\nu_{\tau}|\Gamma(\varpi_{\tau})}\int_{0}^{1}(1-\check{z})^{\varpi_{\tau}-1}|\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}$$

$$+\frac{1}{|\nu_{\tau}|\Gamma(\phi+\varpi_{\tau})}\int_{0}^{1}(1-\check{z})^{\phi+\varpi_{\tau}-1}|\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}$$

$$+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_{\tau}|}\frac{1}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}})}\int_{0}^{1}(1-\check{z})^{\varpi_{\tau}-\varpi_{\check{i}}-1}|\check{a}(\check{z})|d\check{z}\Bigg]$$

$$+\sup_{\aleph\in[0,1]}\left\{\frac{1}{|\nu_{\tau}|\Gamma(\varpi_{\tau})}\int_{0}^{\aleph}(\aleph-\check{z})^{\varpi_{\tau}-1}|\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}\right\}$$

$$+\sup_{\aleph\in[0,1]}\left\{\frac{1}{|\nu_{\tau}|\Gamma(\phi+\varpi_{\tau})}\int_{0}^{\aleph}(\aleph-\check{z})^{\phi+\varpi_{\tau}-1}|\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}\right\}$$

$$+\sup_{\aleph\in[0,1]}\left\{\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_{\tau}|}\frac{1}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}})}\int_{0}^{\aleph}(\aleph-\check{z})^{\varpi_{\tau}-\varpi_{\check{i}}-1}|\check{a}(\check{z})|d\check{z}\right\}. \tag{3.3}$$

Consider the set $\mathcal{A}$ and assume that (A3), (3.3) has follows:

$$\begin{aligned}\|\check{T}\check{a}\| \leq & |\check{a}_0|+\frac{1}{|\gamma_1|}\Bigg[\Bigg|\check{a}_0\Bigg(\check{b}-1-\sum_{\varrho=1}^{\check{r}}\check{h}_{\varrho}\Bigg)\Bigg|+\|\psi_{\check{u}}\|\Bigg[\frac{|\check{b}|}{|\nu_{\tau}|\Gamma(\varpi_{\tau}+2)}\\
&+\frac{|\check{b}|}{|\nu_{\tau}|\Gamma(\phi+\varpi_{\tau}+2)}+\sum_{\varrho=1}^{\check{r}}\frac{|\check{h}_{\varrho}|}{|\nu_{\tau}|\Gamma(\varpi_{\tau}+1)}\pi_{\varrho}^{\varpi_{\tau}}+\sum_{\varrho=1}^{\check{r}}\frac{|\check{h}_{\varrho}|}{|\nu_{\tau}|\Gamma(\phi+\varpi_{\tau}+1)}\pi_{\varrho}^{\phi+\varpi_{\tau}}\\
&+\frac{1}{|\nu_{\tau}||\gamma_1|\Gamma(\varpi_{\tau}+1)}+\frac{1}{|\nu_{\tau}||\gamma_1|\Gamma(\phi+\varpi_{\tau}+1)}+\frac{1}{|\nu_{\tau}|\Gamma(\varpi_{\tau}+1)}+\frac{1}{|\nu_{\tau}|\Gamma(\phi+\varpi_{\tau}+1)}\Bigg]\Bigg]\\
&+\mathfrak{r}_0\Bigg[\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_{\tau}||\gamma_1|}\frac{|\check{b}|}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}}+2)}+\sum_{\varrho=1}^{\check{r}}|\check{h}_{\varrho}|\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_{\tau}||\gamma_1|}\frac{1}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}}+1)}\pi_{\varrho}^{\varpi_{\tau}-\varpi_{\check{i}}}\\
&+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_{\tau}||\gamma_1|}\frac{1}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}}+1)}+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_{\tau}|}\frac{1}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}}+1)}\Bigg].\end{aligned}$$

(2) Now we state that $\check{T}$ is continuous. Let us construct a sequence $\check{a}_{\check{s}}\in\mathcal{A}$ that converges to $\check{a}$ and to prove $\check{T}\check{a}_{\check{s}}\longmapsto\check{T}\check{a}(\aleph)$ as $\check{s}\longmapsto\infty$. In order to achieve this, we take into account:

$$\begin{aligned}\|\check{T}\check{a}_{\check{s}}-\check{T}\check{a}\| \leq & \frac{1}{|\gamma_1|}\Bigg[\frac{|\check{b}|}{|\nu_{\tau}|\Gamma(\varpi_{\tau})}\int_{0}^{1}\int_{0}^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\varpi_{\tau}-1}|\check{u}(\check{z},\check{a}_{\check{s}}(\check{z}),\check{a}_{\check{s}}(\check{w}\check{z}))\\
&-\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}d\mathfrak{s}\\
&+\frac{|\check{b}|}{|\nu_{\tau}|\Gamma(\phi+\varpi_{\tau})}\int_{0}^{1}\int_{0}^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\phi+\varpi_{\tau}-1}|\check{u}(\check{z},\check{a}_{\check{s}}(\check{z}),\check{a}_{\check{s}}(\check{w}\check{z}))\\
&-\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}d\mathfrak{s}\end{aligned}$$

$$+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_{\tau}|}\frac{|\check{b}|}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}})}\int_{0}^{1}\int_{0}^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\varpi_{\tau}-\varpi_{\check{i}}-1}|\check{a}_{\check{s}}(\check{z})-\check{a}(\check{z})|d\check{z}d\mathfrak{s}$$

$$+\sum_{\varrho=1}^{\check{r}}\frac{|\check{h}_{\varrho}|}{|\nu_{\tau}|\Gamma(\varpi_{\tau})}\int_{0}^{\pi_{\varrho}}(\pi_{\varrho}-\check{z})^{\varpi_{\tau}-1}|\check{u}(\check{z},\check{a}_{\check{s}}(\check{z}),\check{a}_{\check{s}}(\check{w}\check{z}))$$

$$-\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}$$

$$+\sum_{\varrho=1}^{\check{r}}\frac{|\check{h}_{\varrho}|}{|\nu_{\tau}|\Gamma(\phi+\varpi_{\tau})}\int_{0}^{\pi_{\varrho}}(\pi_{\varrho}-\check{z})^{\phi+\varpi_{\tau}-1}|\check{u}(\check{z},\check{a}_{\check{s}}(\check{z}),\check{a}_{\check{s}}(\check{w}\check{z}))$$

$$-\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}$$

$$+\sum_{\varrho=1}^{\check{r}}|\check{h}_{\varrho}|\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_{\tau}|}\frac{1}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}})}\int_{0}^{\pi_{\varrho}}(\pi_{\varrho}-\check{z})^{\varpi_{\tau}-\varpi_{\check{i}}-1}|\check{a}_{\check{s}}(\check{z})-\check{a}(\check{z})|d\check{z}$$

$$+\frac{1}{|\nu_{\tau}|\Gamma(\varpi_{\tau})}\int_{0}^{1}(1-\check{z})^{\varpi_{\tau}-1}|\check{u}(\check{z},\check{a}_{\check{s}}(\check{z}),\check{a}_{\check{s}}(\check{w}\check{z}))-\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}$$

$$+\frac{1}{|\nu_{\tau}|\Gamma(\phi+\varpi_{\tau})}\int_{0}^{1}(1-\check{z})^{\phi+\varpi_{\tau}-1}|\check{u}(\check{z},\check{a}_{\check{s}}(\check{z}),\check{a}_{\check{s}}(\check{w}\check{z}))$$

