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Solving Nonlinear Difference Equations: Insights from Three-Dimensional Systems and Numerical Examples

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Abstract. This paper presents a study on nonlinear difference equation systems of $6k+3$ order. The equations are of the form $p_{n+1} = p_{n-(6k+2)}/(\pm 1 \pm q_{n-2k} r_{n-(4k+1)} p_{n-(6k+2)})$, $q_{n+1} = q_{n-(6k+2)}/(\pm 1 \pm r_{n-2k} p_{n-(4k+1)} q_{n-(6k+2)})$, $r_{n+1} = r_{n-(6k+2)}/(\pm 1 \pm p_{n-2k} q_{n-(4k+1)} r_{n-(6k+2)})$, $k \geq 0$ where n is a non-negative integer (belonging to the set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$) and the starting values p_{-l}, q_{-l}, r_{-l}, $l \in \{0, 1, \ldots, 6k+2\}$ are arbitrary nonzero real numbers. We propose a systematic approach to solve this system, introducing a novel technique to find explicit solutions. The main outcomes of our study are the explicit solutions derived from the considered system. The study examines four different cases of this system and provides numerical examples to illustrate the results. The numerical examples demonstrate the behavior of the system for various initial conditions. The study is concluded with graphical representations of the solutions for each case, providing insights into the behavior of the systems.

1. Introduction

Difference equations, or discrete dynamic systems, represent a distinct field of study within mathematics. They are particularly relevant in various scientific disciplines, such as biology, where many systems are naturally described using discrete variables, and in economic modeling (see, for example, [16]- [17] and the references therein). The theory of difference equations is important in applied sciences and numerous other fields. This theory is considered fundamental and is expected to maintain its central role in mathematics. Notably, nonlinear difference equations of higher orders, exceeding one, carry immense significance and cannot be underestimated. They

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have applications and implications that are both broad and profound in various areas of research and problem-solving.

The solvability of nonlinear difference equations and systems has become a focal point of interest among mathematicians recently. This area of study has seen significant attention, with contributions from researchers (see, for example, [1]- [15], [18]- [24] and the references therein). In the realm of nonlinear difference equations, a particularly intriguing aspect is the association with well-known sequences like the Fibonacci sequence. This connection has been thoroughly explored, as exemplified in works such as "*Dynamical behavior and solution of nonlinear difference equation via Fibonacci sequence*" by Elsayed et al. [9]. Notably, Clark and Kulenovic, in particular, have provided valuable insights into the dynamics of solutions in the context of a specific difference equation. Numerous studies have been conducted on the dynamics of solutions in various systems of nonlinear difference equations. Here are some of the notable works and the systems they have investigated:

- Kurbanli et al. [22] examined the behavior of positive solutions in a system described by the equations:

$$x_{n+1}=\frac{x_{n-1}}{1+y_nx_{n-1}},y_{n+1}=\frac{y_{n-1}}{1+x_ny_{n-1}}.$$

- Elsayed [12] found solutions for systems of difference equations with the following form:

$$x_{n+1}=\frac{x_{n-1}}{\pm1+y_nx_{n-1}},y_{n+1}=\frac{y_{n-1}}{\mp1+x_ny_{n-1}}.$$

- Kurbanli et al. [23] discussed a three-dimensional system of difference equations:

$$x_{n+1}=\frac{x_{n-1}}{-1+y_nx_{n-1}},y_{n+1}=\frac{y_{n-1}}{-1+x_ny_{n-1}},z_{n+1}=\frac{z_{n-1}}{-1+y_nz_{n-1}}.$$

- In Elsayed and Gafel [10], solutions for a system of three difference equations were discussed:

$$x_{n+1}=\frac{x_{n-2}}{\pm1+z_ny_{n-1}x_{n-2}},y_{n+1}=\frac{y_{n-2}}{\pm1+x_nz_{n-1}y_{n-2}},z_{n+1}=\frac{z_{n-2}}{\pm1+y_nx_{n-1}z_{n-2}}.$$

This reflects the ongoing efforts to understand and solve nonlinear difference equations, which have practical relevance in a wide range of scientific and mathematical applications. These investigations are critical for advancing our understanding of discrete dynamic systems and their behavior. The investigations mentioned above have motivated the current study to explore the form of solutions in systems involving 3−dimensional rational difference equations

$$p_{n+1}=\frac{p_{n-(6k+2)}}{\kappa+\tau q_{n-2k}r_{n-(4k+1)}p_{n-(6k+2)}},q_{n+1}=\frac{q_{n-(6k+2)}}{\kappa+\tau r_{n-2k}p_{n-(4k+1)}q_{n-(6k+2)}}, \quad (1.1)$$
$$r_{n+1}=\frac{r_{n-(6k+2)}}{\kappa+\tau p_{n-2k}q_{n-(4k+1)}r_{n-(6k+2)}}.$$

The present paper proposes a systematic approach to solving this system and introduces a novel technique to find explicit solutions. The main outcomes of this study is the explicit solutions derived from the considered system. The methodology used in this study builds upon the established techniques introduced in seminal works in the field. Notably, the methods used in this study draws from the following important papers:

- The Recati difference equation system, which is closely related to the system under consideration, was introduced by Recati (1897) and further studied by Brand (1955), Papaschinopoulos and Papadopoulos (2002), Clark and Kulenovic (2005), Stević et al. (2018-2019).
- The method used to find explicit solutions for nonlinear difference equations is based on the works of Elsayed (2012), Kurbanli et al. (2016), and Elsayed and Gafel (2021).

The study examines four different cases of this system and provides numerical examples to illustrate the results. These examples aim to demonstrate the behavior of the system for various initial conditions. The study is concluded with graphical representations of the solutions for each case, providing insights into the behavior of the systems. These studies contribute to a deeper understanding of the behavior and solutions in nonlinear difference equation systems (1.1).

2. Main results

The main results focus on finding solutions to a three-dimensional system of nonlinear difference equations. These equations are represented by the system described in (1.1) and are closely related to the system mentioned. The objective is to analyze and understand the behavior of solutions for this system.

2.1. System (1.1) when $\kappa = \tau = +1$.

In this context, it is assumed that $\{p_n, q_n, r_n\}$ represent a solution to the following system

$$p_{n+1} = \frac{p_{n-2}}{1 + q_n r_{n-1} p_{n-2}},\ q_{n+1} = \frac{q_{n-2}}{1 + r_n p_{n-1} q_{n-2}},\ r_{n+1} = \frac{r_{n-2}}{1 + p_n q_{n-1} r_{n-2}}, \tag{2.1}$$

which is considered as special cases of system (1.1) when $\kappa = \tau = +1$ and $k = 0$. This subsection is dedicated to an in-depth investigation of solutions for the three-dimensional system of nonlinear difference equations (2.1). The investigation employs a methodology that relies on the suggested notations as a fundamental tool. These notations are

$$\alpha_n = p_n q_{n-1} r_{n-2},\ \beta_n = q_n r_{n-1} p_{n-2},\ \gamma_n = r_n p_{n-1} q_{n-2}. \tag{2.2}$$

In many cases, when dealing with systems of difference equations, it can be advantageous to express the system represented by (2.1) as an equivalent system of difference equations. This transformation is typically performed to simplify the study and analysis of the system. Accordingly, our research has resulted in the following equivalent system:

$$\alpha_{n+1} = \frac{\beta_n}{1 + \beta_n},\ \beta_{n+1} = \frac{\gamma_n}{1 + \gamma_n},\ \gamma_{n+1} = \frac{\alpha_n}{1 + \alpha_n}. \tag{2.3}$$

The solutions to the last system of difference equations are provided in the following theorem:

Theorem 2.1. *Let $\{\alpha_n, \beta_n, \gamma_n\}_{n\geq 0}$ be solutions of system* (2.3)*. Then $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ are given by the following formulas for $n = 0, 1, \ldots$*

$$\{\alpha_n\}_{n\geq 0} : \begin{cases} \alpha_{3n} = K_{3n}(\alpha_0) \\ \alpha_{3n+1} = K_{3n+1}(\beta_0) \\ \alpha_{3n+2} = K_{3n+2}(\gamma_0) \end{cases}, \{\beta_n\}_{n\geq 0} : \begin{cases} \beta_{3n} = K_{3n}(\beta_0) \\ \beta_{3n+1} = K_{3n+1}(\gamma_0) \\ \beta_{3n+2} = K_{3n+2}(\alpha_0) \end{cases} \text{ and } \{\gamma_n\}_{n\geq 0} : \begin{cases} \gamma_{3n} = K_{3n}(\gamma_0) \\ \gamma_{3n+1} = K_{3n+1}(\alpha_0) \\ \gamma_{3n+2} = K_{3n+2}(\beta_0) \end{cases}, \tag{2.4}$$

where $K_n(x) = \frac{x}{1+nx}$, and α_0, β_0 and γ_0 are computed from (2.2).

