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Exploring the Influence of Generalized Kernels on Green's Function in Fractional Differential Equations

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Abstract. The basic purpose of this article is to define the Green's function in order to provide the solution of fractional differential equations in the presence of general analytic kernel. Using the technique of Laplace and Fourier transforms, we construct the Green's function for ordinary and partial fractional differential equations. The presented results will provide the generalization of some models existing in the literature. Some examples are also provided to prove the results for some particular cases.

1. Introduction

For the person studying elementary calculus, the idea of differentiation is common but fractional calculus fascinates the mathematicians to think about the differentials of the real order. Since, not a single model of fractional differentiation and integration exist, Riemann-Liouville (RL) [1] approach provides base to the subject of fractional calculus.

RL fractional integral operator $\mathcal{I}$ of arbitrary order p of a function g is defined as [1]

$$ {}^{RL}\mathcal{I}^{p}_{c+}g(t) = \frac{1}{\Gamma(p)}\int_c^t (t-w)^{p-1}g(w)dw, $$

where $Re(p) > 0$ and c is the constant of integration.

Also, the RL fractional differential $\mathcal{D}$ of a function g with p as the order of the differentiation [1] is defined as follow

$$ {}^{RL}\mathcal{D}^{p}_{c+}g(t) = \frac{\mathrm{d}^{\xi}}{\mathrm{d}t^{\xi}}\left({}^{RL}\mathcal{I}^{\xi-p}_{c+}g(t)\right), \tag{1.1} $$

where $Re(p) > 0$, c is the constant of differentiation and $\xi = \lfloor Re(p) \rfloor + 1$.

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RL fractional integral and integral operators are used to solve differintegral problems with initial conditions and have a broad number of applications in the modeling of bioengineering problems [2], chaotic systems [3], controllability [4], Stokes problems [5], thermoelasticity [6], financial modeling [7] and many other complex phenomena.

RL definition was later modified by Caputo [8] by interchanging differential and integral operators in (1.1) which is defined as

$$ {}^{C}\mathcal{D}^{p}_{c+}g(t) = {}^{RL}\mathcal{I}^{\xi-p}_{c+}\left(\frac{\mathrm{d}^{\xi}}{\mathrm{d}t^{\xi}}g(x)\right), \quad Re(p) \geq 0,\ \xi = \lfloor Re(p) \rfloor + 1. $$

Caputo's fractional order model is often used in modeling and analysis [9–11]. Indeed, if the process of describing the time dependance of the point does not have physical interpretation then it may be difficult or impossible to calculate fractional order initial conditions. In most of the cases, initial value conditions illustrate some important properties of the solution at the starting point of the process, e.g, initial value conditions of the fractional differential conditions ensures that the solution is unique [11].

Another special case, for solving fractional integral equations, is Prabhakar fractional model [12].

$$ {}^{P}_{q,v}\mathcal{I}^{p,u}_{c+}g(t) = \int_{c}^{t} (t-w)^{p-1} E^{q}_{q,p}\left(v(t-w)^{q}\right) g(w)dw,\ Re(p) > 0,\ Re(q) > 0. \tag{1.2} $$

The Mittag-Leffler function [13] used in (1.2) is defined as

$$ E^{u}_{q,p}(x) = \sum_{n=0}^{\infty} \frac{\Gamma(u+n)}{\Gamma(u)\Gamma(qn+p)n!}x^{n}. $$

Also Parabhakar fractional differential is defined with same parameters as

$$ {}^{P}_{q,v}D^{p,u}_{c+}g(t) = \frac{\mathrm{d}^{\xi}}{\mathrm{d}t^{\xi}}\left({}^{P}_{q,v}I^{\xi-q,-u}_{c+}g(t)\right), \quad Re(p) > 0,\ Re(q) > 0,\ \xi = Re\lfloor p \rfloor + 1. $$

Although it is older than some of the other models but now it has begun to attract attention, and its application has been discovered, e.g. in theory of dielectrics and stochastic processes [12]. This model also has been generalized [14] to use 4-parameter Mittag-Leffler function and its properties has been inspected in many other papers [15, 16, 25–27].

Atangana-Baleanu fractional model (also known as AB-model) [17] involving Mittag-Leffler function with one parameter (Mittag-Leffler function with one parameter is defined in [18]) is stated below:

$$ \begin{aligned} {}^{ABR}\mathcal{D}^{p}_{c+}g(t) &= \frac{B(p)}{1-p}\frac{\mathrm{d}}{\mathrm{d}t}\int_{c}^{t} E_{p}\left(\frac{-p}{1-p}(t-w)^{p}\right)g(w)dw, \quad 0 < p < 1, \\ {}^{ABC}\mathcal{D}^{p}_{c+}g(t) &= \frac{B(p)}{1-p}\int_{c}^{t} E_{p}\left(\frac{-p}{1-p}(t-w)^{p}\right)g'(w)dw, \quad 0 < p < 1. \end{aligned} $$

where $B(p)$ is a normalization function. Using this model, we are able to describe a different type of physical models (e.g. motion of bodies under certain forces) of complex systems. Non-singular Mittag-Leffler kernel has wide application in fractional calculus and is more easily used from the numerical viewpoint, this has been studied for example in [19].

Another fractional model known as Generalized Proportional Fractional (or GPF) model [20] defines the fractional differential operator with two parameters as:

$$ {}^{GPF}\mathcal{I}^{p,q}_{c+}g(t) = \frac{1}{q^p\Gamma(p)}\int_c^t \exp\left(\frac{q-1}{q}(t-w)\right)(t-w)^{p-1}g(w)dw, $$
$$ 0 < q \leq 1, \; Re(p) > 0. $$

Since, there are multiple ways of defining fractional integral and derivatives: RL, Caputo, tempered, Marchaud, Hilfer, and Atangana-Baleanu, to name but a few [1, 17]. These diverse definitions may be categorized into general classes according to their structure and properties [18].

In 2019, a fractional model proposed by Fernandez et al. [1] which involves general analytic kernels. This model merges above mentioned fractional models due to the analytic kernel. It smooths the way for solving dynamical systems [21]. Basic definitions and important features of this model are defined in the section stated below.

The arrangement of this paper is as follows: since in the first section, some previous models are discussed and physical applications of the models were stated. Now in *Section* 2, some basic definition of the fractional model with GAK are stated. Also some of the useful results for GAK are described. Green's function approach for RL fractional model is defined which provides base to main results of the paper. In *Section* 3, at first, we proof some necessary results that was used in our main results. We define the Green's function for fractional differential operator with MAK and also proof important properties of the Green's function. We provide the general prove for finding the Green's function of n-term ordinary linear fractional differential equation with MAK. An example is stated for result which satisfies RL fractional Green's function for particular values. At the end of the section, fractional Green's function for partial differential equations is stated using Laplace and Fourier transforms. In *Section* 4, we conclude the paper.

2. Materials and Methods

Fractional model with GAK [1] is fractional model with two parameters defined on the analytic disc. These analytic kernels are also known as non singular kernels due to there analytic behavior. Fractional integral with analytic kernel is stated below:

Definition 2.1. *[1] Let $[c,d] \in \mathbb{R}$, p,q be complex parameters with $Re(p) > 0$, $Re(q) > 0$ and $R \in \mathbb{R}^+$ satisfying $R > (d-c)^{Re(q)}$. Let A be the complex function analytic on the disc $D(0,R)$ and defined on the disc by locally uniformly convergent power series*

$$ A(x) = \sum_{m=0}^{\infty} b_m x^m, \tag{2.1} $$

where the coefficients $b_n = b_n(p,q)$ may depend on complex parameters if required.

