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Almost $*$-Ricci Soliton on α-paraSasakian Manifold

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Abstract. It has been noted that if the $*$-Ricci tensor used to define $*$-Ricci soliton is a constant multiple of the metric tensor $g(e_i, e_j)$, for all e_i, e_j orthogonal to characteristic vector field ξ, then the manifold is $*$-Einstein manifold. The metric associated with $*$-Einstein manifold is $*$-Einstein metric, and the $*$-Ricci soliton is its generalization. In this paper we study an almost $*$-Ricci soliton $(\mathbf{g}, W, \lambda)$ and an almost gradient $*$-Ricci soliton $(\mathbf{g}, grad\,(\varrho), \lambda)$ by means of mathematical operators on $(2m+1)$-dimensional α-paraSasakian manifold $\mathbf{S}^{2m+1}$.

1. Introduction

A Ricci soliton is a self-similar solution to the Hamilton's Ricci flow equation. R. S. Hamilton in [1] given the evolution of a Riemannian metric over time t as

$$\frac{\partial}{\partial t}\mathbf{g}_{ij} = -2\mathrm{Ric}_{ij}, \tag{1.1}$$

here Ric_{ij} is the Ricci tensor associated to the metric tensor $\mathbf{g}_{ij}$. This partial differential equation is known as *Ricci flow equation*. Ricci solitons plays a significant role in understanding the singularity of equation (1.1).

Definition 1.1. *Let us consider a differentiable manifold $\mathbb{M}^{2m+1}$ with pseudo-Riemannian metric tensor $\mathbf{g}$, then $(\mathbf{g}, W, \lambda)$ which includes W as a vector field and λ as a smooth function, is known as an almost Ricci*

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soliton if

$$\pounds_W \mathbf{g} + 2\mathrm{Ric} - 2\lambda\, \mathbf{g} = 0, \tag{1.2}$$

here $\pounds_W$ *is the Lie-derivative in the direction of* W *and* Ric *is the Ricci tensor of* $\mathbf{g}$. *An almost Ricci soliton on* $\left(\mathbb{M}^{2m+1}, \mathbf{g}\right)$ *is expanding if* λ *is negative, steady if* λ *is zero or shrinking if* λ *is positive.*

Definition 1.2. *Let us consider a differentiable manifold* $\mathbb{M}^{2m+1}$ *with pseudo-Riemannian metric tensor* $\mathbf{g}$ *and taking* $W = D\varrho$, *for some smooth function* ϱ *on* $\mathbb{M}^{2m+1}$, *in the definition of an almost Ricci soliton, then* $(\mathbf{g}, D\varrho, \lambda)$ *is known as an almost gradient Ricci soliton if*

$$\nabla\nabla\varrho + \mathrm{Ric} - \lambda\, \mathbf{g} = 0. \tag{1.3}$$

Remark 1.1. *If* $W = 0$ *in equation* (1.2) *or* $\varrho = 0$ *in equation* (1.3), *we have* $\mathrm{Ric} = \lambda\mathbf{g}$, *which is the definition of an Einstein metric and soliton constant* λ *becomes an Einstein constant. Thus, Ricci soliton is a generalized notion of an Einstein metric, which has been a subject of intense study in differential geometry.*

Similar to Ricci soliton, $*$-Ricci soliton is a self-similar solution to partial differential equations known as *$*$-Ricci flow equation* and were firstly introduced by G. Kaimakamis and K. Panagiotidou [2], where they replace Ricci tensor in equation (1.2) by $*$-Ricci tensor and it is given as

$$\mathrm{Ric}^*(X_1, X_2) = \frac{1}{2}\left(trace\left\{\varphi \cdot \mathcal{R}(X_1, \varphi X_2)\right\}\right), \tag{1.4}$$

for any vector fields X_1 and X_2 on $\mathbb{M}^{2m+1}$. The concept of $*$-Ricci tensor has been given by S. Tachibana in [3] on almost Hermitian manifolds and again defined on real hypersurfaces in non-flat complex space forms by T. Hamada in [4]. Following that, other authors studied $*$-Ricci tensor and $*$-Ricci soliton in various ambient spaces [5–9, 11, 12, 14].

Almost Ricci solitons and almost gradient Ricci solitons were studied in both Riemannian and pseudo-Riemannian manifolds. Interest of theoretical physicist increases towards the study of Ricci solitons as (1.2) is a special case of an Einstien field equation. Several authors have been studied almost Ricci soliton and almost gradient Ricci soliton on paracontact manifolds [15,20–25]. The study of paracontact manifolds have been started in 1985 [28], and after that focused on paraSasakian manifolds. Recently, the authors of [29] have given the study of $*$-Ricci soliton and almost gradient $*$-Ricci soliton within the frame-work of Sasakian manifold.

In the fields of submanifold theory, soliton theory, tangent bundles, and related topics, numerous geometors have investigated geometric and topological characteristics concerning symmetry. Their works from references ([10, 15–20, 26, 27]) are a great place to start when looking for ideas and a desire to learn more about symmetry.

The above research works give us motivation to study almost $*$-Ricci solitons and almost gradient $*$-Ricci solitons on paracontact geometry, particularly, on α-paraSasakian manifold. α-paraSasakian manifold as a subclass of paracontact manifold have been defined by S. Zamkovoy and G. Nakova in [30].

Sectional study of this paper includes: In Sect.2, we give basic definition of a α-para-Sasakian manifold $\mathbf{S}^{2m+1}$ and its subclasses paraSasakian manifold and paraCosymplectic manifold $\mathbf{C}^{2m+1}$. We also give an example of $\mathbf{S}^{2m+1}$ for better understanding. Further, we find some curvature identities on $\mathbf{S}^{2m+1}$. In Sect.3, firstly we define an almost $*$-Ricci soliton $(\mathbf{g}, W, \lambda)$ on $\mathbf{S}^{2m+1}$ and then discuss some properties of $\mathbf{S}^{2m+1}$ with $(\mathbf{g}, W, \lambda)$. Also, we give an example of $*$-Ricci soliton on $\mathbf{S}^{2m+1}$. In Sect.4, we define Hessian of a smooth function which is used to define an almost gradient $*$-Ricci solitons $(\mathbf{g}, grad\,(\varrho)\,, \lambda)$ on $\mathbf{S}^{2m+1}$ and then discuss some properties of $\mathbf{S}^{2m+1}$ with $(\mathbf{g}, grad\,(\varrho)\,, \lambda)$. In last section, we give physical significance of a $*$-Ricci Soliton. Here are the following results we will focus in the present paper:

Theorem 1.1. *If $\mathbf{S}^{2m+1}$ is a α-paraSasakian manifold with an almost $*$-Ricci soliton $(\mathbf{g}, W, \lambda)$ which includes W as paracontact vector field, then the metric $\mathbf{g}$ is $*$-Ricci soliton.*

Theorem 1.2. *If $\mathbf{S}^{2m+1}$ is a α-paraSasakian manifold admitting a $*$-Ricci soliton $(\mathbf{g}, W, \lambda)$, then either the soliton vector field W is killing or leaves φ invariant.*

Theorem 1.3. *If $\mathbf{S}^{2m+1}$ is a α-paraSasakian manifold admitting an almost gradient $*$-Ricci soliton, then $\mathbf{S}^{2m+1}$ is quasi Einstein manifold.*

2. α-paraSasakian Manifold $\mathbf{S}^{2m+1}$

Consider a differentiable manifold $\mathbb{M}^{2m+1}$, then $\mathbb{M}^{2m+1}$ is known as an *almost paracontact manifold* if it is enriched with (φ, η, ξ)-structure (paracontact structure) and satisfies

$$\left.\begin{aligned} \eta\,(\xi) &= 1, \quad \varphi^2 = \mathcal{I} - \eta \otimes \xi, \\ \varphi\xi &= 0, \quad \eta \circ \varphi = 0. \end{aligned}\right\} \tag{2.1}$$

Also, an endomorphism φ induces an almost paracomplex structure on each fiber of $\mathcal{D} = ker\eta$ (horizontal distribution) *i.e.*, the eigendistribution corresponding to eigenvalues $+1$ and -1, the eigensubbundles $\mathcal{D}^+$ and $\mathcal{D}^-$ have equal dimension m. Here, $\mathcal{I}$ is the identity transformation, φ is a $(1,1)$-*tensor field*, ξ is a *characteristic vector field* and η is a differential *one-form* on $\mathbb{M}^{2m+1}$.
Consider a pseudo-Riemannian metric tensor $\mathbf{g}$, such that

$$\mathbf{g}\,(\varphi X_1, \varphi X_2) = -\mathbf{g}(X_1, X_2) + \eta(X_1)\eta(X_2), \tag{2.2}$$

then $\mathbf{g}$ is compatible with paracontact structure. Here signature of $\mathbf{g}$ is $(m+1, m)$ and $\eta(X_1) = \mathbf{g}\,(X_1, \xi)$, for any vector field X_1 on $\mathbb{M}^{2m+1}$.