Proof. By replacing the expression derived from the last recurrence relation in (2.3) into the second equation, and subsequently incorporating this result into the first recurrence relation in (2.3), we arrive at the following:

$$\alpha_{n+1} = \frac{\alpha_{n-2}}{1 + 3\alpha_{n-2}}, \ n \geq 2.$$

Likewise, we get

$$\beta_{n+1} = \frac{\beta_{n-2}}{1 + 3\beta_{n-2}}, \ \gamma_{n+1} = \frac{\gamma_{n-2}}{1 + 3\gamma_{n-2}}, n \geq 2.$$

Using the follownig notations $\alpha_{n,l} = \alpha_{3n+l}$, $\beta_{n,l} = \beta_{3n+l}$ and $\gamma_{n,l} = \gamma_{3n+l}$, for $n \geq 0$ with $l = 0, 1, 2$, we obtain:

$$\alpha_{n+1,l} = \frac{\alpha_{n,l}}{1 + 3\alpha_{n,l}}, \ \beta_{n+1,l} = \frac{\beta_{n,l}}{1 + 3\beta_{n,l}}, \ \gamma_{n+1,l} = \frac{\gamma_{n,l}}{1 + 3\gamma_{n,l}},$$

for $n \geq 0$ with $l = 0, 1, 2$. Since the three recurrence relations are similar, let's use the first recurrence relation for the next transformation $1 + 3\alpha_{n,l} = \widetilde{\alpha}_n / \widetilde{\alpha}_{n-1}$. By simplifying, we obtain:

$$\widetilde{\alpha}_{n+1} - 2\widetilde{\alpha}_n + \widetilde{\alpha}_{n-1} = 0, \ n \geq 0.$$

So, we have $\widetilde{\alpha}_n = \dfrac{n - (n+1)\widetilde{\alpha}_0}{n(1 - \widetilde{\alpha}_0) - 1}$ and $\alpha_{n,l} = \dfrac{\alpha_{0,l}}{1 + 3n\alpha_{0,l}}$, for $l = 0, 1, 2$. Therefore, the proof is complete. □

Theorem 2.1 provides solutions to the system described by (2.3). This system consists of three difference equations, which govern the sequences α_n, β_n, and γ_n. The theorem states that these sequences can be expressed as follows:

- α_n follows a pattern where α_{3n}, α_{3n+1}, and α_{3n+2} are given by certain expressions involving α_0, β_0, and γ_0.
- β_n follows a similar pattern where β_{3n}, β_{3n+1}, and β_{3n+2} are expressed in terms of α_0, β_0, and γ_0.
- γ_n also follows a similar pattern where γ_{3n}, γ_{3n+1}, and γ_{3n+2} are determined by α_0, β_0, and γ_0.

These expressions provide a comprehensive understanding of the behavior of the system over successive iterations, offering a way to compute α_n, β_n, and γ_n for any $n \geq 0$ based on the initial values α_0, β_0, and γ_0.

Consequently, we obtain the following result.

Theorem 2.2. *Let $\{p_n, q_n, r_n\}_{n\geq -2}$ be solutions of system* (2.1). *Then $\{p_n\}_{n\geq -2}$, $\{q_n\}_{n\geq -2}$ and $\{r_n\}_{n\geq -2}$ are given by the following formulas for $n = 0, 1, \ldots$*

$$p_{3n-k} = p_{-k}\prod_{j=0}^{n-1}\frac{\alpha_{3(n-j)-k}}{\beta_{3(n-j)-(k+1)}},$$

$$q_{3n-k} = q_{-k}\prod_{j=0}^{n-1}\frac{\beta_{3(n-j)-k}}{\gamma_{3(n-j)-(k+1)}},$$

$$r_{3n-k} = r_{-k}\prod_{j=0}^{n-1}\frac{\gamma_{3(n-j)-k}}{\alpha_{3(n-j)-(k+1)}},$$

for $k = 0, 1, 2$, where $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ are the solutions for the system (2.3) *given in* (2.4).

Proof. The solutions of the system described in (2.1) can be determined by using the previously established notations (2.2). The solutions take the following form:

$$p_n = \frac{\alpha_n}{q_{n-1}r_{n-2}},\ q_n = \frac{\beta_n}{r_{n-1}p_{n-2}},\ r_n = \frac{\gamma_n}{p_{n-1}q_{n-2}}. \tag{2.5}$$

Using (2.5), we obviously have

$$p_{3n} = \frac{\alpha_{3n}}{\beta_{3n-1}}p_{3(n-1)}, q_{3n} = \frac{\beta_{3n}}{\gamma_{3n-1}}q_{3(n-1)}, r_{3n} = \frac{\gamma_{3n}}{\alpha_{3n-1}}r_{3(n-1)}, \tag{2.6}$$

$$p_{3n-1} = \frac{\gamma_{3n}r_{3(n-1)}}{r_{3n}\beta_{3n-2}}p_{3(n-1)-1},\ q_{3n-1} = \frac{\alpha_{3n}p_{3(n-1)}}{p_{3n}\gamma_{3n-2}}q_{3(n-1)-1},\ r_{3n-1} = \frac{\beta_{3n}q_{3(n-1)}}{q_{3n}\alpha_{3n-2}}r_{3(n-1)-1}, \tag{2.7}$$

and which system (2.6) (resp. (2.7)) can be solved recursively, yielding:

$$p_{3n} = p_0\prod_{j=0}^{n-1}\frac{\alpha_{3(n-j)}}{\beta_{3(n-j)-1}},\ q_{3n} = q_0\prod_{j=0}^{n-1}\frac{\beta_{3(n-j)}}{\gamma_{3(n-j)-1}},\ r_{3n} = r_0\prod_{j=0}^{n-1}\frac{\gamma_{3(n-j)}}{\alpha_{3(n-j)-1}},$$

$$\text{(resp. } p_{3n-1} = p_{-1}\prod_{j=0}^{n-1}\frac{\gamma_{3(n-j)}r_{3(n-j-1)}}{r_{3(n-j)}\beta_{3(n-j)-2}},\ q_{3n-1} = q_{-1}\prod_{j=0}^{n-1}\frac{\alpha_{3(n-j)}p_{3(n-j-1)}}{p_{3(n-j)}\gamma_{3(n-j)-2}},\ r_{3n-1} = r_{-1}\prod_{j=0}^{n-1}\frac{\beta_{3(n-j)}q_{3(n-j-1)}}{q_{3(n-j)}\alpha_{3(n-j)-2}}).$$

From (2.5), we further have:

$$p_{3n-2} = \frac{\beta_{3n}}{q_{3n}r_{3n-1}},\ q_{3n-2} = \frac{\gamma_{3n}}{r_{3n}p_{3n-1}},\ r_{3n-2} = \frac{\alpha_{3n}}{p_{3n}q_{3n-1}}.$$

□

From Theorems (2.1) – (2.2), we derive the following theorem, elucidating the structure of solutions for the system described by (2.1).

Theorem 2.3. *Assume $\{p_n, q_n, r_n\}$ are solutions to the system represented in* (2.1). *Considering arbitrary nonzero real numbers p_{-l}, q_{-l}, r_{-l} for $l \in \{0, 1, 2\}$ as initial values, the solutions for the system* (2.1) *can be* *expressed as follows:*

$$p_{3n} = p_0 \prod_{j=0}^{n-1} \frac{1 + (3(n-j)-1) p_0 q_{-1} r_{-2}}{1 + 3(n-j) p_0 q_{-1} r_{-2}},$$

$$p_{3n-1} = p_{-1} \prod_{j=0}^{n-1} \frac{1 + (3(n-j)-2) r_0 p_{-1} q_{-2}}{1 + (3(n-j)-1) r_0 p_{-1} q_{-2}},$$

$$p_{3n-2} = p_{-2} \prod_{j=0}^{n-1} \frac{1 + 3(n-j-1) q_0 r_{-1} p_{-2}}{1 + (3(n-j)-2) q_0 r_{-1} p_{-2}},$$

$$q_{3n} = q_0 \prod_{j=0}^{n-1} \frac{1 + (3(n-j)-1) q_0 r_{-1} p_{-2}}{1 + 3(n-j) q_0 r_{-1} p_{-2}},$$

$$q_{3n-1} = q_{-1} \prod_{j=0}^{n-1} \frac{1 + (3(n-j)-2) p_0 q_{-1} r_{-2}}{1 + (3(n-j)-1) p_0 q_{-1} r_{-2}},$$

$$q_{3n-2} = q_{-2} \prod_{j=0}^{n-1} \frac{1 + 3(n-j-1) r_0 p_{-1} q_{-2}}{1 + (3(n-j)-2) r_0 p_{-1} q_{-2}},$$

$$r_{3n} = r_0 \prod_{j=0}^{n-1} \frac{1 + (3(n-j)-1) r_0 p_{-1} q_{-2}}{1 + 3(n-j) r_0 p_{-1} q_{-2}},$$

$$r_{3n-1} = r_{-1} \prod_{j=0}^{n-1} \frac{1 + (3(n-j)-2) q_0 r_{-1} p_{-2}}{1 + (3(n-j)-1) q_0 r_{-1} p_{-2}},$$

$$r_{3n-2} = r_{-2} \prod_{j=0}^{n-1} \frac{1 + 3(n-j-1) p_0 q_{-1} r_{-2}}{1 + (3(n-j)-2) p_0 q_{-1} r_{-2}},$$

for $n \geq 1$*.*

The system under study, as titled in the subsection, is investigated when $k > 0$. The system is represented by the following equations:

$$p_{n+1} = \frac{p_{n-(6k+2)}}{1 + q_{n-2k} r_{n-(4k+1)} p_{n-(6k+2)}}, q_{n+1} = \frac{q_{n-(6k+2)}}{1 + r_{n-2k} p_{n-(4k+1)} q_{n-(6k+2)}},$$
$$r_{n+1} = \frac{r_{n-(6k+2)}}{1 + p_{n-2k} q_{n-(4k+1)} r_{n-(6k+2)}}, \tag{2.8}$$

which is an extension of the system described in (2.1). A scheme for the system (2.8) is presented as follows:

$$n+1 \overset{-2k-1}{\longrightarrow} n-2k \overset{-2k-1}{\longrightarrow} n-(4k+1) \overset{-2k-1}{\longrightarrow} n-(6k+2). \tag{2.9}$$

This scheme allows us to express the system (2.8) as a set of equations for specific indices:

$$p_{(2k+1)(l+1)+t} = \frac{p_{(2k+1)(l-2)+t}}{1+q_{(2k+1)l+t}r_{(2k+1)(l-1)+t}p_{(2k+1)(l-2)+t}},$$
$$q_{(2k+1)(l+1)+t} = \frac{q_{(2k+1)(l-2)+t}}{1+r_{(2k+1)l+t}p_{(2k+1)(l-1)+t}q_{(2k+1)(l-2)+t}},$$
$$r_{(2k+1)(l+1)+t} = \frac{r_{(2k+1)(l-2)+t}}{1+p_{(2k+1)l+t}q_{(2k+1)(l-1)+t}r_{(2k+1)(l-2)+t}},$$

where t ranges from 1 to $2k+1$, and l is a non-negative integer. Using the notations:

$$\psi_l^{(t)} = p_{(2k+1)l+t},$$
$$\xi_l^{(t)} = q_{(2k+1)l+t},$$
$$\varphi_l^{(t)} = r_{(2k+1)l+t}, t \in \{1,\dots,2k+1\},$$

with $l \geq -3$ and t ranging from 1 to $2k+1$, a set of $(2k+1)$-systems analogous to system (2.1) can be derived. This discussion leads to the introduction of the following theorem.