Using analytic kernel (2.1), a modified form of analytic kernel is also defined by Fernendez et al. [1] that is

Definition 2.2. *[1] For any analytic function (2.1), modified analytic function A_Γ is defined as*

$$A_\Gamma(x) = \sum_{m=0}^{\infty} b_m \Gamma(qm+p) x^m. \tag{2.2}$$

The series (2.2) has radius of convergence given by

$$\lim_{m\to\infty} \left| \frac{b_m}{b_{m+1}} (qm+q+p)^{-q} \right|.$$

Remark 2.1. *If the series defined for A_Γ converges then the series for A also converges but converse is not true.*

Definition 2.3. *[1] Fractional integral operator with GAK, operating on the function g from closed interval $[c,d]$ to $\mathbb{R}$ as*

$$^{A}\mathcal{I}^{p,q}_{c+} g(t) = \int_c^t (t-w)^{p-1} A\left((t-w)^q\right) g(w) dw. \tag{2.3}$$

The integral operator defined in (2.3) provides the generalization of RL fractional model [1], Prabhakar fractional model [12], AB fractional model [17] and GPF fractional model [20].

Theorem 2.1. *[1] For terminologies defined in Definition 2.1, for any function $g \in L^1[c,d]$, there exists a locally uniformly convergent power series for the integral operator $^{A}\mathcal{I}^{p,q}_{c+} g$ as a function on $[c,d]$*

$$^{A}\mathcal{I}^{p,q}_{c+} g(t) = \sum_{m=0}^{\infty} b_m \Gamma(qm+p) \, {}^{RL}\mathcal{I}^{qm+p}_{c+} g(t).$$

Moreover, the integral operator defined in (2.3) can also be written in the form of MAK (2.2) as

$$^{A}\mathcal{I}^{p,q}_{c+} g(t) = A_\Gamma\left({}^{RL}\mathcal{I}^{q}_{c+}\right) {}^{RL}\mathcal{I}^{p}_{c+} g(t).$$

Theorem 2.2. *[1] Suppose c,d,A be as in Definition 2.1 and p_1,p_2,q be the fixed complex parameters with positive real parts and $g \in L^1[c,d]$. The semigroup property*

$$^{A}\mathcal{I}^{p_1,q}_{c+} \circ {}^{A}\mathcal{I}^{p_2,q}_{c+} g(t) = {}^{A}\mathcal{I}^{p_1+p_2,q}_{c+} g(t),$$

is uniformly valid (regardless of complex parameters and function g) if and only if the condition stated below holds for all non-negative integers.

$$\sum_{m+n=k} b_m(p_1,q) b_n(p_2,q) B(qm+p_1, qn+p_2) = b_k(p_1+p_2,q).$$

Theorem 2.3. *[1] Suppose c,d,A be as in Definition 2.1 and $p_1,p_2,q_1,q_2 \in \mathbb{C}$ with positive real parts and $g \in L^1[c,d]$. The semigroup property*

$$^{A}\mathcal{I}^{p_1,q_1}_{c+} \circ {}^{A}\mathcal{I}^{p_2,q_2}_{c+} g(t) = {}^{A}\mathcal{I}^{p_1+p_2,q_1+q_2}_{c+} g(t),$$

cannot be uniformly valid for any complex parameters p_1,p_2,q_1,q_2 and g.

Now in correspondence with fractional integral with general analytic kernels, differential operator is also defined as follow:

Definition 2.4. *[1] Fractional derivative with general analytical kernel is defined in both RL and Caputo forms. The operator is defined as follows*

$$\begin{aligned} {}^{A}_{RL}\mathcal{D}^{p,q}_{c+}g(t) &= \frac{d^{\xi}}{dt^{\xi}}\left({}^{A}\mathcal{I}^{p',q'}_{c+}g(t)\right) \\ {}^{A}_{C}\mathcal{D}^{p,q}_{c+}g(t) &= {}^{A}\mathcal{I}^{p',q'}_{c+}\left(\frac{d^{\xi}}{dt^{\xi}}g(t)\right), \end{aligned} \tag{2.4}$$

where $\xi \in \mathbb{N}$. The value of $p^{'}, q^{'}$ may depends of the analytic kernel. For example, Riemann-Liouville operator, we must have $p^{'} = \xi - p, q^{'} = 0$.

Much of applied mathematics is dedicated to the study of differential equations and their solutions. Almost any dynamic process in nature can be modeled by some ordinary or partial differential equation. Since, integral transforms have wide application in solving fractional integral and differential equations so these transformations are also defined for the fractional integral with GAK using convolution property. For the transformation of the functions, Theorem 2.1 is used.

Theorem 2.4. *[1] Assume $c = 0, d \in \mathbb{R}^{+}, p, q$ and A be as in Definition 2.1 and $g \in L^2[c,d]$ with Laplace transform $\overline{g}$. Then Laplace transform of (2.3) is given as*

$$\overline{{}^{A}\mathcal{I}^{p,q}_{0+}g(s)} = s^{-p}A_{\Gamma}(s^{-q})\bar{g}(s),$$

where A_{Γ} is defined in Definition 2.2.

Theorem 2.5. *[1] Using the terminologies defined in Definition 2.1 with $c = -\infty$ and $d \in \mathbb{R}^{+}$ and $g \in L^2[c,d]$ with Fourier transform $\tilde{g}$. Fourier transform for fractional integral (2.3) is defined as*

$$\widetilde{{}^{A}\mathcal{I}^{p,q}_{+}g}(s) = k^{-p}e^{ip\pi/2}A_{\Gamma}(k^{-q}e^{iq\pi/2})\tilde{g}(s),$$

where A_{Γ} is defined in Definition 2.2.

Main purpose of this article is to find the solution of fractional differential equations in a more easier way. Construction of Green's function is one of the important method to get the solutions of the ordinary and partial differential problems. Inspiring from the work of researches in [22–24], we construct the Green's function for the fractional differential problems with MAK which absorbs most of the fractional models in it, so that using fractional differential operator with MAK provides the solution of many differential problems involving other models. For the verification of our results, we gave solution of RL problems as particular case.

