Int. J. Anal. Appl. (2024), 22:186

International Journal of Analysis and Applications

In Vivo Dynamics of HIV-1 Infection With Impaired Antibody Immunity and Three General Infection Mechanisms

Reham H. Halawani[1], **Noura H. AlShamrani**[1,*], **Ahmed M. Elaiw**[2]

[1]*Department of Mathematics and Statistics, College of Science, University of Jeddah, P.O. Box 80327, Jeddah 21589, Saudi Arabia*

[2]*Department of Mathematics, Faculty of Science, King Abdulaziz University, P.O. Box 80203, Jeddah 21589, Saudi Arabia*

**Corresponding author:* nhalshamrani@uj.edu.sa

Abstract. In this paper we investigate two generalized human immunodeficiency virus type-1 (HIV-1) dynamics models with impaired antibody immunity. The models include both latently and actively infected cells. Three infection mechanisms are incorporated into the models, viral infection mechanism (VIM), latent cellular infection mechanism (CIM) and active CIM. The three infection rates are provided by generic nonlinear functions. The second model includes three types of distributed time delays. We find that our models are biologically feasible. The global stability analysis of equilibria are performed and found the basic reproduction ratio ($\Re_0$) as a threshold parameter. Using Lyapunov method we show that, the virus-free equilibrium is globally asymptotically stable when $\Re_0 \leq 1$ and the virus-persistence equilibrium is globally asymptotically stable when $\Re_0 > 1$. Sensitivity analysis on $\Re_0$ is studied. To support our theoretical results we provide some numerical simulations. We have demonstrated that $\Re_0$ is influenced by all three of the infection types, and that if one of them were ignored, $\Re_0$ would be underestimated. This might lead to inadequate medication effectiveness that aims to remove HIV-1 from the body. The effects of time delay and impaired antibody immunity on HIV-1 progression are examined. According to our research, lowered immunity is a significant factor in the infection's growth. Furthermore, time delays might drastically reduce $\Re_0$, which would prevent HIV-1 from replicating. The information provided by our research in this work can improve our comprehension of HIV-1 dynamics within-host and provide guidance for the creation of novel pharmacological treatments.

1. Introduction

Human immunodeficiency virus type-1 (HIV-1) is a class of retrovirus which causes a persistent HIV-1 infection in human body. Once HIV-1 infects the person, a higher viral load follows for

Received: Aug. 27, 2024.

2020 *Mathematics Subject Classification.* 34D20, 34D23, 37N25, 92B05.

Key words and phrases. HIV-1; cellular infection; latently infected cells; immune impairment; global stability; distributed delays; Lyapunov function.

https://doi.org/10.28924/2291-8639-22-2024-186

ISSN: 2291-8639

the first few weeks, and then its replication becomes steady for several years [1]. As a result, the $CD4^+T$ cells (which are the main target cells for HIV-1) decrease considerably. When the $CD4^+T$ cells become below a certain threshold (200 cells/mm^3), the patient develops Acquired Immune Deficiency Syndrome (AIDS) [2]. One of the most important aspects of controlling viral infections is adaptive immunity. The two primary components of the adaptive immunity are (i) Antibodies, which neutralize the virus directly by generating a particular antibody, and (ii) T cells, which are divided into two groups: $CD4^+T$ cells, which orchestrate the adaptive immune response, and $CD8^+T$ cells, which eliminate the infected cells [3]. According to estimates of UNAIDS 2023, there were 39 million HIV-positive individuals worldwide in 2022, 1.3 million new HIV infections, and 630 thousand AIDS-related deaths [4].

Mathematical modeling is among the greatest methods for analyzing the infection's progress and management [5–10]. A standard model of viral infection under the influence of antibody response was introduced in [11] which describes the interaction of four compartment uninfected $CD4^+T$ cells, infected cells, free HIV-1 particles and antibodies. The concept described in reference [11] is based on viral infection mechanism (VIM), where $CD4^+T$ cells get infected upon coming into contact with HIV-1 particles. Numerous studies have demonstrated that HIV-1 may spread directly through the development of virological synapses from an infected cell to an uninfected cell which known as cellular infection mechanism (CIM) (see e.g., [12–19]). CIM can reduce the time it takes for HIV-1 particles to produce by 0.9 times and enhance HIV-1 fitness by 3.9 times [20]. One of the modifications to the conventional viral infection model with antibody response that was introduced in [11] is to include the CIM as [21]:

$$\dot{U}_{\mathcal{J}} = \tau - \iota_{\mathcal{J}} U_{\mathcal{J}} - \left(\mu_{\mathcal{E}} U_{\mathcal{E}} + \mu_{\mathcal{K}} U_{\mathcal{K}}\right) U_{\mathcal{J}}, \tag{1.1}$$

$$\dot{U}_{\mathcal{K}} = \left(\mu_{\mathcal{E}} U_{\mathcal{E}} + \mu_{\mathcal{K}} U_{\mathcal{K}}\right) U_{\mathcal{J}} - \iota_{\mathcal{K}} U_{\mathcal{K}}, \tag{1.2}$$

$$\dot{U}_{\mathcal{E}} = \eta U_{\mathcal{K}} - \iota_{\mathcal{E}} U_{\mathcal{E}} - \lambda U_{\mathcal{L}} U_{\mathcal{E}}, \tag{1.3}$$

$$\dot{U}_{\mathcal{L}} = \delta U_{\mathcal{L}} U_{\mathcal{E}} - \iota_{\mathcal{L}} U_{\mathcal{L}}, \tag{1.4}$$

where $U_{\mathcal{J}} = U_{\mathcal{J}}(t)$, $U_{\mathcal{K}} = U_{\mathcal{K}}(t)$, $U_{\mathcal{E}} = U_{\mathcal{E}}(t)$ and $U_{\mathcal{L}} = U_{\mathcal{L}}(t)$ represent the concentrations of healthy $CD4^+T$ cells, actively infected cells, HIV-1 particles and antibodies at time t, respectively. The production rate of uninfected $CD4^+T$ cells is τ, their death rate is $\iota_{\mathcal{J}} U_{\mathcal{J}}$, and their infection rate is $\left(\mu_{\mathcal{E}} U_{\mathcal{E}} + \mu_{\mathcal{K}} U_{\mathcal{K}}\right) U_{\mathcal{J}}$, where $\mu_{\mathcal{E}} U_{\mathcal{J}} U_{\mathcal{E}}$ and $\mu_{\mathcal{K}} U_{\mathcal{J}} U_{\mathcal{K}}$ represent the infection rates resulting from VIM and CIM, respectively. The rate of decay of the infected cells is considered to be $\iota_{\mathcal{K}} U_{\mathcal{K}}$. The HIV-1 particles are produced from infected cells at rate $\eta U_{\mathcal{K}}$ and are cleared at rate $\iota_{\mathcal{E}} U_{\mathcal{E}}$. The antibody response remove HIV-1 particles at rate $\lambda U_{\mathcal{L}} U_{\mathcal{E}}$. The activation rate of antibodies is $\delta U_{\mathcal{L}} U_{\mathcal{E}}$, whereas their decay rate is $\iota_{\mathcal{L}} U_{\mathcal{L}}$. Model (1.1)-(1.4) has been extended by considering: diffusion [22], [23]; age-structured [24]; time delay [21], [25]; cure of infected cells [26], [27]; and general incidence rates, $\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}}, U_{\mathcal{E}})$ and $\mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}}, U_{\mathcal{K}})$, [22], [23], [25].

Certain viruses have the ability to inhibit or even destroy the immune system in specific situations, particularly when the viral load is excessive. Various conditions can weaken antibodies and decrease their ability to operate [28–30] including: (i) irradiation, malnutrition, trauma, tumors, cytotoxic medications and aging, (ii) immunosuppression by microbes, such as malaria, measles virus but notably HIV-1, and (iii) some disorders, such as diabetes [31]. By considering the impaired antibody immunity, model (1.1)-(1.4) can be modified as [32], [33] and [34]:

$$\dot{U}_{\mathcal{J}} = \tau - \iota_{\mathcal{J}} U_{\mathcal{J}} - \left(\mu_{\mathcal{E}} U_{\mathcal{E}} + \mu_{\mathcal{K}} U_{\mathcal{K}}\right) U_{\mathcal{J}}, \tag{1.5}$$

$$\dot{U}_{\mathcal{K}} = \left(\mu_{\mathcal{E}} U_{\mathcal{E}} + \mu_{\mathcal{K}} U_{\mathcal{K}}\right) U_{\mathcal{J}} - \iota_{\mathcal{K}} U_{\mathcal{K}}, \tag{1.6}$$

$$\dot{U}_{\mathcal{E}} = \eta U_{\mathcal{K}} - \iota_{\mathcal{E}} U_{\mathcal{E}} - \lambda U_{\mathcal{L}} U_{\mathcal{E}}, \tag{1.7}$$

$$\dot{U}_{\mathcal{L}} = \delta U_{\mathcal{E}} - \iota_{\mathcal{L}} U_{\mathcal{L}} - \theta U_{\mathcal{L}} U_{\mathcal{E}}, \tag{1.8}$$

where $\delta U_{\mathcal{E}}$ stands for antibody activation and $\theta U_{\mathcal{L}} U_{\mathcal{E}}$ for antibody impairment rate.

Though there is no medicine to cure AIDS completely till date, but highly active antiretroviral therapy (HAART) have been used for last two decades to treat HIV-1 patients and they are found successful in suppressing HIV-1 replication and reconstituting the immune system in human body [6,35–38]. However, using HAART cannot eradicate the virus completely [39]. An important reason is that HIV-1 provirus can reside in latently infected cells which live long, but can be activated to produce virus by relevant antigens [40], [41]. Moreover, as reported in [42], these cells can contribute in the CIM. In recent works [43], [44] and [45], viral infection models were developed by assuming that the uninfected cells get infected upon coming into contact with viruses (VIM), latently infected cells (latent CIM) and actively infected cells (active CIM). In [46], HIV-1 infection models with impaired antibody immunity and three infection mechanisms was studied. The infection rates due VIM, latent CIM and active CIM were given, respectively, by bilinear incidences, $\mu_{\mathcal{E}} U_{\mathcal{E}} U_{\mathcal{J}}$, $\mu_{\mathcal{P}} U_{\mathcal{P}} U_{\mathcal{J}}$ and $\mu_{\mathcal{K}} U_{\mathcal{K}} U_{\mathcal{J}}$, where $U_{\mathcal{P}}$ is the concentration of the latently infected cells. According to the mass-action principle, this bilinear incidence suggests that the rate of infection is exactly proportional to the product of viral concentrations or infected cells interacting with uninfected cells. In actuality, though, this concept isn't always useful. For example, if there are more viruses or infected cells than there are uninfected cells, the law of mass-action won't be applicable. An increase in the concentration of the virus or infected cells won't cause an increase in infection in such a situation. Papers [47] and [48] studied viral dynamics models with impaired antibody immunity and two infection mechanisms, VIM and active CIM. The infection rates were represented by general incidence functions, $\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}}, U_{\mathcal{E}})$ and $\mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}}, U_{\mathcal{K}})$. However, the contribution of the latently infected cells in the cellular infection was not considered.

The objective of the current work is to propose and study two HIV-1 infection models with impaired antibody immunity, considering three infection mechanisms, VIM, latent CIM and active CIM. The infection rates are given by general incidence functions, $\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}}, U_{\mathcal{E}})$, $\mathcal{F}_{\mathcal{P}}(U_{\mathcal{J}}, U_{\mathcal{P}})$ and $\mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}}, U_{\mathcal{K}})$. The second model includes three different types of distributed-time delays. We show the wellposedness of the proposed models. We demonstrate the existence and stability of the system's equilibria in terms of the basic reproduction ratio ($\mathfrak{R}_0$). To validate the theoretical conclusions, we offer numerical simulations.

2. Model with impaired antibody immunity, general infection rate and three infection mechanisms

2.1. System overview.

We introduce an HIV-1 model that incorporates impaired antibody immunity and takes into account three infection mechanisms as follows:

$$\begin{cases} \dot{U}_{\mathcal{J}} &= \tau - \iota_{\mathcal{J}} U_{\mathcal{J}} - \mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right) - \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right) - \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right), \\ \dot{U}_{\mathcal{P}} &= \mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right) + \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right) - (\alpha + \iota_{\mathcal{P}}) U_{\mathcal{P}}, \\ \dot{U}_{\mathcal{K}} &= \alpha U_{\mathcal{P}} - \iota_{\mathcal{K}} U_{\mathcal{K}}, \\ \dot{U}_{\mathcal{E}} &= \eta U_{\mathcal{K}} - \iota_{\mathcal{E}} U_{\mathcal{E}} - \lambda U_{\mathcal{L}} U_{\mathcal{E}}, \\ \dot{U}_{\mathcal{L}} &= \delta U_{\mathcal{E}} - \iota_{\mathcal{L}} U_{\mathcal{L}} - \theta U_{\mathcal{L}} U_{\mathcal{E}}. \end{cases} \tag{2.1}$$

The general functions $\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right)$, $\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)$ and $\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)$ represent, respectively, the VIM, latent CIM and active CIM. The meanings assigned to all remaining parametersand variables remain in line with explanations provided in Section 1. Model (2.1) is very general because we consider nonlinear incidences ($\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right), \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)$ and $\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)$). It is also important to note that model (2.1) incorporates a large number of pre-existing models (see those from earlier research [43], [44] and [46]). Define $\wp_{\mathcal{X}}\left(U_{\mathcal{J}}\right)$, $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$ as:

$$\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right) = \lim_{U_{\mathcal{E}} \to 0^+} \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right)}{U_{\mathcal{E}}} = \frac{\partial \mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{E}}},$$

$$\wp_{\mathcal{P}}\left(U_{\mathcal{J}}\right) = \lim_{U_{\mathcal{P}} \to 0^+} \frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}{U_{\mathcal{P}}} = \frac{\partial \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{P}}},$$

$$\wp_{\mathcal{K}}\left(U_{\mathcal{J}}\right) = \lim_{U_{\mathcal{K}} \to 0^+} \frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)}{U_{\mathcal{K}}} = \frac{\partial \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{K}}}.$$

The following Hypotheses on the functions $\mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}}, U_{\mathcal{X}}\right)$, $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$ are needed throughout the paper [49], [50]:

Hypothesis H1. $\mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}}, U_{\mathcal{X}}\right)$ is continuously differentiable, $\mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}}, U_{\mathcal{X}}\right) > 0$ and $\mathcal{F}_{\mathcal{X}}\left(0, U_{\mathcal{X}}\right) = \mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}}, 0\right) = 0$ for all $U_{\mathcal{J}} > 0$ and $U_{\mathcal{X}} > 0$, $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$.