Theorem 2.4. *Suppose that $\{p_n, q_n, r_n\}$ are solutions to the system represented in (2.1). Additionally, consider arbitrary nonzero real numbers p_{-l}, q_{-l}, r_{-l} for $l \in \{0,1,\dots,6k+2\}$ as the initial values. In this case, the solutions for the system (2.1) can be expressed as follows:*

$$p_{3(2k+1)l+t} = p_t \prod_{j=0}^{l-1} \frac{1+(3(l-j)-1)\, p_t q_{t-(2k+1)} r_{t-2(2k+1)}}{1+3(l-j)\, p_t q_{t-(2k+1)} r_{t-2(2k+1)}},$$
$$p_{(2k+1)(3l-1)+t} = p_{t-(2k+1)} \prod_{j=0}^{l-1} \frac{1+(3(l-j)-2)\, r_t p_{t-(2k+1)} q_{t-2(2k+1)}}{1+(3(l-j)-1)\, r_t p_{t-(2k+1)} q_{t-2(2k+1)}},$$
$$p_{(2k+1)(3l-2)+t} = p_{t-2(2k+1)} \prod_{j=0}^{l-1} \frac{1+3(l-j-1)\, q_t r_{t-(2k+1)} p_{t-2(2k+1)}}{1+(3(l-j)-2)\, q_t r_{t-(2k+1)} p_{t-2(2k+1)}},$$
$$q_{3(2k+1)l+t} = q_t \prod_{j=0}^{l-1} \frac{1+(3(l-j)-1)\, q_t r_{t-(2k+1)} p_{t-2(2k+1)}}{1+3(l-j)\, q_t r_{t-(2k+1)} p_{t-2(2k+1)}},$$
$$q_{(2k+1)(3l-1)+t} = q_{t-(2k+1)} \prod_{j=0}^{l-1} \frac{1+(3(l-j)-2)\, p_t q_{t-(2k+1)} r_{t-2(2k+1)}}{1+(3(l-j)-1)\, p_t q_{t-(2k+1)} r_{t-2(2k+1)}},$$
$$q_{(2k+1)(3l-2)+t} = q_{t-2(k2+1)} \prod_{j=0}^{l-1} \frac{1+3(l-j-1)\, r_t p_{t-(2k+1)} q_{t-2(2k+1)}}{1+(3(l-j)-2)\, r_t p_{t-(2k+1)} q_{t-2(2k+1)}},$$
$$r_{3(2k+1)l+t} = r_t \prod_{j=0}^{l-1} \frac{1+(3(l-j)-1)\, r_t p_{t-(2k+1)} q_{t-2(2k+1)}}{1+3(l-j)\, r_t p_{t-(2k+1)} q_{t-2(2k+1)}},$$

$$r_{(2k+1)(3l-1)+t} = r_{t-(2k+1)} \prod_{j=0}^{l-1} \frac{1 + (3(l-j)-2)\, q_t r_{t-(2k+1)} p_{t-2(2k+1)}}{1 + (3(l-j)-1)\, q_t r_{t-(2k+1)} p_{t-2(2k+1)}},$$

$$r_{(2k+1)(3l-2)+t} = r_{t-2(2k+1)} \prod_{j=0}^{l-1} \frac{1 + 3(l-j-1)\, p_t q_{t-(2k+1)} r_{t-2(2k+1)}}{1 + (3(l-j)-2)\, p_t q_{t-(2k+1)} r_{t-2(2k+1)}},$$

for $n \geq 1$, $t \in \{1, \ldots, 2k+1\}$.

2.2. **System** (1.1) **when** $\kappa = -\tau = +1$. In this subsection, it is assumed that $\{p_n, q_n, r_n\}$ represent a solution to the following system

$$p_{n+1} = \frac{p_{n-2}}{1 - q_n r_{n-1} p_{n-2}}, q_{n+1} = \frac{q_{n-2}}{1 - r_n p_{n-1} q_{n-2}}, r_{n+1} = \frac{r_{n-2}}{1 - p_n q_{n-1} r_{n-2}}, \tag{2.10}$$

which is considered as special cases of system (1.1) when $\kappa = -\tau = +1$ and $k = 0$. Using the notations (2.2), we obtain the following equivalent system

$$\alpha_{n+1} = \frac{\beta_n}{1-\beta_n}, \beta_{n+1} = \frac{\gamma_n}{1-\gamma_n}, \gamma_{n+1} = \frac{\alpha_n}{1-\alpha_n}. \tag{2.11}$$

The solutions to the last system of difference equations are provided in the following theorem:

Theorem 2.5. *Let* $\{\alpha_n, \beta_n, \gamma_n\}_{n\geq 0}$ *be solutions of system* (2.10). *Then* $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ *and* $\{\gamma_n\}_{n\geq 0}$ *are given by the following formulas for* $n = 0, 1, \ldots$

$$\{\alpha_n\}_{n\geq 0} : \begin{cases} \alpha_{3n} = J_{3n}(\alpha_0) \\ \alpha_{3n+1} = J_{3n+1}(\beta_0) \\ \alpha_{3n+2} = J_{3n+2}(\gamma_0) \end{cases}, \{\beta_n\}_{n\geq 0} : \begin{cases} \beta_{3n} = J_{3n}(\beta_0) \\ \beta_{3n+1} = J_{3n+1}(\gamma_0) \\ \beta_{3n+2} = J_{3n+2}(\alpha_0) \end{cases} \text{and } \{\gamma_n\}_{n\geq 0} : \begin{cases} \gamma_{3n} = J_{3n}(\gamma_0) \\ \gamma_{3n+1} = J_{3n+1}(\alpha_0) \\ \gamma_{3n+2} = J_{3n+2}(\beta_0) \end{cases},$$

where $J_n(x) = \frac{x}{1-nx}$,*and* α_0, β_0 *and* γ_0 *are computed from* (2.2).

Proof. By replacing the expression derived from the last recurrence relation in (2.3) into the second equation, and subsequently incorporating this result into the first recurrence relation in (2.3), we arrive at the following:

$$\alpha_{n+1} = \frac{\alpha_{n-2}}{1 - 3\alpha_{n-2}}, \; n \geq 2.$$

Likewise, we get

$$\beta_{n+1} = \frac{\beta_{n-2}}{1 - 3\beta_{n-2}}, \; \gamma_{n+1} = \frac{\gamma_{n-2}}{1 - 3\gamma_{n-2}}, n \geq 2.$$

Using the follownig notations $\alpha_{n,l} = \alpha_{3n+l}$, $\beta_{n,l} = \beta_{3n+l}$ and $\gamma_{n,l} = \gamma_{3n+l}$, for $n \geq 0$ with $l = 0, 1, 2$, we obtain:

$$\alpha_{n+1,l} = \frac{\alpha_{n,l}}{1 - 3\alpha_{n,l}}, \; \beta_{n+1,l} = \frac{\beta_{n,l}}{1 - 3\beta_{n,l}}, \; \gamma_{n+1,l} = \frac{\gamma_{n,l}}{1 - 3\gamma_{n,l}},$$

for $n \geq 0$ with $l = 0, 1, 2$. Since the three recurrence relations are similar, let's use the first recurrence relation for the next transformation $1 - 3\alpha_{n,l} = \widetilde{\alpha}_n / \widetilde{\alpha}_{n-1}$. By simplifying, we obtain the same difference equation $\widetilde{\alpha}_{n+1} - 2\widetilde{\alpha}_n + \widetilde{\alpha}_{n-1} = 0$, $n \geq 0$. So, we have $\widetilde{\alpha}_n = \dfrac{n - (n+1)\widetilde{\alpha}_0}{n(1 - \widetilde{\alpha}_0) - 1}$ and $\alpha_{n,l} = \dfrac{\alpha_{0,l}}{1 - 3n\alpha_{0,l}}$, for $l = 0, 1, 2$. Therefore, the proof is complete. □

From Theorems (2.2) and (2.5), we establish the following theorem concerning the structure of solutions for the system denoted by (2.10).