2.1. **Green's Function Approach for RL Fractional Differential Operators.** We can find the solution of initial value nonhomogeneous problem for ordinary fractional linear differential equation with constant coefficients using only its Green's function [13]. Due to this result, the solution of homogeneous equation with nonhomogeneous initial value conditions reduced to find only the fractional Green's functions. Fractional Green's function for RL fractional derivative is stated below:

Definition 2.5. *[13] Consider nonhomogeneous fractional differential equation of function $g \in L^1[0,T]$ with homogeneous condition*

$$ {}_0\mathscr{L}_t x(t) = g(t); \quad \left[{}_0\mathcal{D}_t^{\sigma_m - 1} x(t) \right]_{t=0} = 0, \quad m = 1, 2, \cdots, n, \tag{2.5} $$

where

$$ \begin{aligned} {}_0\mathscr{L}_t x(t) &= {}_0\mathcal{D}_t^{\sigma_n} x(t) + \sum_{m=1}^{n-1} f_m(t)\, {}_0\mathcal{D}_t^{\sigma_{n-m}} x(t) + f_n(t) x(t) \\ {}_0\mathcal{D}_t^{\sigma_m} &= {}_0\mathcal{D}_t^{p_m}\, {}_0\mathcal{D}_t^{p_{m-1}} \cdots {}_0\mathcal{D}_t^{p_1}\ {}_0;\ \mathcal{D}_t^{\sigma_m - 1} = {}_0\mathcal{D}_t^{p_m - 1}\, {}_0\mathcal{D}_t^{p_{m-1}} \cdots {}_0\mathcal{D}_t^{p_1} \\ \sigma_m &= \sum_{j-1}^{m} p_j, \quad 0 < p_j < 1, \quad j = 1, 2, \cdots, n. \end{aligned} $$

Then function $G(t,w)$ satisfying the following conditions:

(1) ${}_w\mathscr{L}_t G(t,w) = 0$ *for every $w \in (0,t)$;*
(2) $\lim_{w \to t-0} \left({}_w\mathcal{D}_t^{\sigma_m - 1} G(t,w) \right) = \delta_{m,n},\ m = 0, 1, \cdots, n,$
($\delta_{m,n}$ is Kronecker's delta);
(3) $\lim_{\substack{w,t \to +0 \\ w<t}} \left({}_w\mathcal{D}_t^{\sigma_m} G(t,w) \right) = 0;\ m = 0, 1, \cdots, n-1,$

is called Green's function of (2.5).

To find the fractional Green's function of a nonhomogeneous initial value problem, Laplace transform method is used.

Example 2.1. *[13] Consider RL differential equation with p as the order of differentiation and a is the constant of integration,*

$$ a\, {}_0^{RL}\mathcal{D}_{t+}^{p} x(t) = g(t); \quad x(0) = 0. \tag{2.6} $$

Solution 2.1. *By taking Laplace transform of (2.6), we get*

$$ \bar{x}(s) = \frac{1}{a s^p}. $$

By using convolution property of inverse Laplace transform, we get Green's function of (2.6) as

$$ \begin{aligned} x(t) &= \frac{t^{p-1}}{a\Gamma(p)} \\ x(t) &= \frac{1}{a\Gamma(p)} \int_0^t (t-w)^{p-1} g(w)\, dw, \end{aligned} \tag{2.7} $$

where (2.7) provides the solution of (2.6).

3. Results

In the following section, we will provide some supporting results which will be used to prove properties of Green's function with MAK.

3.1. **Green's Function Approach for Fractional Differential Operator with MAK.** Considering a general nonhomogeneous problem involving modified analytic kernel (2.2), then Green's function for the nonhomogeneous problem (stated below) will be evaluated to provide the solution.

Proposition 3.1. *Let $p, q \in \mathbb{C}$ with $Re(p) > 0$, $Re(q) > 0$. Then for fractional integral (2.3), the equality holds*

$$ {}_{RL}^{A}\mathcal{D}_{0+}^{p,q}\left(\int_0^t k(t,w)dw\right) = \int_0^t {}_{RL}^{A}\mathcal{D}_{w+}^{p,q}k(\eta,w)dw + \lim_{w\to t-0} {}_{RL}^{A}\mathcal{D}_{w+}^{p-1,q}k(\eta,w), $$

Proof. Consider

$$ {}_{RL}^{A}\mathcal{D}_{0+}^{p,q}\left(\int_0^t k(t,w)dw\right) = \frac{\mathrm{d}^{\xi}}{\mathrm{dt}^{\xi}}\left({}^{A}\mathcal{I}_{0+}^{\xi-p,q}\left(\int_0^t k(t,w)dw\right)\right), $$

by putting $\xi = 1$ and using Fubini's Theorem, we get

$$ {}_{RL}^{A}\mathcal{D}_{0+}^{p,q}\left(\int_0^t k(t,w)dw\right) = \frac{\mathrm{d}}{\mathrm{dt}}\int_0^t \tilde{k}(t,w)dw, $$

where

$$ \tilde{k}(t,w) = \int_w^t (t-\eta)^{-p}A\left((t-\eta)^q\right)k(\eta,w)d\eta. $$

Now by using Leibnitz rule and $\tilde{k}(t,w)$ we get the required result as

$$ \begin{aligned} {}_{RL}^{A}\mathcal{D}_{0+}^{p,q}\left(\int_0^t k(t,w)dw\right) &= \int_0^t \frac{\partial}{\partial t}\tilde{k}(t,w)dw + \lim_{w\to t-0}\tilde{k}(t,w) \\ &= \int_0^t {}_{RL}^{A}\mathcal{D}_{w+}^{p,q}k(\eta,w)dw + \lim_{w\to t-0} {}_{RL}^{A}\mathcal{D}_{w+}^{p-1,q}k(\eta,w). \end{aligned} $$

□

Now, to define the Green's function, consider the nonhomogeneous equation subject to the homogeneous constraints.

$$ {}_{0}\mathscr{L}_{t}x(t) = g(t) \tag{3.1} $$

$$ \left[{}_{0}^{A}\mathcal{D}_{t}^{\sigma_m-1,q}x(t)\right]_{t=0} = 0; \quad m = 1,2,\cdots,n, \tag{3.2} $$

where

$$ \begin{aligned} {}_{0}\mathscr{L}_{t}x(t) &= {}_{0}^{A}\mathcal{D}_{t}^{\sigma_n,q}x(t) + \sum_{m=1}^{n-1} f_m(t)\, {}_{0}^{A}\mathcal{D}_{t}^{\sigma_{n-m},q}x(t) + f_n(t)x(t) \\ {}_{0}^{A}\mathcal{D}_{t}^{\sigma_m,q} &= {}_{0}^{A}\mathcal{D}_{t}^{p_m,q}\, {}_{0}^{A}\mathcal{D}_{t}^{p_{m-1},q} \cdots {}_{0}^{A}\mathcal{D}_{t}^{p_1,q} \\ {}_{0}^{A}\mathcal{D}_{t}^{\sigma_m-1,q} &= {}_{0}^{A}\mathcal{D}_{t}^{p_m-1,q}\, {}_{0}^{A}\mathcal{D}_{t}^{p_{m-1},q} \cdots {}_{0}^{A}\mathcal{D}_{t}^{p_1,q} \\ \sigma_m &= \sum_{j-1}^{m} p_j, \quad 0 < p_j < 1, \quad j = 1,2,\cdots,n, \end{aligned} $$

and $g(t) \in L^1(0,T)$, i.e.

$$ \int_0^T |g(t)|dt < \infty. $$

For the simplicity of notation we can also write $f(t) = 0$ for $t > T$.

Definition 3.1. *The function $G(t, w)$ satisfying the following conditions*

(C1) ${}_{w}\mathscr{L}_{t}G(t,w) = 0$ *for every* $w \in (0, t)$;
(C2) $\lim_{w\to t-0}\left({}_{w}^{A}\mathcal{D}_{t}^{\sigma_m-1,q}G(t,w)\right) = \delta_{m,n}$, $m = 1, 2, \cdots n$,
($\delta_{m,n}$ is Kronecker's delta).
(C3) $\lim_{\substack{w,t\to+0\\ w<t}}\left({}_{w}^{A}\mathcal{D}_{t}^{\sigma_m,q}G(t,w)\right) = 0,\ m = 1, 2, \cdots, n-1,$
is called the Green's function of (3.1).