Definition 2.1. *A differentiable manifold $\left(\mathbb{M}^{2m+1}, \mathbf{g}\right)$ is called $(2m+1)$- dimensional almost paracontact metric manifold if $\mathbf{g}$ is compatible with (φ, η, ξ)-structure.*

From now on, we are taking $\mathbb{M}^{2m+1}$ as an almost paracontact metric manifold throughout this paper.

Definition 2.2. *A manifold $\mathbb{M}^{2m+1}$ is called $(2m+1)$-dimensional paracontact metric manifold if it satisfies $\Psi = d\eta$, where Ψ is the fundamental 2-form given by $\Psi(X_1, X_2) = \mathbf{g}(\varphi X_1, X_2)$, for any vector fields X_1 and X_2 on $\mathbb{M}^{2m+1}$ [30].*

Also, for a manifold $\mathbb{M}^{2m+1}$ we can find φ-basis which is a local orthonormal basis $\{X_i, \varphi X_i, \xi\}$ such that $\mathbf{g}(X_i, X_i) = 1$ and $\mathbf{g}(\varphi X_i, \varphi X_i) = -1$, $i = 1, \cdots, m$.
Next, we give the definition as well as example of a α-paraSasakian manifold which is a subclass of $\mathbb{M}^{2m+1}$:

Definition 2.3. *A manifold $\mathbb{M}^{2m+1}$ is said to be α-paraSasakian manifold $\mathbf{S}^{2m+1}$ if*

$$(\nabla_{X_2}\varphi) X_1 = \alpha \mathbf{g}(X_2, X_1)\xi - \alpha\eta(X_1) X_2, \tag{2.3}$$

where X_1 and X_2 are vector fields on $\mathbb{M}^{2m+1}$ and $\alpha\ (\neq 0)$ is a constant [30].

Example 2.1. *Let $\mathcal{S}^3 := R^3(\varphi, \eta, \xi)$, where $\xi = e_3$ and given the one-form η and an endomorphism φ as: $\varphi e_1 = e_2$, $\varphi e_2 = e_1 - ye_3$, $\varphi e_3 = 0$, $\eta = ydx + dz$ ((x, y, z), being the cartesian coordinates and $e_1 = \frac{\partial}{\partial x}$, $e_2 = \frac{\partial}{\partial y}$, $e_3 = \frac{\partial}{\partial z}$ and the metric tensor $\mathbf{g} = dx^2 - dy^2 + \eta \otimes \eta$. Then $\mathcal{S}^3$ is α-paraSasakian manifold $\mathbf{S}^3$. By further computation we have the following coefficients of Levi-Civita connection as*

$$\begin{aligned} &\nabla_{e_1} e_1 = ye_2,\ \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = \frac{1}{2} ye_1 + \frac{1}{2}\left(1 - y^2\right) e_3,\ \nabla_{e_2} e_2 = 0, \\ &\nabla_{e_1} e_3 = \nabla_{e_3} e_1 = \frac{1}{2} e_2,\ \nabla_{e_2} e_3 = \nabla_{e_3} e_2 = \frac{1}{2}(e_1 - ye_3),\ \nabla_{e_3} e_3 = 0. \end{aligned} \tag{2.4}$$

Using Eqs. (2.3) *and* (2.4), *we have $\alpha = \frac{1}{2}$.*

Remark 2.1. *If $\alpha = 1$ in equation* (2.3) *and $\Psi(X_1, X_2) = d\eta(X_1, X_2)$ for any vector fields X_1 and X_2 on $\mathbf{S}^{2m+1}$, then α-paraSasakian manifold $\mathbf{S}^{2m+1}$ is called paraSasakian manifold.*

Remark 2.2. *If $\alpha = 0$ in equation* (2.3), *then α-paraSasakian manifold $\mathbf{S}^{2m+1}$ is called paracosymplectic manifold $\mathbf{C}^{2m+1}$.*

Proposition 2.1. *For a α-paraSasakian manifold $\mathbf{S}^{2m+1}$, we have*

$$\nabla_{X_1}\xi = \alpha\varphi X_1, \tag{2.5}$$

for any vector field X_1 on $\mathbf{S}^{2m+1}$.

Proof. By using equation (2.3), we get the required result. □

Proposition 2.2. *For a α-paraSasakian manifold $\mathbf{S}^{2m+1}$, we have*

$$(\nabla_{X_3}\eta) X_1 = \alpha \mathbf{g}(X_1, \varphi X_3), \tag{2.6}$$

for any vector fields X_1 and X_3 on $\mathbf{S}^{2m+1}$.

Proof. By using equation (2.5), we get the required result. □

2.1. **Curvature Properties of $\mathbf{S}^{2m+1}$.** The curvature tensor $\mathcal{R}$ on a manifold $\mathbf{S}^{2m+1}$ with pseudo-Riemannian metric $\mathbf{g}$ is given as

$$\mathcal{R}(X_1, X_2) X_3 = [\nabla_{X_1}, \nabla_{X_2}] X_3 - \nabla_{[X_1,X_2]} X_3, \tag{2.7}$$

where X_1, X_2 and X_3 are vector fields on $\mathbf{S}^{2m+1}$.

Proposition 2.3. *For a α-paraSasakian manifold $\mathbf{S}^{2m+1}$, we have the following curvature properties*

$$\mathcal{R}(X_1, X_2)\xi = \alpha^2 (\eta(X_1) X_2 - \eta(X_2) X_1), \tag{2.8}$$

$$\mathcal{R}(X_1, \xi) X_2 = \alpha^2 (g(X_1, X_2)\xi - \eta(X_2) X_1), \tag{2.9}$$

$$\mathcal{R}(X_1, \xi)\xi = \alpha^2 (\eta(X_1)\xi - X_1), \tag{2.10}$$

$$\mathrm{Ric}(X_1, \xi) = -2m\alpha^2 \eta(X_1), \tag{2.11}$$

$$Q\xi = -2m\alpha^2 \xi, \tag{2.12}$$

where X_1, X_2 and X_3 are vector fields on $\mathbf{S}^{2m+1}$.

Proof. By using equation (2.3), (2.5) and (2.7), we get the required expressions for curvature tensor $\mathcal{R}$. □

Proposition 2.4. *For a α-paraSasakian manifold $\mathbf{S}^{2m+1}$, we have*

$$\begin{aligned} &\mathcal{R}(X_1, X_2, \varphi X_3, X_4) + \mathcal{R}(X_1, X_2, X_3, \varphi X_4) = \alpha\mathbf{g}(X_2, X_3)\, d\eta(X_1, X_4) \\ &- \alpha\mathbf{g}(X_2, X_4)\, d\eta(X_1, X_3) + \alpha\mathbf{g}(X_1, X_4)\, d\eta(X_2, X_3) - \alpha\mathbf{g}(X_1, X_3)\, d\eta(X_2, X_4), \end{aligned} \tag{2.13}$$

$$\begin{aligned} &\mathcal{R}(\varphi X_1, \varphi X_2, \varphi X_3, \varphi X_4) - \mathcal{R}(X_1, X_2, X_3, X_4) = \alpha^2\mathbf{g}(X_2, X_3)\,\eta(X_1)\,\eta(X_4) \\ &- \alpha^2\mathbf{g}(X_2, X_4)\,\eta(X_1)\,\eta(X_3) + \alpha^2\mathbf{g}(X_1, X_4)\,\eta(X_2)\,\eta(X_3) - \alpha^2\mathbf{g}(X_1, X_3)\,\eta(X_2)\,\eta(X_4), \end{aligned} \tag{2.14}$$

where X_1, X_2, X_3 and X_4 are vector fields on $\mathbf{S}^{2m+1}$.

Proof. By using the definition of curvature tensor and equation (2.3) and (2.5), we get

$$\begin{aligned} \mathcal{R}(X_1, X_2)\varphi X_3 =& \alpha^2\mathbf{g}(X_2, X_3)\varphi X_1 - \alpha^2\mathbf{g}(\varphi X_1, X_3) X_2 + \alpha^2\mathbf{g}(\varphi X_2, X_3) X_1 \\ & - \alpha^2\mathbf{g}(X_1, X_3)\varphi X_2 + \varphi\mathcal{R}(X_1, X_2) X_3, \end{aligned} \tag{2.15}$$

and scalar product of the above equation with X_4 gives equation (2.13). Futher, replacing $X_1 \longrightarrow \varphi X_1$, $X_2 \longrightarrow \varphi X_2$ and $X_4 \longrightarrow \varphi X_4$ in equation (2.13), then the use of equation (2.2), (2.9) and (2.15) gives equation (2.14). □

Proposition 2.5. *For a α-paraSasakian manifold $\mathbf{S}^{2m+1}$, we have*

$$\mathrm{Ric}(X_1, \varphi X_2) + \mathrm{Ric}(\varphi X_1, X_2) = -\alpha d\eta(X_1, X_2). \tag{2.16}$$

Also for $X_i \perp \xi$, $i = 1 \cdots 2m$

$$\mathrm{Ric}(X_1, \varphi X_2) + \mathrm{Ric}(\varphi X_1, X_2) = 0. \tag{2.17}$$

Proof. By using Proposition 2.4, we get the required result *i.e.* equation (2.16). Also for equation (2.17), taking $\{X_i,\ i = 1 \cdots 2m\}$ orthogonal to ξ in equation (2.14). □

Proposition 2.6. *On a α-paraSasakian manifold $\mathbf{S}^{2m+1}$, we have*

$$Q\varphi = \varphi Q - \alpha^2\varphi, \tag{2.18}$$

where Q is a Ricci operator.