Theorem 2.6. *Suppose that $\{p_n, q_n, r_n\}$ are solutions to the system represented in (2.10). Considering arbitrary nonzero real numbers p_{-l}, q_{-l}, r_{-l} for $l \in \{0,1,2\}$ as the initial values, the solutions for the system (2.10) can be expressed as follows:*

$$p_{3n} = p_0 \prod_{j=0}^{n-1} \frac{1-(3(n-j)-1)\,p_0 q_{-1} r_{-2}}{1-3(n-j)\,p_0 q_{-1} r_{-2}},$$

$$p_{3n-1} = p_{-1} \prod_{j=0}^{n-1} \frac{1-(3(n-j)-2)\,r_0 p_{-1} q_{-2}}{1-(3(n-j)-1)\,r_0 p_{-1} q_{-2}},$$

$$p_{3n-2} = p_{-2} \prod_{j=0}^{n-1} \frac{1-3(n-j-1)\,q_0 r_{-1} p_{-2}}{1-(3(n-j)-2)\,q_0 r_{-1} p_{-2}},$$

$$q_{3n} = q_0 \prod_{j=0}^{n-1} \frac{1-(3(n-j)-1)\,q_0 r_{-1} p_{-2}}{1-3(n-j)\,q_0 r_{-1} p_{-2}},$$

$$q_{3n-1} = q_{-1} \prod_{j=0}^{n-1} \frac{1-(3(n-j)-2)\,p_0 q_{-1} r_{-2}}{1-(3(n-j)-1)\,p_0 q_{-1} r_{-2}},$$

$$q_{3n-2} = q_{-2} \prod_{j=0}^{n-1} \frac{1-3(n-j-1)\,r_0 p_{-1} q_{-2}}{1-(3(n-j)-2)\,r_0 p_{-1} q_{-2}},$$

$$r_{3n} = r_0 \prod_{j=0}^{n-1} \frac{1-(3(n-j)-1)\,r_0 p_{-1} q_{-2}}{1-3(n-j)\,r_0 p_{-1} q_{-2}},$$

$$r_{3n-1} = r_{-1} \prod_{j=0}^{n-1} \frac{1-(3(n-j)-2)\,q_0 r_{-1} p_{-2}}{1-(3(n-j)-1)\,q_0 r_{-1} p_{-2}},$$

$$r_{3n-2} = r_{-2} \prod_{j=0}^{n-1} \frac{1-3(n-j-1)\,p_0 q_{-1} r_{-2}}{1-(3(n-j)-2)\,p_0 q_{-1} r_{-2}},$$

for $n \geq 1$.

The system under study, as titled in the subsection, is investigated when $k > 0$. The system is represented by the following equations:

$$p_{n+1} = \frac{p_{n-(6k+2)}}{1-q_{n-2k} r_{n-(4k+1)} p_{n-(6k+2)}}, q_{n+1} = \frac{q_{n-(6k+2)}}{1-r_{n-2k} p_{n-(4k+1)} q_{n-(6k+2)}}, \tag{2.12}$$

$$r_{n+1} = \frac{r_{n-(6k+2)}}{1-p_{n-2k} q_{n-(4k+1)} r_{n-(6k+2)}},$$

which is an extension of the system described in (2.10). Utilizing the scheme (2.9), the system (2.12) is reformulated as follows:

$$p_{(2k+1)(l+1)+t} = \frac{p_{(2k+1)(l-2)+t}}{1 - q_{(2k+1)l+t}r_{(2k+1)(l-1)+t}p_{(2k+1)(l-2)+t}},$$
$$q_{(2k+1)(l+1)+t} = \frac{q_{(2k+1)(l-2)+t}}{1 - r_{(2k+1)l+t}p_{(2k+1)(l-1)+t}q_{(2k+1)(l-2)+t}},$$
$$r_{(2k+1)(l+1)+t} = \frac{r_{(2k+1)(l-2)+t}}{1 - p_{(2k+1)l+t}q_{(2k+1)(l-1)+t}r_{(2k+1)(l-2)+t}},$$

where t ranges from 1 to $2k+1$, and l is a non-negative integer. This discussion leads to the introduction of the following theorem.

Theorem 2.7. *Suppose that $\{p_n, q_n, r_n\}$ are solutions to the system represented in (2.10). Additionally, consider arbitrary nonzero real numbers p_{-l}, q_{-l}, r_{-l} for $l \in \{0, 1, \dots, 6k+2\}$ as the initial values. In this case, the solutions for the system (2.10) can be expressed as follows:*

$$p_{3(2k+1)l+t} = p_t \prod_{j=0}^{l-1} \frac{1 - (3(l-j)-1)\, p_t q_{t-(2k+1)} r_{t-2(2k+1)}}{1 - 3(l-j)\, p_t q_{t-(2k+1)} r_{t-2(2k+1)}},$$
$$p_{(2k+1)(3l-1)+t} = p_{t-(2k+1)} \prod_{j=0}^{l-1} \frac{1 - (3(l-j)-2)\, r_t p_{t-(2k+1)} q_{t-2(2k+1)}}{1 - (3(l-j)-1)\, r_t p_{t-(2k+1)} q_{t-2(2k+1)}},$$
$$p_{(2k+1)(3l-2)+t} = p_{t-2(2k+1)} \prod_{j=0}^{l-1} \frac{1 - 3(l-j-1)\, q_t r_{t-(2k+1)} p_{t-2(2k+1)}}{1 - (3(l-j)-2)\, q_t r_{t-(2k+1)} p_{t-2(2k+1)}},$$

$$q_{3(2k+1)l+t} = q_t \prod_{j=0}^{l-1} \frac{1 - (3(l-j)-1)\, q_t r_{t-(2k+1)} p_{t-2(2k+1)}}{1 - 3(l-j)\, q_t r_{t-(2k+1)} p_{t-2(2k+1)}},$$
$$q_{(2k+1)(3l-1)+t} = q_{t-(2k+1)} \prod_{j=0}^{l-1} \frac{1 - (3(l-j)-2)\, p_t q_{t-(2k+1)} r_{t-2(2k+1)}}{1 - (3(l-j)-1)\, p_t q_{t-(2k+1)} r_{t-2(2k+1)}},$$
$$q_{(2k+1)(3l-2)+t} = q_{t-2(k2+1)} \prod_{j=0}^{l-1} \frac{1 - 3(l-j-1)\, r_t p_{t-(2k+1)} q_{t-2(2k+1)}}{1 - (3(l-j)-2)\, r_t p_{t-(2k+1)} q_{t-2(2k+1)}},$$

$$r_{3(2k+1)l+t} = r_t \prod_{j=0}^{l-1} \frac{1 - (3(l-j)-1)\, r_t p_{t-(2k+1)} q_{t-2(2k+1)}}{1 - 3(l-j)\, r_t p_{t-(2k+1)} q_{t-2(2k+1)}},$$
$$r_{(2k+1)(3l-1)+t} = r_{t-(2k+1)} \prod_{j=0}^{l-1} \frac{1 - (3(l-j)-2)\, q_t r_{t-(2k+1)} p_{t-2(2k+1)}}{1 - (3(l-j)-1)\, q_t r_{t-(2k+1)} p_{t-2(2k+1)}},$$
$$r_{(2k+1)(3l-2)+t} = r_{t-2(2k+1)} \prod_{j=0}^{l-1} \frac{1 - 3(l-j-1)\, p_t q_{t-(2k+1)} r_{t-2(2k+1)}}{1 - (3(l-j)-2)\, p_t q_{t-(2k+1)} r_{t-2(2k+1)}},$$

for $n \geq 1$, $t \in \{1, \dots, 2k+1\}$.

2.3. **System** (1.1) **when** $\kappa = -\tau = -1$. In this subsection, it is assumed that $\{p_n, q_n, r_n\}$ represent a solution to the following system

$$p_{n+1} = \frac{p_{n-2}}{-1 + q_n r_{n-1} p_{n-2}}, q_{n+1} = \frac{q_{n-2}}{-1 + r_n p_{n-1} q_{n-2}}, r_{n+1} = \frac{r_{n-2}}{-1 + p_n q_{n-1} r_{n-2}}, \tag{2.13}$$

which is considered as special cases of system (1.1) when $\kappa = -\tau = -1$ and $k = 0$. Using the notations (2.2), we obtain the following equivalent system

$$\alpha_{n+1} = \frac{\beta_n}{-1 + \beta_n}, \beta_{n+1} = \frac{\gamma_n}{-1 + \gamma_n}, \gamma_{n+1} = \frac{\alpha_n}{-1 + \alpha_n}. \tag{2.14}$$

The solutions to the last system of difference equations are provided in the following theorem:

Theorem 2.8. *Let* $\{\alpha_n, \beta_n, \gamma_n\}_{n\geq 0}$ *be solutions of system* (2.13). *Then* $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ *and* $\{\gamma_n\}_{n\geq 0}$ *are given by the following formulas for* $n = 0, 1, \ldots$

$$\{\alpha_n\}_{n\geq 0} : \begin{cases} \alpha_{3n} = I_n^{(1)}(\alpha_0) \\ \alpha_{3n+1} = I_n^{(2)}(\beta_0) \\ \alpha_{3n+2} = I_n^{(1)}(\gamma_0) \end{cases}, \{\beta_n\}_{n\geq 0} : \begin{cases} \beta_{3n} = I_n^{(1)}(\beta_0) \\ \beta_{3n+1} = I_n^{(2)}(\gamma_0) \\ \beta_{3n+2} = I_n^{(1)}(\alpha_0) \end{cases} \text{ and } \{\gamma_n\}_{n\geq 0} : \begin{cases} \gamma_{3n} = I_n^{(1)}(\gamma_0) \\ \gamma_{3n+1} = I_n^{(2)}(\alpha_0) \\ \gamma_{3n+2} = I_n^{(1)}(\beta_0) \end{cases},$$

where $I_n^{(1)}(x) = \dfrac{x}{(-1)^n + (n - 2[n/2])x}$, $I_n^{(2)}(x) = \dfrac{x}{(-1)^{n+1} - (n - 1 - 2[n/2])x}$, α_0, β_0 *and* γ_0 *are computed from* (2.2).