Theorem 3.1. *The function $x(t) = \int_0^t G(t, w)g(w)dw$ is the solution of the problem (3.1) and (3.2).*

Proof. Using Proposition 3.1 and (C2), we have

$$\begin{aligned}
{}_{RL}^{A}\mathcal{D}_{0+}^{\sigma_n,q}x(t) &= {}_{RL}^{A}\mathcal{D}_{0+}^{p_n}\int_0^t {}_{RL}^{A}\mathcal{D}_{w+}^{\sigma_{n-1},q}G(t,w)g(w)dw \\
&= \int_0^t {}_{RL}^{A}\mathcal{D}_{w+}^{p_n,q}\left({}_{RL}^{A}\mathcal{D}_{w+}^{\sigma_{n-1},q}G(t,w)g(w)\right)dw + \\
&\quad \lim_{w\to t-0} {}_{RL}^{A}\mathcal{D}_{w+}^{p_n,q}\left({}_{RL}^{A}\mathcal{D}_{w+}^{\sigma_{n-1},q}G(t,w)g(w)\right) \\
&= \int_0^t {}_{RL}^{A}\mathcal{D}_{w+}^{\sigma_n,q}G(t,w)g(w)dw + g(t).
\end{aligned}$$

Consider

$$\begin{aligned}
{}_{0}\mathscr{L}_{t}x(t) =\ & {}_{0}^{A}\mathcal{D}_{t}^{\sigma_n,q}x(t) + \sum_{m=1}^{n-1} f_n(t)\, {}_{0}^{A}\mathcal{D}_{t}^{\sigma_{n-m},q}x(t) + f_n(t)x(t) \\
=\ & \int_0^t {}_{RL}^{A}\mathcal{D}_{w+}^{\sigma_n,q}G(t,w)g(w)dw + \sum_{m=1}^{n-1} f_m(t)\int_0^t {}_{RL}^{A}\mathcal{D}_{w+}^{\sigma_{n-m},q}G(t,w)g(w)dw \\
& + f_n(t)\int_0^t G(t,w)g(w)dw + g(t), \qquad\qquad w \in (0,1) \\
=\ & g(t).
\end{aligned}$$

□

Theorem 3.2. *For fractional differential equation (2.4) with constant coefficients, we have $G(t, w) = G(t - w)$.*

Proof. This is obvious because in such a case Green's function can be obtained by using Laplace transform. □

Theorem 3.3. *Appropriate derivatives of the Green's function $G(t - w)$ form a set of linearly independent solutions of a homogeneous equation $g(t) \equiv 0$ in (3.1) with initial condition (3.2) equals to b_m where $m = 1, 2, \cdots, n$ defined in Definition 3.1.*

Proof. Assume $0 < p < \sigma_n$, then

$$ {}_0\mathscr{L}_t x_{p,q}(t) = {}_0\mathscr{L}_t \left({}^{A}_{RL}\mathcal{D}^{p,q}_{0+} G(t,w) \right). $$

By using condition (C3) and (C1), we have

$$ \begin{aligned} {}_0\mathscr{L}_t x_{p,q}(t) &= {}^{A}_{RL}\mathcal{D}^{p,q}_{0+} \left({}_0\mathscr{L}_t G(t,w) \right) \\ &= 0. \end{aligned} $$

Also,

$$ \begin{aligned} {}^{A}\mathcal{D}^{\sigma_n - p - 1}_{0+} x_{p,q}(t) &= {}^{A}\mathcal{D}^{\sigma_n - p - 1, q}_{0+}\, {}^{A}\mathcal{D}^{p,q}_{0+} G(t,w) \\ &= {}^{A}\mathcal{D}^{\sigma_n - 1, q}_{0+} G(t,w), \end{aligned} $$

by using condition (C2), we have

$$ {}^{A}\mathcal{D}^{\sigma_n - p - 1}_{0+} x_{p,q}(t)\Big|_{t=0} = {}^{A}\mathcal{D}^{\sigma_n - 1, q}_{0+} G(t,w)\Big|_{t=0} = 1. $$

□

Laplace transform of fractional differential operator with GAK [1] is defined as follow:

Theorem 3.4. *Assume $c = 0, d > 0, p, q, A$ be as in Definition 2.1, and let $x \in L^2[a,b]$. The function ${}^{A}\mathcal{D}^{p,q}_{a+} x(t)$ has a Laplace transform given by the following form.*

$$ {}^{A}\mathcal{D}^{p,q}_{a+} x(s) = s^{n-p'} A_{\Gamma}(s^{-q'}) x\hat{(s)}. \tag{3.3} $$

Proof. Consider (2.4) and taking Laplace transform on both sides, we have

$$ \begin{aligned} \mathcal{L}\left({}^{A}_{RL}\mathcal{D}^{p,q}_{a+} x(t) \right) &= \mathcal{L}\left[\frac{d^{\xi}}{dt^{\xi}} \left({}^{A}\mathcal{I}^{p',q'}_{a+} x(t) \right) \right] \\ &= s^n \mathcal{L}\left({}^{A}\mathcal{I}^{p',q'}_{a+} x(t) \right) - \left[\sum_{k=0}^{n-1} s^k \left({}^{A}\mathbf{I}^{p',q'}_{a+} x(t) \right)^{n-k-1} \right]_{t=0}, \end{aligned} $$

by using Laplace transform (2.5), we have

$$ \mathcal{L}\left({}^{A}_{RL}\mathcal{D}^{p,q}_{a+} x(t) \right) = s^n \left(s^{-p'} A_{\Gamma}(s^{-q'}) \overline{x(s)} \right). $$

□

Green's function is one from the important methods to find the solution of the differential equations. Here, method for finding the Green's function of fractional differential equation with MAK (stated in (2.5)) using Laplace transform is introduced.

Theorem 3.5. *For $p, q \in \mathbb{C}$ with positive real parts and the arbitrary constant coefficients b_i's, the nth-term fractional differential equation*

$$ b_n\, {}^{A}_{RL}\mathcal{D}^{p_n,q}_{0+} x(t) + b_{n-1}\, {}^{A}_{RL}\mathcal{D}^{p_{n-1},q}_{0+} x(t) + \cdots + b_0\, {}^{A}_{RL}\mathcal{D}^{p_0,q}_{0+} x(t) = g(t), \tag{3.4} $$

has Green's function.

Proof. To find the Green's function, consider homogeneous part of (3.4) as

$$b_n \, {}^{A}_{RL}\mathcal{D}^{p_n,q}_{0+}x(t) + b_{n-1} \, {}^{A}_{RL}\mathcal{D}^{p_{n-1},q}_{0+}x(t) + \cdots + b_0 \, {}^{A}_{RL}\mathcal{D}^{p_0,q}_{0+}x(t) = 1.$$