$$-\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}$$

$$+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_{\tau}|}\frac{1}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}})}\int_{0}^{1}(1-\check{z})^{\varpi_{\tau}-\varpi_{\check{i}}-1}|\check{a}_{\check{s}}(\check{z})-\check{a}(\check{z})|d\check{z}\Bigg]$$

$$+\sup_{\aleph\in[0,1]}\Bigg\{\frac{1}{|\nu_{\tau}|\Gamma(\varpi_{\tau})}\int_{0}^{\aleph}(\aleph-\check{z})^{\varpi_{\tau}-1}|\check{u}(\check{z},\check{a}_{\check{s}}(\check{z}),\check{a}_{\check{s}}(\check{w}\check{z}))$$

$$-\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}\Bigg\}$$

$$+\sup_{\aleph\in[0,1]}\Bigg\{\frac{1}{|\nu_{\tau}|\Gamma(\phi+\varpi_{\tau})}\int_{0}^{\aleph}(\aleph-\check{z})^{\phi+\varpi_{\tau}-1}|\check{u}(\check{z},\check{a}_{\check{s}}(\check{z}),\check{a}_{\check{s}}(\check{w}\check{z}))$$

$$-\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))|d\check{z}\Bigg\}$$

$$+\sup_{\aleph\in[0,1]}\Bigg\{\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}}\frac{|\nu_{\check{i}}|}{|\nu_{\tau}|}\frac{1}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}})}\int_{0}^{\aleph}(\aleph-\check{z})^{\varpi_{\tau}-\varpi_{-}1}|\check{a}_{\check{s}}(\check{z})-\check{a}(\check{z})|d\check{z}\Bigg\}.$$

Thus, we get that according to the Lebesgue dominated convergent theorem as, $\|\check{T}\check{a}_{\check{s}}-\check{T}\check{a}\|\longmapsto 0$ as $\check{s}\longmapsto\infty$.

(3) We establish the bounded set is mapped into equicontinuous functions by $\wp$. Let $\aleph_1 \leq \aleph_2$, if follows from (A3) that

$$
\begin{aligned}
|\check{T}\breve{a}(\aleph_1) - \check{T}\breve{a}(\aleph_2)| \leq & \frac{|\aleph_1^{\check{x}-1} - \aleph_2^{\check{x}-1}|}{|\gamma_1|}\Bigg[\Bigg|\breve{a}_0\Bigg(\check{b} - 1 - \sum_{\varrho=1}^{\check{r}} \check{h}_\varrho\Bigg)\Bigg| \\
& + \|\psi_{\breve{u}}\| \frac{|\check{b}|}{|\nu_\tau|\Gamma(\varpi_\tau + 2)} + \|\psi_{\breve{u}}\| \frac{|\check{b}|}{|\nu_\tau|\Gamma(\phi + \varpi_\tau + 2)} \\
& + \mathfrak{r}_0 \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{|\nu_{\check{i}}|}{|\nu_\tau|} \frac{|\check{b}|}{\Gamma(\varpi_\tau - \varpi_{\check{i}} + 2)} + \|\psi_{\breve{u}}\| \sum_{\varrho=1}^{\check{r}} \frac{|\check{h}_\varrho|}{|\nu_\tau|\Gamma(\varpi_\tau + 1)} \pi_\varrho^{\varpi_\tau} \\
& + \|\psi_{\breve{u}}\| \sum_{\varrho=1}^{\check{r}} \frac{|\check{h}_\varrho|}{|\nu_\tau|\Gamma(\phi + \varpi_\tau + 1)} \pi_\varrho^{\phi+\varpi_\tau} \\
& + \mathfrak{r}_0 \sum_{\varrho=1}^{\check{r}} |\check{h}_\varrho| \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{|\nu_{\check{i}}|}{|\nu_\tau|} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}} + 1)} \pi_\varrho^{\varpi_\tau - \varpi_{\check{i}}} \\
& + \|\psi_{\breve{u}}\| \frac{|\check{h}_\varrho|}{|\nu_\tau|\Gamma(\varpi_\tau + 1)} + \|\psi_{\breve{u}}\| \frac{|\check{h}_\varrho|}{|\nu_\tau|\Gamma(\phi + \varpi_\tau + 1)} \\
& + \mathfrak{r}_0 \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{|\nu_{\check{i}}|}{|\nu_\tau|} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}} + 1)}\Bigg] \\
& + \|\psi_{\breve{u}}\| \frac{|\check{h}_\varrho|}{|\nu_\tau|\Gamma(\varpi_\tau + 1)}\Big[\aleph_2^{\varpi_\tau} - (\aleph_2 - \aleph_1)^{\varpi_\tau} - \aleph_1^{\varpi_\tau} + (\aleph_2 - \aleph_1)^{\varpi_\tau}\Big] \\
& + \|\psi_{\breve{u}}\| \frac{|\check{h}_\varrho|}{|\nu_\tau|\Gamma(\phi + \varpi_\tau + 1)}\Big[\aleph_2^{\phi+\varpi_\tau} - (\aleph_2 - \aleph_1)^{\phi+\varpi_\tau} - \aleph_1^{\phi+\varpi_\tau} + (\aleph_2 - \aleph_1)^{\phi+\varpi_\tau}\Big] \\
& + \mathfrak{r}_0 \sum_{\check{i}=1, \check{i}\neq\tau}^{\check{x}} \frac{|\nu_{\check{i}}|}{|\nu_\tau|} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\check{i}} + 1)}\Big[\aleph_2^{\varpi_\tau - \varpi_{\check{i}}} - (\aleph_2 - \aleph_1)^{\varpi_\tau - \varpi_{\check{i}}} - \aleph_1^{\varpi_\tau - \varpi_{\check{i}}} \\
& + (\aleph_2 - \aleph_1)^{\varpi_\tau - \varpi_{\check{i}}}\Big].
\end{aligned}
$$

Clearly, $\|\check{T}\breve{a}(\aleph_1) - \check{T}\breve{a}(\aleph_2)\| \to 0$ as $\aleph_1 \to \aleph_2$.

(4) Now, we claim that we can find an open set $\mathcal{U} \subseteq \Lambda([0,1], \mathbb{R})$ satisfying $\breve{a} \neq \lambda_* \check{T}(\breve{a}(\aleph))$ for $\lambda_* \in (0,1)$ and $\breve{a} \in \partial\mathcal{U}$. For this, let $\breve{a} \in \Lambda([0,1], \mathbb{R})$ be such that $\breve{a} = \lambda_* \check{T}(\breve{a}(\aleph))$ for $\lambda_* \in (0,1)$. Then, for $\aleph \in [0,1]$, we derive that