Proof. By using the definition of Ricci operator *i.e.*, $\mathbf{g}(QX_1, X_2) = \mathrm{Ric}(X_1, X_2)$ and equation (2.6) and (2.16), we have

$$-\varphi QX_1 + Q\varphi X_1 = -\alpha^2\varphi X_1,$$

which gives the required result. □

Remark 2.3. *For a α-paraSasakian manifold $\mathbf{S}^{2m+1}$ with $\{X_i \perp \xi,\ i = 1\cdots 2m\}$, the Ricci operator Q commutes with an endomorphism φ.*

Proposition 2.7. *For a α-paraSasakian manifold $\mathbf{S}^{2m+1}$, we have*

$$(\nabla_{X_1}Q)\,\xi = -\,\alpha\left(Q\varphi X_1 + 2m\alpha^2\varphi X_1\right), \tag{2.19}$$

$$(\nabla_{\xi}Q) = \alpha^3\varphi, \tag{2.20}$$

for any vector field X_1 on $\mathbf{S}^{2m+1}$ and Q is the Ricci operator.

Proof. Since,

$$0 = \pounds_{\xi}(QX_1) - Q(\pounds_{\xi}X_1) = \nabla_{\xi}QX_1 - \nabla_{QX_1}\xi - Q(\nabla_{\xi}X_1) + Q(\nabla_{X_1}\xi),$$

then equation (2.5) and (2.18) gives equation (2.20).
Further, covariant differentiation of equation (2.12) along an arbitrary vector field X_1 and (2.5) gives equation (2.19). □

Definition 2.4. *A manifold $\mathbf{S}^{2m+1}$ with pseudo-Riemannian metric $\mathbf{g}$ is known as $*$-quasi Einstein manifold if*

$$\mathrm{Ric}^*(X_1, X_2) = a_1\,\mathbf{g}(X_1, X_2) + a_2\,\eta(X_1)\,\eta(X_2), \tag{2.21}$$

here a_1 and a_2 are given as non-zero functions, η as a one-form and Ric^* as a $*$-Ricci tensor which is defined in equation (1.4). If $a_2 = 0$ in equation (2.21) then $\mathbf{S}^{2m+1}$ is a $*$-Einstein manifold.

3. Almost $*$-Ricci Solitons on $\mathbf{S}^{2m+1}$

Similar to Ricci soliton, $*$-Ricci soliton is a generalized notion of $*$-Einstein metric and it is self-similar solution to the partial differential equations known as $*$-Ricci flow equation.
Also, $*$-Ricci flow on a pseudo-Riemannian manifold $\mathbf{S}^{2m+1}$ will be defined as:

$$\frac{\partial \mathbf{g}}{\partial t} = -2\mathrm{Ric}^*(X_1, X_2), \tag{3.1}$$

for any vector fields X_1 and X_2 on $\mathbf{S}^{2m+1}$. Here $\mathbf{g}$ is a smooth symmetric metric tensor and Ric^* is a $*$-Ricci tensor given in equation (1.4).

Now, consider a differentiable manifold $\mathbf{S}^{2m+1}$ with pseudo-Riemannian metric $\mathbf{g}$ then, $(\mathbf{g}, W, \lambda)$ is called an *almost* $*$*-Ricci soliton* on $\mathbf{S}^{2m+1}$ if

$$(\pounds_W \mathbf{g} + 2\mathrm{Ric}^* - 2\lambda \mathbf{g})(X_1, X_2) = 0, \tag{3.2}$$

for any vector fields X_1 and X_2 on $\mathbf{S}^{2m+1}$. Here $\pounds_W$: Lie derivation along W, Ric^*: $*$-Ricci tensor, which is given in equation (1.4) and λ: a smooth function.

Remark 3.1. *An almost $*$-Ricci soliton on $\mathbf{S}^{2m+1}$ is expanding if λ is negative, steady if λ is zero or shrinking if λ is positive.*

Theorem 3.1. *For a α-paraSasakian manifold $\mathbf{S}^{2m+1}$, the $*$-Ricci tensor can be expressed as*

$$\mathrm{Ric}^*(X_1, X_2) = -\mathrm{Ric}(X_1, X_2) + \left(\frac{\alpha^2}{2} - \alpha^2(2m-1)\right)\mathbf{g}(X_1, X_2) - \frac{3\alpha^2}{2}\eta(X_1)\eta(X_2), \tag{3.3}$$

where X_1 and X_2 are vector fields on $\mathbf{S}^{2m+1}$.

Proof. Covariant differentiation of eq. (2.8) in the direction of X_3 on $\mathbf{S}^{2m+1}$ and the use of equation (2.5) gives

$$(\nabla_{X_3}\mathcal{R})(X_1, X_2)\xi + \alpha\mathcal{R}(X_1, X_2)\varphi X_3 = \alpha^3\mathbf{g}(X_1, \varphi X_3)X_2 - \alpha^3\mathbf{g}(X_2, \varphi X_3)X_1, \tag{3.4}$$

contracting equation (3.4) w.r.t an orthonormal frame $\{e_i\}$ of $\mathcal{T}\mathbf{S}^{2m+1}$, we left with

$$(div\mathcal{R})(X_1, X_2)\xi + \alpha\mathbf{g}(\mathcal{R}(X_1, X_2)\varphi e_i, e_i) = -2\alpha^3\mathbf{g}(\varphi X_1, X_2).$$

Now, by using contracted Bianchi identity the above equation becomes

$$\mathbf{g}((\nabla_{X_1}Q)X_2 - (\nabla_{X_2}Q)X_1, \xi) + \alpha\mathbf{g}(\mathcal{R}(X_1, X_2)\varphi e_i, e_i) = -2\alpha^3\mathbf{g}(\varphi X_1, X_2).$$

By virtue of equation (2.19) it follows from the above equation that

$$\mathbf{g}(\mathcal{R}(X_1, X_2)\varphi e_i, e_i) = 2\alpha^2(2m-1)\mathbf{g}(\varphi X_1, X_2) - \mathbf{g}(X_1, Q\varphi X_2) - \mathbf{g}(\varphi Q X_2, X_1).$$

Replacing $X_2 \longrightarrow \varphi X_2$ and using equation (1.4), we get

$$2\mathrm{Ric}^*(X_1, X_2) = 2\alpha^2(2m-1)\mathbf{g}(\varphi X_1, \varphi X_2) + \mathbf{g}(Q\varphi X_1, \varphi X_2) + \mathbf{g}(\varphi Q X_1, \varphi X_2).$$

Since $Q\varphi = \varphi Q - \alpha^2\varphi$, and the use of equation (2.1) and (2.12), gives the required expression for $*$-Ricci tensor, *i.e.* equation (3.3). □

Corollary 3.1. *For a α-paraSasakian manifold $\mathbf{S}^{2m+1}$, $*$-Ricci operator and $*$-scalar curvature can be expressed as*

$$Q^*X_1 = -QX_1 + \left(\frac{\alpha^2}{2} - \alpha^2(2m-1)\right)X_1 - \frac{3\alpha^2}{2}\eta(X_1)\xi, \tag{3.5}$$

$$\tau^* = -\tau - 4m^2\alpha^2 + m\alpha^2, \tag{3.6}$$

where X_1 and X_2 are vector fields on $\mathbf{S}^{2m+1}$.

Proof. The expression for $*$-Ricci operator *i.e.* equation (3.5) can be easily obtained by using equation (3.3). Now, by contracting equation (3.3) we get the required expession for the $*$-scalar curvature, i.e., equation (3.6). □

Corollary 3.2. *For a α-paraSasakian manifold $\mathbf{S}^{2m+1}$ with $\{X_i \perp \xi,\ i = 1 \cdots 2m\}$, the $*$-Ricci tensor is given as*

$$\mathrm{Ric}^*(X_1, X_2) = -\mathrm{Ric}(X_1, X_2) - \alpha^2(2m-1)\,\mathbf{g}(X_1, X_2) - \alpha^2\eta(X_1)\,\eta(X_2), \tag{3.7}$$

where X_1 and X_2 are vector fields on $\mathbf{S}^{2m+1}$.

Proof. With the help of equation (2.17) in the proof of Theorem 3.1, we get the required expression for $*$-Ricci tensor. □

Proposition 3.1. *For a α-paraSasakian manifold $\mathbf{S}^{2m+1}$, we have*

$$(\nabla_{X_1}Q^*)\,\xi = \alpha\left(Q\varphi X_1 - \left(\frac{\alpha^2}{2} - \alpha^2(2m-1)\right)\varphi X_1\right), \tag{3.8}$$

$$(\nabla_\xi Q^*) = -\alpha^3\varphi, \tag{3.9}$$

for any vector field X_1 on $\mathbf{S}^{2m+1}$. Here Q^* is the $*$-Ricci operator.