Taking Laplace transform defined in (3.3) and simplifying, we get

$$b_n\mathcal{L}({}^{A}_{RL}\mathcal{D}^{p_n,q}_{0+}x(t)) + b_{n-1}\mathcal{L}({}^{A}_{RL}\mathcal{D}^{p_{n-1},q}_{0+}x(t)) + \cdots + b_0\mathcal{L}({}^{A}_{RL}\mathcal{D}^{p_0,q}_{0+}x(t)) = \mathcal{L}(\delta)$$

$$\begin{bmatrix} b_n s^n \left(s^{-p_n'}A_\Gamma(s^{-q'})\right) + b_{n-1}s^n \left(s^{-p_{n-1}'}A_\Gamma(s^{-q'})\right) \\ + \cdots + b_0 s^n \left(s^{-p_0'}A_\Gamma(s^{-q'})\right) \end{bmatrix} \overline{G}_n(s) = 1$$

$$\overline{G}_n(s) = \begin{bmatrix} b_n s^n \left(s^{-p_n'}A_\Gamma(s^{-q'})\right) + b_{n-1}s^n \left(s^{-p_{n-1}'}A_\Gamma(s^{-q'})\right) \\ + \cdots + b_0 s^n \left(s^{-p_0'}A_\Gamma(s^{-q'})\right) \end{bmatrix}^{-1}, \tag{3.5}$$

by using the (2.2), we have (3.5) as

$$\overline{G}_n(s) = \begin{bmatrix} b_n s^{n-p_n'}A_\Gamma(s^{-q'}) + b_{n-1}s^{n-p_{n-1}'}A_\Gamma(s^{-q'}) \\ + \cdots + b_0 s^{n-p_0'}A_\Gamma(s^{-q'}) \end{bmatrix}^{-1}$$

By using Definition 2.2, we have

$$\begin{aligned}
\overline{G}_n(s) &= \begin{bmatrix} b_n \sum_{r=0}^{\infty} a_r \Gamma(q'r + p_n') s^{\xi + p_n' + rq'} + \\ b_{n-1} \sum_{r=0}^{\infty} a_r \Gamma(q'r + p'_{(n-1)}) s^{\xi + p'_{(n-1)} + rq'} \\ + \cdots + b_0 \sum_{r=0}^{\infty} a_r \Gamma(q'r + p_0') s^{\xi + p_0' + rq'} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} b_n \sum_{r=1}^{\infty} a_r \Gamma(q'r + p_n') s^{\xi + p_n' + rq'} + \\ \sum_{k=0}^{n-1} b_k \sum_{r=0}^{\infty} a_r \Gamma(q'r + p_k') s^{\xi + p_k' + rq'} \end{bmatrix}^{-1} \\
&= \begin{bmatrix} b_n a_0 \Gamma(p_n') s^{\xi + p_n'} + b_n a_1 \Gamma(q' + p_n') s^{\xi + p_n' + q'} + \\ b_n \sum_{r=2}^{\infty} a_r \Gamma(q'r + p_n') s^{\xi + p_n' + rq'} + \\ \sum_{k=0}^{n-1} b_k \sum_{r=0}^{\infty} a_r \Gamma(q'r + p_k') s^{\xi + p_k' + rq'} \end{bmatrix}^{-1} \\
&= \frac{s^{-\xi + p_n' + q'}}{b_n a_0 \Gamma(p_n')} \begin{bmatrix} s^{-q'} + \frac{a_1 \Gamma(q' + p_n')}{a_0 \Gamma(p_n')} + \frac{\sum_{r=2}^{\infty} a_r \Gamma(q'r + p_n')}{a_0 \Gamma(p_n')} s^{(r-1)q'} + \\ \sum_{k=0}^{n-1} \frac{\sum_{r=0}^{\infty} b_k a_r \Gamma(q'r + p_k')}{b_n a_0 \Gamma(p_n')} s^{p_k' - p_n' + (r-1)q'} \end{bmatrix}^{-1}.
\end{aligned}$$

Assuming $\frac{a_1\Gamma(q'+p_n')}{a_0\Gamma(p_n')} = A$, we have the form

$$\begin{aligned}
\overline{G}_n(s) &= \frac{s^{-\xi + p_n' + q'}}{b_n a_0 \Gamma(p_n')} \frac{1}{s^{-q'} + A} \begin{bmatrix} 1 + \sum_{r=2}^{\infty} \frac{[a_r \Gamma(q'r + p_n')/a_0 \Gamma(p_n')] s^{(r-1)q'}}{s^{-q'} + A} + \\ \sum_{k=0}^{n-1} \sum_{r=0}^{\infty} \frac{[b_k a_r \Gamma(q'r + p_k')/b_n a_0 \Gamma(p_n')] s^{p_k' - p_n' + (r-1)q'}}{s^{-q'} + A} \end{bmatrix}^{-1} \\
&= \sum_{m=0}^{\infty} \frac{s^{-\xi + p_n' + q'}}{b_n a_0 \Gamma(p_n')} \frac{(-1)^m}{(s^{-q'} + A)^{m+1}} \begin{bmatrix} \sum_{r=2}^{\infty} \frac{a_r \Gamma(q'r + p_n')}{a_0 \Gamma(p_n')} s^{(r-1)q'} + \\ \sum_{k=0}^{n-1} \sum_{r=0}^{\infty} \frac{b_k a_r \Gamma(q'r + p_k')}{b_n a_0 \Gamma(p_n')} s^{p_k' - p_n' + (r-1)q'} \end{bmatrix}^{m} \\
&= \sum_{m=0}^{\infty} \frac{s^{-\xi + p_n' + q'}}{b_n a_0 \Gamma(p_n')} \frac{(-1)^m}{(s^{-q'} + A)^{m+1}} \sum_{k=0}^{m} \left[\sum_{r=2}^{\infty} \frac{a_r \Gamma(q'r + p_n')}{a_0 \Gamma(p_n')} s^{(r-1)q'} \right]^{k}
\end{aligned} \tag{3.6}$$

$$\times\left[\sum_{k=0}^{n-1}\sum_{r=0}^{\infty}\frac{b_k a_r\Gamma(q'r+p'_k)}{b_n a_0\Gamma(p'_n)}s^{p'_k-p'_n+(r-1)q'}\right]^{k-m}. \tag{3.7}$$

Since the series $\sum_{r=0}^{\infty}\frac{b_k a_r\Gamma(q'r+p'_k)}{b_n a_0\Gamma(p'_n)}s^{p'_k-p'_n+(r-1)q'}$ is convergent in disc with radius of convergence defined as

$$0<\lim_{r\to\infty}\left|\frac{a_r(q'r+p'+q')^{-q'}}{a_{r+1}}\right|<\frac{1}{s^{q'}}$$

Now from (3.7), after simplification, we have

$$\begin{aligned}\overline{G}_n(s)=&\sum_{m=0}^{\infty}\frac{(-1)^m}{m!b_n a_0\Gamma(p'_n)}\sum_{k=0}^{m}\left[\sum_{r=2}^{\infty}\frac{a_r\Gamma(q'r+p'_n)}{a_0\Gamma(p'_n)}\right]^k\left[\sum_{k=0}^{n-1}\sum_{r=0}^{\infty}\frac{b_k a_r\Gamma(q'r+p'_k)}{b_n a_0\Gamma(p'_n)}\right]^{k-m}\\&\times\frac{m!s^{-q'-\left(\xi-p'_n-2q'-k(r-1)q'-(k-m)(p'_k-p'_n+(r-1)q')\right)}}{\left(s^{-q'}+A\right)^{m+1}}.\end{aligned}$$