$$
\begin{aligned}
|\breve{a}(\aleph)| = & |\lambda_* \check{T}\breve{a}(\aleph)| \\
\leq & |\breve{a}_0| + \frac{1}{|\gamma_1|}\Bigg|\breve{a}_0\Bigg(\check{b} - 1 - \sum_{\varrho=1}^{\check{r}} \check{h}_\varrho\Bigg)\Bigg| \\
& + \|\psi_{\breve{u}}\|\Bigg[\frac{|\check{b}|}{|\nu_\tau|\Gamma(\varpi_\tau + 2)} + \frac{|\check{b}|}{|\nu_\tau|\Gamma(\phi + \varpi_\tau + 2)}
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{\varrho=1}^{\check{u}} \frac{|\check{h}_{\varrho}|}{|\nu_{\tau}||\gamma_1|\Gamma(\varpi_{\tau}+1)}\pi_{\varrho}^{\varpi_{\tau}} \\
&+\sum_{\varrho=1}^{\check{u}} \frac{|\check{h}_{\varrho}|}{|\nu_{\tau}||\gamma_1|\Gamma(\phi+\varpi_{\tau}+1)}\pi_{\varrho}^{\phi+\varpi_{\tau}} \\
&+\frac{1}{|\nu_{\tau}||\gamma_1|\Gamma(\varpi_{\tau}+1)}+\frac{1}{|\nu_{\tau}||\gamma_1|\Gamma(\phi+\varpi_{\tau}+1)} \\
&+\frac{1}{|\nu_{\tau}|\Gamma(\varpi_{\tau}+1)}+\frac{1}{|\nu_{\tau}|\Gamma(\phi+\varpi_{\tau}+1)}\Bigg] \\
&+\mathfrak{r}_0\Bigg[\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}} \frac{|\nu_{\check{i}}|}{|\nu_{\tau}||\gamma_1|}\frac{|\check{b}|}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}}+2)} \\
&+\sum_{\varrho=1}^{\check{r}} |\check{h}_{\varrho}| \sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}} \frac{|\nu_{\check{i}}|}{|\nu_{\tau}||\gamma_1|}\frac{1}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}}+1)}\pi_{\varrho}^{\varpi_{\tau}-\varpi_{\check{i}}} \\
&+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}} \frac{|\nu_{\check{i}}|}{|\nu_{\tau}||\gamma_1|}\frac{1}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}}+1)} \\
&+\sum_{\check{i}=1,\check{i}\neq\tau}^{\check{x}} \frac{|\nu_{\check{i}}|}{|\nu_{\tau}|}\frac{1}{\Gamma(\varpi_{\tau}-\varpi_{\check{i}}+1)}\Bigg]=\mathcal{M}.
\end{aligned}
$$

Set $\mathcal{U}=\{\check{a}\in\Lambda([0,1],\mathbb{R})\colon \|\check{a}\|\leq\mathcal{M}+1\}$. Hence, for any $\lambda_*\in(0,1)$, there is no $\check{a}\in\partial\mathcal{U}$ satisfying $\check{a}=\lambda_*\check{T}\check{a}(\aleph)$. Therefore, $\check{T}$ has at least one $\mathcal{FP}$ in $\bar{\mathcal{U}}$. □

4. Stability results

In this part, we explore the stability criteria, includes $\mathcal{UH}$, $\mathcal{GUH}$, $\mathcal{UHR}$, and $\mathcal{GUHR}$stability, equation (1.1).

Definition 4.1. *The solution of our considered equation* (1.1) *is said to be a $\mathcal{UH}$-stable if there exist constants $\mathcal{A}_1\geq 0$ and $\eta\geq 0$ which imlies $\check{a}\in\Lambda([0,1],\mathbb{R})$ such that:*

$$
\begin{aligned}
&\left|\sum_{\check{i}=1}^{\check{x}} \nu_{\check{i}}^{\mathcal{R}}\ \mathcal{D}^{\varpi_{\check{i}}}\check{a}(\aleph)-\check{u}(\aleph,\check{a}(\aleph),\check{a}(\check{w}\aleph))-\int_0^{\aleph}\frac{(\aleph-\check{z})^{\phi-1}\check{u}(\check{z},\check{a}(\check{z}),\check{a}(\check{w}\check{z}))}{\Gamma(\phi)}d\check{z}\right|\leq\eta, \\
&\text{where}\quad \aleph\in[0,1],\ \nu_{\check{i}}\in\mathbb{R},
\end{aligned}
\tag{4.1}
$$

we have that $|\check{a}-\check{a}^*|\leq\mathcal{A}_1\eta$, where $\check{a}^*(\aleph)$ is a unique solution of equation (1.1)*. Additionally, if thre exist a map $\kappa\colon\mathbb{R}^+\longmapsto\mathbb{R}^+$ with $\kappa(0)=0$ satisfying $|\check{a}-\check{a}^*|\leq\mathcal{A}_1\kappa(\eta)$, then the solution $\check{a}(\aleph)$ of equation* (1.1) *is said to be a $\mathcal{GUH}$-stable.*

Definition 4.2. *The solution of our considered equation* (1.1) *is $\mathcal{UHR}$-stable and $\rho \in \mathcal{U}$ be a continuous function if we can find a constants $\psi, \mathcal{A}_2 > 0$ and $\eta > 0$ which implies $\breve{a} \in \Lambda([0,1],\mathbb{R})$ such that:*

$$\Bigg|\sum_{\check{i}=1}^{\check{x}} \nu_{\check{i}}^{\mathcal{R}}\ \mathcal{D}^{\varpi_{\check{i}}}\breve{a}(\aleph) - \breve{u}(\aleph,\breve{a}(\aleph),\breve{a}(\breve{w}\aleph)) - \int_0^{\aleph} \frac{(\aleph-\check{z})^{\phi-1}\breve{u}(\check{z},\breve{a}(\check{z}),\breve{a}(\breve{w}\check{z}))}{\Gamma(\phi)}d\check{z}\Bigg| \leq (\rho(\aleph)+\psi)\eta, \tag{4.2}$$

we have that $|\breve{a}-\breve{a}^*| \leq \mathcal{A}_2(\rho(\aleph)+\psi)\eta$, where $\breve{a}^* \in \Lambda([0,1],\mathbb{R})$ is a unique solution.

Definition 4.3. *The solution of our considered equation* (1.1) *is $\mathcal{GUHR}$-stable and $\rho \in \mathcal{U}$ be a continuous function and ψ be a positive constant, if there exists $\mathcal{A}_2 > 0$ is constant which implies $\breve{a} \in \Lambda([0,1],\mathbb{R})$ such that:*

$$\Bigg|\sum_{\check{i}=1}^{\check{x}} \nu_{\check{i}}^{\mathcal{R}}\ \mathcal{D}^{\varpi_{\check{i}}}\breve{a}(\aleph) - \breve{u}(\aleph,\breve{a}(\aleph),\breve{a}(\breve{w}\aleph)) - \int_0^{\aleph} \frac{(\aleph-\check{z})^{\phi-1}\breve{u}(\check{z},\breve{a}(\check{z}),\breve{a}(\breve{w}\check{z}))}{\Gamma(\phi)}d\check{z}\Bigg| \leq \rho(\aleph)+\psi, \tag{4.3}$$

we have that $|\breve{a}-\breve{a}^*| \leq \mathcal{A}_2\rho(\aleph)\eta$, where $\breve{a}^* \in \Lambda([0,1],\mathbb{R})$ is a unique solution.