Proof. Replacing $X_1 \longrightarrow \xi$ in equation (3.5),

$$Q^*\xi = -Q\xi + \left(\frac{\alpha^2}{2} - \alpha^2(2m-1)\right)\xi - \frac{3\alpha^2}{2}\xi, \tag{3.10}$$

covariant differentiation of equation (3.10) along vector field X_1 gives

$$(\nabla_{X_1}Q^*)\,\xi = -(\nabla_{X_1}Q)\,\xi - \frac{3\alpha^3}{2}\varphi X_1,$$

then equation (2.19) gives equation (3.8).
Further, covariant differentiation of equation (3.10) along vector field ξ gives

$$(\nabla_\xi Q^*)\,\xi = -(\nabla_\xi Q)\,\xi,$$

then equation (2.20) gives equation (3.9). □

Proposition 3.2. *For a α-paraSasakian manifold $\mathbf{S}^{2m+1}$ admitting an almost $*$-Ricci soliton, we have*

$$(\pounds_W\eta)(\xi) = \lambda = -\eta(\pounds_W\xi), \tag{3.11}$$

where λ is a smooth function.

Proof. By using Proposition 3.1, equation (3.2) can be written as

$$(\pounds_W\mathbf{g})(X_1, X_2) = 2\mathrm{Ric}(X_1, X_2) + \left\{2\alpha^2(2m-1) + 2\lambda - \alpha^2\right\}\mathbf{g}(X_1, X_2) + 3\alpha^2\eta(X_1)\,\eta(X_2).$$

Put $X_2 = \xi$ in the above equation and using equation (2.11) it follows that

$$(\pounds_W\mathbf{g})(X_1, \xi) = 2\lambda\eta(X_1). \tag{3.12}$$

Next, Lie-differentiating equation $\eta(X_1) = \mathbf{g}(X_1, \xi)$ along W, we have

$$(\pounds_W \eta)(X_1) - \mathbf{g}(\pounds_W \xi, X_1) - (\pounds_W \mathbf{g})(X_1, \xi) = 0. \tag{3.13}$$

Again, Lie-differentiating equation $\mathbf{g}(\xi, \xi) = 1$ along W, we have

$$\mathbf{g}(\pounds_W \xi, \xi) = -\lambda. \tag{3.14}$$

Now, using equation (3.12), (3.13) and (3.14), we get the required result. □

Definition 3.1. *A vector field W on an almost paracontact pseudo-Riemannian manifold $\mathbb{M}^{2m+1}$ is called an infinitesimal paracontact transformation if*

$$\pounds_W \eta = \varrho \eta, \tag{3.15}$$

for a scalar function ϱ on $\mathbb{M}^{2m+1}$ and $\pounds_W$ is the Lie differentiation along W [31].

Remark 3.2. *If ϱ in equation* (3.15) *is identically zero, then a vector field W on $\mathbb{M}^{2m+1}$ is infinitesimal strict paracontact transformation.*

Theorem 3.2. *Let us consider a α-paraSasakian manifold $\mathbf{S}^{2m+1}$ with an almost $*$-Ricci Soliton $(\mathbf{g}, W, \lambda)$ and potential vector field W is an infinitesimal paracontact transformation which leaves both $*$-Ricci tensor and Ricci tensor invariant, then W is an infinitesimal strict paracontact transformation if and only if an almost $*$-Ricci Soliton on $\mathbf{S}^{2m+1}$ is steady.*

Proof. Since infinitesimal paracontact transformation W leaves both $*$-Ricci tensor and Ricci tensor invariant, we have

$$(\pounds_W \mathrm{Ric}^*)(X_1, X_2) = 0, \quad \text{and} \quad (\pounds_W \mathrm{Ric})(X_1, X_2) = 0. \tag{3.16}$$

Also,

$$(\pounds_W \mathrm{Ric}^*)(X_1, \xi) = 0, \quad \text{and} \quad (\pounds_W \mathrm{Ric})(X_1, \xi) = 0. \tag{3.17}$$

Taking Lie-derivation of equation (3.3), then using equation (2.11), (3.15) and (3.17), we get

$$\begin{aligned}\mathrm{Ric}^*(X_1, \pounds_W \xi) = -\mathrm{Ric}(X_1, \pounds_W \xi) + \left\{\frac{3\alpha^2}{2} - 2m\alpha^2\right\} \mathbf{g}(X_1, \pounds_W \xi) \\ + \left\{\frac{9\lambda\alpha^2}{2} - 3\varrho\alpha^2 - 4m\lambda\alpha^2\right\} \eta(X_1).\end{aligned}$$

Replacing $X_1 \longrightarrow \xi$ and using equation (2.11), (3.3) and (3.14), we get

$$3\varrho = \lambda(3 - 4m),$$

which gives the required result. □

Proof of Theorem 1.1. Consider a α-paraSasakian manifold $\mathbf{S}^{2m+1}$ with an almost $*$-Ricci soliton $(\mathbf{g}, W, \lambda)$ which includes W as a paracontact vector field, then the defining property of W gives

$$£_W d\eta = dL_W\eta = d\varrho\eta = d\varrho \wedge \eta + \varrho\,(d\eta)\,. \tag{3.18}$$

Further, $\nu = \eta \wedge (d\eta)^m \neq 0$ on $\mathbf{S}^{2m+1}$, then the Lie-differentiation of this along W gives

$$£_W \nu = (m+1)\,\varrho\nu. \tag{3.19}$$

Also, the formula $£_W\nu = (\operatorname{div} W)\,\nu$ and above equation gives

$$\operatorname{div} W = (m+1)\,\varrho. \tag{3.20}$$

On the other hand, the trace of equation (3.2) gives

$$\operatorname{div} W = \lambda\,(2m+1) + \tau - m\alpha^2 + 4m^2\alpha^2. \tag{3.21}$$

Now, from equation (3.20) and (3.21), we have

$$\tau = -\lambda\,(2m+1) + m\alpha^2 - 4m^2\alpha^2 + (m+1)\,\varrho.$$

Next, the Lie-differentiation of $\eta\,(X_1) = \mathbf{g}\,(X_1, \xi)$ along W, and the use of equation (3.3) and (3.15) gives

$$£_W\xi = (\varrho - 2\lambda)\,\xi, \tag{3.22}$$

taking scalar product of equation (3.22) with ξ, we have

$$\mathbf{g}\,(£_W\xi, \xi) = (\varrho - 2\lambda)\,,$$

from equation (3.14), we get $\varrho = \lambda$. Using this in equation (3.15) and (3.22), we have

$$\left.\begin{aligned} £_W\eta &= \lambda\eta \\ £_W\xi &= -\lambda\xi \end{aligned}\right\}. \tag{3.23}$$

Now, the Lie-differentiation of $d\eta\,(X_1, X_2) = \alpha\,\mathbf{g}\,(\varphi X_1, X_2)$ along W, and the use of equation (3.3) and (3.15) gives

$$2\,(£_W\varphi)\,X_1 = -4Q\varphi X_1 + 2\left\{\varrho - 2\lambda + 3\alpha^2 - 4m\alpha^2\right\}\varphi X_1 + (X_1\varrho)\,\xi - \eta\,(X_1)\,D\varrho. \tag{3.24}$$

Replacing $X_1 \longrightarrow \xi$, we get

$$2\,(£_W\varphi)\,\xi = (X_1\varrho)\,\xi - \eta\,(X_1)\,D\varrho. \tag{3.25}$$

Now, the Lie-differentiation of $\varphi\xi = 0$ along W, and the use of equation (3.23) gives

$$(£_W\varphi)\,\xi = 0, \tag{3.26}$$

using the above equation in equation (3.25), we get

$$d\varrho = (\xi\varrho)\,\eta. \tag{3.27}$$

Taking exterior derivative of the above equation and using $d^2 = 0$ and $\eta \wedge \eta = 0$, we get $d\varrho = 0$, which implies ϱ is constant, and hence λ is also constant. □

3.1. **$*$-Ricci solitons on $\mathbf{S}^{2m+1}$.** An almost $*$-Ricci soliton on $\mathbf{S}^{2m+1}$ is called a *$*$-Ricci soliton* if λ in equation (3.2) is constant.

Theorem 3.3. *Consider a α-paraSasakian manifold $\mathbf{S}^{2m+1}$ with $*$-Ricci Soliton $(\mathbf{g}, W, \lambda)$, then $\mathbf{S}^{2m+1}$ is a quasi-Einstein manifold and expression for the Ricci tensor is given as*

$$\mathrm{Ric}(X_1, X_2) = -\left(\frac{\lambda}{2} + \alpha^2(2m-1)\right)\mathbf{g}(X_1, X_2) + \left(\frac{\lambda}{2} - \alpha^2\right)\eta(X_1)\eta(X_2), \tag{3.28}$$

where X_1 and X_2 are vector fields on $\mathbf{S}^{2m+1}$.