Within the disc we have the singularities at $s^{-q'}=-A$. Thus at the singular points, we have the solution of the Laplace transform as

$$\begin{aligned}G_n(t)=&\sum_{m=0}^{\infty}\frac{(-1)^m}{m!b_n a_0\Gamma(p'_n)}\sum_{k=0}^{m}\left[\sum_{r=2}^{\infty}\frac{a_r\Gamma(q'r+p'_n)}{a_0\Gamma(p'_n)}\right]^k\left[\sum_{k=0}^{n-1}\sum_{r=0}^{\infty}\frac{b_k a_r\Gamma(q'r+p'_k)}{b_n a_0\Gamma(p'_n)}\right]^{k-m}\\&\times t^{-mq'+\left(\xi-p'_n-2q'-k(r-1)q'-(k-m)(p'_k-p'_n+(r-1)q')\right)-1}\\&\times E^{(m)}_{-q',\xi-p'_n-2q'-k(r-1)q'-(k-m)(p'_k-p'_n+(r-1)q')}(-At^{-q'}).\end{aligned}$$

□

Remark 3.1. *For $n=0,1,2,...$ we will get one-term differential equation, two-term differential equation and so on.*

Example 3.1. *The nonhomogeneous fractional differential equation is given by*

$${}^{A}\mathcal{D}^{p,q}x(t)+{}^{A}\mathcal{D}^{h,q}x(t)=g(t);\quad t>0, \tag{3.8}$$

with initial condition

$${}^{A}_{RL}\mathcal{D}^{p+qn-1}x(0)=0,\quad\forall\, n\in(0,\infty),$$

where $0<p\le 1$ and q is fixed, has unique Green's function.

Proof. Taking homogeneous part of (3.8) and applying Laplace transform, (3.8) gives

$$\begin{aligned}\left[s^{\xi-p'}A_\Gamma(s^{-q'})+s^{\xi-h'}A_\Gamma(s^{q'})\right]\bar{x}(s)=&1\\ \bar{G}(s)\left[\sum_{n=0}^{\infty}a_n\Gamma(p'+q'n)s^{\xi-p'-q'n}+\sum_{k=0}^{\infty}a_k\Gamma(h'+q'k)s^{\xi-h'-q'k}\right]=&1\end{aligned}$$

$$
\begin{aligned}
\overline{G}(s) &= \left[\sum_{n=0}^{\infty} a_n \Gamma(p'+q'n) s^{\xi-p'-q'n} + \sum_{k=0}^{\infty} a_k \Gamma(h'+q'k) s^{\xi-h'-q'k}\right]^{-1} \\
&= \left[\begin{array}{c} a_0\Gamma(p')s^{\xi-p'} + a_1\Gamma(p'+q')s^{\xi-p'-q'} + \sum_{n=2}^{\infty} a_n\Gamma(p'+q'n)s^{\xi-p'+q'n} \\ \sum_{k=0}^{\infty} a_k\Gamma(h'+q'k)s^{\xi-h'-q'k} \end{array}\right]^{-1} \\
&= \frac{s^{-\xi+p'+q'}}{a_0\Gamma(p')} \left[s^{-q'} + \frac{a_1\Gamma(p'+q')}{a_0\Gamma(p')} + \frac{\sum_{n=2}^{\infty} a_n\Gamma(p'+q'n)s^{\xi-p'-q'n} + \sum_{k=0}^{\infty} a_k\Gamma(h'+q'k)s^{\xi-h'-q'k}}{a_0\Gamma(p')} \right]^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\overline{G}(s) &= \frac{1}{a_0\Gamma(p')} \frac{s^{-\xi+p'+q'}}{s^{-q'} + \frac{a_1\Gamma(p'+q')}{a_0\Gamma(p)}} \left[1 + \frac{\left[\sum_{n=2}^{\infty} a_n\Gamma(p'+q'n)s^{q'(n-1)} + \sum_{k=0}^{\infty} a_k\Gamma(h'+q'k)s^{\xi-h'-q'k}\right]/[a_0\Gamma(p')]}{s^{-q'} + [a_1\Gamma(p'+q')]/[a_0\Gamma(p')]} \right]^{-1} \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{[a_0\Gamma(p')]^{m+1}} \frac{s^{-\xi+p'+q'}}{\left(s^{-q'} + \frac{a_1\Gamma(p'+q')}{a_0\Gamma(p')}\right)^{m+1}} \\
&\quad \times \left[\sum_{n=2}^{\infty} a_n\Gamma(p'+q'n)s^{-q'(n-1)} + \sum_{k=0}^{\infty} a_k\Gamma(h'+q'k)s^{\xi-h'-q'k}\right]^m \\
&= \sum_{m=0}^{\infty} \frac{(-1)^m}{[a_0\Gamma(p')]^{m+1}} \frac{s^{-\xi+p'+q'}}{\left(s^{-q'} + \frac{a_1\Gamma(p'+q')}{a_0\Gamma(p')}\right)^{m+1}} \sum_{r=0}^{m} \binom{m}{r} \\
&\quad \times \left[\sum_{n=2}^{\infty} a_n\Gamma(p'+q'n)s^{-q'(n-1)}\right]^r \left[\sum_{k=0}^{\infty} a_k\Gamma(h'+q'k)s^{\xi-h'-q'k}\right]^{m-r}.
\end{aligned}
$$

After simplification, we have the following expression of Laplace transform of Mittag-Leffler function:

$$
\begin{aligned}
\overline{G}(s) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m![a_0\Gamma(p')]^{m+1}} \sum_{r=0}^{m} \binom{m}{r} \left[\sum_{n=2}^{\infty} a_n\Gamma(p'+q'n)\right]^r \\
&\quad \times \left[\sum_{k=0}^{\infty} a_k\Gamma(h'+q'k)\right]^{m-r} \frac{m! s^{-\xi+p'+q'+-q' r(n-1)(m-r)(\xi-h'-q'k)}}{\left(s^{-q'} + \frac{a_1\Gamma(p'+q')}{a_0\Gamma(p')}\right)^{m+1}}.
\end{aligned}
$$

By taking the inverse Laplace transform, we have the Green's function of (3.8), that is

$$
\begin{aligned}
g(t) &= \sum_{m=0}^{\infty} \frac{(-1)^m}{m![a_0\Gamma(p')]^{m+1}} \sum_{r=0}^{m} \binom{m}{r} \left[\sum_{n=2}^{\infty} a_n\Gamma(p'+q'n)\right]^r \left[\sum k = 0^{\infty} a_k\Gamma(h'+q'k)\right]^{m-r} \\
&\quad \times t^{-q'm+\xi-p'-2q'+q'(n-1)-(\xi-h'-kq')(m-r)-1} \\
&\quad \times E^{(m)}_{-q',\xi-p'-2q'+q'(n-1)-(\xi-h'-kq')(m-r)} \left(\frac{-a_1\Gamma(p'+q')}{a_0\Gamma(p')} t^{-q'} \right).
\end{aligned}
$$

□

Remark 3.2. *In Example 3.1, if we have $m = 0$ them there must be $r = 0$, Also by substituting $-q' = p$ and $\frac{a_1\Gamma(p'+q')}{a_0\Gamma(p')} = \frac{b}{a}$, we have the solution of two-term RL fractional differential equation (see P-154, [13]).*

3.2. **Green's Function for Partial Differential Equations with MAK.** Since, the number of problems in multiple fields of science and engineering have been solved through fractional partial differential equations using different fractional operators. In this section, we will provide a new approach for solving partial differential problems using differential operator with GAK.