Hypothesis H2. $\dfrac{\partial \mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}}, U_{\mathcal{X}}\right)}{\partial U_{\mathcal{J}}} > 0$, $\dfrac{\partial \mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}}, U_{\mathcal{X}}\right)}{\partial U_{\mathcal{X}}} > 0$ for all $U_{\mathcal{J}} > 0$ and $U_{\mathcal{X}} > 0$, $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$.

Hypothesis H3. $\wp_{\mathcal{X}}\left(U_{\mathcal{J}}\right) > 0$ and $\wp'_{\mathcal{X}}\left(U_{\mathcal{J}}\right) > 0$ for all $U_{\mathcal{J}} > 0$, $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$.

Hypothesis H4. $\dfrac{\mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}}, U_{\mathcal{X}}\right)}{U_{\mathcal{X}}}$ is non-increasing with respect to $U_{\mathcal{X}}$ for all $U_{\mathcal{X}} > 0$, $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$.

2.2. Fundamental characteristics.

2.2.1. *The well-posedness of the system.*

Proposition 2.1. *The system (2.1) is considered with Hypothesis **H1**. Then, a positive constants Γ_i for $i = 1, 2, 3$ exist, ensuring the following set:*

$$\Delta = \left\{\left(U_{\mathcal{J}}, U_{\mathcal{P}}, U_{\mathcal{K}}, U_{\mathcal{E}}, U_{\mathcal{L}}\right) \in \mathbb{R}^5_{\geq 0} : 0 \leq U_{\mathcal{J}}(t), U_{\mathcal{P}}(t), U_{\mathcal{K}}(t) \leq \Gamma_1, 0 \leq U_{\mathcal{E}}(t) \leq \Gamma_2, 0 \leq U_{\mathcal{L}}(t) \leq \Gamma_3\right\},$$

is positively invariant.

Proof. To guarantee nonnegativity in the solutions, we handle system (2.1) in the following manner

$$\begin{aligned}
&\dot{U}_{\mathcal{J}} \mid_{U_{\mathcal{J}}=0} = \tau > 0,\\
&\dot{U}_{\mathcal{P}} \mid_{U_{\mathcal{P}}=0} = \mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right) \geq 0, \text{ for all } U_{\mathcal{J}}, U_{\mathcal{E}}, U_{\mathcal{K}} \geq 0,\\
&\dot{U}_{\mathcal{K}} \mid_{U_{\mathcal{K}}=0} = \alpha U_{\mathcal{P}} \geq 0, \text{ for all } U_{\mathcal{P}} \geq 0,\\
&\dot{U}_{\mathcal{E}} \mid_{U_{\mathcal{E}}=0} = \eta U_{\mathcal{K}} \geq 0, \text{ for all } U_{\mathcal{K}} \geq 0,\\
&\dot{U}_{\mathcal{L}} \mid_{U_{\mathcal{L}}=0} = \delta U_{\mathcal{E}} \geq 0, \text{ for all } U_{\mathcal{E}} \geq 0.
\end{aligned}$$

Therefore, for all $t \geq 0$ we conclude that $(U_{\mathcal{J}}(t), U_{\mathcal{P}}(t), U_{\mathcal{K}}(t), U_{\mathcal{E}}(t), U_{\mathcal{L}}(t)) \in \mathbb{R}^5_{\geq 0}$, whenever $(U_{\mathcal{J}}(0), U_{\mathcal{P}}(0), U_{\mathcal{K}}(0),$
$U_{\mathcal{E}}(0), U_{\mathcal{L}}(0)) \in \mathbb{R}^5_{\geq 0}$. Let

$$\Phi = U_{\mathcal{J}} + U_{\mathcal{P}} + U_{\mathcal{K}} + \frac{\iota_{\mathcal{K}}}{2\eta} U_{\mathcal{E}} + \frac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}{4\eta\delta} U_{\mathcal{L}}.$$

Then, we have

$$\begin{aligned}
\dot{\Phi} &= \dot{U}_{\mathcal{J}} + \dot{U}_{\mathcal{P}} + \dot{U}_{\mathcal{K}} + \frac{\iota_{\mathcal{K}}}{2\eta}\dot{U}_{\mathcal{E}} + \frac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}{4\eta\delta}\dot{U}_{\mathcal{L}}\\
&= \tau - \iota_{\mathcal{J}} U_{\mathcal{J}} - \iota_{\mathcal{P}} U_{\mathcal{P}} - \frac{\iota_{\mathcal{K}}}{2} U_{\mathcal{K}} - \frac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}{4\eta} U_{\mathcal{E}} - \left(\frac{\iota_{\mathcal{K}}\lambda}{2\eta} + \frac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}\theta}{4\eta\delta}\right) U_{\mathcal{L}} U_{\mathcal{E}} - \frac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}\iota_{\mathcal{L}}}{4\eta\delta} U_{\mathcal{L}}\\
&\leq \tau - \iota_{\mathcal{J}} U_{\mathcal{J}} - \iota_{\mathcal{P}} U_{\mathcal{P}} - \frac{\iota_{\mathcal{K}}}{2} U_{\mathcal{K}} - \frac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}{4\eta} U_{\mathcal{E}} - \frac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}\iota_{\mathcal{L}}}{4\eta\delta} U_{\mathcal{L}}\\
&\leq \tau - \varepsilon\left(U_{\mathcal{J}} + U_{\mathcal{P}} + U_{\mathcal{K}} + \frac{\iota_{\mathcal{K}}}{2\eta} U_{\mathcal{E}} + \frac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}{4\eta\delta} U_{\mathcal{L}}\right) = \tau - \varepsilon\Phi,
\end{aligned}$$

where $\varepsilon = \min\left\{\iota_{\mathcal{J}}, \iota_{\mathcal{P}}, \frac{\iota_{\mathcal{K}}}{2}, \frac{\iota_{\mathcal{E}}}{2}, \iota_{\mathcal{L}}\right\}$. Hence

$$\Phi(t) \leq e^{-\varepsilon t}\left(\Phi(0) - \frac{\tau}{\varepsilon}\right) + \frac{\tau}{\varepsilon}.$$

This yields $0 \leq \Phi(t) \leq \Gamma_1$ if $\Phi(0) \leq \Gamma_1$, where $\Gamma_1 = \dfrac{\tau}{\varepsilon}$.
Considering that all state variables have non-negative values, $0 \leq U_{\mathcal{J}}(t), U_{\mathcal{P}}(t), U_{\mathcal{K}}(t) \leq \Gamma_1$, $0 \leq U_{\mathcal{E}}(t) \leq \Gamma_2$, and $0 \leq U_{\mathcal{L}}(t) \leq \Gamma_3$, for all $t \geq 0$ if $U_{\mathcal{J}}(0) + U_{\mathcal{P}}(0) + U_{\mathcal{K}}(0) + \dfrac{\iota_{\mathcal{K}}}{2\eta} U_{\mathcal{E}}(0) + \dfrac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}{4\eta\delta} U_{\mathcal{L}}(0) \leq \Gamma_1$, where $\Gamma_2 = \dfrac{2\eta\Gamma_1}{\iota_{\mathcal{K}}}$ and $\Gamma_3 = \dfrac{4\eta\delta\Gamma_1}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}$. To sum up, the boundedness of $U_{\mathcal{J}}(t)$,$U_{\mathcal{P}}(t)$,$U_{\mathcal{K}}(t)$,$U_{\mathcal{E}}(t)$ and $U_{\mathcal{L}}(t)$ implies that Δ is a positively invariant and compact set concerning system (2.1). □

2.2.2. *The reproduction ratio and equilibria of the system.*

Proposition 2.2. *Assuming that Hypotheses **H1-H4** are met, there exists a positive basic reproduction ratio* $\Re_0 = \frac{\eta\alpha\wp_{\mathcal{E}}(\bar{U}^0_{\mathcal{J}}) + \iota_{\mathcal{E}}\iota_{\mathcal{K}}\wp_{\mathcal{P}}(\bar{U}^0_{\mathcal{J}}) + \alpha\iota_{\mathcal{E}}\wp_{\mathcal{K}}(\bar{U}^0_{\mathcal{J}})}{\iota_{\mathcal{E}}\iota_{\mathcal{K}}(\alpha+\iota_{\mathcal{P}})}$ *for system (2.1) in a way that*

(i) ensure the system consistently maintains a virus-free equilibrium, denoted as $\bar{O}^0$*, and*

(ii) if $\Re_0 > 1$*, the system additionally possesses a virus-persistence equilibrium, denoted as* $\bar{O}^1$.

Proof. Using the next-generation method given in [51] we can calculate the basic reproduction ratio as:

$$\Re_0 = \frac{\eta\alpha\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right) + \iota_{\mathcal{E}}\iota_{\mathcal{K}}\wp_{\mathcal{P}}\left(\bar{U}^0_{\mathcal{J}}\right) + \alpha\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\iota_{\mathcal{E}}\iota_{\mathcal{K}}\left(\alpha+\iota_{\mathcal{P}}\right)} = \Re_{0\mathcal{E}} + \Re_{0\mathcal{P}} + \Re_{0\mathcal{K}}, \tag{2.2}$$

where

$$\Re_{0\mathcal{E}} = \frac{\eta\alpha\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\iota_{\mathcal{E}}\iota_{\mathcal{K}}\left(\alpha+\iota_{\mathcal{P}}\right)}, \qquad \Re_{0\mathcal{P}} = \frac{\wp_{\mathcal{P}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\alpha+\iota_{\mathcal{P}}}, \qquad \Re_{0\mathcal{K}} = \frac{\alpha\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\iota_{\mathcal{K}}\left(\alpha+\iota_{\mathcal{P}}\right)}, \qquad \bar{U}^0_{\mathcal{J}} = \frac{\tau}{\iota_{\mathcal{J}}}.$$

The clinical relevance of the parameter $\Re_0$ holds significance as it plays a crucial role in determining whether the HIV-1 infection will progress into a chronic state. Within this framework, $\Re_{0\mathcal{X}}$ symbolizes the average quantities of secondary infected cells generated from engagements with viruses and infected cells, where $\mathcal{X} \in \{\mathcal{E},\mathcal{P},\mathcal{K}\}$. To extend our exploration beyond $\bar{O}^0$, we scrutinize $(U_{\mathcal{J}}, U_{\mathcal{P}}, U_{\mathcal{K}}, U_{\mathcal{E}}, U_{\mathcal{L}})$ as a potential equilibrium governed by the following set of algebraic equations:

$$0 = \tau - \iota_{\mathcal{J}}U_{\mathcal{J}} - \mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right) - \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right) - \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right), \tag{2.3}$$

$$0 = \mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right) + \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right) - \left(\alpha+\iota_{\mathcal{P}}\right)U_{\mathcal{P}}, \tag{2.4}$$

$$0 = \alpha U_{\mathcal{P}} - \iota_{\mathcal{K}}U_{\mathcal{K}}, \tag{2.5}$$

$$0 = \eta U_{\mathcal{K}} - \iota_{\mathcal{E}}U_{\mathcal{E}} - \lambda U_{\mathcal{L}}U_{\mathcal{E}}, \tag{2.6}$$

$$0 = \delta U_{\mathcal{E}} - \iota_{\mathcal{L}}U_{\mathcal{L}} - \theta U_{\mathcal{L}}U_{\mathcal{E}}. \tag{2.7}$$

From Eqs. (2.5) and (2.7), we get

$$U_{\mathcal{P}} = \frac{\iota_{\mathcal{K}}U_{\mathcal{K}}}{\alpha}, \quad U_{\mathcal{L}} = \frac{\delta U_{\mathcal{E}}}{\iota_{\mathcal{L}} + \theta U_{\mathcal{E}}}. \tag{2.8}$$

The following outcome is obtained after inserting Eq. (2.8) into Eq. (2.6):

$$U_{\mathcal{K}} = \Im_1\left(U_{\mathcal{E}}\right) = \frac{\iota_{\mathcal{L}}\iota_{\mathcal{E}}U_{\mathcal{E}} + \left(\lambda\delta + \iota_{\mathcal{E}}\theta\right)U^2_{\mathcal{E}}}{\eta\left(\iota_{\mathcal{L}} + \theta U_{\mathcal{E}}\right)}. \tag{2.9}$$