Proof. Consider a α-paraSasakian manifold $\mathbf{S}^{2m+1}$ admitting a $*$-Ricci Soliton $(\mathbf{g}, W, \lambda)$. With the help of Proposition 3.1, equation (3.2) can be written as

$$\begin{aligned}(\pounds_W \mathbf{g})(X_1, X_2) = 2\mathrm{Ric}(X_1, X_2) + \left\{2\alpha^2(2m-1) + 2\lambda - \alpha^2\right\}\mathbf{g}(X_1, X_2)\\ +3\alpha^2\eta(X_1)\eta(X_2),\end{aligned} \tag{3.29}$$

covariant differentiation of equation (3.29) in the direction of X_3 on $\mathbf{S}^{2m+1}$ gives

$$\begin{aligned}(\nabla_{X_3}\pounds_W \mathbf{g})(X_1, X_2) = 2(\nabla_{X_3}\mathrm{Ric})(X_1, X_2) + 3\alpha^3\mathbf{g}(X_1, \varphi X_3)\eta(X_2)\\ +3\alpha^3\mathbf{g}(X_2, \varphi X_3)\eta(X_1).\end{aligned} \tag{3.30}$$

According to [32], we have

$$\begin{aligned}\left(\pounds_W \nabla_{X_3}\mathbf{g} - \nabla_{X_3}\pounds_W \mathbf{g} - \nabla_{[W,X_3]}\mathbf{g}\right)(X_1, X_3) = -\mathbf{g}((\pounds_W \nabla)(X_3, X_1), X_2)\\ -\mathbf{g}((\pounds_W \nabla)(X_3, X_2), X_1).\end{aligned}$$

By the parallelism of pseudo-Riemannian metric the above equation gives

$$(\nabla_{X_3}\pounds_W \mathbf{g})(X_1, X_2) = \mathbf{g}((\pounds_W \nabla)(X_3, X_1), X_2) + \mathbf{g}((\pounds_W \nabla)(X_3, X_2), X_1). \tag{3.31}$$

Now, using equation (3.30) in (3.31), we have

$$\begin{aligned}\mathbf{g}((\pounds_W \nabla)(X_3, X_1), X_2) + \mathbf{g}((\pounds_W \nabla)(X_3, X_2), X_1) = 2(\nabla_{X_3}\mathrm{Ric})(X_1, X_2)\\ + 3\alpha^3\mathbf{g}(X_1, \varphi X_3)\eta(X_2) + 3\alpha^3\mathbf{g}(X_2, \varphi X_3)\eta(X_1).\end{aligned}$$

By a straight forward combinatorial combination equation gives

$$\begin{aligned}\mathbf{g}((\pounds_W \nabla)(X_1, X_2), X_3) = -(\nabla_{X_3}\mathrm{Ric})(X_1, X_2) + (\nabla_{X_1}\mathrm{Ric})(X_2, X_3)\\ + (\nabla_{X_2}\mathrm{Ric})(X_3, X_1) - 3\alpha^3\mathbf{g}(X_1, \varphi X_3)\eta(X_2) - 3\alpha^3\mathbf{g}(X_2, \varphi X_3)\eta(X_1).\end{aligned} \tag{3.32}$$

Now, replacing $X_2 \longrightarrow \xi$ and using equation (2.1),

$$(\pounds_W \nabla)(X_1, \xi) = -2\alpha Q\varphi X_1 - 2\alpha^3\left(2m - \frac{3}{2}\right)\varphi X_1. \tag{3.33}$$

Further, covariant differentiation of equation (3.33) in the direction of X_2 on $\mathbf{S}^{2m+1}$ gives

$$\begin{aligned}(\nabla_{X_2}(\pounds_W \nabla))(X_1, \xi) = -(\pounds_W \nabla)(X_1, \alpha\varphi X_2) - 2\alpha(\nabla_{X_2}Q)\varphi X_1 + 2\alpha^2\eta(X_1)QX_2\\ + 2\alpha^4\left(2m - \frac{3}{2}\right)\eta(X_1)X_2 + 3\alpha^4\mathbf{g}(X_1, X_2)\xi.\end{aligned}$$

Now, using this equation in the given commutation formula [32]

$$(\pounds_W \mathcal{R})(X_1, X_2) X_3 = (\nabla_{X_1} (\pounds_W \nabla))(X_2, X_3) - (\nabla_{X_2} (\pounds_W \nabla))(X_1, X_3).$$

Replacing $X_3 \longrightarrow \xi$,

$$\begin{aligned}(\pounds_W \mathcal{R})(X_1, X_2)\xi = {} & (\pounds_W \nabla)(X_1, \alpha\varphi X_2) - (\pounds_W \nabla)(X_2, \alpha\varphi X_1) - 2\alpha (\nabla_{X_1} Q)\varphi X_2 \\ & + 2\alpha^2 \eta(X_2) Q X_1 + 2\alpha^4 \left(2m - \frac{3}{2}\right)\eta(X_2) X_1 + 2\alpha (\nabla_{X_2} Q)\varphi X_1 \\ & - 2\alpha^2 \eta(X_1) Q X_2 - 2\alpha^4 \left(2m - \frac{3}{2}\right)\eta(X_1) X_2. \end{aligned} \tag{3.34}$$

Replacing $X_2 \longrightarrow \xi$ and using equation (2.1), (2.20)

$$(\pounds_W \mathcal{R})(X_1, \xi)\xi = 4\alpha^2 \left\{ Q X_1 + (2m-1)\alpha^2 X_1 + \alpha^2 \eta(X_1)\xi \right\}. \tag{3.35}$$

Taking Lie-derivative of (2.10) along W and using equation (2.9), (2.8) and (2.10)

$$(\pounds_W \mathcal{R})(X_1, \xi)\xi = -\alpha^2 \mathbf{g}(X_1, \pounds_W \xi)\xi + 2\alpha^2 \eta(\pounds_W \xi) X_1 + \alpha^2 (\pounds_W \eta)(X_1)\xi.$$

Now with the help of Proposition 3.2 and equation (3.35), (3.12) and (3.13), we get

$$\mathrm{Ric}(X_1, X_2) = -\left(\frac{\lambda}{2} + \alpha^2(2m-1)\right)\mathbf{g}(X_1, X_2) + \left(\frac{\lambda}{2} - \alpha^2\right)\eta(X_1)\eta(X_2), \tag{3.36}$$

where X_1 and X_2 are vector fields on $\mathbf{S}^{2m+1}$ and $\mathbf{S}^{2m+1}$ is quasi Einstein manifold, which is the required result. □

Proof of Theorem 1.2. Consider a α-paraSasakian manifold $\mathbf{S}^{2m+1}$ with $*$-Ricci Soliton $(\mathbf{g}, W, \lambda)$. With the help of equation (3.36), equation (3.29) reduces to

$$(\pounds_W \mathbf{g})(X_1, X_2) = \lambda \mathbf{g}(X_1, X_2) + \lambda \eta(X_1)\eta(X_2). \tag{3.37}$$

Taking covariant differentiation of equation (3.36) along X_3 on $\mathbf{S}^{2m+1}$ and using equation (2.5), we get

$$(\nabla_{X_3} \mathrm{Ric})(X_1, X_2) = \left(\frac{\lambda}{2} - \alpha^2\right)\{\alpha \mathbf{g}(X_1, \varphi X_3)\eta(X_2) + \alpha \mathbf{g}(X_2, \varphi X_3)\eta(X_2)\},$$

with the help of above equation, equation (3.32) reduces to

$$(\pounds_W \nabla)(X_1, X_2) = \lambda\alpha \{\varphi X_1 \eta(X_2) + \varphi X_2 \eta(X_1)\}. \tag{3.38}$$

Covariant differentiation of equation (3.38) in the direction of X_3 on $\mathbf{S}^{2m+1}$ and equation (2.5) gives

$$\begin{aligned}(\nabla_{X_3} \pounds_W \nabla)(X_1, X_2) = {} & \lambda\alpha^2 \mathbf{g}(X_1, \varphi X_3)\varphi X_2 + \lambda\alpha^2 \mathbf{g}(X_2, \varphi X_3)\varphi X_1 \\ & -2\lambda\alpha^2 \eta(X_1)\eta(X_2) X_3 + \lambda\alpha^2 \eta(X_1)\mathbf{g}(X_3, X_2)\xi + \lambda\alpha^2 \eta(X_2)\mathbf{g}(X_3, X_1)\xi. \end{aligned}$$

By using the above equation in equation (3.34), we get

$$\begin{aligned}(\pounds_W \mathcal{R})(X_3, X_1) X_2 = {} & 2\lambda\alpha^2 \mathbf{g}(X_1, \varphi X_3)\varphi X_2 + \lambda\alpha^2 \mathbf{g}(X_2, \varphi X_3)\varphi X_1 - \lambda\alpha^2 \mathbf{g}(X_2, \varphi X_1)\varphi X_3 \\ & + \lambda\alpha^2 \eta(X_1)\mathbf{g}(X_3, X_2)\xi - \lambda\alpha^2 \eta(X_3)\mathbf{g}(X_1, X_2)\xi \\ & + 2\lambda\alpha^2 \eta(X_2)\eta(X_3) X_1 - 2\lambda\alpha^2 \eta(X_1)\eta(X_2) X_3. \end{aligned}$$