Theorem 3.6. *For $p, q \in \mathbb{C}$ with positive real parts, the arbitrary constant coefficients $b_i's$ and $g(x,t) \in L^1(0,1)$, the nth-term fractional differential equation*

$$\begin{array}{c} b_n \frac{\partial^{p_n,q}}{\partial t^{p_n,q}} u(x,t) + b_{n-1} \frac{\partial^{p_{n-1},q}}{\partial t^{p_{n-1},q}} u(x,t) + \cdots + b_0 \frac{\partial^{p_0,q}}{\partial t^{p_0,q}} u(x,t) \\ + c_j \frac{\partial^{h_j,q}}{\partial x^{h_j,q}} u(x,t) + c_{j-1} \frac{\partial^{h_{j-1},q}}{\partial x^{h_{j-1},q}} u(x,t) + \cdots + c_0 \frac{\partial^{h_0,q}}{\partial x^{h_0,q}} u(x,t) \end{array} = g(x,t), \tag{3.9}$$

has Green's function.

Proof. To find the Green's function, we can write (3.9) as

$$\begin{array}{c} b_n \frac{\partial^{p_n,q}}{\partial t^{p_n,q}} G(x,t) + b_{n-1} \frac{\partial^{p_{n-1},q}}{\partial t^{p_{n-1},q}} G(x,t) + \cdots + b_0 \frac{\partial^{p_0,q}}{\partial t^{p_0,q}} G(x,t) \\ + \sum_{v=0}^{j} c_v \frac{\partial^{h_v,q}}{\partial x^{h_v,q}} G(x,t) \end{array} = g(x,t).$$

Applying Laplace transform with respect to t on above equation and simplifying, we get

$$\left[\begin{array}{c} b_n \sum_{r=0}^{\infty} a_r \Gamma(q'r + p'_n) s^{\xi - p'_n - rq'} \overline{G}(x,s) + \\ b_{n-1} \sum_{r=0}^{\infty} a_r \Gamma(q'r + p'_{n-1}) s^{\xi - p'_{n-1} - rq'} \overline{G}(x,s) + \cdots + \\ b_0 \sum_{r=0}^{\infty} a_r \Gamma(q'r + p'_0) s^{\xi - p'_0 - rq'} \overline{G}(x,s) + \sum_{v=0}^{j} c_v \frac{d^{h_v,q}}{dx^{h_v,q}} \overline{G}(x,s) \end{array} \right] = \overline{g}(x,s),$$

applying Fourier transform with respect to x on above equation, we have

$$\left[\begin{array}{c} b_n \sum_{r=0}^{\infty} a_r \Gamma(q'r + p'_n) s^{\xi - p'_n - rq'} \widetilde{\overline{G}}(\omega,s) + \\ b_{n-1} \sum_{r=0}^{\infty} a_r \Gamma(q'r + p'_{n-1}) s^{\xi - p'_{n-1} - rq'} \widetilde{\overline{G}}(\omega,s) \\ + \cdots + b_0 \sum_{r=0}^{\infty} a_r \Gamma(q'r + p'_0) s^{p'_0 + rq'} \widetilde{\overline{G}}(\omega,s) \\ + \sum_{v=0}^{j} c_v \omega^{\xi - h'_v} e^{ih'_v \pi/2} A_\Gamma(\omega^{-q'} e^{iq'\pi/2}) \widetilde{\overline{G}}(\omega,s) \end{array} \right] = \widetilde{\overline{g}}(\omega,s)$$

$$\begin{aligned} \widetilde{\overline{G}}(\omega,s) &= \left[\begin{array}{c} b_n \sum_{r=1}^{\infty} a_r \Gamma(q'r + p'_n) s^{\xi - p'_n - rq'} + \\ \sum_{k=0}^{n-1} b_k \sum_{r=0}^{\infty} a_r \Gamma(q'r + p'_k) s^{\xi - p'_k - rq'} \\ + \sum_{v=0}^{j} c_v \omega^{\xi - h'_v} e^{ih'_v \pi/2} A_\Gamma(\omega^{-q'} e^{iq'\pi/2}) \end{array} \right]^{-1} \widetilde{\overline{g}}(\omega,s) \\ &= \left[\begin{array}{c} b_n a_0 \Gamma(p'_n) s^{\xi - p'_n} + b_n a_1 \Gamma(q' + p'_n) s^{\xi - p'_n - q'} + \\ b_n \sum_{r=2}^{\infty} a_r \Gamma(q'r + p'_n) s^{\xi - p'_n - rq'} + \\ \sum_{k=0}^{n-1} b_k \sum_{r=0}^{\infty} a_r \Gamma(q'r + p'_k) s^{\xi - p'_k - rq'} + \\ + \sum_{v=0}^{j} c_v \omega^{\xi - h'_v} e^{ih'_v \pi/2} A_\Gamma(\omega^{-q'} e^{iq'\pi/2}) \end{array} \right]^{-1} \widetilde{\overline{g}}(\omega,s) \\ &= \frac{s^{-\xi + p'_n + q'}}{b_n a_0 \Gamma(p'_n)} \left[\begin{array}{c} s^{-q'} + \frac{a_1 \Gamma(q' + p'_n)}{a_0 \Gamma(p'_n)} + \frac{\sum_{r=2}^{\infty} a_r \Gamma(q'r + p'_n)}{a_0 \Gamma(p'_n)} s^{(r-1)q'} + \\ \sum_{k=0}^{n-1} \frac{\sum_{r=0}^{\infty} b_k a_r \Gamma(q'r + p'_k)}{b_n a_0 \Gamma(p'_n)} s^{p'_k - p'_n + (r-1)q'} + \\ + \sum_{v=0}^{j} \frac{c_v \omega^{\xi - h'_v} e^{ih_v \pi/2} A_\Gamma(\omega^{-q'} e^{iq'\pi/2})}{b_n a_0 \Gamma(p'_n)} s^{-\xi + p'_n + q'} \end{array} \right]^{-1} \widetilde{\overline{g}}(\omega,s). \end{aligned}$$

Assuming $\frac{a_1\Gamma(q'+p'_n)}{a_0\Gamma(p'_n)}=A$, we have the form

$$\begin{aligned}\widetilde{\widetilde{G}}(\omega,s)=&\frac{s^{-\xi+p'_n+q'}}{b_na_0\Gamma(p'_n)}\frac{1}{s^{-q'}+A}\\&\times\left[\begin{array}{c}1+\sum_{r=2}^{\infty}\frac{[a_r\Gamma(q'r+p'_n)/a_0\Gamma(p'_n)]s^{(r-1)q'}}{s^{-q'}+A}+\\\sum_{k=0}^{n-1}\sum_{r=0}^{\infty}\frac{[b_ka_r\Gamma(q'r+p'_k)/b_na_0\Gamma(p_n)]s^{p'_k-p'_n+(r-1)q'}}{s^{-q'}+A}\\+\frac{\sum_{v=0}^{j}[c_v\omega^{\xi-h'_v}e^{ih'_v\pi/2}A_\Gamma(\omega^{-q'}e^{iq'\pi/2})]/[b_na_0\Gamma(p'_n)]}{s^{-q'}+A}s^{-p'_n-q'}\end{array}\right]^{-1}\widetilde{\overline{g}}(\omega,s)\\=&\sum_{m=0}^{\infty}\frac{s^{-\xi+p'_n+q'}}{[b_na_0\Gamma(p'_n)]^{m+1}}\frac{(-1)^m}{(s^{-q'}+A)^{m+1}}\\&\times\left[\begin{array}{c}\sum_{r=2}^{\infty}a_r\Gamma(q'r+p'_n)s^{(r-1)q'}+\\\sum_{k=0}^{n-1}\sum_{r=0}^{\infty}b_ka_r\Gamma(q'r+p'_k)s^{p'_k-p'_n+(r-1)q'}\\+\sum_{v=0}^{j}c_v\omega^{\xi-h'_v}e^{ih'_v\pi/2}A_\Gamma(\omega^{-q'}e^{iq'\pi/2})s^{-\xi+p'_n+q'}\end{array}\right]^{m}\widetilde{\overline{g}}(\omega,s)\end{aligned}$$