It is clear that $\Im_1(0) = 0$. From Eqs. (2.9) and (2.8), we get

$$U_{\mathcal{P}} = \Im_2\left(U_{\mathcal{E}}\right) = \frac{\iota_{\mathcal{K}}\left(\iota_{\mathcal{L}}\iota_{\mathcal{E}}U_{\mathcal{E}} + \left(\lambda\delta + \iota_{\mathcal{E}}\theta\right)U^2_{\mathcal{E}}\right)}{\alpha\eta\left(\iota_{\mathcal{L}} + \theta U_{\mathcal{E}}\right)}. \tag{2.10}$$

It is clear that $\Im_2(0) = 0$. The following outcome is obtained after inserting Eq. (2.3) into Eq. (2.4):

$$\tau - \iota_{\mathcal{J}}U_{\mathcal{J}} = \left(\alpha + \iota_{\mathcal{P}}\right)U_{\mathcal{P}}. \tag{2.11}$$

From Eqs. (2.10) and (2.11), we get

$$U_{\mathcal{J}} = \Im_3\left(U_{\mathcal{E}}\right) = \frac{1}{\iota_{\mathcal{J}}}\left(\tau - \frac{\iota_{\mathcal{K}}\left(\alpha+\iota_{\mathcal{P}}\right)\left(\iota_{\mathcal{L}}\iota_{\mathcal{E}}U_{\mathcal{E}} + \left(\lambda\delta + \iota_{\mathcal{E}}\theta\right)U^2_{\mathcal{E}}\right)}{\alpha\eta\left(\iota_{\mathcal{L}} + \theta U_{\mathcal{E}}\right)}\right). \tag{2.12}$$

Note that $\mathfrak{I}_3(0) = \bar{U}^0_{\mathcal{J}}$. Upon substitution of Eqs. (2.9), (2.10) and (2.12) into Eq. (2.4), the result is as follows:

$$\mathcal{F}_{\mathcal{E}}\left(\mathfrak{I}_3\left(U_{\mathcal{E}}\right), U_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{P}}\left(\mathfrak{I}_3\left(U_{\mathcal{E}}\right), \mathfrak{I}_2\left(U_{\mathcal{E}}\right)\right) + \mathcal{F}_{\mathcal{K}}\left(\mathfrak{I}_3\left(U_{\mathcal{E}}\right), \mathfrak{I}_1\left(U_{\mathcal{E}}\right)\right) - \frac{\iota_{\mathcal{K}}\left(\alpha + \iota_{\mathcal{P}}\right)\left(\iota_{\mathcal{L}}\iota_{\mathcal{E}}U_{\mathcal{E}} + \left(\lambda\delta + \iota_{\mathcal{E}}\theta\right)U^2_{\mathcal{E}}\right)}{\alpha\eta\left(\iota_{\mathcal{L}} + \theta U_{\mathcal{E}}\right)} = 0. \tag{2.13}$$

From Eq. (2.13), we have

(1) When $U_{\mathcal{E}} = 0$, the virus-free equilibrium $\bar{O}^0 = \left(\bar{U}^0_{\mathcal{J}}, 0, 0, 0, 0\right)$ is derived from Eqs. (2.8)-(2.12).

(2) When $U_{\mathcal{E}} \neq 0$, let us define a function $\Theta\left(U_{\mathcal{E}}\right)$ on $[0, \infty)$ as:

$$\Theta\left(U_{\mathcal{E}}\right) = \mathcal{F}_{\mathcal{E}}\left(\mathfrak{I}_3\left(U_{\mathcal{E}}\right), U_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{P}}\left(\mathfrak{I}_3\left(U_{\mathcal{E}}\right), \mathfrak{I}_2\left(U_{\mathcal{E}}\right)\right) + \mathcal{F}_{\mathcal{K}}\left(\mathfrak{I}_3\left(U_{\mathcal{E}}\right), \mathfrak{I}_1\left(U_{\mathcal{E}}\right)\right) - \frac{\iota_{\mathcal{K}}\left(\alpha + \iota_{\mathcal{P}}\right)\left(\iota_{\mathcal{L}}\iota_{\mathcal{E}}U_{\mathcal{E}} + \left(\lambda\delta + \iota_{\mathcal{E}}\theta\right)U^2_{\mathcal{E}}\right)}{\alpha\eta\left(\iota_{\mathcal{L}} + \theta U_{\mathcal{E}}\right)}.$$

We have $\Theta(0) = 0$. Let $\hat{U}_{\mathcal{E}}$ be such that $\mathfrak{I}_3\left(\hat{U}_{\mathcal{E}}\right) = 0$, i.e.,

$$\bar{U}^0_{\mathcal{J}} - \frac{\iota_{\mathcal{K}}\left(\alpha + \iota_{\mathcal{P}}\right)\left(\iota_{\mathcal{L}}\iota_{\mathcal{E}}\hat{U}_{\mathcal{E}} + \left(\lambda\delta + \iota_{\mathcal{E}}\theta\right)\hat{U}^2_{\mathcal{E}}\right)}{\iota_{\mathcal{J}}\alpha\eta\left(\iota_{\mathcal{L}} + \theta\hat{U}_{\mathcal{E}}\right)} = 0,$$

which implies that

$$\iota_{\mathcal{K}}\left(\alpha + \iota_{\mathcal{P}}\right)\left(\lambda\delta + \iota_{\mathcal{E}}\theta\right)\hat{U}^2_{\mathcal{E}} + \left(\iota_{\mathcal{K}}\iota_{\mathcal{L}}\iota_{\mathcal{E}}\left(\alpha + \iota_{\mathcal{P}}\right) - \iota_{\mathcal{J}}\alpha\eta\theta\bar{U}^0_{\mathcal{J}}\right)\hat{U}_{\mathcal{E}} - \iota_{\mathcal{J}}\alpha\eta\iota_{\mathcal{L}}\bar{U}^0_{\mathcal{J}} = 0, \tag{2.14}$$

Therefore, Eq. (2.14) has a positive solution

$$\hat{U}_{\mathcal{E}} = \frac{-B + \sqrt{B^2 - 4AC}}{2A},$$

where

$$A = \iota_{\mathcal{K}}\left(\alpha + \iota_{\mathcal{P}}\right)\left(\lambda\delta + \iota_{\mathcal{E}}\theta\right), \quad B = \iota_{\mathcal{K}}\iota_{\mathcal{L}}\iota_{\mathcal{E}}\left(\alpha + \iota_{\mathcal{P}}\right) - \iota_{\mathcal{J}}\alpha\eta\theta\bar{U}^0_{\mathcal{J}}, \quad C = -\iota_{\mathcal{J}}\alpha\eta\iota_{\mathcal{L}}\bar{U}^0_{\mathcal{J}}.$$

We can see that

$$\begin{aligned}\Theta\left(\hat{U}_{\mathcal{E}}\right) &= \mathcal{F}_{\mathcal{E}}\left(0, \hat{U}_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{P}}\left(0, \mathfrak{I}_2\left(\hat{U}_{\mathcal{E}}\right)\right) + \mathcal{F}_{\mathcal{K}}\left(0, \mathfrak{I}_1\left(\hat{U}_{\mathcal{E}}\right)\right) \\ &\quad - \frac{\iota_{\mathcal{K}}\left(\alpha + \iota_{\mathcal{P}}\right)\left(\iota_{\mathcal{L}}\iota_{\mathcal{E}}\hat{U}_{\mathcal{E}} + \left(\lambda\delta + \iota_{\mathcal{E}}\theta\right)\hat{U}^2_{\mathcal{E}}\right)}{\alpha\eta\left(\iota_{\mathcal{L}} + \theta\hat{U}_{\mathcal{E}}\right)} \\ &= -\frac{\iota_{\mathcal{K}}\left(\alpha + \iota_{\mathcal{P}}\right)\left(\iota_{\mathcal{L}}\iota_{\mathcal{E}}\hat{U}_{\mathcal{E}} + \left(\lambda\delta + \iota_{\mathcal{E}}\theta\right)\hat{U}^2_{\mathcal{E}}\right)}{\alpha\eta\left(\iota_{\mathcal{L}} + \theta\hat{U}_{\mathcal{E}}\right)} < 0.\end{aligned}$$

In addition

$$\begin{aligned}\Theta'\left(U_{\mathcal{E}}\right) &= \mathfrak{I}'_3\left(U_{\mathcal{E}}\right)\frac{\partial\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right)}{\partial U_{\mathcal{J}}} + \frac{\partial\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right)}{\partial U_{\mathcal{E}}} + \mathfrak{I}'_3\left(U_{\mathcal{E}}\right)\frac{\partial\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}{\partial U_{\mathcal{J}}} \\ &\quad + \mathfrak{I}'_2\left(U_{\mathcal{E}}\right)\frac{\partial\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}{\partial U_{\mathcal{P}}} + \mathfrak{I}'_3\left(U_{\mathcal{E}}\right)\frac{\partial\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)}{\partial U_{\mathcal{J}}} + \mathfrak{I}'_1\left(U_{\mathcal{E}}\right)\frac{\partial\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)}{\partial U_{\mathcal{K}}}\end{aligned}$$

$$-\frac{\iota_{\mathcal{K}}\left(\alpha+\iota_{\mathcal{P}}\right)}{\alpha\eta}\left(\iota_{\mathcal{E}}+\frac{\lambda\delta\left(2\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)U_{\mathcal{E}}}{\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)^{2}}\right).$$

Hypothesis **H1** implies that $\frac{\partial \mathcal{F}_{\mathcal{X}}\left(\bar{U}^0_{\mathcal{J}},0\right)}{\partial U_{\mathcal{J}}}=0$, $\mathcal{X}\in\{\mathcal{E},\mathcal{P},\mathcal{K}\}$. Then

$$\begin{aligned}
\Theta'(0) &= \frac{\partial \mathcal{F}_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}},0\right)}{\partial U_{\mathcal{E}}}+\mathfrak{I}_2'(0)\frac{\partial \mathcal{F}_{\mathcal{P}}\left(\bar{U}^0_{\mathcal{J}},0\right)}{\partial U_{\mathcal{P}}}+\mathfrak{I}_1'(0)\frac{\partial \mathcal{F}_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}},0\right)}{\partial U_{\mathcal{K}}}-\frac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}\left(\alpha+\iota_{\mathcal{P}}\right)}{\alpha\eta}\\
&= \frac{\partial \mathcal{F}_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}},0\right)}{\partial U_{\mathcal{E}}}+\frac{\iota_{\mathcal{E}}\iota_{\mathcal{K}}}{\alpha\eta}\frac{\partial \mathcal{F}_{\mathcal{P}}\left(\bar{U}^0_{\mathcal{J}},0\right)}{\partial U_{\mathcal{P}}}+\frac{\iota_{\mathcal{E}}}{\eta}\frac{\partial \mathcal{F}_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}},0\right)}{\partial U_{\mathcal{K}}}-\frac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}\left(\alpha+\iota_{\mathcal{P}}\right)}{\alpha\eta}\\
&= \wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)+\frac{\iota_{\mathcal{E}}\iota_{\mathcal{K}}\wp_{\mathcal{P}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\alpha\eta}+\frac{\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\eta}-\frac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}\left(\alpha+\iota_{\mathcal{P}}\right)}{\alpha\eta}\\
&= \frac{\iota_{\mathcal{K}}\iota_{\mathcal{E}}\left(\alpha+\iota_{\mathcal{P}}\right)}{\alpha\eta}\left(\mathfrak{R}_0-1\right),
\end{aligned}$$

where $\mathfrak{R}_0$ is defined in Eq. (2.2). Therefore, if $\mathfrak{R}_0>1$ then $\Theta'(0)>0$ and there exists $\bar{U}^1_{\mathcal{E}}\in\left(0,\hat{U}_{\mathcal{E}}\right)$ such that $\Theta\left(\bar{U}^1_{\mathcal{E}}\right)=0$. Let $U_{\mathcal{E}}=\bar{U}^1_{\mathcal{E}}$ in Eq. (2.3) and define

$$\aleph\left(U_{\mathcal{J}}\right)=\tau-\iota_{\mathcal{J}}U_{\mathcal{J}}-\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)-\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},\mathfrak{I}_2\left(\bar{U}^1_{\mathcal{E}}\right)\right)-\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},\mathfrak{I}_1\left(\bar{U}^1_{\mathcal{E}}\right)\right).$$

Subsequently, based on Hypothesis **H1**, we obtain $\aleph(0)=\tau>0$ and

$$\aleph\left(\bar{U}^0_{\mathcal{J}}\right)=-\left(\mathcal{F}_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(\bar{U}^0_{\mathcal{J}},\mathfrak{I}_2\left(\bar{U}^1_{\mathcal{E}}\right)\right)+\mathcal{F}_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}},\mathfrak{I}_1\left(\bar{U}^1_{\mathcal{E}}\right)\right)\right)<0.$$

Under Hypothesis **H2** it follows that $\aleph\left(U_{\mathcal{J}}\right)$ decreases strictly as a function of $U_{\mathcal{J}}$. Consequently, there is a unique value $\bar{U}^1_{\mathcal{J}}$ within the interval $\left(0,\bar{U}^0_{\mathcal{J}}\right)$ for which $\aleph\left(\bar{U}^1_{\mathcal{J}}\right)$ equals zero. Moreover, considering Eqs. (2.8)-(2.10), we find

$$\bar{U}^1_{\mathcal{P}}=\frac{\iota_{\mathcal{K}}\left(\iota_{\mathcal{L}}\iota_{\mathcal{E}}\bar{U}^1_{\mathcal{E}}+\left(\lambda\delta+\iota_{\mathcal{E}}\theta\right)\left(\bar{U}^1_{\mathcal{E}}\right)^2\right)}{\alpha\eta\left(\iota_{\mathcal{L}}+\theta\bar{U}^1_{\mathcal{E}}\right)},\quad \bar{U}^1_{\mathcal{K}}=\frac{\iota_{\mathcal{L}}\iota_{\mathcal{E}}\bar{U}^1_{\mathcal{E}}+\left(\lambda\delta+\iota_{\mathcal{E}}\theta\right)\left(\bar{U}^1_{\mathcal{E}}\right)^2}{\eta\left(\iota_{\mathcal{L}}+\theta\bar{U}^1_{\mathcal{E}}\right)},\quad \bar{U}^1_{\mathcal{L}}=\frac{\delta\bar{U}^1_{\mathcal{E}}}{\iota_{\mathcal{L}}+\theta\bar{U}^1_{\mathcal{E}}}.$$

The presence of the virus-persistence equilibrium $\bar{O}^1=\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{P}},\bar{U}^1_{\mathcal{K}},\bar{U}^1_{\mathcal{E}},\bar{U}^1_{\mathcal{L}}\right)$ becomes evident when $\mathfrak{R}_0>1$. □

2.2.3. *Global stability of equilibria.* The forthcoming theorems explore the global asymptotic stability of both virus-free and virus-persistence equilibria. The development of the Lyapunov function will adhere to the approach outlined in [49] and [52].