Now, contraction of the above equation over X_3 gives

$$(\pounds_W \mathrm{Ric})(X_1, X_2) = 2\lambda\alpha^2 \mathbf{g}(X_1, X_2) - 2\lambda\alpha^2 (2m+1)\eta(X_1)\eta(X_2). \tag{3.39}$$

Lie-differentiating equation (3.36) in the direction of W and using equation (3.37), we get

$$(\pounds_W \mathrm{Ric})(X_1, X_2) = \left(\frac{\lambda}{2} - \alpha^2\right)(\pounds_W \eta)(X_1)\eta(X_2) + \left(\frac{\lambda}{2} - \alpha^2\right)(\pounds_W \eta)(X_2)\eta(X_1)$$
$$- \lambda\left(\frac{\lambda}{2} + \alpha^2(2m-1)\right)\mathbf{g}(X_1, X_2) - \lambda\left(\frac{\lambda}{2} + \alpha^2(2m-1)\right)\eta(X_1)\eta(X_2) \tag{3.40}$$

Now, comparison of equation (3.39) with (3.40) and use of equation (3.37) gives

$$\left(\frac{\lambda}{2} - \alpha^2\right)(\pounds_W \eta)(X_1)\eta(X_2) + \left(\frac{\lambda}{2} - \alpha^2\right)(\pounds_W \eta)(X_2)\eta(X_1)$$
$$= \lambda\left(\frac{\lambda}{2} + \alpha^2(2m+1)\right)\mathbf{g}(X_1, X_2) + \lambda\left(\frac{\lambda}{2} - \alpha^2(2m+3)\right)\eta(X_1)\eta(X_2). \tag{3.41}$$

Replacing $X_1 \longrightarrow \varphi^2 X_1$ and $X_2 \longrightarrow \varphi X_2$ in the above equation then we have

$$\lambda\left(\frac{\lambda}{2\alpha} + \alpha(2m+1)\right)d\eta(X_1, X_2) = 0.$$

Since $d\eta$ is non-vanishing everywhere on $\mathbf{S}^{2m+1}$

$$\lambda\left(\frac{\lambda}{2\alpha} + \alpha(2m+1)\right) = 0. \tag{3.42}$$

Then, we have either $\lambda = 0$ or $\lambda = -2(2m+1)\alpha^2$.

Case I : If $\lambda = 0$, then equation (3.37) gives W is Killing vector field and equation (3.36) gives $\mathbf{S}^{2m+1}$ is quasi-Einstein i.e.

$$\mathrm{Ric}(X_1, X_2) = -\alpha^2(2m-1)\mathbf{g}(X_1, X_2) - \alpha^2\eta(X_1)\eta(X_2).$$

Case II : If $\lambda = -2(2m+1)\alpha^2$, then using this value of λ in (3.41), we have

$$\left(\frac{\lambda}{2} - \alpha^2\right)(\pounds_W \eta)(X_1)\eta(X_2) + \left(\frac{\lambda}{2} - \alpha^2\right)(\pounds_W \eta)(X_2)\eta(X_1)$$
$$= -4\lambda\alpha^2(m+1)\eta(X_1)\eta(X_2).$$

Replacing $X_2 \longrightarrow \xi$ and $X_1 \longrightarrow \varphi X_1$, we have

$$\left(\frac{\lambda}{2} - \alpha^2\right)(\pounds_W \eta)(\varphi X_1) = 0.$$

Since $\lambda = -2(2m+1)\alpha^2$ then $\lambda \neq 2\alpha^2$, which implies

$$(\pounds_W \eta)(\varphi X_1) = 0. \tag{3.43}$$

Further using $\lambda = -2(2m+1)\alpha^2$ in equation (3.36), we have

$$\mathrm{Ric}(X_1, X_2) = 2\alpha^2\mathbf{g}(X_1, X_2) - 2\alpha^2(m+1)\eta(X_1)\eta(X_2). \tag{3.44}$$

Replacing $X_1 \longrightarrow \varphi X_1$ in equation (3.43), we have

$$(\pounds_W \eta)(X_1) = -2(2m+1)\alpha^2\eta(X_1). \tag{3.45}$$

Also,

$$\pounds_W \xi = 2(2m+1)\alpha^2 \xi.$$

Moreover, operating d in equation (3.45). Note that d commutes with $\pounds_W$, we have

$$(\pounds_W d\eta)(X_1, X_2) = -2(2m+1)\alpha^3 \mathbf{g}(\varphi X_1, X_2). \tag{3.46}$$

Next, the Lie-derivative of equation $d\eta(X_1, X_2) = \alpha \mathbf{g}(\varphi X_1, X_2)$ along the vector field W and the use of equation (3.37) gives

$$(\pounds_W d\eta)(X_1, X_2) = -2(2m+1)\alpha^3 \mathbf{g}(\varphi X_1, X_2) + \alpha \mathbf{g}((\pounds_W \varphi) X_1, X_2), \tag{3.47}$$

comparing the above equation with (3.46), we get $(\pounds_W \varphi) = 0$. Thus, from case I and case II, we get either the soliton vector field W is killing or leaves φ invariant, which is the required result. □

Example 3.1. *Consider a paraSasakian manifold $\mathcal{S}^{2m+1}$ of dimension $(2m+1)$, $m > 1$ and if the paraholomorphic sectional curvature does not depend on the paraholomorphic section at a point then the curvature tensor is given by*

$$\begin{aligned}\mathcal{R}(X_1, X_2) X_3 = & \frac{k-3}{4}\{\mathbf{g}(X_2, X_3) X_1 - \mathbf{g}(X_1, X_3) X_2\} + \frac{k+1}{4}\{\eta(X_1)\eta(X_3) X_2 \\ & - \eta(X_2)\eta(X_3) X_1 - \mathbf{g}(X_2, X_3)\eta(X_1)\xi + \mathbf{g}(X_1, X_3)\eta(X_2)\xi \\ & + \mathbf{g}(X_2, \varphi X_3)\varphi X_1 - \mathbf{g}(X_1, \varphi X_3)\varphi X_2 + 2\mathbf{g}(\varphi X_1, X_2)\varphi X_3\}.\end{aligned} \tag{3.48}$$

where X_1, X_2 and X_3 are vector fields on $\mathcal{S}^{2m+1}$.
Next, contraction of the above equation over X_1 gives

$$\mathrm{Ric}(X_1, X_2) = \frac{m(k-3)}{2}\{\mathbf{g}(X_1, X_2)\} - \frac{k+1}{2}\{m\,\eta(X_1)\eta(X_2) + \mathbf{g}(\varphi X_1, \varphi X_2)\}. \tag{3.49}$$

As we know, a paraSasakian manifold with constant paraholomorphic sectional curvature is a paraSasakian space form, and we can find the expression of $*$-Ricci tensor on such space form.
Now, taking $\mathcal{S}^{2m+1}$ with constant paraholomorphic sectional curvature k and using equation (3.49) *in equation* (3.3), *we get*

$$\mathrm{Ric}^*(X_1, X_2) = \{(m+1)k + (m-2)\}\mathbf{g}(\varphi X_1, \varphi X_2), \tag{3.50}$$

for any vector fields X_1 and X_2 on $\mathcal{S}^{2m+1}$. Here, if we choose $k = \frac{2-m}{m+1}$ then $\mathcal{S}^{2m+1}$ becomes $*$-Ricci flat.
Again, using equation (3.49) *in equation* (3.7), *we get*

$$\mathrm{Ric}^*(X_1, X_2) = \{(m+1)k + (m-1)\}\mathbf{g}(X_1, X_2), \tag{3.51}$$

for any vector fields $X_1, X_2 \perp \xi$ on $\mathcal{S}^{2m+1}$ and $\mathcal{S}^{2m+1}$ becomes $*$-Einstein. Thus, any paraSasakian space form $\mathcal{S}^{2m+1}(k)$ with $\{X_i \perp \xi, i = 1 \cdots 2m\}$ on $\mathcal{S}^{2m+1}$ is $*$-Einstein.

4. Almost Gradient $*$-Ricci Solitons on $\mathbf{S}^{2m+1}$

In this section, firstly we define mathematical operators gradient and Hessian on $\mathbf{S}^{2m+1}$. So, consider a manifold $\mathbf{S}^{2m+1}$ with pseudo-Riemannian metric $\mathbf{g}$ and $\varrho : \mathbf{S}^{2m+1} \longrightarrow R$ is a smooth function over $\mathbf{S}^{2m+1}$. Then, the gradient (first order differential operator) $\nabla : C^1\left(\mathbf{S}^{2m+1}\right) \longrightarrow \Gamma\left(\mathcal{T}\mathbf{S}^{2m+1}\right)$ of a function ϱ is given as:

$$\mathbf{g}\left(\nabla\varrho\left(x\right), X_1\right) = X_1\varrho\left(x\right),$$

for any vector field X_1 on $\mathbf{S}^{2m+1}$ and Hessian (covariant derivative of the gradient operator) of a function ϱ is given as:

$$\nabla^2\varrho\left(X_1, X_2\right) = X_1X_2\varrho - \left(\nabla_{X_1}X_2\right)\varrho,$$

for any vector fields X_1 and X_2 on $\mathbf{S}^{2m+1}$.