Proof. By replacing the expression derived from the last recurrence relation in (2.3) into the second equation, and subsequently incorporating this result into the first recurrence relation in (2.3), we arrive at the following:

$$\alpha_{n+1} = \frac{\alpha_{n-2}}{-1 + \alpha_{n-2}}, \; n \geq 2.$$

Likewise, we get

$$\beta_{n+1} = \frac{\beta_{n-2}}{-1 + \beta_{n-2}}, \; \gamma_{n+1} = \frac{\gamma_{n-2}}{-1 + \gamma_{n-2}}, n \geq 2.$$

Using the follownig notations $\alpha_{n,l} = \alpha_{3n+l}$, $\beta_{n,l} = \beta_{3n+l}$ and $\gamma_{n,l} = \gamma_{3n+l}$, for $n \geq 0$ with $l = 0, 1, 2$, we obtain:

$$\alpha_{n+1,l} = \frac{\alpha_{n,l}}{-1 + \alpha_{n,l}}, \; \beta_{n+1,l} = \frac{\beta_{n,l}}{-1 + \beta_{n,l}}, \; \gamma_{n+1,l} = \frac{\gamma_{n,l}}{-1 + \gamma_{n,l}},$$

for $n \geq 0$ with $l = 0, 1, 2$. Since the three recurrence relations are similar, let's use the first recurrence relation for the next transformation $\alpha_{n,l} - 1 = \widetilde{\alpha}_n / \widetilde{\alpha}_{n-1}$. By simplifying, we obtain:

$$\widetilde{\alpha}_{n+1} - \widetilde{\alpha}_{n-1} = 0, \; n \geq 0.$$

So, we have $\widetilde{\alpha}_n = \dfrac{\widetilde{\alpha}_0 + (\widetilde{\alpha}_0 - 1)(-1)^n + 1}{\widetilde{\alpha}_0 - (\widetilde{\alpha}_0 - 1)(-1)^n + 1}$ and $\alpha_{n,l} = \dfrac{\alpha_{0,l}}{(-1)^n + \left(\sum\limits_{j=0}^{n-1} (-1)^j\right)\alpha_{0,l}}$, for $l = 0, 1, 2$. Therefore, the proof is complete. □

From Theorems (2.2) and (2.8), the following theorem is established concerning the structure of solutions for the system denoted by (2.13).

Theorem 2.9. *Assume* $\{p_n, q_n, r_n\}$ *are solutions to the system represented in* (2.13)*. Additionally, consider arbitrary nonzero real numbers* p_{-l}*,* q_{-l}*,* r_{-l} *for* $l \in \{0,1,2\}$ *as the initial values. In this case, the solutions for the system* (2.13) *can be expressed as follows:*

$$p_{3n} = p_0 \prod_{j=0}^{n-1} \frac{(-1)^{n-j-1} + (n-j-1-2[(n-j-1)/2])\, p_0q_{-1}r_{-2}}{(-1)^{n-j} + (n-j-2[(n-j)/2])\, p_0q_{-1}r_{-2}},$$

$$p_{3n-1} = p_{-1} \prod_{j=0}^{n-1} \frac{(-1)^{n-j} - (n-j-2-2[(n-j-1)/2])\, r_0p_{-1}q_{-2}}{(-1)^{n-j-1} + (n-j-1-2[(n-j-1)/2])\, r_0p_{-1}q_{-2}},$$

$$p_{3n-2} = p_{-2} \prod_{j=0}^{n-1} \frac{(-1)^{n-j-1} + (n-j-1-2[(n-j-1)/2])\, q_0r_{-1}p_{-2}}{(-1)^{n-j} - (n-j-2-2[(n-j-1)/2])\, q_0r_{-1}p_{-2}},$$

$$q_{3n} = q_0 \prod_{j=0}^{n-1} \frac{(-1)^{n-j-1} + (n-j-1-2[(n-j-1)/2])\, q_0r_{-1}p_{-2}}{(-1)^{n-j} + (n-j-2[(n-j)/2])\, q_0r_{-1}p_{-2}},$$

$$q_{3n-1} = q_{-1} \prod_{j=0}^{n-1} \frac{(-1)^{n-j} - (n-j-2-2[(n-j-1)/2])\, p_0q_{-1}r_{-2}}{(-1)^{n-j-1} + (n-j-1-2[(n-j-1)/2])\, p_0q_{-1}r_{-2}},$$

$$q_{3n-2} = q_{-2} \prod_{j=0}^{n-1} \frac{(-1)^{n-j-1} + (n-j-1-2[(n-j-1)/2])\, r_0p_{-1}q_{-2}}{(-1)^{n-j} - (n-j-2-2[(n-j-1)/2])\, r_0p_{-1}q_{-2}},$$

$$r_{3n} = r_0 \prod_{j=0}^{n-1} \frac{(-1)^{n-j-1} + (n-j-1-2[(n-j-1)/2])\, r_0p_{-1}q_{-2}}{(-1)^{n-j} + (n-j-2[(n-j)/2])\, r_0p_{-1}q_{-2}},$$

$$r_{3n-1} = r_{-1} \prod_{j=0}^{n-1} \frac{(-1)^{n-j} - (n-j-2-2[(n-j-1)/2])\, q_0r_{-1}p_{-2}}{(-1)^{n-j-1} + (n-j-1-2[(n-j-1)/2])\, q_0r_{-1}p_{-2}},$$

$$r_{3n-2} = r_{-2} \prod_{j=0}^{n-1} \frac{(-1)^{n-j-1} + (n-j-1-2[(n-j-1)/2])\, p_0q_{-1}r_{-2}}{(-1)^{n-j} - (n-j-2-2[(n-j-1)/2])\, p_0q_{-1}r_{-2}},$$

for $n \geq 1$.

The system under study, as titled in the subsection, is investigated when $k > 0$. The system is represented by the following equations:

$$p_{n+1} = \frac{p_{n-(6k+2)}}{-1 + q_{n-2k}r_{n-(4k+1)}p_{n-(6k+2)}}, q_{n+1} = \frac{q_{n-(6k+2)}}{-1 + r_{n-2k}p_{n-(4k+1)}q_{n-(6k+2)}}, \quad (2.15)$$

$$r_{n+1} = \frac{r_{n-(6k+2)}}{-1 + p_{n-2k}q_{n-(4k+1)}r_{n-(6k+2)}},$$

which is an extension of the system described in (2.13). Utilizing the scheme (2.9), the system (2.15) is reformulated as follows:

$$p_{(2k+1)(l+1)+t} = \frac{p_{(2k+1)(l-2)+t}}{-1 + q_{(2k+1)l+t} r_{(2k+1)(l-1)+t} p_{(2k+1)(l-2)+t}},$$
$$q_{(2k+1)(l+1)+t} = \frac{q_{(2k+1)(l-2)+t}}{-1 + r_{(2k+1)l+t} p_{(2k+1)(l-1)+t} q_{(2k+1)(l-2)+t}},$$
$$r_{(2k+1)(l+1)+t} = \frac{r_{(2k+1)(l-2)+t}}{-1 + p_{(2k+1)l+t} q_{(2k+1)(l-1)+t} r_{(2k+1)(l-2)+t}},$$

where t ranges from 1 to $2k+1$, and l is a non-negative integer. This discussion leads to the introduction of the following theorem.

Theorem 2.10. *Suppose that $\{p_n, q_n, r_n\}$ are solutions to the system represented in (2.13). Additionally, consider arbitrary nonzero real numbers p_{-l}, q_{-l}, r_{-l} for $l \in \{0, 1, \ldots, 6k+2\}$ as the initial values. In this case, the solutions for the system (2.13) can be expressed as follows:*

$$p_{3(2k+1)l+t} = p_t \prod_{j=0}^{l-1} \frac{(-1)^{l-j-1} + (l-j-1-2\left[(l-j-1)/2\right]) p_t q_{t-(2k+1)} r_{t-2(2k+1)}}{(-1)^{l-j} + (l-j-2\left[(l-j)/2\right]) p_t q_{t-(2k+1)} r_{t-2(2k+1)}},$$
$$p_{(2k+1)(3l-1)+t} = p_{t-(2k+1)} \prod_{j=0}^{l-1} \frac{(-1)^{l-j} - (l-j-2-2\left[(l-j-1)/2\right]) r_t p_{t-(2k+1)} q_{t-2(2k+1)}}{(-1)^{l-j-1} + (l-j-1-2\left[(l-j-1)/2\right]) r_t p_{t-(2k+1)} q_{t-2(2k+1)}},$$
$$p_{(2k+1)(3l-2)+t} = p_{t-2(2k+1)} \prod_{j=0}^{l-1} \frac{(-1)^{l-j-1} + (l-j-1-2\left[(l-j-1)/2\right]) q_t r_{t-(2k+1)} p_{t-2(2k+1)}}{(-1)^{l-j} - (l-j-2-2\left[(l-j-1)/2\right]) q_t r_{t-(2k+1)} p_{t-2(2k+1)}},$$

$$q_{3(2k+1)l+t} = q_t \prod_{j=0}^{l-1} \frac{(-1)^{l-j-1} + (l-j-1-2\left[(l-j-1)/2\right]) q_t r_{t-(2k+1)} p_{t-2(2k+1)}}{(-1)^{l-j} + (l-j-2\left[(l-j)/2\right]) q_t r_{t-(2k+1)} p_{t-2(2k+1)}},$$
$$q_{(2k+1)(3l-1)+t} = q_{t-(2k+1)} \prod_{j=0}^{l-1} \frac{(-1)^{l-j} - (l-j-2-2\left[(l-j-1)/2\right]) p_t q_{t-(2k+1)} r_{t-2(2k+1)}}{(-1)^{l-j-1} + (l-j-1-2\left[(l-j-1)/2\right]) p_t q_{t-(2k+1)} r_{t-2(2k+1)}},$$
$$q_{(2k+1)(3l-2)+t} = q_{t-2(k2+1)} \prod_{j=0}^{l-1} \frac{(-1)^{l-j-1} + (l-j-1-2\left[(l-j-1)/2\right]) r_t p_{t-(2k+1)} q_{t-2(2k+1)}}{(-1)^{l-j} - (l-j-2-2\left[(l-j-1)/2\right]) r_t p_{t-(2k+1)} q_{t-2(2k+1)}},$$

$$r_{3(2k+1)l+t} = r_t \prod_{j=0}^{l-1} \frac{(-1)^{l-j-1} + (l-j-1-2\left[(l-j-1)/2\right]) r_t p_{t-(2k+1)} q_{t-2(2k+1)}}{(-1)^{l-j} + (l-j-2\left[(l-j)/2\right]) r_t p_{t-(2k+1)} q_{t-2(2k+1)}},$$
$$r_{(2k+1)(3l-1)+t} = r_{t-(2k+1)} \prod_{j=0}^{l-1} \frac{(-1)^{l-j} - (l-j-2-2\left[(l-j-1)/2\right]) q_t r_{t-(2k+1)} p_{t-2(2k+1)}}{(-1)^{l-j-1} + (l-j-1-2\left[(l-j-1)/2\right]) q_t r_{t-(2k+1)} p_{t-2(2k+1)}},$$
$$r_{(2k+1)(3l-2)+t} = r_{t-2(2k+1)} \prod_{j=0}^{l-1} \frac{(-1)^{l-j-1} + (l-j-1-2\left[(l-j-1)/2\right]) p_t q_{t-(2k+1)} r_{t-2(2k+1)}}{(-1)^{l-j} - (l-j-2-2\left[(l-j-1)/2\right]) p_t q_{t-(2k+1)} r_{t-2(2k+1)}},$$

for $n \geq 1$, $t \in \{1, \ldots, 2k+1\}$.