In the remaining work, we focus on a function $\Xi_i\left(U_{\mathcal{J}},U_{\mathcal{P}},U_{\mathcal{K}},U_{\mathcal{E}},U_{\mathcal{L}}\right)$, and we characterize M_i' as the largest invariant subset of

$$M_i=\left\{\left(U_{\mathcal{J}},U_{\mathcal{P}},U_{\mathcal{K}},U_{\mathcal{E}},U_{\mathcal{L}}\right):\frac{d\Xi_i}{dt}=0\right\},\quad i=0,1,...,3.$$

Let us define $\Upsilon=\left\{\left(U_{\mathcal{J}},U_{\mathcal{P}},U_{\mathcal{K}},U_{\mathcal{E}},U_{\mathcal{L}}\right)\in\mathbb{R}^5_{\geq 0}:U_{\mathcal{J}}\geq 0\right\}$. In order to examine whether the equilibrium $\bar{O}^0$, is globally asymptotically stable (G.A.S), we necessitate adherence to Hypothesis **H5** as outlined below [50]:

Hypothesis H5. The supremum of $\frac{\wp_{\mathcal{X}}(U_{\mathcal{J}})}{\wp_{\mathcal{E}}(U_{\mathcal{J}})}$ on $(0, \tau/\iota_{\mathcal{J}}]$ is attained at $U_{\mathcal{J}} = \dfrac{\tau}{\iota_{\mathcal{J}}}$, where $\mathcal{X} \in \{\mathcal{P}, \mathcal{K}\}$.

Theorem 2.1. *For system (2.1), let $\mathfrak{R}_0 \leq 1$ and Hypotheses **H1-H5** are satisfied, then $\bar{O}^0$ is G.A.S in Δ.*

Proof. Let's contemplate a potential Lyapunov function as:

$$\Xi_0 = U_{\mathcal{J}} - \bar{U}^0_{\mathcal{J}} - \int_{\bar{U}^0_{\mathcal{J}}}^{U_{\mathcal{J}}} \frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}(\gamma)} d\gamma + U_{\mathcal{P}} + \frac{\eta\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right) + \iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}} U_{\mathcal{K}} + \frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\iota_{\mathcal{E}}} U_{\mathcal{E}} + \frac{\lambda\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{2\delta\iota_{\mathcal{E}}} U^2_{\mathcal{L}}.$$

Evidently, $\Xi_0(U_{\mathcal{J}}, U_{\mathcal{P}}, U_{\mathcal{K}}, U_{\mathcal{E}}, U_{\mathcal{L}}) > 0$ for every $U_{\mathcal{J}}, U_{\mathcal{P}}, U_{\mathcal{K}}, U_{\mathcal{E}}, U_{\mathcal{L}} > 0$, as well as $\Xi_0(\bar{U}^0_{\mathcal{J}}, 0, 0, 0, 0) = 0$. Computing $\frac{d\Xi_0}{dt}$ we derive

$$\begin{aligned}
\frac{d\Xi_0}{dt} &= \left(1 - \frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\right)\dot{U}_{\mathcal{J}} + \dot{U}_{\mathcal{P}} + \frac{\eta\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right) + \iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\dot{U}_{\mathcal{K}} \\
&\quad + \frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\iota_{\mathcal{E}}}\dot{U}_{\mathcal{E}} + \frac{\lambda\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\delta\iota_{\mathcal{E}}} U_{\mathcal{L}}\dot{U}_{\mathcal{L}} \\
&= \left(1 - \frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\right)\left(\tau - \iota_{\mathcal{J}}U_{\mathcal{J}} - \mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right) - \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right) - \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)\right) \\
&\quad + \mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right) + \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right) - \left(\alpha + \iota_{\mathcal{P}}\right)U_{\mathcal{P}} \\
&\quad + \frac{\eta\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right) + \iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\left(\alpha U_{\mathcal{P}} - \iota_{\mathcal{K}}U_{\mathcal{K}}\right) + \frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\iota_{\mathcal{E}}}\left(\eta U_{\mathcal{K}} - \iota_{\mathcal{E}}U_{\mathcal{E}} - \lambda U_{\mathcal{L}}U_{\mathcal{E}}\right) \\
&\quad + \frac{\lambda\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\delta\iota_{\mathcal{E}}} U_{\mathcal{L}}\left(\delta U_{\mathcal{E}} - \iota_{\mathcal{L}}U_{\mathcal{L}} - \theta U_{\mathcal{L}}U_{\mathcal{E}}\right) \\
&= \left(1 - \frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\right)\left(\tau - \iota_{\mathcal{J}}U_{\mathcal{J}}\right) + \left(\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right)}{U_{\mathcal{E}}} - \wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)\right)U_{\mathcal{E}} \\
&\quad + \left(\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}{U_{\mathcal{P}}} - \left(\alpha + \iota_{\mathcal{P}}\right) + \frac{\alpha\left(\eta\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right) + \iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\right)U_{\mathcal{P}} \\
&\quad + \left(\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)}{U_{\mathcal{K}}} - \wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)\right)U_{\mathcal{K}} - \frac{\lambda\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\delta\iota_{\mathcal{E}}}\left(\iota_{\mathcal{L}} + \theta U_{\mathcal{E}}\right)U^2_{\mathcal{L}}.
\end{aligned} \tag{2.15}$$

From Hypothesis **H4**, we have

$$\begin{aligned}
\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right)}{U_{\mathcal{E}}} &\leq \lim_{U_{\mathcal{E}} \to 0^+} \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right)}{U_{\mathcal{E}}} = \wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right), \qquad \text{for all } U_{\mathcal{J}}, U_{\mathcal{E}} > 0, \\
\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}{U_{\mathcal{P}}} &\leq \lim_{U_{\mathcal{P}} \to 0^+} \frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}{U_{\mathcal{P}}} = \wp_{\mathcal{P}}\left(U_{\mathcal{J}}\right), \qquad \text{for all } U_{\mathcal{J}}, U_{\mathcal{P}} > 0, \\
\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)}{U_{\mathcal{K}}} &\leq \lim_{U_{\mathcal{K}} \to 0^+} \frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)}{U_{\mathcal{K}}} = \wp_{\mathcal{K}}\left(U_{\mathcal{J}}\right), \qquad \text{for all } U_{\mathcal{J}}, U_{\mathcal{K}} > 0.
\end{aligned} \tag{2.16}$$

Then

$$\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{U_{\mathcal{E}}}-\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)\leq\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)-\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)=0.$$

Further, Hypotheses **H4** and **H5** in case of $\bar{U}^0_{\mathcal{J}}=\dfrac{\tau}{\iota_{\mathcal{J}}}$ imply that

$$\begin{aligned}
&\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{U_{\mathcal{P}}}-\left(\alpha+\iota_{\mathcal{P}}\right)+\frac{\alpha\left(\eta\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\\
&\leq\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)\wp_{\mathcal{P}}\left(U_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}-\left(\alpha+\iota_{\mathcal{P}}\right)+\frac{\alpha\left(\eta\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\\
&\leq\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)\wp_{\mathcal{P}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}-\left(\alpha+\iota_{\mathcal{P}}\right)+\frac{\alpha\left(\eta\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\\
&=\wp_{\mathcal{P}}\left(\bar{U}^0_{\mathcal{J}}\right)-\left(\alpha+\iota_{\mathcal{P}}\right)+\frac{\alpha\left(\eta\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\\
&=\frac{\alpha\left(\eta\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)\right)+\iota_{\mathcal{K}}\iota_{\mathcal{E}}\wp_{\mathcal{P}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}-\left(\alpha+\iota_{\mathcal{P}}\right)\\
&=\left(\alpha+\iota_{\mathcal{P}}\right)\left(\Re_0-1\right).
\end{aligned}$$

Furthermore

$$\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{U_{\mathcal{K}}}-\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)\leq\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)\wp_{\mathcal{K}}\left(U_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}-\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)\leq\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}-\wp_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}\right)=0.$$

Upon conducting a computation and substituting the value $\bar{U}^0_{\mathcal{J}}=\tau/\iota_{\mathcal{J}}$, Eq. (2.15) transforms into the following expression:

$$\frac{d\Xi_0}{dt}\leq\tau\left(1-\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{\bar{U}^0_{\mathcal{J}}}\right)+\left(\alpha+\iota_{\mathcal{P}}\right)\left(\Re_0-1\right)U_{\mathcal{P}}-\frac{\lambda\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\delta\iota_{\mathcal{E}}}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)U^2_{\mathcal{L}}.$$

From Hypothesis **H3**, we have

$$\left(1-\frac{\wp_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{\bar{U}^0_{\mathcal{J}}}\right)\leq 0.$$

Clearly, $\frac{d\Xi_0}{dt}\leq 0$ when $\Re_0\leq 1$ and $\frac{d\Xi_0}{dt}=0$ when $U_{\mathcal{J}}=\bar{U}^0_{\mathcal{J}}$ and $U_{\mathcal{P}}=U_{\mathcal{E}}=U_{\mathcal{L}}=0$. All solutions convey to M'_0, wherein every element fulfills $U_{\mathcal{J}}(t)=\bar{U}^0_{\mathcal{J}}$ and $U_{\mathcal{P}}(t)=U_{\mathcal{E}}(t)=U_{\mathcal{L}}(t)=0$ for all t [53]. Following that, the first equation of system (2.1) in conjunction with Hypothesis **H1** results in

$$0=\dot{U}_{\mathcal{J}}=\tau-\iota_{\mathcal{J}}\bar{U}^0_{\mathcal{J}}-\mathcal{F}_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}},U_{\mathcal{K}}(t)\right)\Longrightarrow\mathcal{F}_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}},U_{\mathcal{K}}(t)\right)=0\Longrightarrow U_{\mathcal{K}}(t)=0,\ \text{for all } t.$$

Therefore, with $M'_0=\left\{\bar{O}^0\right\}$, our conclusion asserts that $\bar{O}^0$ is G.A.S under the condition $\Re_0\leq 1$, in accordance with LaSalle's invariance principle (L.I.P) [54]. □

In investigating the global stability of $\bar{O}^1$, we defined $F(a) = a - 1 - \ln(a)$. Beside, the following remarks and Hypothesis will be essential.

Remark 2.1. *Considering Hypotheses* ***H2*** *and* ***H4****, it is established that, for every positive* $U_{\mathcal{J}}, U_{\mathcal{E}}, U_{\mathcal{E}}^*$ *we have*

$$\left(\frac{\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}}, U_{\mathcal{E}})}{U_{\mathcal{E}}} - \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}^*\right)}{U_{\mathcal{E}}^*}\right)\left(\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}}, U_{\mathcal{E}}) - \mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}^*\right)\right) \leq 0,$$

which implies that

$$\left(\frac{\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}}, U_{\mathcal{E}})}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}^*\right)} - \frac{U_{\mathcal{E}}}{U_{\mathcal{E}}^*}\right)\left(1 - \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}^*\right)}{\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}}, U_{\mathcal{E}})}\right) \leq 0, \tag{2.17}$$

where $U_{\mathcal{E}}^* \in \left\{\bar{U}_{\mathcal{E}}^1, U_{\mathcal{E}}^1\right\}$.