Definition 4.1. *An almost $*$-Ricci soliton is known as an almost gradient $*$-Ricci soliton if W of equation* (3.2) *is of gradient type,* i.e.. $W = \text{grad}\left(\%\right)$ *and satisfies:*

$$\left(Hess\left(\varrho\right) + \text{Ric}^* - \lambda\,\mathbf{g}\right)\left(X_1, X_2\right) = 0, \tag{4.1}$$

where X_1, X_2 and X_3 are vector fields on $\mathbf{S}^{2m+1}$ and the Hessian of ϱ is given as: $Hess\left(\varrho\right)\left(X_1, X_2\right) := \mathbf{g}\left(\nabla_{X_1}\xi, X_2\right)$.

Proof of Theorem 1.3. Consider a α-paraSasakian manifold $\mathbf{S}^{2m+1}$ admitting an almost gradient $*$-Ricci soliton, then equation (4.1) gives

$$\nabla_{X_1}D\varrho + Q^*X_1 - \lambda X_1 = 0, \tag{4.2}$$

here Q^* is $*$-Ricci operator and D is gradient operator of metric $\mathbf{g}$. By using the expression of $*$-Ricci tensor the above equation reduces to

$$\nabla_{X_1}D\varrho = QX_1 + \left\{\alpha^2\left(2m-1\right) + \lambda - \frac{\alpha^2}{2}\right\}X_1 + \frac{3\alpha^2}{2}\eta\left(X_1\right)\xi. \tag{4.3}$$

Covariant differentiation of equation (4.3) in the direction of X_2 on $\mathbf{S}^{2m+1}$ gives

$$\begin{aligned}\nabla_{X_2}\nabla_{X_1}D\varrho &= \left(\nabla_{X_2}Q\right)X_1 + Q\left(\nabla_{X_2}X_1\right) + \left\{\alpha^2\left(2m-1\right) + \lambda - \frac{\alpha^2}{2}\right\}\nabla_{X_2}X_1 \\ &+ \left(X_2\lambda\right)X_1 + \frac{3\alpha^3}{2}\mathbf{g}\left(X_1, \varphi X_2\right)\xi + \frac{3\alpha^2}{2}\eta\left(\nabla_{X_2}X_1\right)\xi + \frac{3\alpha^3}{2}\eta\left(X_1\right)\varphi X_2,\end{aligned} \tag{4.4}$$

using differential equations (4.4) and (4.3) in the expression of the curvature tensor given in (2.7), we get

$$\begin{aligned}\mathcal{R}\left(X_1, X_2\right)D\varrho &= \left(\nabla_{X_1}Q\right)X_2 - \left(\nabla_{X_2}Q\right)X_1 + \left(X_1\lambda\right)X_2 - \left(X_2\lambda\right)X_1 \\ &+ 3\alpha^3\mathbf{g}\left(X_2, \varphi X_1\right)\xi - \frac{3\alpha^3}{2}\eta\left(X_1\right)\varphi X_2 + \frac{3\alpha^3}{2}\eta\left(X_2\right)\varphi X_1.\end{aligned} \tag{4.5}$$

Now, the scalar product of equation (4.5) with ξ and the use of equation (2.19) gives

$$\mathbf{g}\left(\mathcal{R}\left(X_1, X_2\right) D\varrho, \xi\right) = \alpha\mathbf{g}\left(Q\varphi X_2, X_1\right) - \alpha\mathbf{g}\left(Q\varphi X_1, X_2\right) + \left(X_1\lambda\right)\eta\left(X_2\right) - \left(X_2\lambda\right)\eta\left(X_1\right) + 3\alpha^3\mathbf{g}\left(X_2, \varphi X_1\right). \tag{4.6}$$

Replacing $X_2 \longrightarrow \xi$ in (4.6) and using (2.8) and (2.12), we get

$$\xi\left(\lambda - \alpha^2\varrho\right)\eta\left(X_1\right) = X_1\left(\lambda - \alpha^2\varrho\right),$$

writing the above equation as

$$\xi\left(\lambda - \alpha^2\varrho\right)\eta = d\left(\lambda - \alpha^2\varrho\right),$$

operating exterior differentiation operator d in the above equation and using $d^2 = 0$, we have

$$\left(\xi\left(\lambda - \alpha^2\varrho\right)\right)d\eta = d\left(\xi\left(\lambda - \alpha^2\varrho\right)\right)\eta, \tag{4.7}$$

Also, wedge product of equation (4.7) with one-form η and the use of $\eta \wedge \eta = 0$ gives

$$\left(\xi\left(\lambda - \alpha^2\varrho\right)\right)d\eta \wedge \eta = 0.$$

As $\eta \wedge d\eta \neq 0$ everywhere on $\mathbf{S}^{2m+1}$, we have $\xi\left(\lambda - \alpha^2\varrho\right) = 0$. Which implies

$$\lambda - \alpha^2\varrho = c, \quad \text{c is constant.} \tag{4.8}$$

Setting $X_1 = \xi$ in equation (4.5), then the scalar product of resulting equation with X_1 and the use of (2.19) and (2.20) gives

$$\mathbf{g}\left(\mathcal{R}\left(\xi, X_2\right) D\varrho, X_1\right) = \mathbf{g}\left(Q\varphi X_2, X_1\right) + \left(\xi\lambda\right)\mathbf{g}\left(X_1, X_2\right) - \left(X_2\lambda\right)\eta\left(X_1\right) + \alpha^3\left(2m - \frac{1}{2}\right)\mathbf{g}\left(\varphi X_2, X_1\right),$$

using equation (2.8) in the above equation, we have

$$\alpha^2\left(\xi\varrho\right)\mathbf{g}\left(X_1, X_2\right) - \alpha^2\eta\left(X_1\right)\left(X_2\varrho\right) = \mathbf{g}\left(Q\varphi X_2, X_1\right) + \left(\xi\lambda\right)\mathbf{g}\left(X_1, X_2\right) - \left(X_2\lambda\right)\eta\left(X_1\right) + \alpha^3\left(2m - \frac{1}{2}\right)\mathbf{g}\left(\varphi X_2, X_1\right),$$

using equation (4.8) and (2.18) in the above equation, we have

$$\alpha Q\varphi X_1 + \alpha^3\left(2m - \frac{3}{2}\right)\varphi X_1 = 0.$$

Replacing $X_1 \longrightarrow \varphi X_1$ and the use of equation (2.12) gives

$$\mathrm{Ric}\left(X_1, X_2\right) = -\alpha^2\left(2m - \frac{3}{2}\right)\mathbf{g}\left(X_1, X_2\right) - \frac{3\alpha^2}{2}\eta\left(X_1\right)\eta\left(X_2\right), \tag{4.9}$$

which implies, $\mathbf{S}^{2m+1}$ is quasi-Einstein. By using equation (4.8) and (4.9) in (4.3), we have

$$\nabla_{X_1} D\lambda = \alpha^2\lambda X_1,$$

for any vector field X_1 on $\mathbf{S}^{2m+1}$. □

4.1. **Gradient $*$-Ricci solitons on $\mathbf{S}^{2m+1}$.** An almost gradient $*$-Ricci soliton on $\mathbf{S}^{2m+1}$ is called a *gradient $*$-Ricci soliton* if λ in equation (4.1) is constant.

Theorem 4.1. *Consider a α-paraSasakian manifold $\mathbf{S}^{2m+1}$ with gradient $*$-Ricci soliton, then $\mathbf{S}^{2m+1}$ is $*$-Einstein manifold.*

Proof. Consider a α-paraSasakian manifold $\mathbf{S}^{2m+1}$ with gradient $*$-Ricci soliton. By using equation (4.2) in the definition of curvature tensor $\mathcal{R}$ given in (2.7), we get

$$\mathcal{R}(X_1, X_2) D\varrho = (\nabla_{X_2} Q^*) X_1 - (\nabla_{X_1} Q^*) X_2. \tag{4.10}$$

Replacing $X_1 \longrightarrow \xi$ and $X_2 \longrightarrow X_1$ in (4.10), we get

$$\mathcal{R}(\xi, X_1) D\varrho = (\nabla_{X_1} Q^*) \xi - (\nabla_{\xi} Q^*) X_1. \tag{4.11}$$

Also, taking scalar product of equation (4.10) with ξ, we get

$$\mathbf{g}(\mathcal{R}(\xi, X_1) D\varrho, \xi) = \mathbf{g}((\nabla_{X_1} Q^*) \xi, \xi) - \mathbf{g}((\nabla_{\xi} Q^*) X_1, \xi). \tag{4.12}$$

By using Proposition 3.1, we get

$$\mathbf{g}(\mathcal{R}(\xi, X_1) D\varrho, \xi) = 0. \tag{4.13}$$

Now, equation (3.5) and (4.13) gives

$$X_1 \varrho = X_1(\xi \varrho).$$

Therefore, either $\varrho = 0$ or ϱ is constant. Thus, equation (4.2) gives

$$\mathrm{Ric}^*(X_1, X_2) = \lambda \mathbf{g}(X_1, X_2),$$

which is the definition of $*$-Einstein manifold. □

5. Conclusion

In differential geometry as well as in physics, Ricci soliton plays a very important role as they are the generalized notion of an Einstein metric on Riemannian and pseudo-Riemannian manifolds. Similar to Ricci soliton, a new notion have been defined by replacing Ricci tensor in soliton equation to $*$-Ricci tensor, which is *$*$-Ricci soliton*. In physics literature, $*$-Ricci solitons were first introduced as $*$-Einstein metric on Riemannian and pseudo-Riemannian manifolds.