Remark 4.1. *Let $\breve{a} \in \Lambda([0,1],\mathbb{R})$ be the solution of equation* (4.1) *iff we can find a $\zeta \in \Lambda([0,1],\mathbb{R})$, depends on $\breve{a}$, satisfying*

(1) $\eta \geq \zeta(\aleph)$, *where* $\aleph \in [0,1]$,
(2) $\sum_{\check{i}=1}^{\check{x}} \nu_{\check{i}}^{\mathcal{R}}\ \mathcal{D}^{\varpi_{\check{i}}}\breve{a}(\aleph) - \breve{u}(\aleph,\breve{a}(\aleph),\breve{a}(\breve{w}\aleph)) - \int_0^{\aleph} \frac{(\aleph-\check{z})^{\phi-1}\breve{u}(\check{z},\breve{a}(\check{z}),\breve{a}(\breve{w}\check{z}))}{\Gamma(\phi)}d\check{z} - \zeta(\aleph) = 0.$

Remark 4.2. *Let $\breve{a} \in \Lambda([0,1],\mathbb{R})$ be the solution of equation* (4.2) *iff we can find a $\zeta \in \Lambda([0,1],\mathbb{R})$, depends on $\breve{a}$, satisfying*

(1) $\zeta(\aleph) \leq \rho(\aleph)\eta$ *and* $\zeta(\aleph) \leq \psi\eta, \quad \aleph \in [0,1]$,
(2) $\sum_{\check{i}=1}^{\check{x}} \nu_{\check{i}}^{\mathcal{R}}\ \mathcal{D}^{\varpi_{\check{i}}}\breve{a}(\aleph) - \breve{u}(\aleph,\breve{a}(\aleph),\breve{a}(\breve{w}\aleph)) - \int_0^{\aleph} \frac{(\aleph-\check{z})^{\phi-1}\breve{u}(\check{z},\breve{a}(\check{z}),\breve{a}(\breve{w}\check{z}))}{\Gamma(\phi)}d\check{z} - \zeta(\aleph) = 0.$

Remark 4.3. *Let $\breve{a} \in \Lambda([0,1],\mathbb{R})$ be the solution of equation* (4.3) *iff we can find a $\zeta \in \Lambda([0,1],\mathbb{R})$, depends on $\breve{a}$, satisfying*

(1) $\zeta(\aleph) \leq \rho(\aleph)$ *and* $\rho \leq \psi, \quad \aleph \in [0,1]$,
(2) $\sum_{\check{i}=1}^{\check{x}} \nu_{\check{i}}^{\mathcal{R}}\ \mathcal{D}^{\varpi_{\check{i}}}\breve{a}(\aleph) - \breve{u}(\aleph,\breve{a}(\aleph),\breve{a}(\breve{w}\aleph)) - \int_0^{\aleph} \frac{(\aleph-\check{z})^{\phi-1}\breve{u}(\check{z},\breve{a}(\check{z}),\breve{a}(\breve{w}\check{z}))}{\Gamma(\phi)}d\check{z} - \zeta(\aleph) = 0.$

Lemma 4.1. *Let $\breve{a} \in \Lambda([0,1],\mathbb{R})$ be a solution satisfies as follows:*

$$\begin{cases} \sum_{\check{i}=1}^{\check{x}} \nu_{\check{i}}^{\mathcal{R}}\ \mathcal{D}^{\varpi_{\check{i}}}\breve{a}(\aleph) = \breve{u}(\aleph,\breve{a}(\aleph),\breve{a}(\breve{w}\aleph)) + \int_0^{\aleph} \frac{(\aleph-\check{z})^{\phi-1}\breve{u}(\check{z},\breve{a}(\check{z}),\breve{a}(\breve{w}\check{z}))}{\Gamma(\phi)}d\check{z} + \zeta(\aleph), \\ \check{x} - \check{i} < \varpi_{\check{i}} \leq \check{x} + 1 - \check{i},\ \breve{w} \in (0,1), \phi \in (0,1], \nu_{\check{i}} \in \mathbb{R}, \aleph \in [0,1], \\ \breve{a}(0) = \breve{a}_0,\ \frac{d^{\varrho}\breve{a}(0)}{d\aleph^{\varrho}} = 0, \\ \breve{a}(1) = \check{b}\int_0^1 \breve{a}(\mathfrak{s})d\mathfrak{s} - \sum_{\varrho=1}^{\check{r}} \check{h}_{\varrho}\breve{a}(\pi_{\varrho}),\ \check{h}_{\varrho} \in \mathbb{R}, \\ \pi_{\varrho} \in (0,1), \varrho = 1,2,\cdots,\check{x}-2\ \&\ \check{i} = 1,\cdots,\check{r}. \end{cases}$$

Then, $\breve{a}$ satisfies the following relation:

$$|\breve{a}(\aleph) - \Im\breve{a}(\aleph)| \leq \mathcal{A}_1\eta,$$

where

$$\begin{aligned}
\Im\breve{a}(\aleph) =& \breve{a}_0 + \frac{\aleph^{\breve{x}-1}}{\gamma_1}\Bigg[\breve{a}_0\Bigg(\breve{b} - 1 - \sum_{\varrho=1}^{\breve{r}} \breve{h}_\varrho\Bigg) \\
&+ \frac{\breve{b}}{\nu_\tau\Gamma(\varpi_\tau)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \breve{z})^{\varpi_\tau - 1}\breve{u}(\breve{z}, \breve{y}(\breve{z}), \breve{y}(\breve{w}\breve{z}))d\breve{z}d\mathfrak{s} \\
&+ \frac{\breve{b}}{\nu_\tau\Gamma(\phi + \varpi_\tau)} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \breve{z})^{\phi+\varpi_\tau - 1}\breve{u}(\breve{z}, \breve{y}(\breve{z}), \breve{y}(\breve{w}\breve{z}))d\breve{z}d\mathfrak{s} \\
&- \sum_{\breve{i}=1, \breve{i}\neq\tau}^{\breve{x}} \frac{\nu_{\breve{i}}}{\nu_\tau} \frac{\breve{b}}{\Gamma(\varpi_\tau - \varpi_{\breve{i}})} \int_0^1 \int_0^{\mathfrak{s}} (\mathfrak{s} - \breve{z})^{\varpi_\tau - \varpi_{\breve{i}} - 1}\breve{a}(\breve{z})d\breve{z}d\mathfrak{s} \\
&- \sum_{\varrho=1}^{\breve{r}} \frac{\breve{h}_\varrho}{\nu_\tau\Gamma(\varpi_\tau)} \int_0^{\pi_\varrho} (\pi_\varrho - \breve{z})^{\varpi_\tau - 1}\breve{u}(\breve{z}, \breve{y}(\breve{z}), \breve{y}(\breve{w}\breve{z}))d\breve{z} \\
&- \sum_{\varrho=1}^{\breve{r}} \frac{\breve{h}_\varrho}{\nu_\tau\Gamma(\phi + \varpi_\tau)} \int_0^{\pi_\varrho} (\pi_\varrho - \breve{z})^{\phi+\varpi_\tau - 1}\breve{u}(\breve{z}, \breve{y}(\breve{z}), \breve{y}(\breve{w}\breve{z}))d\breve{z} \\
&+ \sum_{\varrho=1}^{\breve{r}} \breve{h}_\varrho \sum_{\breve{i}=1, \breve{i}\neq\tau}^{\breve{x}} \frac{\nu_{\breve{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\breve{i}})} \int_0^{\pi_\varrho} (\pi_\varrho - \breve{z})^{\varpi_\tau - \varpi_{\breve{i}} - 1}\breve{a}(\breve{z})d\breve{z} \\
&- \frac{1}{\nu_\tau\Gamma(\varpi_\tau)} \int_0^1 (1 - \breve{z})^{\varpi_\tau - 1}\breve{u}(\breve{z}, \breve{y}(\breve{z}), \breve{y}(\breve{w}\breve{z}))d\breve{z} \\
&- \frac{1}{\nu_\tau\Gamma(\phi + \varpi_\tau)} \int_0^1 (1 - \breve{z})^{\phi+\varpi_\tau - 1}\breve{u}(\breve{z}, \breve{y}(\breve{z}), \breve{y}(\breve{w}\breve{z}))d\breve{z} \\
&+ \sum_{\breve{i}=1, \breve{i}\neq\tau}^{\breve{x}} \frac{\nu_{\breve{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\breve{i}})} \int_0^1 (1 - \breve{z})^{\varpi_\tau - \varpi_{\breve{i}} - 1}\breve{a}(\breve{z})d\breve{z}\Bigg] \\
&+ \frac{1}{\nu_\tau} \int_0^{\aleph} \frac{(\aleph - \breve{z})^{\varpi_\tau - 1}}{\Gamma(\varpi_\tau)}\breve{u}(\breve{z}, \breve{y}(\breve{z}), \breve{y}(\breve{w}\breve{z}))d\breve{z} \\
&+ \frac{1}{\nu_\tau} \int_0^{\aleph} \frac{(\aleph - \breve{z})^{\phi+\varpi_\tau - 1}\breve{u}(\breve{z}, \breve{y}(\breve{z}), \breve{y}(\breve{w}\breve{z}))}{\Gamma(\phi + \varpi_\tau)}d\breve{z} \\
&- \sum_{\breve{i}=1, \breve{i}\neq\tau}^{\breve{x}} \frac{\nu_{\breve{i}}}{\nu_\tau} \frac{1}{\Gamma(\varpi_\tau - \varpi_{\breve{i}})} \int_0^{\aleph} (\aleph - \breve{z})^{\varpi_\tau - \varpi_{\breve{i}} - 1}\breve{a}(\breve{z})d\breve{z},
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{A}_1 =& \frac{1}{|\gamma_1||\nu_\tau|\Gamma(\varpi_\tau + 1)}\Bigg(\frac{|\breve{b}|}{(\varpi_\tau + 1)} + \sum_{\varrho=1}^{\breve{r}} |\breve{h}_\varrho|\pi_\varrho^{\varpi_\tau} + 1 + |\gamma_1|\Bigg) \\
&+ \frac{1}{|\gamma_1||\nu_\tau|\Gamma(\phi + \varpi_\tau + 1)}\Bigg(\frac{|\breve{b}|}{(\phi + \varpi_\tau + 1)} + \sum_{\varrho=1}^{\breve{r}} |\breve{h}_\varrho|\pi_\varrho^{\phi+\varpi_\tau} + 1 + |\gamma_1|\Bigg). \qquad (4.4)
\end{aligned}$$

Proof. Assume that $\check{a} \in \Lambda([0,1],\mathbb{R})$ then,

$$\begin{aligned}\check{a}(\aleph) =&\Im\check{a}(\aleph) + \frac{\aleph^{\check{x}-1}}{\gamma_1}\Big[\frac{\check{b}}{\nu_\tau\Gamma(\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\varpi_\tau-1}\zeta(\check{z})d\check{z}d\mathfrak{s}\\
&+\frac{\check{b}}{\nu_\tau\Gamma(\phi+\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\phi+\varpi_\tau-1}\zeta(\check{z})d\check{z}d\mathfrak{s}\\
&-\sum_{\varrho=1}^{\check{r}}\frac{\check{h}_\varrho}{\nu_\tau\Gamma(\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\check{z})^{\varpi_\tau-1}\zeta(\check{z})d\check{z}\\
&-\sum_{\varrho=1}^{\check{r}}\frac{\check{h}_\varrho}{\nu_\tau\Gamma(\phi+\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\check{z})^{\phi+\varpi_\tau-1}\zeta(\check{z})d\check{z}\\
&-\frac{1}{\nu_\tau\Gamma(\varpi_\tau)}\int_0^1(1-\check{z})^{\varpi_\tau-1}\zeta(\check{z})d\check{z}\\
&-\frac{1}{\nu_\tau\Gamma(\phi+\varpi_\tau)}\int_0^1(1-\check{z})^{\phi+\varpi_\tau-1}\zeta(\check{z})d\check{z}\Big]\\
&+\frac{1}{\nu_\tau\Gamma(\varpi_\tau)}\int_0^{\aleph}(\aleph-\check{z})^{\varpi_\tau-1}\zeta(\check{z})d\check{z}\\
&+\frac{1}{\nu_\tau\Gamma(\phi+\varpi_\tau)}\int_0^{\aleph}(\aleph-\check{z})^{\phi+\varpi_\tau-1}\zeta(\check{z})d\check{z}.\end{aligned}$$

With the use of Remark 4.1 in (4.5) and the computation technique used to validate the findings of the previous section, we obtain

$$\begin{aligned}|\check{a}(\aleph)-\Im\check{a}(\aleph)| \leq&\eta\sup_{\aleph\in[0,1]}\Big\{\frac{1}{|\gamma_1|}\Big[\frac{|\check{b}|}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\varpi_\tau-1}d\check{z}d\mathfrak{s}\\
&+\frac{|\check{b}|}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\check{z})^{\phi+\varpi_\tau-1}d\check{z}d\mathfrak{s}\\
&+\sum_{\varrho=1}^{\check{r}}\frac{|\check{h}_\varrho|}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\check{z})^{\varpi_\tau-1}d\check{z}\\
&+\sum_{\varrho=1}^{\check{r}}\frac{|\check{h}_\varrho|}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\check{z})^{\phi+\varpi_\tau-1}d\check{z}\\
&+\frac{1}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^1(1-\check{z})^{\varpi_\tau-1}d\check{z}\\
&+\frac{1}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^1(1-\check{z})^{\phi+\varpi_\tau-1}d\check{z}\Big]\\
&+\frac{1}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^{\aleph}(\aleph-\check{z})^{\varpi_\tau-1}d\check{z}\\
&+\frac{1}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^{\aleph}(\aleph-\check{z})^{\phi+\varpi_\tau-1}d\check{z}\Big\}\end{aligned}$$

$$|\breve{a}(\aleph) - \Im\breve{a}(\aleph)| \leq \frac{\eta}{|\gamma_1||\nu_\tau|\Gamma(\varpi_\tau + 1)}\left(\frac{|\breve{b}|}{(\varpi_\tau + 1)} + \sum_{\varrho=1}^{\breve{r}} |\breve{h}_\varrho|\pi_\varrho^{\varpi_\tau} + 1 + |\gamma_1|\right)$$
$$+ \frac{\eta}{|\gamma_1||\nu_\tau|\Gamma(\phi + \varpi_\tau + 1)}\left(\frac{|\breve{b}|}{(\phi + \varpi_\tau + 1)} + \sum_{\varrho=1}^{\breve{r}} |\breve{h}_\varrho|\pi_\varrho^{\phi+\varpi_\tau} + 1 + |\gamma_1|\right). \tag{4.5}$$