Hypothesis H6. For any $U_{\mathcal{J}}$ within the interval $(0, \tau/\iota_{\mathcal{J}})$, and positive values of $U_{\mathcal{P}}$ and $U_{\mathcal{K}}$, the following Hypothesis is satisfied

(i) $\left(\frac{\mathcal{F}_{\mathcal{P}}(U_{\mathcal{J}}, U_{\mathcal{P}})}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}^*\right)U_{\mathcal{P}}} - \frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^*, U_{\mathcal{P}}^*\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^*, U_{\mathcal{E}}^*\right)U_{\mathcal{P}}^*}\right)\left(\frac{\mathcal{F}_{\mathcal{P}}(U_{\mathcal{J}}, U_{\mathcal{P}})}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}^*\right)} - \frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^*, U_{\mathcal{P}}^*\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^*, U_{\mathcal{E}}^*\right)}\right) \leq 0,$

(ii) $\left(\frac{\mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}}, U_{\mathcal{K}})}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}^*\right)U_{\mathcal{K}}} - \frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^*, U_{\mathcal{K}}^*\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^*, U_{\mathcal{E}}^*\right)U_{\mathcal{K}}^*}\right)\left(\frac{\mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}}, U_{\mathcal{K}})}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}^*\right)} - \frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^*, U_{\mathcal{K}}^*\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^*, U_{\mathcal{E}}^*\right)}\right) \leq 0,$

where $U_{\mathcal{J}}^* \in \left\{\bar{U}_{\mathcal{J}}^1, U_{\mathcal{J}}^1\right\} > 0$, $U_{\mathcal{P}}^* \in \left\{\bar{U}_{\mathcal{P}}^1, U_{\mathcal{P}}^1\right\} > 0$, $U_{\mathcal{K}}^* \in \left\{\bar{U}_{\mathcal{K}}^1, U_{\mathcal{K}}^1\right\} > 0$, and $U_{\mathcal{E}}^* \in \left\{\bar{U}_{\mathcal{E}}^1, U_{\mathcal{E}}^1\right\} > 0$.

Remark 2.2. *From Hypothesis* ***H6****, for any* $U_{\mathcal{J}}$ *within the interval* $(0, \tau/\iota_{\mathcal{J}})$*, and positive values of* $U_{\mathcal{P}}$ *and* $U_{\mathcal{K}}$*, we get*

$$\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^*, U_{\mathcal{E}}^*\right)\mathcal{F}_{\mathcal{P}}(U_{\mathcal{J}}, U_{\mathcal{P}})}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}^*\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^*, U_{\mathcal{P}}^*\right)} - \frac{U_{\mathcal{P}}}{U_{\mathcal{P}}^*}\right)\left(1 - \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}^*\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^*, U_{\mathcal{P}}^*\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^*, U_{\mathcal{E}}^*\right)\mathcal{F}_{\mathcal{P}}(U_{\mathcal{J}}, U_{\mathcal{P}})}\right) \leq 0, \tag{2.18}$$

$$\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^*, U_{\mathcal{E}}^*\right)\mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}}, U_{\mathcal{K}})}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}^*\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^*, U_{\mathcal{K}}^*\right)} - \frac{U_{\mathcal{K}}}{U_{\mathcal{K}}^*}\right)\left(1 - \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}^*\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^*, U_{\mathcal{K}}^*\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^*, U_{\mathcal{E}}^*\right)\mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}}, U_{\mathcal{K}})}\right) \leq 0, \tag{2.19}$$

where $U_{\mathcal{J}}^* \in \left\{\bar{U}_{\mathcal{J}}^1, U_{\mathcal{J}}^1\right\} > 0$*,* $U_{\mathcal{P}}^* \in \left\{\bar{U}_{\mathcal{P}}^1, U_{\mathcal{P}}^1\right\} > 0$*,* $U_{\mathcal{K}}^* \in \left\{\bar{U}_{\mathcal{K}}^1, U_{\mathcal{K}}^1\right\} > 0$*, and* $U_{\mathcal{E}}^* \in \left\{\bar{U}_{\mathcal{E}}^1, U_{\mathcal{E}}^1\right\} > 0$.

Theorem 2.2. *Given* $\Re_0 > 1$ *and the fulfillment of Hypotheses* ***H1-H4*** *and* ***H6****, it can be concluded that the equilibrium* $\bar{O}^1$ *for system (2.1) is guaranteed to be G.A.S in* $\Delta \backslash \Upsilon$.

Proof. We formulate $\Xi_1(U_{\mathcal{J}}, U_{\mathcal{P}}, U_{\mathcal{K}}, U_{\mathcal{E}}, U_{\mathcal{L}})$ given by Eq. (2.20) as:

$$\Xi_1 = U_{\mathcal{J}} - \bar{U}_{\mathcal{J}}^1 - \int_{\bar{U}_{\mathcal{J}}^1}^{U_{\mathcal{J}}} \frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{E}}^1\right)}{\mathcal{F}_{\mathcal{E}}\left(\gamma, \bar{U}_{\mathcal{E}}^1\right)} d\gamma + \bar{U}_{\mathcal{P}}^1 F\left(\frac{U_{\mathcal{P}}}{\bar{U}_{\mathcal{P}}^1}\right)$$
$$+ \frac{\eta \bar{U}_{\mathcal{K}}^1 \mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{E}}^1\right) + \bar{U}_{\mathcal{E}}^1\left(\iota_{\mathcal{E}} + \lambda \bar{U}_{\mathcal{L}}^1\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{K}}^1\right)}{\iota_{\mathcal{K}} \bar{U}_{\mathcal{E}}^1\left(\iota_{\mathcal{E}} + \lambda \bar{U}_{\mathcal{L}}^1\right)} F\left(\frac{U_{\mathcal{K}}}{\bar{U}_{\mathcal{K}}^1}\right)$$

$$+\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}}F\left(\frac{U_{\mathcal{E}}}{\bar{U}^1_{\mathcal{E}}}\right)+\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{2\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\left(\delta-\theta\bar{U}^1_{\mathcal{L}}\right)}\left(U_{\mathcal{L}}-\bar{U}^1_{\mathcal{L}}\right)^2. \tag{2.20}$$

Equilibrium condition Eq. (2.7) guarantees that $\delta-\theta\bar{U}^1_{\mathcal{L}}=\dfrac{\iota_{\mathcal{L}}\bar{U}^1_{\mathcal{L}}}{\bar{U}^1_{\mathcal{E}}}>0$. Clearly, Ξ_1 is positive definite. Calculating $\frac{d\Xi_1}{dt}$:

$$\begin{aligned}
\frac{d\Xi_1}{dt}&=\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}\right)\dot{U}_{\mathcal{J}}+\left(1-\frac{\bar{U}^1_{\mathcal{P}}}{U_{\mathcal{P}}}\right)\dot{U}_{\mathcal{P}}\\
&\quad+\frac{\eta\bar{U}^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)}{\iota_{\mathcal{K}}\bar{U}^1_{\mathcal{K}}\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\left(1-\frac{\bar{U}^1_{\mathcal{K}}}{U_{\mathcal{K}}}\right)\dot{U}_{\mathcal{K}}\\
&\quad+\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\left(1-\frac{\bar{U}^1_{\mathcal{E}}}{U_{\mathcal{E}}}\right)\dot{U}_{\mathcal{E}}+\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\left(\delta-\theta\bar{U}^1_{\mathcal{L}}\right)}\left(U_{\mathcal{L}}-\bar{U}^1_{\mathcal{L}}\right)\dot{U}_{\mathcal{L}}\\
&=\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}\right)\left(\tau-\iota_{\mathcal{J}}U_{\mathcal{J}}-\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)-\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)-\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\right)\\
&\quad+\left(1-\frac{\bar{U}^1_{\mathcal{P}}}{U_{\mathcal{P}}}\right)\left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)-\left(\alpha+\iota_{\mathcal{P}}\right)U_{\mathcal{P}}\right)\\
&\quad+\frac{\eta\bar{U}^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)}{\iota_{\mathcal{K}}\bar{U}^1_{\mathcal{K}}\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\left(1-\frac{\bar{U}^1_{\mathcal{K}}}{U_{\mathcal{K}}}\right)\left(\alpha U_{\mathcal{P}}-\iota_{\mathcal{K}}U_{\mathcal{K}}\right)\\
&\quad+\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\left(1-\frac{\bar{U}^1_{\mathcal{E}}}{U_{\mathcal{E}}}\right)\left(\eta U_{\mathcal{K}}-\iota_{\mathcal{E}}U_{\mathcal{E}}-\lambda U_{\mathcal{L}}U_{\mathcal{E}}\right)\\
&\quad+\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\left(\delta-\theta\bar{U}^1_{\mathcal{L}}\right)}\left(U_{\mathcal{L}}-\bar{U}^1_{\mathcal{L}}\right)\left(\delta U_{\mathcal{E}}-\iota_{\mathcal{L}}U_{\mathcal{L}}-\theta U_{\mathcal{L}}U_{\mathcal{E}}\right)\\
&=\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}\right)\left(\tau-\iota_{\mathcal{J}}U_{\mathcal{J}}\right)+\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}\left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\right)\\
&\quad-\left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\right)\frac{\bar{U}^1_{\mathcal{P}}}{U_{\mathcal{P}}}-\left(\alpha+\iota_{\mathcal{P}}\right)U_{\mathcal{P}}+\left(\alpha+\iota_{\mathcal{P}}\right)\bar{U}^1_{\mathcal{P}}\\
&\quad+\frac{\alpha\left(\eta\bar{U}^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)\right)}{\iota_{\mathcal{K}}\bar{U}^1_{\mathcal{K}}\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}U_{\mathcal{P}}\\
&\quad-\frac{\alpha\left(\eta\bar{U}^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)\right)}{\iota_{\mathcal{K}}\bar{U}^1_{\mathcal{K}}\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\frac{U_{\mathcal{P}}\bar{U}^1_{\mathcal{K}}}{U_{\mathcal{K}}}\\
&\quad-\frac{\eta\bar{U}^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)}{\bar{U}^1_{\mathcal{K}}\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\left(U_{\mathcal{K}}-\bar{U}^1_{\mathcal{K}}\right)\\
&\quad+\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)\frac{\eta U_{\mathcal{K}}}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}-\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)\frac{\eta U_{\mathcal{K}}}{U_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}-\frac{\iota_{\mathcal{E}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\left(U_{\mathcal{E}}-\bar{U}^1_{\mathcal{E}}\right)
\end{aligned}$$

$$-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}U_{\mathcal{L}}\left(U_{\mathcal{E}}-\bar{U}^1_{\mathcal{E}}\right)+\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\left(\delta-\theta\bar{U}^1_{\mathcal{L}}\right)}\left(U_{\mathcal{L}}-\bar{U}^1_{\mathcal{L}}\right)\left(\delta U_{\mathcal{E}}-\iota_{\mathcal{L}}U_{\mathcal{L}}-\theta U_{\mathcal{L}}U_{\mathcal{E}}\right). \tag{2.21}$$

Using the following equilibrium conditions for $\bar{O}^1$

$$\tau=\iota_{\mathcal{J}}\bar{U}^1_{\mathcal{J}}+\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right),$$
$$\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)=\left(\alpha+\iota_{\mathcal{P}}\right)\bar{U}^1_{\mathcal{P}},$$
$$\bar{U}^1_{\mathcal{K}}=\frac{\alpha\bar{U}^1_{\mathcal{P}}}{\iota_{\mathcal{K}}},\quad \bar{U}^1_{\mathcal{E}}=\frac{\eta\bar{U}^1_{\mathcal{K}}}{\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}},\quad \delta\bar{U}^1_{\mathcal{E}}=\iota_{\mathcal{L}}\bar{U}^1_{\mathcal{L}}+\theta\bar{U}^1_{\mathcal{L}}\bar{U}^1_{\mathcal{E}},$$

we get

$$\begin{aligned}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)&=\frac{\eta\bar{U}^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\\&=\frac{\alpha\bar{U}^1_{\mathcal{P}}\left(\eta\bar{U}^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)\right)}{\iota_{\mathcal{K}}\bar{U}^1_{\mathcal{K}}\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}.\end{aligned}$$

Consequently, the representation of Eq. (2.21) will be as follows:

$$\begin{aligned}\frac{d\Xi_1}{dt}=&\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}\right)\left(\iota_{\mathcal{J}}\bar{U}^1_{\mathcal{J}}-\iota_{\mathcal{J}}U_{\mathcal{J}}\right)+\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}\right)\left(\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{P}}\right)\right.\\&\left.+\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)\right)+\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}+\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{P}}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{P}}\right)}\\&+\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)}-\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)\bar{U}^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)U_{\mathcal{P}}}\\&-\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{P}}\right)\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)\bar{U}^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{P}}\right)U_{\mathcal{P}}}-\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\bar{U}^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)U_{\mathcal{P}}}-\left(\alpha+\iota_{\mathcal{P}}\right)\bar{U}^1_{\mathcal{P}}\frac{U_{\mathcal{P}}}{\bar{U}^1_{\mathcal{P}}}\\&+\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)\\&+\frac{\alpha\bar{U}^1_{\mathcal{P}}\left(\eta\bar{U}^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)\right)}{\iota_{\mathcal{K}}\bar{U}^1_{\mathcal{K}}\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\frac{U_{\mathcal{P}}}{\bar{U}^1_{\mathcal{P}}}\\&-\frac{\alpha\bar{U}^1_{\mathcal{P}}\left(\eta\bar{U}^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)\right)}{\iota_{\mathcal{K}}\bar{U}^1_{\mathcal{K}}\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\frac{U_{\mathcal{P}}\bar{U}^1_{\mathcal{K}}}{\bar{U}^1_{\mathcal{P}}U_{\mathcal{K}}}\\&-\frac{\eta\bar{U}^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)+\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{K}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\left(\frac{U_{\mathcal{K}}}{\bar{U}^1_{\mathcal{K}}}-1\right)\\&+\frac{\eta\bar{U}^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\frac{U_{\mathcal{K}}}{\bar{U}^1_{\mathcal{K}}}-\frac{\eta\bar{U}^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}}\frac{U_{\mathcal{K}}}{\bar{U}^1_{\mathcal{K}}U_{\mathcal{E}}}-\frac{\iota_{\mathcal{E}}\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}},\bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda\bar{U}^1_{\mathcal{L}}\right)}\left(U_{\mathcal{E}}-\bar{U}^1_{\mathcal{E}}\right)\end{aligned}$$