2.4. **System** (1.1) **when** $\kappa = \tau = -1$. In this subsection, it is assumed that $\{p_n, q_n, r_n\}$ represent a solution to the following system

$$p_{n+1} = \frac{p_{n-2}}{-1 - q_n r_{n-1} p_{n-2}}, q_{n+1} = \frac{q_{n-2}}{-1 - r_n p_{n-1} q_{n-2}}, r_{n+1} = \frac{r_{n-2}}{-1 - p_n q_{n-1} r_{n-2}}, \tag{2.16}$$

which is considered as special cases of system (1.1) when $\kappa = \tau = -1$ and $k = 0$. Using the notations (2.2), we obtain the following equivalent system

$$\alpha_{n+1} = \frac{\beta_n}{-1 - \beta_n}, \beta_{n+1} = \frac{\gamma_n}{-1 - \gamma_n}, \gamma_{n+1} = \frac{\alpha_n}{-1 - \alpha_n}. \tag{2.17}$$

The solutions to the last system of difference equations are provided in the following theorem:

Theorem 2.11. *Let $\{\alpha_n, \beta_n, \gamma_n\}_{n\geq 0}$ be solutions of system* (2.16). *Then $\{\alpha_n\}_{n\geq 0}$, $\{\beta_n\}_{n\geq 0}$ and $\{\gamma_n\}_{n\geq 0}$ are given by the following formulas for $n = 0, 1, \ldots$*

$$\{\alpha_n\}_{n\geq 0} : \begin{cases} \alpha_{3n} = F_n^{(1)}(\alpha_0) \\ \alpha_{3n+1} = F_n^{(2)}(\beta_0) \\ \alpha_{3n+2} = F_n^{(1)}(\gamma_0) \end{cases}, \{\beta_n\}_{n\geq 0} : \begin{cases} \beta_{3n} = F_n^{(1)}(\beta_0) \\ \beta_{3n+1} = F_n^{(2)}(\gamma_0) \\ \beta_{3n+2} = F_n^{(1)}(\alpha_0) \end{cases} \text{and } \{\gamma_n\}_{n\geq 0} : \begin{cases} \gamma_{3n} = F_n^{(1)}(\gamma_0) \\ \gamma_{3n+1} = F_n^{(2)}(\alpha_0) \\ \gamma_{3n+2} = F_n^{(1)}(\beta_0) \end{cases},$$

where $F_n^{(1)}(x) = \dfrac{x}{(-1)^n - (n - 2[n/2])\, x}$, $F_n^{(2)}(x) = \dfrac{x}{(-1)^{n+1} + (n - 1 - 2[n/2])\, x}$, and α_0, β_0 and γ_0 are computed from (2.2).

Proof. By replacing the expression derived from the last recurrence relation in (2.3) into the second equation, and subsequently incorporating this result into the first recurrence relation in (2.3), we arrive at the following:

$$\alpha_{n+1} = \frac{\alpha_{n-2}}{-1 - \alpha_{n-2}},\ n \geq 2.$$

Likewise, we get

$$\beta_{n+1} = \frac{\beta_{n-2}}{-1 - \beta_{n-2}},\ \gamma_{n+1} = \frac{\gamma_{n-2}}{-1 - \gamma_{n-2}}, n \geq 2.$$

Using the follownig notations $\alpha_{n,l} = \alpha_{3n+l}$, $\beta_{n,l} = \beta_{3n+l}$ and $\gamma_{n,l} = \gamma_{3n+l}$, for $n \geq 0$ with $l = 0, 1, 2$, we obtain:

$$\alpha_{n+1,l} = \frac{\alpha_{n,l}}{-1 - \alpha_{n,l}},\ \beta_{n+1,l} = \frac{\beta_{n,l}}{-1 - \beta_{n,l}},\ \gamma_{n+1,l} = \frac{\gamma_{n,l}}{-1 - \gamma_{n,l}},$$

for $n \geq 0$ with $l = 0, 1, 2$. Since the three recurrence relations are similar, let's use the first recurrence relation for the next transformation $-1 - \alpha_{n,l} = \widetilde{\alpha}_n / \widetilde{\alpha}_{n-1}$. By simplifying, we obtain the same difference equation: $\widetilde{\alpha}_{n+1} - \widetilde{\alpha}_{n-1} = 0$, $n \geq 0$. So, we have $\widetilde{\alpha}_n = \dfrac{\widetilde{\alpha}_0 + (\widetilde{\alpha}_0 - 1)(-1)^n + 1}{\widetilde{\alpha}_0 - (\widetilde{\alpha}_0 - 1)(-1)^n + 1}$ and $\alpha_{n,l} = \dfrac{\alpha_{0,l}}{(-1)^n - \left(\sum_{j=0}^{n-1} (-1)^j\right) \alpha_{0,l}}$, for $l = 0, 1, 2$. Therefore, the proof is complete. □

From Theorems (2.2) and (2.11), the following theorem is established concerning the structure of solutions for the system denoted by (2.16).

Theorem 2.12. *Assume $\{p_n, q_n, r_n\}$ are solutions to the system represented in (2.16). Additionally, consider arbitrary nonzero real numbers p_{-l}, q_{-l}, r_{-l} for $l \in \{0,1,2\}$ as the initial values. In this case, the solutions for the system (2.16) can be expressed as follows:*

$$p_{3n} = p_0 \prod_{j=0}^{n-1} \frac{(-1)^{n-j-1} - (n-j-1-2\left[(n-j-1)/2\right]) p_0 q_{-1} r_{-2}}{(-1)^{n-j} - (n-j-2\left[(n-j)/2\right]) p_0 q_{-1} r_{-2}},$$

$$p_{3n-1} = p_{-1} \prod_{j=0}^{n-1} \frac{(-1)^{n-j} + (n-j-2-2\left[(n-j-1)/2\right]) r_0 p_{-1} q_{-2}}{(-1)^{n-j-1} - (n-j-1-2\left[(n-j-1)/2\right]) r_0 p_{-1} q_{-2}},$$

$$p_{3n-2} = p_{-2} \prod_{j=0}^{n-1} \frac{(-1)^{n-j-1} - (n-j-1-2\left[(n-j-1)/2\right]) q_0 r_{-1} p_{-2}}{(-1)^{n-j} + (n-j-2-2\left[(n-j-1)/2\right]) q_0 r_{-1} p_{-2}},$$

$$q_{3n} = q_0 \prod_{j=0}^{n-1} \frac{(-1)^{n-j-1} - (n-j-1-2\left[(n-j-1)/2\right]) q_0 r_{-1} p_{-2}}{(-1)^{n-j} - (n-j-2\left[(n-j)/2\right]) q_0 r_{-1} p_{-2}},$$

$$q_{3n-1} = q_{-1} \prod_{j=0}^{n-1} \frac{(-1)^{n-j} + (n-j-2-2\left[(n-j-1)/2\right]) p_0 q_{-1} r_{-2}}{(-1)^{n-j-1} - (n-j-1-2\left[(n-j-1)/2\right]) p_0 q_{-1} r_{-2}},$$

$$q_{3n-2} = q_{-2} \prod_{j=0}^{n-1} \frac{(-1)^{n-j-1} - (n-j-1-2\left[(n-j-1)/2\right]) r_0 p_{-1} q_{-2}}{(-1)^{n-j} + (n-j-2-2\left[(n-j-1)/2\right]) r_0 p_{-1} q_{-2}},$$

$$r_{3n} = r_0 \prod_{j=0}^{n-1} \frac{(-1)^{n-j-1} - (n-j-1-2\left[(n-j-1)/2\right]) r_0 p_{-1} q_{-2}}{(-1)^{n-j} - (n-j-2\left[(n-j)/2\right]) r_0 p_{-1} q_{-2}},$$

$$r_{3n-1} = r_{-1} \prod_{j=0}^{n-1} \frac{(-1)^{n-j} + (n-j-2-2\left[(n-j-1)/2\right]) q_0 r_{-1} p_{-2}}{(-1)^{n-j-1} - (n-j-1-2\left[(n-j-1)/2\right]) q_0 r_{-1} p_{-2}},$$

$$r_{3n-2} = r_{-2} \prod_{j=0}^{n-1} \frac{(-1)^{n-j-1} - (n-j-1-2\left[(n-j-1)/2\right]) p_0 q_{-1} r_{-2}}{(-1)^{n-j} + (n-j-2-2\left[(n-j-1)/2\right]) p_0 q_{-1} r_{-2}},$$

for $n \geq 1$.