Applying (4.4) in (4.5), we obtian

$$|\breve{a}(\aleph) - \Im\breve{a}(\aleph)| \leq \mathcal{A}_1\eta,$$

this completes the proof. □

Theorem 4.1. *Let us assume (A1) and (A4), of equation* (1.1) *is $\mathcal{UH}$-stable and $\mathcal{GUH}$-stable if $\theta < 1$, where θ value is in* (3.2)*.*

Proof. Let us assume $\breve{a} \in \Lambda([0,1],\mathbb{R})$ and $\breve{a}^*$ be a unique solution of equation (1.1), thus

$$\begin{aligned}\|\breve{a} - \breve{a}^*\| &= \|\breve{a}(\aleph) - \Im\breve{a}^*(\aleph)\| = \|\breve{a}(\aleph) - \Im\breve{a}(\aleph) + \Im\breve{a}(\aleph) - \Im\breve{a}^*(\aleph)\| \\ &\leq \|\breve{a}(\aleph) - \Im\breve{a}(\aleph)\| + \|\Im\breve{a}(\aleph) - \Im\breve{a}^*(\aleph)\|.\end{aligned}$$

By Lemma 4.1 and Theorem 3.1, along with the computation technique used to generate the results of the preceding section, allow us to deduce that

$$\|\breve{a}(\aleph) - \breve{a}^*(\aleph)\| \leq \mathcal{A}_1\eta + \theta\|\breve{a} - \breve{a}^*\|,$$

it is also possible to write as

$$\|\breve{a} - \breve{a}^*\| \leq \frac{\mathcal{A}_1}{1-\theta}\eta.$$

Let $\mathcal{B}_1 = \frac{\mathcal{A}_1}{1-\theta}$, then the solution of the equation (1.1) is $\mathcal{UH}$-stable. Moreover, if we take $\kappa(\eta) = \eta$, then the equation (1.1) is $\mathcal{GUH}$-stable. □

Lemma 4.2. *Let us assume* ((A4)) *holds, then any solution $\breve{a} \in \Lambda([0,1],\mathbb{R})$ such that*

$$\begin{cases}\sum_{\breve{i}=1}^{\breve{x}} \nu_{\breve{i}}^{\Lambda}\ \mathcal{D}^{\varpi_{\breve{i}}}\breve{a}(\aleph) = \breve{u}(\aleph, \breve{a}(\aleph), \breve{a}(\breve{w}\aleph)) + \int_0^{\aleph} \frac{(\aleph-\breve{z})^{\phi-1}\breve{u}(\breve{z},\breve{a}(\breve{z}),\breve{a}(\breve{w}\breve{z}))}{\Gamma(\phi)}d\breve{z} + \zeta(\aleph), \\ \breve{x} - \breve{i} < \varpi_{\breve{i}} \leq \breve{x} + 1 - \breve{i},\ \breve{w} \in (0,1), \phi \in (0,1], \nu_{\breve{i}} \in \mathbb{R}, \aleph \in [0,1], \\ \breve{a}(0) = \breve{a}_0,\ \frac{d^{\varrho}\breve{a}(0)}{d\aleph^{\varrho}} = 0, \\ \breve{a}(1) = \breve{b}\int_0^1 \breve{a}(\mathfrak{s})d\mathfrak{s} - \sum_{\varrho=1}^{\breve{r}} \breve{h}_\varrho \breve{a}(\pi_\varrho),\ \breve{h}_\varrho \in \mathbb{R},\ \pi_\varrho \in (0,1), \\ \varrho = 1, 2, \cdots, \breve{x} - 2\ \&\ \breve{i} = 1, \cdots, \breve{r}.\end{cases}$$

satisfies the following relation:

$$|\breve{a}(\aleph)-\Im\breve{a}(\aleph)|\leq\eta\Bigg(\Bigg(\frac{|\breve{b}|(\varpi_\tau+1)^{-1}+\sum_{\varrho=1}^{\breve{r}}|\breve{h}_\varrho|\pi_\varrho^{\varpi_\tau}+1}{|\gamma_1||\nu_\tau|\Gamma(\varpi_\tau+1)}+\frac{|\breve{b}|(\phi+\varpi_\tau+1)^{-1}+\sum_{\varrho=1}^{\breve{r}}|\breve{h}_\varrho|\pi_\varrho^{\phi+\varpi_\tau}+1}{|\gamma_1||\nu_\tau|\Gamma(\phi+\varpi_\tau+1)}\Bigg)\psi+\frac{\rho(\aleph)}{|\nu_\tau|}(2\breve{q}+1)\Bigg).$$

Proof. Let $\breve{a}\in\Lambda([0,1],\mathbb{R})$ be a solution of (1.1); then,

$$\begin{aligned}\breve{a}(\aleph)=&\Im\breve{a}(\aleph)+\frac{\aleph^{\breve{x}-1}}{\gamma_1}\Bigg[\frac{\breve{b}}{\nu_\tau\Gamma(\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\breve{z})^{\varpi_\tau-1}\zeta(\breve{z})d\breve{z}d\mathfrak{s}\\&+\frac{\breve{b}}{\nu_\tau\Gamma(\phi+\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\breve{z})^{\phi+\varpi_\tau-1}\zeta(\breve{z})d\breve{z}d\mathfrak{s}\\&-\sum_{\varrho=1}^{\breve{r}}\frac{\breve{h}_\varrho}{\nu_\tau\Gamma(\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\breve{z})^{\varpi_\tau-1}\zeta(\breve{z})d\breve{z}\\&-\sum_{\varrho=1}^{\breve{r}}\frac{\breve{h}_\varrho}{\nu_\tau\Gamma(\phi+\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\breve{z})^{\phi+\varpi_\tau-1}\zeta(\breve{z})d\breve{z}\\&-\frac{1}{\nu_\tau\Gamma(\varpi_\tau)}\int_0^1(1-\breve{z})^{\varpi_\tau-1}\zeta(\breve{z})d\breve{z}\\&-\frac{1}{\nu_\tau\Gamma(\phi+\varpi_\tau)}\int_0^1(1-\breve{z})^{\phi+\varpi_\tau-1}\zeta(\breve{z})d\breve{z}\Bigg]\\&+\frac{1}{\nu_\tau\Gamma(\varpi_\tau)}\int_0^{\aleph}(\aleph-\breve{z})^{\varpi_\tau-1}\zeta(\breve{z})d\breve{z}\\&+\frac{1}{\nu_\tau\Gamma(\phi+\varpi_\tau)}\int_0^{\aleph}(\aleph-\breve{z})^{\phi+\varpi_\tau-1}\zeta(\breve{z})d\breve{z}.\end{aligned}\tag{4.6}$$