$$
\begin{aligned}
&- \frac{\lambda \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}} + \lambda \bar{U}^1_{\mathcal{L}}\right)} U_{\mathcal{L}}\left(U_{\mathcal{E}} - \bar{U}^1_{\mathcal{E}}\right) + \frac{\lambda \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}} + \lambda \bar{U}^1_{\mathcal{L}}\right)} \bar{U}^1_{\mathcal{L}}\left(U_{\mathcal{E}} - \bar{U}^1_{\mathcal{E}}\right) - \frac{\lambda \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}} + \lambda \bar{U}^1_{\mathcal{L}}\right)} \bar{U}^1_{\mathcal{L}}\left(U_{\mathcal{E}} - \bar{U}^1_{\mathcal{E}}\right) \\
&+ \frac{\lambda \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}} + \lambda \bar{U}^1_{\mathcal{L}}\right)\left(\delta - \theta \bar{U}^1_{\mathcal{L}}\right)} \left(U_{\mathcal{L}} - \bar{U}^1_{\mathcal{L}}\right)\left(\delta U_{\mathcal{E}} - \iota_{\mathcal{L}} U_{\mathcal{L}} - \theta U_{\mathcal{L}} U_{\mathcal{E}} - \delta \bar{U}^1_{\mathcal{E}} + \iota_{\mathcal{L}} \bar{U}^1_{\mathcal{L}} \right. \\
&\left. + \theta \bar{U}^1_{\mathcal{L}} \bar{U}^1_{\mathcal{E}} + \theta \bar{U}^1_{\mathcal{L}} U_{\mathcal{E}} - \theta \bar{U}^1_{\mathcal{L}} U_{\mathcal{E}}\right) \\
= {}& \iota_{\mathcal{J}} \bar{U}^1_{\mathcal{J}} \left(1 - \frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}\right)\left(1 - \frac{U_{\mathcal{J}}}{\bar{U}^1_{\mathcal{J}}}\right) + \left(2 - \frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}\right)\left(\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right)\right. \\
&\left. + \mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right)\right) + \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right) \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)} + \mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right) \frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right)} \\
&+ \mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right) \frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right)} - \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right) \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right)\bar{U}^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)U_{\mathcal{P}}} \\
&- \mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right) \frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)\bar{U}^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right)U_{\mathcal{P}}} - \mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right) \frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)\bar{U}^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right)U_{\mathcal{P}}} \\
&- \left(\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right) + \mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right)\right) \frac{U_{\mathcal{P}}}{\bar{U}^1_{\mathcal{P}}} \\
&+ \left(\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right)\right) \frac{U_{\mathcal{P}}}{\bar{U}^1_{\mathcal{P}}} - \left(\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right)\right) \frac{U_{\mathcal{P}}\bar{U}^1_{\mathcal{K}}}{\bar{U}^1_{\mathcal{P}}U_{\mathcal{K}}} \\
&- \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right) \frac{U_{\mathcal{K}}}{\bar{U}^1_{\mathcal{K}}} + \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right) - \mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right) \frac{U_{\mathcal{K}}}{\bar{U}^1_{\mathcal{K}}} + \mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right) + \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right) \frac{U_{\mathcal{K}}}{\bar{U}^1_{\mathcal{K}}} \\
&- \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right) \frac{U_{\mathcal{K}}\bar{U}^1_{\mathcal{E}}}{\bar{U}^1_{\mathcal{K}}U_{\mathcal{E}}} - \frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}} + \lambda \bar{U}^1_{\mathcal{L}}\right)} \left(\iota_{\mathcal{E}} + \lambda \bar{U}^1_{\mathcal{L}}\right)\left(U_{\mathcal{E}} - \bar{U}^1_{\mathcal{E}}\right) \\
&- \frac{\lambda \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}} + \lambda \bar{U}^1_{\mathcal{L}}\right)} \left(U_{\mathcal{L}} - \bar{U}^1_{\mathcal{L}}\right)\left(U_{\mathcal{E}} - \bar{U}^1_{\mathcal{E}}\right) \\
&+ \frac{\lambda \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}} + \lambda \bar{U}^1_{\mathcal{L}}\right)\left(\delta - \theta \bar{U}^1_{\mathcal{L}}\right)} \left(\delta - \theta \bar{U}^1_{\mathcal{L}}\right)\left(U_{\mathcal{L}} - \bar{U}^1_{\mathcal{L}}\right)\left(U_{\mathcal{E}} - \bar{U}^1_{\mathcal{E}}\right) \\
&- \frac{\lambda \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\bar{U}^1_{\mathcal{E}}\left(\iota_{\mathcal{E}} + \lambda \bar{U}^1_{\mathcal{L}}\right)\left(\delta - \theta \bar{U}^1_{\mathcal{L}}\right)} \left(\iota_{\mathcal{L}} + \theta U_{\mathcal{E}}\right)\left(U_{\mathcal{L}} - \bar{U}^1_{\mathcal{L}}\right)^2.
\end{aligned}
$$

This implies that

$$
\begin{aligned}
\frac{d\Xi_1}{dt} = {}& \iota_{\mathcal{J}} \bar{U}^1_{\mathcal{J}} \left(1 - \frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}\right)\left(1 - \frac{U_{\mathcal{J}}}{\bar{U}^1_{\mathcal{J}}}\right) + \left(2 - \frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}\right)\left(\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right)\right. \\
&\left. + \mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right)\right) + \mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right) \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)} + \mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right) \frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right)}
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)}-\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)U_{\mathcal{P}}}\\
&-\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}-\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)U_{\mathcal{P}}}\\
&-\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)\frac{U_{\mathcal{P}}}{\bar{U}_{\mathcal{P}}^{1}}-\left(\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)+\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)\right)\frac{U_{\mathcal{P}}\bar{U}_{\mathcal{K}}^{1}}{\bar{U}_{\mathcal{P}}^{1}U_{\mathcal{K}}}\\
&+2\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)-\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)\frac{U_{\mathcal{K}}}{\bar{U}_{\mathcal{K}}^{1}}+\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)-\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\frac{U_{\mathcal{K}}\bar{U}_{\mathcal{E}}^{1}}{\bar{U}_{\mathcal{K}}^{1}U_{\mathcal{E}}}\\
&-\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\frac{U_{\mathcal{E}}}{\bar{U}_{\mathcal{E}}^{1}}-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\bar{U}_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda\bar{U}_{\mathcal{L}}^{1}\right)\left(\delta-\theta\bar{U}_{\mathcal{L}}^{1}\right)}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)\left(U_{\mathcal{L}}-\bar{U}_{\mathcal{L}}^{1}\right)^{2}\\
=&\,\iota_{\mathcal{J}}\bar{U}_{\mathcal{J}}^{1}\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{\bar{U}_{\mathcal{J}}^{1}}\right)+\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}-\frac{U_{\mathcal{E}}}{\bar{U}_{\mathcal{E}}^{1}}\right)\\
&+\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\left(4-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)U_{\mathcal{P}}}-\frac{U_{\mathcal{P}}\bar{U}_{\mathcal{K}}^{1}}{\bar{U}_{\mathcal{P}}^{1}U_{\mathcal{K}}}-\frac{U_{\mathcal{K}}\bar{U}_{\mathcal{E}}^{1}}{\bar{U}_{\mathcal{K}}^{1}U_{\mathcal{E}}}\right)\\
&+\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)}-\frac{U_{\mathcal{P}}}{\bar{U}_{\mathcal{P}}^{1}}\right)\\
&+\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)\left(2-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}-\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}\right)\\
&+\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)}-\frac{U_{\mathcal{K}}}{\bar{U}_{\mathcal{K}}^{1}}\right)\\
&+\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)\left(3-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}-\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)U_{\mathcal{P}}}-\frac{U_{\mathcal{P}}\bar{U}_{\mathcal{K}}^{1}}{\bar{U}_{\mathcal{P}}^{1}U_{\mathcal{K}}}\right)\\
&-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\bar{U}_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda\bar{U}_{\mathcal{L}}^{1}\right)\left(\delta-\theta\bar{U}_{\mathcal{L}}^{1}\right)}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)\left(U_{\mathcal{L}}-\bar{U}_{\mathcal{L}}^{1}\right)^{2}.
\end{aligned}
$$

Then

$$
\begin{aligned}
\frac{d\Xi_1}{dt}=&\,\iota_{\mathcal{J}}\bar{U}_{\mathcal{J}}^{1}\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{\bar{U}_{\mathcal{J}}^{1}}\right)\\
&+\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}-\frac{U_{\mathcal{E}}}{\bar{U}_{\mathcal{E}}^{1}}-1+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)U_{\mathcal{E}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)\bar{U}_{\mathcal{E}}^{1}}\right)\\
&+\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\left(5-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)U_{\mathcal{P}}}-\frac{U_{\mathcal{P}}\bar{U}_{\mathcal{K}}^{1}}{\bar{U}_{\mathcal{P}}^{1}U_{\mathcal{K}}}-\frac{U_{\mathcal{K}}\bar{U}_{\mathcal{E}}^{1}}{\bar{U}_{\mathcal{K}}^{1}U_{\mathcal{E}}}-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)U_{\mathcal{E}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)\bar{U}_{\mathcal{E}}^{1}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)}-\frac{U_{\mathcal{P}}}{\bar{U}_{\mathcal{P}}^{1}}-1+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)\bar{U}_{\mathcal{P}}^{1}}\right)\\
&+\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)\left(3-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}-\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)\bar{U}_{\mathcal{P}}^{1}}\right)\\
&+\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)}-\frac{U_{\mathcal{K}}}{\bar{U}_{\mathcal{K}}^{1}}-1+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)U_{\mathcal{K}}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\bar{U}_{\mathcal{K}}^{1}}\right)\\
&+\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)\left(4-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}-\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)U_{\mathcal{P}}}-\frac{U_{\mathcal{P}}\bar{U}_{\mathcal{K}}^{1}}{\bar{U}_{\mathcal{P}}^{1}U_{\mathcal{K}}}-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)U_{\mathcal{K}}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\bar{U}_{\mathcal{K}}^{1}}\right)\\
&-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\bar{U}_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda\bar{U}_{\mathcal{L}}^{1}\right)\left(\delta-\theta\bar{U}_{\mathcal{L}}^{1}\right)}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)\left(U_{\mathcal{L}}-\bar{U}_{\mathcal{L}}^{1}\right)^{2}\\
=&\ \iota_{\mathcal{J}}\bar{U}_{\mathcal{J}}^{1}\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{\bar{U}_{\mathcal{J}}^{1}}\right)+\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}-\frac{U_{\mathcal{E}}}{\bar{U}_{\mathcal{E}}^{1}}\right)\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}\right)\\
&+\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\left(5-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)U_{\mathcal{P}}}-\frac{U_{\mathcal{P}}\bar{U}_{\mathcal{K}}^{1}}{\bar{U}_{\mathcal{P}}^{1}U_{\mathcal{K}}}-\frac{U_{\mathcal{K}}\bar{U}_{\mathcal{E}}^{1}}{\bar{U}_{\mathcal{K}}^{1}U_{\mathcal{E}}}-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)U_{\mathcal{E}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)\bar{U}_{\mathcal{E}}^{1}}\right)\\
&+\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)}-\frac{U_{\mathcal{P}}}{\bar{U}_{\mathcal{P}}^{1}}\right)\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}\right)\\
&+\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)\left(3-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}-\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)\bar{U}_{\mathcal{P}}^{1}}\right)\\
&+\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)}-\frac{U_{\mathcal{K}}}{\bar{U}_{\mathcal{K}}^{1}}\right)\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}\right)\\
&+\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)\left(4-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}-\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)U_{\mathcal{P}}}-\frac{U_{\mathcal{P}}\bar{U}_{\mathcal{K}}^{1}}{\bar{U}_{\mathcal{P}}^{1}U_{\mathcal{K}}}-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)U_{\mathcal{K}}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\bar{U}_{\mathcal{K}}^{1}}\right)\\
&-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\bar{U}_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda\bar{U}_{\mathcal{L}}^{1}\right)\left(\delta-\theta\bar{U}_{\mathcal{L}}^{1}\right)}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)\left(U_{\mathcal{L}}-\bar{U}_{\mathcal{L}}^{1}\right)^{2}.
\end{aligned}
$$

From Hypothesis **H2**, we have

$$\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{\bar{U}_{\mathcal{J}}^{1}}\right)\leq 0.$$

Under the AM-GM inequality, we discern that

$$
\begin{aligned}
5&\leq\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)U_{\mathcal{P}}}+\frac{U_{\mathcal{P}}\bar{U}_{\mathcal{K}}^{1}}{\bar{U}_{\mathcal{P}}^{1}U_{\mathcal{K}}}+\frac{U_{\mathcal{K}}\bar{U}_{\mathcal{E}}^{1}}{\bar{U}_{\mathcal{K}}^{1}U_{\mathcal{E}}}+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)U_{\mathcal{E}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)\bar{U}_{\mathcal{E}}^{1}},\\
4&\leq\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)}+\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\bar{U}_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)U_{\mathcal{P}}}+\frac{U_{\mathcal{P}}\bar{U}_{\mathcal{K}}^{1}}{\bar{U}_{\mathcal{P}}^{1}U_{\mathcal{K}}}+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{K}}^{1}\right)U_{\mathcal{K}}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^{1},\bar{U}_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\bar{U}_{\mathcal{K}}^{1}},
\end{aligned}
$$

$$3 \leq \frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)} + \frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)\bar{U}^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right)U_{\mathcal{P}}} + \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right)U_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)\bar{U}^1_{\mathcal{P}}}.$$

Moreover, from Eqs. (2.17)-(2.19), we conclude

$$\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)} - \frac{U_{\mathcal{E}}}{\bar{U}^1_{\mathcal{E}}}\right)\left(1 - \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right)}\right) \leq 0,$$

$$\left(\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right)} - \frac{U_{\mathcal{P}}}{\bar{U}^1_{\mathcal{P}}}\right)\left(1 - \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}\right) \leq 0,$$

$$\left(\frac{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right)} - \frac{U_{\mathcal{K}}}{\bar{U}^1_{\mathcal{K}}}\right)\left(1 - \frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}^1_{\mathcal{J}}, \bar{U}^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)}\right) \leq 0.$$

Therefore, when $\Re_0 > 1$, it implies that $\frac{d\Xi_1}{dt} \leq 0$ for all $U_{\mathcal{J}}, U_{\mathcal{P}}, U_{\mathcal{K}}, U_{\mathcal{E}}, U_{\mathcal{L}} > 0$. Moreover, $\frac{d\Xi_1}{dt} = 0$ when $U_{\mathcal{J}} = \bar{U}^1_{\mathcal{J}}, U_{\mathcal{P}} = \bar{U}^1_{\mathcal{P}}, U_{\mathcal{K}} = \bar{U}^1_{\mathcal{K}}, U_{\mathcal{E}} = \bar{U}^1_{\mathcal{E}}$ and $U_{\mathcal{L}} = \bar{U}^1_{\mathcal{L}}$. Consequently, $M'_1 = \left\{\bar{O}^1\right\}$. Applying the L.I.P allows us to infer that if $\Re_0 > 1$, the equilibrium $\bar{O}^1$ is G.A.S [53], [54]. □

3. HIV-1 model with distributed delays

This section enhances the previously introduced model by integrating three varieties of distributed time delays.