The system under study, as titled in the subsection, is investigated when $k > 0$. The system is represented by the following equations:

$$p_{n+1} = \frac{p_{n-(6k+2)}}{-1 - q_{n-2k} r_{n-(4k+1)} p_{n-(6k+2)}}, q_{n+1} = \frac{q_{n-(6k+2)}}{-1 - r_{n-2k} p_{n-(4k+1)} q_{n-(6k+2)}}, \quad (2.18)$$
$$r_{n+1} = \frac{r_{n-(6k+2)}}{-1 - p_{n-2k} q_{n-(4k+1)} r_{n-(6k+2)}},$$

which is an extension of the system described in (2.16). Utilizing the scheme (2.9), the system (2.18) is reformulated as follows:

$$p_{(2k+1)(l+1)+t} = \frac{p_{(2k+1)(l-2)+t}}{-1 - q_{(2k+1)l+t} r_{(2k+1)(l-1)+t} p_{(2k+1)(l-2)+t}},$$

$$q_{(2k+1)(l+1)+t} = \frac{q_{(2k+1)(l-2)+t}}{-1 - r_{(2k+1)l+t}p_{(2k+1)(l-1)+t}q_{(2k+1)(l-2)+t}},$$
$$r_{(2k+1)(l+1)+t} = \frac{r_{(2k+1)(l-2)+t}}{-1 - p_{(2k+1)l+t}q_{(2k+1)(l-1)+t}r_{(2k+1)(l-2)+t}},$$

where t ranges from 1 to $2k+1$, and l is a non-negative integer. This discussion leads to the introduction of the following theorem.

Theorem 2.13. *Suppose that $\{p_n, q_n, r_n\}$ are solutions to the system represented in (2.16). Additionally, consider arbitrary nonzero real numbers p_{-l}, q_{-l}, r_{-l} for $l \in \{0, 1, \ldots, 6k+2\}$ as the initial values. In this case, the solutions for the system (2.16) can be expressed as follows:*

$$p_{3(2k+1)l+t} = p_t \prod_{j=0}^{l-1} \frac{(-1)^{l-j-1} - (l-j-1-2\left[(l-j-1)/2\right]) p_t q_{t-(2k+1)} r_{t-2(2k+1)}}{(-1)^{l-j} - (l-j-2\left[(l-j)/2\right]) p_t q_{t-(2k+1)} r_{t-2(2k+1)}},$$
$$p_{(2k+1)(3l-1)+t} = p_{t-(2k+1)} \prod_{j=0}^{l-1} \frac{(-1)^{l-j} + (l-j-2-2\left[(l-j-1)/2\right]) r_t p_{t-(2k+1)} q_{t-2(2k+1)}}{(-1)^{l-j-1} - (l-j-1-2\left[(l-j-1)/2\right]) r_t p_{t-(2k+1)} q_{t-2(2k+1)}},$$
$$p_{(2k+1)(3l-2)+t} = p_{t-2(2k+1)} \prod_{j=0}^{l-1} \frac{(-1)^{l-j-1} - (l-j-1-2\left[(l-j-1)/2\right]) q_t r_{t-(2k+1)} p_{t-2(2k+1)}}{(-1)^{l-j} + (l-j-2-2\left[(l-j-1)/2\right]) q_t r_{t-(2k+1)} p_{t-2(2k+1)}},$$

$$q_{3(2k+1)l+t} = q_t \prod_{j=0}^{l-1} \frac{(-1)^{l-j-1} - (l-j-1-2\left[(l-j-1)/2\right]) q_t r_{t-(2k+1)} p_{t-2(2k+1)}}{(-1)^{l-j} - (l-j-2\left[(l-j)/2\right]) q_t r_{t-(2k+1)} p_{t-2(2k+1)}},$$
$$q_{(2k+1)(3l-1)+t} = q_{t-(2k+1)} \prod_{j=0}^{l-1} \frac{(-1)^{l-j} + (l-j-2-2\left[(l-j-1)/2\right]) p_t q_{t-(2k+1)} r_{t-2(2k+1)}}{(-1)^{l-j-1} - (l-j-1-2\left[(l-j-1)/2\right]) p_t q_{t-(2k+1)} r_{t-2(2k+1)}},$$
$$q_{(2k+1)(3l-2)+t} = q_{t-2(k2+1)} \prod_{j=0}^{l-1} \frac{(-1)^{l-j-1} - (l-j-1-2\left[(l-j-1)/2\right]) r_t p_{t-(2k+1)} q_{t-2(2k+1)}}{(-1)^{l-j} + (l-j-2-2\left[(l-j-1)/2\right]) r_t p_{t-(2k+1)} q_{t-2(2k+1)}},$$

$$r_{3(2k+1)l+t} = r_t \prod_{j=0}^{l-1} \frac{(-1)^{l-j-1} - (l-j-1-2\left[(l-j-1)/2\right]) r_t p_{t-(2k+1)} q_{t-2(2k+1)}}{(-1)^{l-j} - (l-j-2\left[(l-j)/2\right]) r_t p_{t-(2k+1)} q_{t-2(2k+1)}},$$
$$r_{(2k+1)(3l-1)+t} = r_{t-(2k+1)} \prod_{j=0}^{l-1} \frac{(-1)^{l-j} + (l-j-2-2\left[(l-j-1)/2\right]) q_t r_{t-(2k+1)} p_{t-2(2k+1)}}{(-1)^{l-j-1} - (l-j-1-2\left[(l-j-1)/2\right]) q_t r_{t-(2k+1)} p_{t-2(2k+1)}},$$
$$r_{(2k+1)(3l-2)+t} = r_{t-2(2k+1)} \prod_{j=0}^{l-1} \frac{(-1)^{l-j-1} - (l-j-1-2\left[(l-j-1)/2\right]) p_t q_{t-(2k+1)} r_{t-2(2k+1)}}{(-1)^{l-j} + (l-j-2-2\left[(l-j-1)/2\right]) p_t q_{t-(2k+1)} r_{t-2(2k+1)}},$$

for $n \geq 1$, $t \in \{1, \ldots, 2k+1\}$.

3. Numerical Examples

In this section, several numerical examples are provided to illustrate and support the theoretical results from the previous section.

Example 3.1. *The first example focuses on the difference equation system*

$$p_{n+1} = \frac{p_{n-8}}{1 + q_{n-2}r_{n-5}p_{n-8}}, q_{n+1} = \frac{q_{n-8}}{1 + r_{n-2}p_{n-5}q_{n-8}}, r_{n+1} = \frac{r_{n-8}}{1 + p_{n-2}q_{n-5}r_{n-8}}, \tag{3.1}$$

$n = 0, 1, \ldots$ *The starting point of the sequence is determined by the initial conditions, which are specified as follows:* $p_{-j} = 1.2j,$, $q_{-j} = 0.6j - 1$ *and* $r_{-j} = 1 - 0.09j$, $j = 0, \ldots, 8$. *The plot of the system* (3.1) *is shown in Figure* 1.

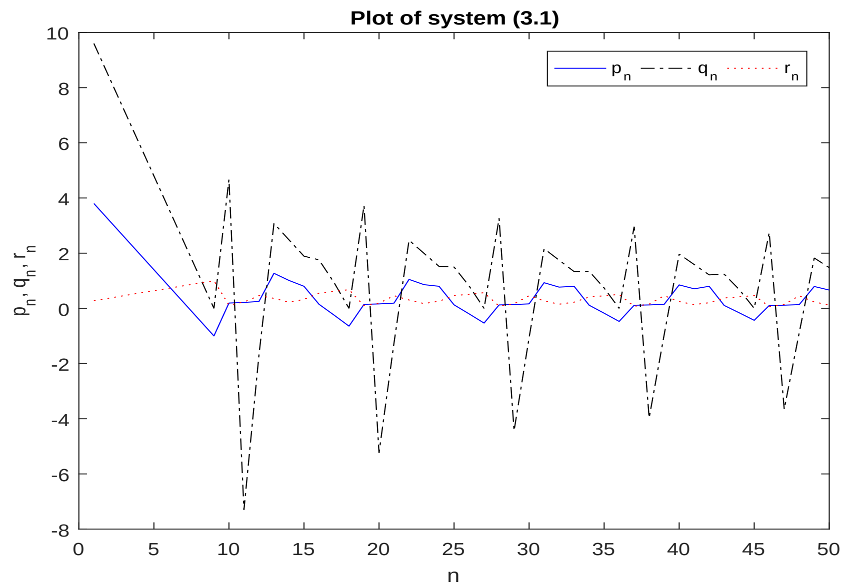

Figure1.This figure shows the solutions of the system represented by (3.1)

when $p_{-j} = 1.2j, , q_{-j} = 0.6j - 1$ *and* $r_{-j} = 1 - 0.09j, \ j = 0, \ldots, 8.$

Example 3.2. *The second example focuses on the difference equation system*

$$p_{n+1} = \frac{p_{n-8}}{1 - q_{n-2}r_{n-5}p_{n-8}}, q_{n+1} = \frac{q_{n-8}}{1 - r_{n-2}p_{n-5}q_{n-8}}, r_{n+1} = \frac{r_{n-8}}{1 - p_{n-2}q_{n-5}r_{n-8}}, \tag{3.2}$$

$n = 0, 1, \ldots$ *The starting point of the sequence is determined by the initial conditions, which are specified as follows:* $p_{-j} = 1.2j,$, $q_{-j} = -0.6j - 1$ *and* $r_{-j} = 1 + 0.09j$, $j = 0, \ldots, 8$. *The plot of the system* (3.2) *is shown in Figure* 2.