As it is known, the concept of $*$-Ricci tensor has been defined only on complex and contact manifolds. But in the literature, some categorizations are also available in terms of $*$-Ricci tensor. Over the last few years, several authors have been studied $*$-Ricci soliton on different ambient spaces. So, we study $*$-Ricci soliton on paracontact manifold, particularly, on α-paraSasakian manifold. Also, the results we have found are playing a significant role in differential geometry and in mathematical physics.

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References

[1] R.S. Hamilton, The Ricci flow on surfaces, Mathematics and General Relativity, Contemp. Math. 71 (1988), 237–261. https://cir.nii.ac.jp/crid/1573105975542168832.

[2] G. Kaimakamis, K. Panagiotidou, $*$-Ricci Solitons of Real Hypersurfaces in Non-Flat Complex Space Forms, J. Geom. Phys. 86 (2014), 408–413. https://doi.org/10.1016/j.geomphys.2014.09.004.

[3] S. Tachibana, On Almost-Analytic Vectors in Almost-Kählerian Manifolds, Tohoku Math. J. 11 (1959), 247–265. https://doi.org/10.2748/tmj/1178244584.

[4] T. Hamada, Real Hypersurfaces of Complex Space Forms in Terms of Ricci $*$-Tensor, Tokyo J. Math. 25 (2002), 473–483. https://doi.org/10.3836/tjm/1244208866.

[5] K. De, A Note on Gradient $*$-Ricci Solitons, Math. Sci. Appl. E-Notes 8 (2020), 79–85. https://doi.org/10.36753/mathenot.727083.

[6] X. Chen, Real Hypersurfaces With $*$-Ricci Tensors in Complex Two-Plane Grassmannians, Bull. Korean Math. Soc. 54 (2017), 975–992. https://doi.org/10.4134/BKMS.B160414.

[7] R.S. Gupta, S. Rani, $*$-Ricci Soliton on GSSF with Sasakian Metric, Note Mat. 42 (2022), 95–108. https://doi.org/10.1285/i15900932v42n1p95.

[8] R. Ma, D. Pei, $*$-Ricci Tensor on (κ, μ)-Contact Manifolds, AIMS Math. 7 (2022), 11519–11528. https://doi.org/10.3934/math.2022642.

[9] M. Ben Ayed, K. El Mehdi, Multiplicity Results for the Scalar Curvature Problem on Half Spheres, Discr. Contin. Dyn. Syst. 44 (2024), 1878–1900. https://doi.org/10.3934/dcds.2024013.

[10] S. Elsaeed, O. Moaaz, K.S. Nisar, M. Zakarya, E.M. Elabbasy, Sufficient Criteria for Oscillation of Even-Order Neutral Differential Equations With Distributed Deviating Arguments, AIMS Math. 9 (2024), 15996–16014. https://doi.org/10.3934/math.2024775.

[11] B.B. Chaturvedi, P. Bhagat, M.N.I. Khan, Novel Theorems for a Bochner Flat Lorentzian Kahler Space-Time Manifold with η-Ricci-Yamabe Solitons, Chaos Solitons Fractals: X 11 (2023), 100097. https://doi.org/10.1016/j.csfx.2023.100097.

[12] A. Sardar, M.N.I. Khan, U.C. De, η-$*$-Ricci Solitons and Almost co-Kähler Manifolds, Mathematics 9 (2021), 3200. https://doi.org/10.3390/math9243200.

[13] N. Omar, S. Serra-Capizzano, B. Qaraad, F. Alharbi, O. Moaaz, E.M. Elabbasy, More Effective Criteria for Testing the Oscillation of Solutions of Third-Order Differential Equations, Axioms 13 (2024), 139. https://doi.org/10.3390/axioms13030139.

[14] K. De M.N.I. Khan, U.C. De, Almost co-Kähler Manifolds and (m, ρ)-Quasi-Einstein Solitons, Chaos Solitons Fractals, 167 (2023), 113050. https://doi.org/10.1016/j.chaos.2022.113050.

[15] A. Haseeb, S.K. Chaubey, F. Mofarreh, A.A.H. Ahmadini, A Solitonic Study of Riemannian Manifolds Equipped with a Semi-Symmetric Metric ξ-Connection, Axioms 12 (2023), 809. https://doi.org/10.3390/axioms12090809.

[16] D.G. Prakasha, M.R. Amruthalakshmi, F. Mofarreh, A. Haseeb, Generalized Lorentzian Sasakian-Space-Forms with M-Projective Curvature Tensor, Mathematics 10 (2022), 2869. https://doi.org/10.3390/math10162869.

[17] S. Kumar, M. Bilal, R. Prasad, A. Haseeb, Z. Chen, V-Quasi-Bi-Slant Riemannian Maps, Symmetry 14 (2022), 1360. https://doi.org/10.3390/sym14071360.

[18] A. Haseeb, M. Bilal, S.K. Chaubey, A.A.H. Ahmadini, ξ-Conformally Flat LP-Kenmotsu Manifolds and Ricci-Yamabe Solitons, Mathematics 11 (2022), 212. https://doi.org/10.3390/math11010212.

[19] U.C. De, Y.J. Suh, S.K. Chaubey, Semi-Symmetric Curvature Properties of Robertson–Walker Spacetimes, Z. Mat. Fiz. Anal. Geom. 18 (2022), 368–381. https://doi.org/10.15407/mag18.03.368.

[20] J.T. Cho, M. Kimura, Ricci Solitons and Real Hypersurfaces in a Complex Space Form, Tohoku Math. J. 61 (2009), 205–212. https://doi.org/10.2748/tmj/1245849443.

[21] G. Calvaruso, A. Perrone, Ricci Solitons in Three-Dimensional Paracontact Geometry, J. Geom. Phys. 98 (2015), 1–12. https://doi.org/10.1016/j.geomphys.2015.07.021.

[22] U.C. De, K. Mandal, Ricci Almost Solitons and Gradient Ricci Almost Solitons in (k,μ)-Paracontact Geometry, Bol. Soc. Paran. Mat. 37 (2017), 119–130. https://doi.org/10.5269/bspm.v37i3.33027.

[23] D.S. Patra, Ricci Solitons and Paracontact Geometry, Mediterr. J. Math. 16 (2019), 137. https://doi.org/10.1007/s00009-019-1419-6.

[24] B.Y. Chen, Some Results on Concircular Vector Fields and Their Applications to Ricci Solitons, Bull. Korean Math. Soc. 52 (2015), 1535–1547. https://doi.org/10.4134/BKMS.2015.52.5.1535.

[25] R. Sharma, A. Ghosh, Sasakian 3-Manifold as a Ricci Soliton Represents the Heisenberg Group, Int. J. Geom. Methods Mod. Phys. 08 (2011), 149–154. https://doi.org/10.1142/s021988781100504x.

[26] M.N.I. Khan, S.K. Chaubey, N. Fatima, A. Al Eid, Metallic Structures for Tangent Bundles over Almost Quadratic ϕ-Manifolds, Mathematics 11 (2023), 4683. https://doi.org/10.3390/math11224683.

[27] M.N.I. Khan, U.C. De, L.S. Velimirovic, Lifts of a Quarter-Symmetric Metric Connection from a Sasakian Manifold to Its Tangent Bundle, Mathematics 11 (2022), 53. https://doi.org/10.3390/math11010053.

[28] S. Kaneyuki, F.L. Williams, Almost Paracontact and Parahodge Structures on Manifolds, Nagoya Math. J. 99 (1985), 173–187. https://doi.org/10.1017/s0027763000021565.

[29] A. Ghosh, D.S. Patra, $*$-Ricci Soliton Within the Frame-Work of Sasakian and (k,μ)-Contact Manifold, Int. J. Geom. Methods Mod. Phys. 15 (2018), 1850120. https://doi.org/10.1142/s0219887818501207.

[30] S. Zamkovoy, G. Nakova, The Decomposition of Almost Paracontact Metric Manifolds in Eleven Classes Revisited, J. Geom. 109 (2018), 18. https://doi.org/10.1007/s00022-018-0423-5.

[31] K. Matsumoto, Conformal Killing Vector Fields in a P-Sasakian Manifold, J. Korean Math. Soc. 14 (1977), 135–142.

[32] K. Yano, Integral Formulas in Riemannian Geometry, Marcel Dekker, New York, 1970.