3.1. **System overview.**

The system of delay differential equations (DDEs) presented below will be studied.

$$\begin{cases} \dot{U}_{\mathcal{J}} &= \tau - \iota_{\mathcal{J}} U_{\mathcal{J}} - \mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right) - \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right) - \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right), \\ \dot{U}_{\mathcal{P}} &= \int_0^{f_1} \pi_1(\ell) e^{-b_1 \ell}\left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}(t-\ell), U_{\mathcal{E}}(t-\ell)\right) + \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}(t-\ell), U_{\mathcal{P}}(t-\ell)\right)\right. \\ &\quad \left. + \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}(t-\ell), U_{\mathcal{K}}(t-\ell)\right)\right) d\ell - (\alpha + \iota_{\mathcal{P}}) U_{\mathcal{P}}, \\ \dot{U}_{\mathcal{K}} &= \alpha \int_0^{f_2} \pi_2(\ell) e^{-b_2 \ell} U_{\mathcal{P}}(t-\ell)\, d\ell - \iota_{\mathcal{K}} U_{\mathcal{K}}, \\ \dot{U}_{\mathcal{E}} &= \eta \int_0^{f_3} \pi_3(\ell) e^{-b_3 \ell} U_{\mathcal{K}}(t-\ell)\, d\ell - \iota_{\mathcal{E}} U_{\mathcal{E}} - \lambda U_{\mathcal{L}} U_{\mathcal{E}}, \\ \dot{U}_{\mathcal{L}} &= \delta U_{\mathcal{E}} - \iota_{\mathcal{L}} U_{\mathcal{L}} - \theta U_{\mathcal{L}} U_{\mathcal{E}}. \end{cases} \tag{3.1}$$

Here, system (3.1) includes the following assumptions:

- At the moment t, healthy cells encountering either HIV-1 particles or infected cells transition to a latent infection state ℓ units of time later. The emergence of latently infected cells at time t depends on the count of cells recently contacted at time $t - \ell$, which persist until time t.
- After a latency period of ℓ units of time after infection, cells that were initially latent become actively infected. The emergence of actively infected cells at time t depends on the count of cells recently transitioned into latent infection at time $t - \ell$, which persist until time t.

- Newly formed mature HIV-1 particles emerge from actively infected cells ℓ units of time after infection. The occurrence of HIV-1 particle production at time t relies on the count of cells that recently transitioned into actively infected status at time $t-\ell$ that remain viable until time t.

As such, the likelihood of surviving the time-frame from $t-\ell$ to t is represented by $\pi_i(\ell)\, e^{-b_i \ell}$. Here, $1/b_1$ is the average lifetime of the cell during the period of formation of a latently infected cell [55], $1/b_2$ is the average lifetime of the cell during the period of a latently infected cell's reactivation [56], $1/b_3$ is the average lifetime of an immature virus [55]. Besides, we choose the delay parameter, ℓ, at random. It is drawn from a probability distribution function $\pi_i(\ell)$, which spans the interval $[0, f_i]$, in which f_i refers to the maximum delay duration. The functions $\pi_i(\ell)$, for $i = 1,2,3$ adhere to and fulfill the following specified conditions:

$$\pi_i(\ell) > 0, \quad \int_0^{f_i} \pi_i(\ell)\, d\ell = 1, \quad \text{and} \quad \int_0^{f_i} \pi_i(\ell)\, e^{-\rho \ell} d\ell < \infty, \quad \text{where} \quad \rho > 0.$$

Assuming that

$$\bar{\Pi}_i(\ell) = \pi_i(\ell) e^{-b_i \ell}, \qquad \Pi_i = \int_0^{f_i} \bar{\Pi}_i(\ell) d\ell, i = 1,2,3,$$

implies the fact that $0 < \Pi_i \leq 1$, $i = 1,2,3$.
System (3.1) has the outlined initial conditions below:

$$\begin{cases} (U_{\mathcal{J}}(v), U_{\mathcal{P}}(v), U_{\mathcal{K}}(v), U_{\mathcal{E}}(v), U_{\mathcal{L}}(v)) = (k_1(v), k_2(v), k_3(v), k_4(v), k_5(v)), \\ k_i(v) \geq 0, \quad i = 1,2,...,5, \quad v \in [-f, 0], \quad f = \max\{f_1, f_2, f_3\}, \end{cases} \tag{3.2}$$

where $k_i(v) \in C([-f,0], \mathbb{R}_{\geq 0})$, $i = 1,2,...,5$ and $C = C([-f,0], \mathbb{R}_{\geq 0})$ is the Banach space comprising continuous functions, and it is equipped with the norm $\|k_i\| = \sup_{-f \leq \zeta \leq 0} |k_i(\zeta)|$ for all $k_i \in C$. Thus, a unique solution for system (3.1) under the specified initial conditions (3.2) is guaranteed, as affirmed by references [53] and [57]. The meanings assigned to all remaining parameters and variables remain in line with the explanations provided earlier. We presume that functions $\mathcal{F}_{\mathcal{X}}(U_{\mathcal{J}}, U_{\mathcal{X}})$, $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$, satisfy Hypotheses **H1**-**H6** as presented previously.

3.2. **Fundamental characteristics.**

3.2.1. *The well-posedness of the system.*

Proposition 3.1. *Assuming the fulfillment of Hypothesis* ***H1****, for system (3.1) and its initial conditions (3.2). Then, a positive constants $\hat{\Gamma}_i$ for $i = 1,2,3,4$ exist, ensuring the following compact set:*

$$\hat{\Delta} = \Big\{ (U_{\mathcal{J}}, U_{\mathcal{P}}, U_{\mathcal{K}}, U_{\mathcal{E}}, U_{\mathcal{L}}) \in C^5_{\geq 0} : \|U_{\mathcal{J}}(t)\| \leq \hat{\Gamma}_1, \|U_{\mathcal{P}}(t)\| \leq \hat{\Gamma}_1, \\ \|U_{\mathcal{K}}(t)\| \leq \hat{\Gamma}_2, \|U_{\mathcal{E}}(t)\| \leq \hat{\Gamma}_3, \|U_{\mathcal{L}}(t)\| \leq \hat{\Gamma}_4 \Big\},$$

is positively invariant.

Proof. As $\dot{U}_{\mathcal{J}} \mid_{U_{\mathcal{J}}=0} = \tau > 0$, it can be inferred that $U_{\mathcal{J}}(t)$ is always positive for every $t \geq 0$. Furthermore, other equations within system (3.1) will be expressed as:

$$\begin{aligned}
&\dot{U}_{\mathcal{P}} + (\alpha + \iota_{\mathcal{P}})\, U_{\mathcal{P}} = \int_0^{f_1} \bar{\Pi}_1(\ell) \left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}(t-\ell), U_{\mathcal{E}}(t-\ell)\right) + \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}(t-\ell), U_{\mathcal{P}}(t-\ell)\right)\right. \\
&\qquad\qquad\qquad\qquad \left. + \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}(t-\ell), U_{\mathcal{K}}(t-\ell)\right)\right) d\ell \\
&\Longrightarrow U_{\mathcal{P}}(t) = k_2(0) e^{-(\alpha+\iota_{\mathcal{P}})t} + \int_0^t e^{-(\alpha+\iota_{\mathcal{P}})(t-\gamma)} \int_0^{f_1} \bar{\Pi}_1(\ell) \left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}(\gamma-\ell), U_{\mathcal{E}}(\gamma-\ell)\right)\right. \\
&\qquad\qquad \left. + \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}(\gamma-\ell), U_{\mathcal{P}}(\gamma-\ell)\right) + \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}(\gamma-\ell), U_{\mathcal{K}}(\gamma-\ell)\right)\right)) d\ell d\gamma \geq 0. \\
&\dot{U}_{\mathcal{K}} + \iota_{\mathcal{K}} U_{\mathcal{K}} = \alpha \int_0^{f_2} \bar{\Pi}_2(\ell)\, U_{\mathcal{P}}(t-\ell)\, d\ell \\
&\Longrightarrow U_{\mathcal{K}}(t) = k_3(0) e^{-\iota_{\mathcal{K}} t} + \alpha \int_0^t e^{-\iota_{\mathcal{K}}(t-\gamma)} \int_0^{f_2} \bar{\Pi}_2(\ell)\, U_{\mathcal{P}}(\gamma-\ell)\, d\ell d\gamma \geq 0. \\
&\dot{U}_{\mathcal{E}} + (\iota_{\mathcal{E}} + \lambda U_{\mathcal{L}})\, U_{\mathcal{E}} = \eta \int_0^{f_3} \bar{\Pi}_3(\ell) U_{\mathcal{K}}(t-\ell)\, d\ell \\
&\Longrightarrow U_{\mathcal{E}}(t) = k_4(0) e^{-\int_0^t (\iota_{\mathcal{E}} + \lambda U_{\mathcal{L}}(u)) du} + \eta \int_0^t e^{-\int_\gamma^t (\iota_{\mathcal{E}} + \lambda U_{\mathcal{L}}(u)) du} \int_0^{f_3} \bar{\Pi}_3(\ell)\, U_{\mathcal{K}}(\gamma-\ell)\, d\ell d\gamma \geq 0. \\
&\dot{U}_{\mathcal{L}} + (\iota_{\mathcal{L}} + \theta U_{\mathcal{E}})\, U_{\mathcal{L}} = \delta U_{\mathcal{E}} \\
&\Longrightarrow U_{\mathcal{L}}(t) = k_5(0) e^{-\int_0^t (\iota_{\mathcal{L}} + \theta U_{\mathcal{E}}(u)) du} + \delta \int_0^t e^{-\int_\gamma^t (\iota_{\mathcal{L}} + \theta U_{\mathcal{E}}(u)) du} U_{\mathcal{E}}(\ell)\, d\ell \geq 0,
\end{aligned}$$

for every $t \in [0, f]$. By employing a recursive argumentation approach, it can be shown that $U_{\mathcal{J}}(t), U_{\mathcal{P}}(t), U_{\mathcal{K}}(t), U_{\mathcal{E}}(t)$ and $U_{\mathcal{L}}(t)$ remain nonnegative for every $t \geq 0$. As a result, system (3.1) only admits solutions where $(U_{\mathcal{J}}(t), U_{\mathcal{P}}(t), U_{\mathcal{K}}(t), U_{\mathcal{E}}(t), U_{\mathcal{L}}(t)) \in \mathbb{R}^5_{\geq 0}$, for every $t \geq 0$.