$$\begin{aligned}\widetilde{\widetilde{G}}(\omega,s)=&\sum_{m=0}^{\infty}\frac{s^{-\xi+p'_n+q'}}{[b_na_0\Gamma(p'_n)]^{m+1}}\frac{(-1)^m}{(s^{-q'}+A)^{m+1}}\sum_{k_1+k_2+k_3=m}\binom{m}{k_1,k_2,k_3}\\&\times\left[\sum_{r=2}^{\infty}a_r\Gamma(q'r+p'_n)s^{(r-1)q'}\right]^{k_1}\left[\sum_{k=0}^{n-1}\sum_{r=0}^{\infty}b_ka_r\Gamma(q'r+p'_k)s^{p'_k-p'_n+(r-1)q'}\right]^{k_2}\\&\times\left[\sum_{v=0}^{j}c_v\omega^{\xi-h'_v}e^{ih'_v\pi/2}A_\Gamma(\omega^{-q'}e^{iq'\pi/2})s^{-\xi+p'_n+q'}\right]^{k_3}\widetilde{\overline{g}}(\omega,s)\\=&\sum_{m=0}^{\infty}\frac{(-1)^m}{[b_na_0\Gamma(p'_n)]^{m+1}}\sum_{k_1+k_2+k_3=m}\binom{m}{k_1,k_2,k_3}\left[\sum_{r=2}^{\infty}a_r\Gamma(q'r+p'_n)\right]^{k_1}\\&\times\left[\sum_{k=0}^{n-1}\sum_{r=0}^{\infty}b_ka_r\Gamma(q'r+p'_k)\right]^{k_2}\left[\sum_{v=0}^{j}c_v\omega^{\xi-h'_v}e^{ih'_v\pi/2}A_\Gamma(\omega^{-q'}e^{iq'\pi/2})\right]^{k_3}\\&\times\frac{s^{-q'-(\xi-p'_n-2q'-(r-1)q'k_1-(p'_k-p'_n+(r-1)q')k_2-(-\xi+p'_n+q')k_3)}}{(s^{-q'}+A)^{m+1}}\widetilde{\overline{g}}(\omega,s)\end{aligned}$$

By taking inverse Laplace transform with respect to t and then inverse Fourier transform with respect to x, we have the unique solution of the differential equation. □

Example 3.2. *The nonhomogeneous fractional wave equation*

$$\frac{{}^{A}\partial^{p,q}u(x,t)}{\partial t^{p,q}}-c^2\frac{\partial^2u(x,t)}{\partial^2x}=g(x,t);\quad x\in\mathbb{R},\quad t>0,\tag{3.10}$$

with initial conditions

$$\begin{aligned}{}^{A}_{RL}\mathcal{D}^{p+qn-1}u(x,0)&=0\\{}^{A}_{RL}\mathcal{D}^{p+qn-2}u(x,0)&=0,\quad\forall\, n\in(0,\infty),x\in\mathbb{R},\end{aligned}$$

where c is a constant and $1 < p \leq 2$, has unique Green's function.

Proof. Applying Laplace transform with respect to t on equation (3.10) gives

$$\overline{g}(x,s) = s^{\xi-p'}A_{\Gamma}(s^{-q'})\overline{G}(x,s) - c^2\frac{d^2}{du^2}\overline{G}(x,s). \tag{3.11}$$

By taking Fourier transform with respect to x of (3.11), we get

$$\widetilde{\overline{G}}(w,s)\left[\sum_{n=0}^{\infty} a_n\Gamma(p'+q'n)s^{\xi-p'-q'n} + c^2w^2\right] = \widetilde{\overline{g}}(w,s),$$

which implies

$$\begin{aligned}
\widetilde{\overline{G}}(w,s) =& \frac{\widetilde{\overline{g}}(w,s)}{\sum_{n=0}^{\infty} a_n\Gamma(p'+q'n)s^{\xi-p'-q'n} + c^2w^2} \\
=& \frac{1}{a_0\Gamma(p')}\left(\frac{(s^{-\xi+p'+q'})\widetilde{\overline{g}}(w,s)}{s^{-q'} + \frac{a_1\Gamma(p'+q')}{a_0\Gamma(p')}}\right)\left[\ 1 + \frac{\sum_{n=2}^{\infty}\frac{a_n\Gamma(p'+q'n)}{a_0\Gamma(p')}s^{q'(n-1)} + \frac{c^2w^2}{a_0\Gamma(p')}}{s^{-q'} + \frac{a_1\Gamma(p'+q')}{a_0\Gamma(p')}}s^{-\xi+p'+q'}\ \right]^{-1} \\
=& \sum_{m=0}^{\infty}\frac{(-1)^m}{a_0\Gamma(p')}\frac{s^{-\xi+p'+q'}}{\left(s^{-q'} + \frac{a_1\Gamma(p'+q')}{a_0\Gamma(p')}\right)^{m+1}}\left[\sum_{n=2}^{\infty}\frac{a_n\Gamma(p'+q'n)}{a_0\Gamma(p')}s^{q'(n-1)} + \frac{c^2w^2}{a_0\Gamma(p')}s^{-\xi+p'+q'}\right]^m \\
&\times\widetilde{\overline{g}}(w,s) \\
=& \sum_{m=0}^{\infty}\frac{(-1)^m}{[a_0\Gamma(p')]^{m+1}}\frac{s^{-\xi+p'+q'}}{\left(s^{-q'} + \frac{a_1\Gamma(p'+q')}{a_0\Gamma(p')}\right)^{m+1}}\sum_{k=0}^{m}\binom{m}{k}\left[c^2w^2s^{-\xi+p+q}\right]^{m-k} \\
&\times\left[\sum_{n=2}^{\infty} a_n\Gamma(p'+q'n)s^{q'(n-1)}\right]^k\widetilde{\overline{g}}(w,s).
\end{aligned}$$

Taking the inverse Laplace transform, we have Green's function for (3.10). □

4. Discussion

Fractional calculus is a field enlightening the multiple areas of science and engineering. RL provides an origin to researchers for exploring the space of fractional calculus but there is not a single general formula defining fractional differentiation. In 2019, Fernandez [1] gave a generalization of fractional operators involving analytic kernel. In this article, we used the integral and differential operators involving general analytic kernels to solve ordinary and partial differential problems.

5. Conclusions

In this work, we introduced Green's function for the fractional differential operators involving MAK, Green's function for ordinary linear fractional differential equations is provided by the means of Laplace transform and for partial differential equations by the means for joint Laplace and Fourier transform generalizing the classical ways. The Laplace transform method is used to evaluate the complex fractional differential equations, so that we can easily calculate Green's function of the fractional differential equation. In order to obtain fractional term obtained by Laplace transform in a more simple form, multinomial and binomial theorems allowed us to have the required forms.

Since this model absorbs some previously introduced special cases in the course of fractional calculus, it provides Green's function for RL fractional model [1], Prabhakar fractional model [12], AB fractional model [17] and GPF fractional model [20].

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