With the use of Remark 4.2 and (A4) in (4.6), and the computation technique used to validate the findings of the previous section, we obtain

$$\begin{aligned}|\breve{a}(\aleph)-\Im\breve{a}(\aleph)|\leq&\eta\Bigg\{\frac{1}{|\gamma_1|}\Bigg[\frac{|\breve{b}|}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\breve{z})^{\varpi_\tau-1}\rho(\breve{z})d\breve{z}d\mathfrak{s}\\&+\frac{|\breve{b}|}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^1\int_0^{\mathfrak{s}}(\mathfrak{s}-\breve{z})^{\phi+\varpi_\tau-1}\rho(\breve{z})d\breve{z}d\mathfrak{s}\\&+\sum_{\varrho=1}^{\breve{r}}\frac{|\breve{h}_\varrho|}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\breve{z})^{\varpi_\tau-1}\rho(\breve{z})d\breve{z}\\&+\sum_{\varrho=1}^{\breve{r}}\frac{|\breve{h}_\varrho|}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^{\pi_\varrho}(\pi_\varrho-\breve{z})^{\phi+\varpi_\tau-1}\rho(\breve{z})d\breve{z}\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^1(1-\check{z})^{\varpi_\tau-1}\rho(\check{z})d\check{z}\\
&+\frac{1}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^1(1-\check{z})^{\phi+\varpi_\tau-1}\rho(\check{z})d\check{z}\Bigg]\\
&+\frac{1}{|\nu_\tau|\Gamma(\varpi_\tau)}\int_0^{\aleph}(\aleph-\check{z})^{\varpi_\tau-1}\rho(\check{z})d\check{z}\\
&+\frac{1}{|\nu_\tau|\Gamma(\phi+\varpi_\tau)}\int_0^{\aleph}(\aleph-\check{z})^{\phi+\varpi_\tau-1}\rho(\check{z})d\check{z}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
|\check{a}(\aleph)-\mathfrak{I}\check{a}(\aleph)|\leq&\eta\Bigg(\frac{|\check{b}|(\varpi_\tau+1)^{-1}+\sum_{\varrho=1}^{\check{r}}|\check{h}_\varrho|\pi_\varrho^{\varpi_\tau}+1}{|\gamma_1||\nu_\tau|\Gamma(\varpi_\tau+1)}\psi+\frac{\check{q}}{|\nu_\tau|}\rho(\aleph)\\
&+\frac{|\check{b}|(\phi+\varpi_\tau+1)^{-1}+\sum_{\varrho=1}^{\check{r}}|\check{h}_\varrho|\pi_\varrho^{\phi+\varpi_\tau}+1}{|\gamma_1||\nu_\tau|\Gamma(\phi+\varpi_\tau+1)}\psi+\frac{\check{q}+1}{|\nu_\tau|}\rho(\aleph)\Bigg)\\
=&\eta\Bigg(\Bigg(\frac{|\check{b}|(\varpi_\tau+1)^{-1}+\sum_{\varrho=1}^{\check{r}}|\check{h}_\varrho|\pi_\varrho^{\varpi_\tau}+1}{|\gamma_1||\nu_\tau|\Gamma(\varpi_\tau+1)}\\
&+\frac{|\check{b}|(\phi+\varpi_\tau+1)^{-1}+\sum_{\varrho=1}^{\check{r}}|\check{h}_\varrho|\pi_\varrho^{\phi+\varpi_\tau}+1}{|\gamma_1||\nu_\tau|\Gamma(\phi+\varpi_\tau+1)}\Bigg)\psi+\frac{\rho(\aleph)}{|\nu_\tau|}(2\check{q}+1)\Bigg),
\end{aligned}
$$

which is the desired condition. □

Theorem 4.2. *If the assumptions (A1), (A2) and (A4) are satisfied, then equation* (1.1) *is $\mathcal{UHR}$-stable and $\mathcal{GUHR}$-stable if $\theta < 1$, where θ value is in* (3.2).

Proof. Let us assume $\check{a} \in \Lambda([0,1],\mathbb{R})$ and $\check{a}^*$ be a unique solution of equation (1.1), thus

$$
\begin{aligned}
\|\check{a}(\aleph)-\check{a}^*(\aleph)\| &= \|\check{a}(\aleph)-\mathfrak{I}\check{a}^*(\aleph)\| = \|\check{a}(\aleph)-\mathfrak{I}\check{a}(\aleph)+\mathfrak{I}\check{a}(\aleph)-\mathfrak{I}\check{a}^*(\aleph)\|\\
&\leq \|\check{a}(\aleph)-\mathfrak{I}\check{a}(\aleph)\|+\|\mathfrak{I}\check{a}(\aleph)-\mathfrak{I}\check{a}^*(\aleph)\|.
\end{aligned}
$$

By Lemma 4.2 and Theorem 3.1 in the above inequality, we get

$$
\|\check{a}(\aleph)-\check{a}^*(\aleph)\| \leq \Bigg(\mathcal{A}_2\psi+\frac{2\check{q}+1}{|\nu_\tau|}\rho(\aleph)\Bigg)\eta+\theta\|\check{a}-\check{a}^*\|,
$$

where

$$
\begin{aligned}
\mathcal{A}_2 = &\frac{|\check{b}|(\varpi_\tau+1)^{-1}+\sum_{\varrho=1}^{\check{r}}|\check{h}_\varrho|\pi_\varrho^{\varpi_\tau}+1}{|\gamma_1||\nu_\tau|\Gamma(\varpi_\tau+1)}\\
&+\frac{|\check{b}|(\phi+\varpi_\tau+1)^{-1}+\sum_{\varrho=1}^{\check{r}}|\check{h}_\varrho|\pi_\varrho^{\phi+\varpi_\tau}+1}{|\gamma_1||\nu_\tau|\Gamma(\phi+\varpi_\tau+1)}
\end{aligned}
$$

Alternatively, we have

$$
\|\check{a}(\aleph)-\check{a}^*(\aleph)\| \leq \frac{1}{1-\theta}\Bigg(\mathcal{A}_2\psi+\frac{2\check{q}+1}{|\nu_\tau|}\rho(\aleph)\Bigg)\eta.
$$

Letting $\mathcal{B}_2 = \frac{1}{1-\theta}\max\{\mathcal{A}_2, \frac{2\check{q}+1}{|\nu_\tau|}\}$, the solution of equation (1.1) is $\mathcal{UHR}$-stable. The answer to equation in (1.1) is $\mathcal{UHR}$-stable when $\eta = 1$. Hence proof is finished. □

5. Examples

The purpose of this section is to illustrate the conclusions.

Example 5.1. *Examine the multipoint-integral boundary value of the $\mathcal{FDE}$ with multiple delays that are provided by*

$$\begin{aligned}&\frac{13}{7}{}^{\mathcal{R}}\mathcal{D}^{0.8}\breve{a}(\aleph)-\frac{7}{2}{}^{\mathcal{R}}\mathcal{D}^{0.65}\breve{a}(\aleph)+13{}^{\mathcal{R}}\mathcal{D}^{6.5}\breve{a}(\aleph)-\frac{14}{17}{}^{\mathcal{R}}\mathcal{D}^{0.46}\breve{a}(\aleph)+\frac{4}{5}{}^{\mathcal{R}}\mathcal{D}^{0.75}\breve{a}(\aleph)\\&=\breve{u}(\aleph,\breve{a}(\aleph),\breve{a}(\breve{w}\aleph))+\int_0^{\aleph}\frac{(\aleph-\check{z})^{\frac{-5}{6}}\breve{u}(\check{z},\breve{a}(\check{z}),\breve{a}(\breve{w}\check{z}))}{\Gamma(\frac{1}{6})}d\check{z},\aleph\in[0,1],\\&\breve{a}(0)=2,\frac{d^{\varrho}\breve{a}(0)}{d\aleph^{\varrho}}=0,\breve{a}(1)=5\int_0^1\breve{a}(\mathfrak{s})d\mathfrak{s}+4\breve{a}(\frac{1}{3})-3\breve{a}(\frac{1}{5})-7\breve{a}(\frac{2}{7}),\end{aligned}\tag{5.1}$$