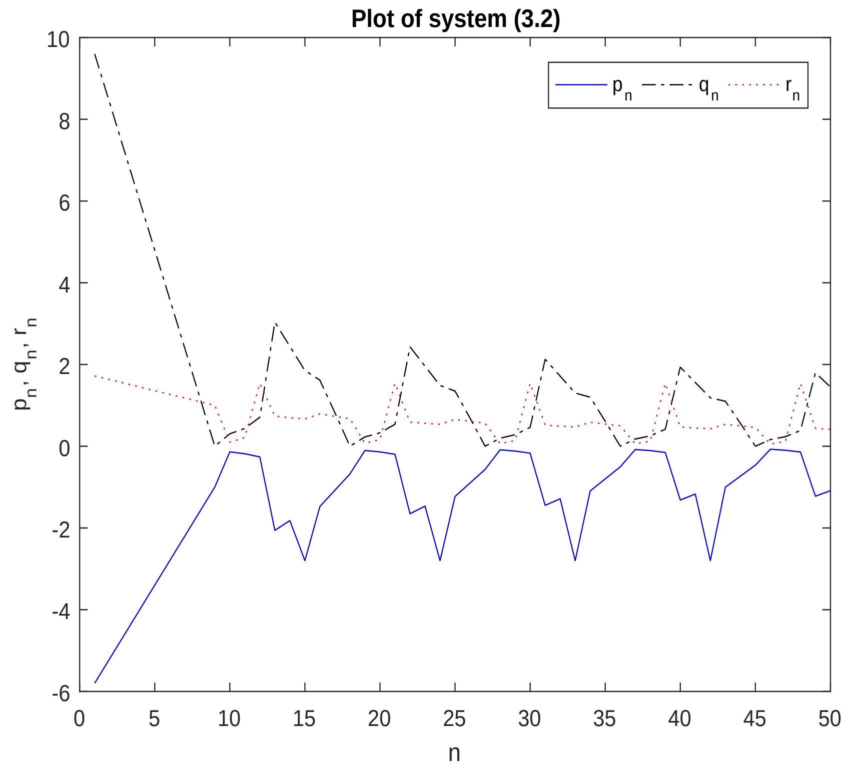

Figure2. This figure shows the solutions of the system represented by (3.2) *when* $p_{-j} = 1.2j, , q_{-j} = -0.6j - 1$ *and* $r_{-j} = 1 + 0.09j,\ j = 0, \ldots, 8$.

Example 3.3. *The third example focuses on the difference equation system*

$$p_{n+1} = \frac{p_{n-8}}{-1 + q_{n-2}r_{n-5}p_{n-8}}, q_{n+1} = \frac{q_{n-8}}{-1 + r_{n-2}p_{n-5}q_{n-8}}, r_{n+1} = \frac{r_{n-8}}{-1 + p_{n-2}q_{n-5}r_{n-8}}, \tag{3.3}$$

$n = 0, 1, \ldots$ *The starting point of the sequence is determined by the initial conditions, which are specified as follows:* $p_{-j} = 1.2j$,, $q_{-j} = 0.6j + 1$ *and* $r_{-j} = 1 - 0.09j$, $j = 0, \ldots, 8$. *The plot of the system* (3.3) *is shown in Figure* 3.

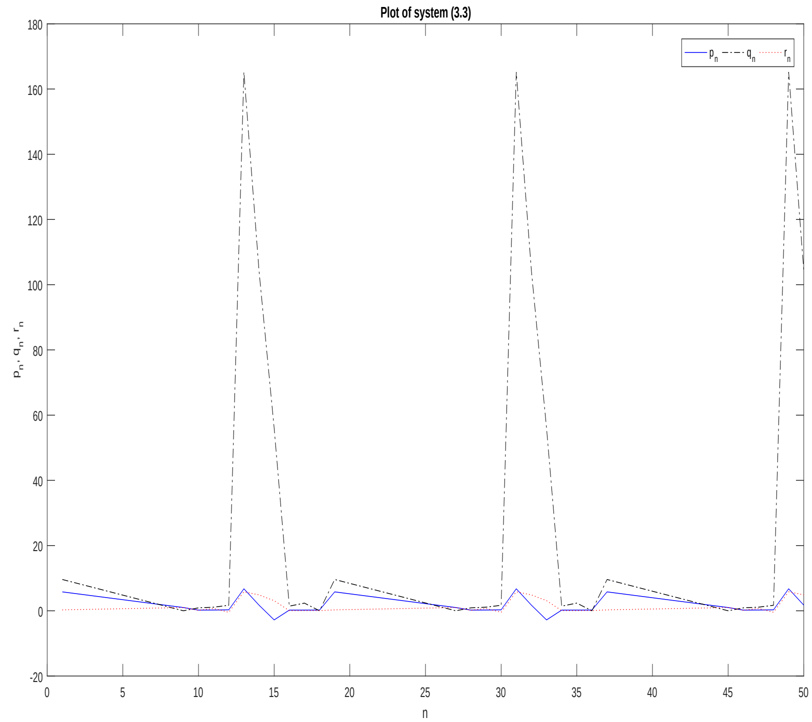

Figure3. This figure shows the solutions of the system represented by (3.3) *when* $p_{-j} = 1.2j, , q_{-j} = 0.6j + 1$ *and* $r_{-j} = 1 - 0.09j,\ j = 0, \ldots, 8.$

Example 3.4. *The fourth example focuses on the difference equation system*

$$p_{n+1} = \frac{p_{n-8}}{-1 - q_{n-2}r_{n-5}p_{n-8}}, q_{n+1} = \frac{q_{n-8}}{-1 - r_{n-2}p_{n-5}q_{n-8}}, r_{n+1} = \frac{r_{n-8}}{-1 - p_{n-2}q_{n-5}r_{n-8}}, \tag{3.4}$$

$n = 0, 1, \ldots$ *The starting point of the sequence is determined by the initial conditions, which are specified as follows:* $p_{-j} = 0.7j,$, $q_{-j} = -0.6j + 1$ *and* $r_{-j} = 1 - 0.09j$, $j = 0, \ldots, 8$. *The plot of the system* (3.4) *is shown in Figure* 4.

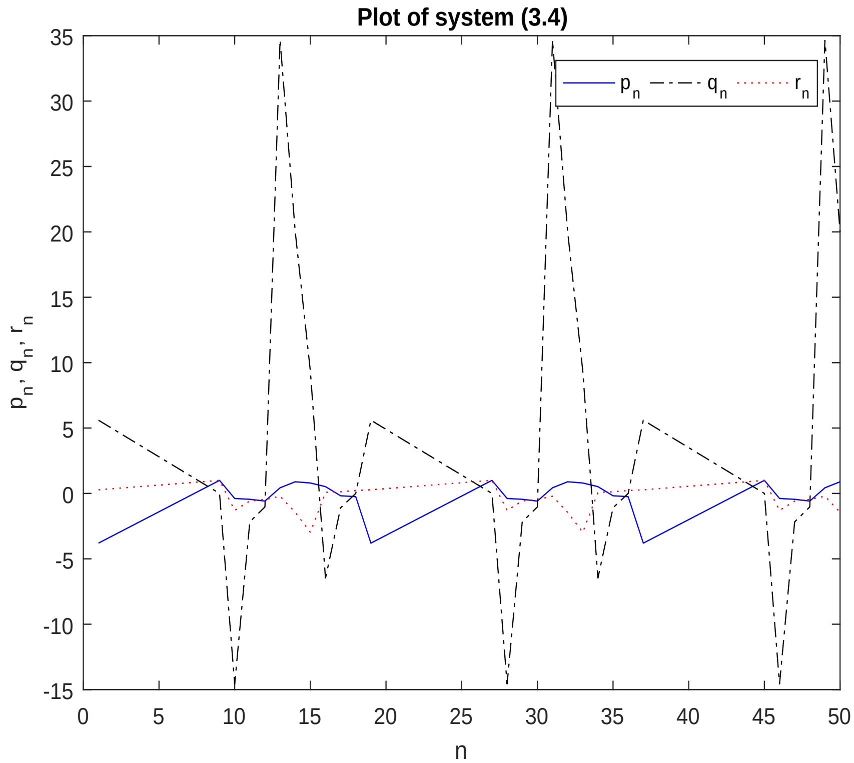

Figure4. This figure shows the solutions of the system represented by (3.4)

when $p_{-j} = 0.7j, , q_{-j} = -0.6j + 1$ *and* $r_{-j} = 1 - 0.09j, \; j = 0, \ldots, 8.$

4. Conclusion

This paper has formulated solutions to several systems of nonlinear difference equations in three dimensions. The obtained formulas are expressed as solutions to homogeneous linear difference equations with constant coefficients associated with the respective systems. It's worth noting that these methods can be extended to equations more general than those discussed in (1.1). For instance, the approach can be applied to $s-$dimensional systems of nonlinear difference equations,

$$\omega^{(1)}_{n+1} = \frac{\omega^{(1)}_{n-(6k+2)}}{\pm 1 \pm \omega^{(2)}_{n-2k}\omega^{(3)}_{n-(4k+1)}\omega^{(1)}_{n-(6k+2)}}, \omega^{(2)}_{n+1} = \frac{\omega^{(2)}_{n-(6k+2)}}{\pm 1 \pm \omega^{(3)}_{n-2k}\omega^{(4)}_{n-(4k+1)}\omega^{(2)}_{n-(6k+2)}}, \ldots,$$

$$\omega^{(s)}_{n+1} = \frac{\omega^{(s)}_{n-(6k+2)}}{\pm 1 \pm \omega^{(1)}_{n-2k}\omega^{(2)}_{n-(4k+1)}\omega^{(s)}_{n-(6k+2)}}, n \in \mathbb{N}_0, s \in \mathbb{N}^*,$$

with initial values $\omega^{(j)}_{-i}$, $i \in \{0, 1, \ldots, 3k+2\}$, $j \in \{1, 2, \ldots, s\}$ are arbitrary nonzero real numbers. In the context of open problems, it is worth noting that the solution of this type of nonlinear difference equations is closely related to the Fibonacci sequence. This intriguing connection introduces avenues for further investigation into the interplay between nonlinear dynamics and the well-known Fibonacci sequence, presenting open problems that could deepen our understanding of these mathematical structures.

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