where $\varpi_1=0.8,\varpi_2=0.65,\varpi_{\tau=3}=6.5,\varpi_4=0.46,\varpi_5=0.75,\phi=0.16,\nu_1=\frac{13}{7},\nu_2=\frac{-7}{2},\nu_{\tau=3}=13,\nu_4=\frac{-14}{17},\nu_5=\frac{4}{5},\check{h}_1=-4,\check{h}_2=3,\check{h}_3=7,\pi_1=\frac{1}{3},\pi_2=\frac{1}{5},\pi_3=\frac{2}{7},\check{b}=4,\breve{w}=\frac{1}{8}$, *and*

$$\breve{u}(\aleph,\breve{a}(\aleph),\breve{a}(\breve{w}\aleph))=\frac{1}{\aleph^2+e^{2\aleph}}\Bigg(\frac{3\breve{a}(\aleph)}{2\aleph^2+1+\breve{a}(\aleph)}-\frac{\breve{a}(\frac{\aleph}{4})}{2+\aleph^2e^{\aleph}+\breve{a}(\frac{\aleph}{4})}-\sin(\aleph)\Bigg).$$

Observe that

$$\Bigg|\breve{u}(\aleph,\breve{a}_1(\aleph),\breve{a}_1(\breve{w}\aleph))-\breve{u}(\aleph,\breve{a}_2(\aleph),\breve{a}_2(\breve{w}\aleph))\Bigg|\leq 3\|\breve{a}_1-\breve{a}_2\|+\|\breve{a}_1(\frac{\aleph}{4})-\breve{a}_2(\frac{\aleph}{4})\|.$$

Clearly, $\mathcal{L}_1+\mathcal{L}_2=4=\mathcal{L}_3$. *With the given data, it is found that* $\gamma_1=0.2021\neq 0$ *and* $\theta<1$. *Thus, we conclude that there is only one solution of equation* (5.1) *on* $[0,1]$ *based on the result of Theorem 3.1.*

Example 5.2. *Examine the four-point integral fractional-order boundary-value problem with many terms provided by*

$$\begin{aligned}&13{}^{\mathcal{R}}\mathcal{D}^{1.7}\breve{a}(\aleph)+0.6{}^{\mathcal{R}}\mathcal{D}^{0.8}\breve{a}(\aleph)+0.03{}^{\mathcal{R}}\mathcal{D}^{0.55}\breve{a}(\aleph)\\&=\breve{u}(\aleph,\breve{a}(\aleph),\breve{a}(\frac{\aleph}{5}))+\int_0^{\aleph}\frac{(\aleph-\check{z})^{\frac{-8}{9}}\breve{u}(\check{z},\breve{a}(\check{z}),\breve{a}(\breve{w}\check{z}))}{\Gamma(\frac{1}{9})}d\check{z},\aleph\in[0,1],\\&\breve{a}(0)=2,\frac{d^{\varrho}\breve{a}(0)}{d\aleph^{\varrho}}=0,\breve{a}(1)=3\int_0^1\breve{a}(\mathfrak{s})d\mathfrak{s}-4\breve{a}(\frac{1}{3})-2\breve{a}(\frac{1}{7}),\end{aligned}\tag{5.2}$$

where

$$\begin{aligned}\breve{u}(\aleph,\breve{a}(\aleph),\breve{a}(\frac{\aleph}{5}))=\frac{\aleph^2}{e^{\aleph}}\Bigg(&\frac{\aleph}{\aleph^2e^{\aleph}+2|\breve{a}(\aleph)|+\sin\aleph|\breve{a}(\frac{\aleph}{5})|}\\&+\frac{e^{\aleph}}{\aleph e^{-2\aleph}+3|\breve{a}(\aleph)|+|\breve{a}(\frac{\aleph}{5})|}+\cos(\aleph)e^{-2\aleph}\Bigg),\end{aligned}$$

$\check{h}_1 = 4, \check{h}_2 = 2, \tau = 1, \pi_1 = \frac{1}{3}, \pi_2 = \frac{1}{7}, \phi = 0.11, \varpi_\tau = 1.7, \varpi_2 = 0.8, \varpi_3 = 0.55, \nu_\tau = 13, \nu_2 = 0.6, \nu_3 = 0.03, \check{b} = 3, \check{a}_0 = 2$. *Observe that*

$$\begin{aligned}
|\check{u}(\aleph, \check{a}(\aleph), \check{a}(\frac{\aleph}{5}))| &= \left|\frac{\aleph^2}{e^{\aleph}}\left(\frac{\aleph}{\aleph^2 e^{\aleph} + 2|\check{a}(\aleph)| + \sin\aleph|\check{a}(\frac{\aleph}{5})|}\right.\right. \\
&\quad \left.\left. + \frac{e^{\aleph}}{\aleph e^{-2\aleph} + 3|\check{a}(\aleph)| + |\check{a}(\frac{\aleph}{5})|} + \cos(\aleph)e^{-2\aleph}\right)\right| \\
&\leq \left|\frac{\aleph^2}{e^{\aleph}}\right|\left|\left(\frac{\aleph}{\aleph^2 e^{\aleph} + 2|\check{a}(\aleph)| + \sin\aleph|\check{a}(\frac{\aleph}{5})|}\right.\right. \\
&\quad \left.\left. + \frac{e^{\aleph}}{\aleph e^{-2\aleph} + 3|\check{a}(\aleph)| + |\check{a}(\frac{\aleph}{5})|} + \cos(\aleph)e^{-2\aleph}\right)\right| \\
&\leq \frac{\aleph^2}{e^{\aleph}}\left(\frac{\aleph}{\aleph^2 e^{\aleph}} + \frac{e^{\aleph}}{\aleph e^{-2\aleph}} + e^{-2\aleph}\right) \\
&\leq \aleph e^{-2\aleph} + \aleph e^{2\aleph} + \aleph^2 e^{-3\aleph} = \psi_\hbar(\aleph),
\end{aligned}$$

We utilized information that $\sin\aleph$ *is a positive dereasing function on* $[0,1]$. *Also, we have that* $\|\psi_\hbar\| = 7.5742, \gamma_1 = 0.4853$. *Therefore, the hypothesis of Theorem 3.2 is clearly fulfilled. Hence equation* (5.2) *has required solution on* $[0,1]$.

6. Conclusions

In this paper, we looked at a novel type of nonlocal boundary-value problem with multipoint-integral boundary conditions and multiterm caputo $\mathcal{FDE}$. At the first, we discovered an integral operator linked with the situation. We used the Banach fixed-point theorem and $\mathcal{LST}$ nonlinear alternative to determine the existence and uniqueness of solutions to the given problem. Our stability criteria for this problem include $\mathcal{UH}$, $\mathcal{GUH}$, $\mathcal{UHR}$, and $\mathcal{GUHR}$ stability. Our findings are consistent with those reported in [32]. According to our obtained results are based on integral boundary conditions and these results are novel. This effort will provide fresh insights on the problem's inclusions and impulsive forms. We want to expand our research to include multipoint-integral boundary conditions on coupled systems of multi-term caputo fractional differentials.

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