Obviously, $\lim\limits_{t\to\infty} \sup U_{\mathcal{J}}(t) \leq \dfrac{\tau}{\iota_{\mathcal{J}}}$, as deduced from the first equation in system (3.1). Following that, we proceed with defining Φ_1 as follows:

$$\Phi_1 = \int_0^{f_1} \bar{\Pi}_1(\ell)\, U_{\mathcal{J}}(t-\ell)\, d\ell + U_{\mathcal{P}}.$$

Therefore

$$\begin{aligned}
\dot{\Phi}_1 &= \int_0^{f_1} \bar{\Pi}_1(\ell) \frac{dU_{\mathcal{J}}(t-\ell)}{dt} d\ell + \dot{U}_{\mathcal{P}} \\
&= \int_0^{f_1} \bar{\Pi}_1(\ell) \left(\tau - \iota_{\mathcal{J}} U_{\mathcal{J}}(t-\ell)\right) d\ell - (\alpha + \iota_{\mathcal{P}})\, U_{\mathcal{P}} \\
&= \tau \Pi_1 - \iota_{\mathcal{J}} \int_0^{f_1} \bar{\Pi}_1(\ell) U_{\mathcal{J}}(t-\ell)\, d\ell - (\alpha + \iota_{\mathcal{P}})\, U_{\mathcal{P}} \\
&\leq \tau - \varepsilon_1 \left(\int_0^{f_1} \bar{\Pi}_1(\ell) U_{\mathcal{J}}(t-\ell)\, d\ell + U_{\mathcal{P}} \right) = \tau - \varepsilon_1 \Phi_1,
\end{aligned}$$

where $\varepsilon_1 = \min\{\iota_{\mathcal{J}}, \alpha + \iota_{\mathcal{P}}\}$. This indicates that $\limsup_{t\to\infty} \Phi_1(t) \le \dfrac{\tau}{\varepsilon_1} = \hat{\Gamma}_1$. Based on the nonnegativity of $\int_0^{f_1} \bar{\Pi}_1(\ell)\, U_{\mathcal{J}}(t-\ell)\, d\ell$ and $U_{\mathcal{P}}$, then $\limsup_{t\to\infty} U_{\mathcal{P}}(t) \le \hat{\Gamma}_1$ is confirmed. As a result of the third equation in system (3.1), one can acquire

$$\dot{U}_{\mathcal{K}} = \alpha \int_0^{f_2} \bar{\Pi}_2(\ell) U_{\mathcal{P}}(t-\ell)\, d\ell - \iota_{\mathcal{K}} U_{\mathcal{K}} \le \alpha \Pi_2 \hat{\Gamma}_1 - \iota_{\mathcal{K}} U_{\mathcal{K}} \le \alpha \hat{\Gamma}_1 - \iota_{\mathcal{K}} U_{\mathcal{K}}.$$

Then, $\limsup_{t\to\infty} U_{\mathcal{K}}(t) \le \frac{\alpha \hat{\Gamma}_1}{\iota_{\mathcal{K}}} = \hat{\Gamma}_2$. Finally, we let

$$\Phi_2(t) = U_{\mathcal{E}} + \frac{\iota_{\mathcal{E}}}{2\delta} U_{\mathcal{L}}.$$

This produces

$$\begin{aligned}
\dot{\Phi}_2 &= \dot{U}_{\mathcal{E}} + \frac{\iota_{\mathcal{E}}}{2\delta} \dot{U}_{\mathcal{L}} \\
&= \eta \int_0^{f_3} \bar{\Pi}_3(\ell)\, U_{\mathcal{K}}(t-\ell)\, d\ell - \iota_{\mathcal{E}} U_{\mathcal{E}} - \lambda U_{\mathcal{L}} U_{\mathcal{E}} + \frac{\iota_{\mathcal{E}}}{2\delta}(\delta U_{\mathcal{E}} - \iota_{\mathcal{L}} U_{\mathcal{L}} - \theta U_{\mathcal{L}} U_{\mathcal{E}}) \\
&= \eta \int_0^{f_3} \bar{\Pi}_3(\ell)\, U_{\mathcal{K}}(t-\ell)\, d\ell - \frac{\iota_{\mathcal{E}}}{2} U_{\mathcal{E}} - \frac{\iota_{\mathcal{E}} \iota_{\mathcal{L}}}{2\delta} U_{\mathcal{L}} - (\lambda + \frac{\iota_{\mathcal{E}} \theta}{2\delta}) U_{\mathcal{L}} U_{\mathcal{E}} \\
&\le \eta \int_0^{f_3} \bar{\Pi}_3(\ell)\, U_{\mathcal{K}}(t-\ell)\, d\ell - \frac{\iota_{\mathcal{E}}}{2} U_{\mathcal{E}} - \frac{\iota_{\mathcal{E}} \iota_{\mathcal{L}}}{2\delta} U_{\mathcal{L}} \\
&\le \eta \hat{\Gamma}_2 - \varepsilon_2 (U_{\mathcal{E}} + \frac{\iota_{\mathcal{E}}}{2\delta} U_{\mathcal{L}}) = \eta \hat{\Gamma}_2 - \varepsilon_2 \Phi_2,
\end{aligned}$$

where $\varepsilon_2 = \min\{\frac{\iota_{\mathcal{E}}}{2}, \iota_{\mathcal{L}}\}$. Thus, $\limsup_{t\to\infty} \Phi_2(t) \le \frac{\eta \hat{\Gamma}_2}{\varepsilon_2} = \hat{\Gamma}_3$. Based on the nonnegativity of $U_{\mathcal{E}}$ and $U_{\mathcal{L}}$, one can guarantee that $\limsup_{t\to\infty} U_{\mathcal{E}}(t) \le \hat{\Gamma}_3$, and $\limsup_{t\to\infty} U_{\mathcal{L}}(t) \le \frac{2\delta \hat{\Gamma}_3}{\iota_{\mathcal{E}}} = \hat{\Gamma}_4$. Consequently, it can be inferred that $U_{\mathcal{J}}(t), U_{\mathcal{P}}(t), U_{\mathcal{K}}(t), U_{\mathcal{E}}(t)$ and $U_{\mathcal{L}}(t)$ are being ultimately bounded. This, in turn, ensures the positive invariance of the compact set $\hat{\Delta}$ under the dynamics of system (3.1). □

3.2.2. *The reproduction ratio and equilibria of the system.*

Proposition 3.2. *Assuming that Hypotheses **H1**-**H4** are met, there exists a positive basic reproduction ratio* $\tilde{\mathfrak{R}}_0 = \frac{\Pi_1\left(\eta\alpha\Pi_2\Pi_3\wp_{\mathcal{E}}\left(U^0_{\mathcal{J}}\right) + \iota_{\mathcal{E}}\iota_{\mathcal{K}}\wp_{\mathcal{P}}\left(U^0_{\mathcal{J}}\right) + \alpha\iota_{\mathcal{E}}\Pi_2\wp_{\mathcal{K}}\left(U^0_{\mathcal{J}}\right)\right)}{\iota_{\mathcal{E}}\iota_{\mathcal{K}}(\alpha + \iota_{\mathcal{P}})}$ *for system (3.1) in a way that*

(i) ensures the system consistently maintains a virus-free equilibrium, denoted as O^0, and

(ii) if $\tilde{\mathfrak{R}}_0 > 1$, the system additionally possesses a virus-persistence equilibrium, denoted as O^1.

The proof is omitted as it is similar to the proof of Proposition 2.2.

3.2.3. *Global stability of equilibria.* The forthcoming theorems explore the global asymptotic stability of both virus-free and virus-persistence equilibria. For clarity's sake, let's demonstrate $(U_{\mathcal{J}}(t-\ell), U_{\mathcal{P}}(t-\ell),$
$U_{\mathcal{K}}(t-\ell), U_{\mathcal{E}}(t-\ell), U_{\mathcal{L}}(t-\ell))$ by $\left(U_{\mathcal{J}_\ell}, U_{\mathcal{P}_\ell}, U_{\mathcal{K}_\ell}, U_{\mathcal{E}_\ell}, U_{\mathcal{L}_\ell}\right)$

Theorem 3.1. *For system (3.1), let $\tilde{\mathfrak{R}}_0 \le 1$ and Hypotheses **H1**-**H5** are satisfied, then O^0 is G.A.S in $\hat{\Delta}$.*

Proof. Let's contemplate a potential Lyapunov function, $\Xi_2(U_{\mathcal{J}}, U_{\mathcal{P}}, U_{\mathcal{K}}, U_{\mathcal{E}}, U_{\mathcal{L}})$, in the following manner:

$$\begin{aligned}
\Xi_2 = & U_{\mathcal{J}} - U_{\mathcal{J}}^0 - \int_{U_{\mathcal{J}}^0}^{U_{\mathcal{J}}} \frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\wp_{\mathcal{E}}\left(\gamma\right)} d\gamma + \frac{1}{\Pi_1} U_{\mathcal{P}} + \frac{\eta \Pi_3 \wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right) + \iota_{\mathcal{E}} \wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)}{\iota_{\mathcal{K}} \iota_{\mathcal{E}}} U_{\mathcal{K}} \\
& + \frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\iota_{\mathcal{E}}} U_{\mathcal{E}} + \frac{\lambda \wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{2\delta \iota_{\mathcal{E}}} U_{\mathcal{L}}^2 + \frac{1}{\Pi_1} \int_0^{f_1} \bar{\Pi}_1(\ell) \int_{t-\ell}^{t} \left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}\left(\gamma\right), U_{\mathcal{E}}\left(\gamma\right)\right)\right. \\
& \left. + \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}\left(\gamma\right), U_{\mathcal{P}}\left(\gamma\right)\right) + \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}\left(\gamma\right), U_{\mathcal{K}}\left(\gamma\right)\right)\right) d\gamma d\ell \\
& + \frac{\alpha\left(\eta \Pi_3 \wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right) + \iota_{\mathcal{E}} \wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)\right)}{\iota_{\mathcal{K}} \iota_{\mathcal{E}}} \int_0^{f_2} \bar{\Pi}_2(\ell) \int_{t-\ell}^{t} U_{\mathcal{P}}(\gamma) d\gamma d\ell \\
& + \frac{\eta \wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\iota_{\mathcal{E}}} \int_0^{f_3} \bar{\Pi}_3(\ell) \int_{t-\ell}^{t} U_{\mathcal{K}}(\gamma) d\gamma d\ell.
\end{aligned}$$

Evidently, $\Xi_2(U_{\mathcal{J}}, U_{\mathcal{P}}, U_{\mathcal{K}}, U_{\mathcal{E}}, U_{\mathcal{L}}) > 0$ for every $U_{\mathcal{J}}, U_{\mathcal{P}}, U_{\mathcal{K}}, U_{\mathcal{E}}, U_{\mathcal{L}} > 0$, as well as $\Xi_2\left(U_{\mathcal{J}}^0, 0, 0, 0, 0\right) = 0$. Further, $\frac{d\Xi_2}{dt}$ is given by:

$$\begin{aligned}
\frac{d\Xi_2}{dt} = & \left(1 - \frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\right) \dot{U}_{\mathcal{J}} + \frac{1}{\Pi_1} \dot{U}_{\mathcal{P}} + \frac{\eta \Pi_3 \wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right) + \iota_{\mathcal{E}} \wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)}{\iota_{\mathcal{K}} \iota_{\mathcal{E}}} \dot{U}_{\mathcal{K}} + \frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\iota_{\mathcal{E}}} \dot{U}_{\mathcal{E}} \\
& + \frac{\lambda \wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\delta \iota_{\mathcal{E}}} U_{\mathcal{L}} \dot{U}_{\mathcal{L}} + \mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right) + \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right) + \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right) \\
& - \frac{1}{\Pi_1} \int_0^{f_1} \bar{\Pi}_1(\ell) \left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell}, U_{\mathcal{E}_\ell}\right) + \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell}, U_{\mathcal{P}_\ell}\right)\right. \\
& \left. + \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell}, U_{\mathcal{K}_\ell}\right)\right) d\ell + \frac{\alpha \Pi_2 \left(\eta \Pi_3 \wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right) + \iota_{\mathcal{E}} \wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)\right)}{\iota_{\mathcal{K}} \iota_{\mathcal{E}}} U_{\mathcal{P}} \\
& - \frac{\alpha\left(\eta \Pi_3 \wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right) + \iota_{\mathcal{E}} \wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)\right)}{\iota_{\mathcal{K}} \iota_{\mathcal{E}}} \int_0^{f_2} \bar{\Pi}_2(\ell) U_{\mathcal{P}_\ell} d\ell + \frac{\eta \Pi_3 \wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\iota_{\mathcal{E}}} U_{\mathcal{K}} \\
& - \frac{\eta \wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\iota_{\mathcal{E}}} \int_0^{f_3} \bar{\Pi}_3(\ell) U_{\mathcal{K}_\ell} d\ell \\
= & \left(1 - \frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\right) \left(\tau - \iota_{\mathcal{J}} U_{\mathcal{J}} - \mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, U_{\mathcal{E}}\right) - \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right) - \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)\right) \\
& + \frac{1}{\Pi_1} \int_0^{f_1} \bar{\Pi}_1(\ell) \left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell}, U_{\mathcal{E}_\ell}\right) + \mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell}, U_{\mathcal{P}_\ell}\right)\right. \\
& \left. + \mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell}, U_{\mathcal{K}_\ell}\right)\right) d\ell - \frac{\alpha + \iota_{\mathcal{P}}}{\Pi_1} U_{\mathcal{P}} \\
& + \frac{\eta \Pi_3 \wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right) + \iota_{\mathcal{E}} \wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)}{\iota_{\mathcal{K}} \iota_{\mathcal{E}}} \left(\alpha \int_0^{f_2} \bar{\Pi}_2(\ell) U_{\mathcal{P}_\ell} d\ell - \iota_{\mathcal{K}} U_{\mathcal{K}}\right)
\end{aligned}$$

$$+\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\iota_{\mathcal{E}}}\left(\eta\int_0^{f_3}\bar{\Pi}_3(\ell)U_{\mathcal{K}_\ell}d\ell-\iota_{\mathcal{E}}U_{\mathcal{E}}-\lambda U_{\mathcal{L}}U_{\mathcal{E}}\right)+\frac{\lambda\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\delta\iota_{\mathcal{E}}}U_{\mathcal{L}}\left(\delta U_{\mathcal{E}}-\iota_{\mathcal{L}}U_{\mathcal{L}}-\theta U_{\mathcal{L}}U_{\mathcal{E}}\right)$$
$$+\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)-\frac{1}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)\right.$$
$$\left.+\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)\right)d\ell$$
$$+\frac{\alpha\Pi_2\left(\eta\Pi_3\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}U_{\mathcal{P}}-\frac{\alpha\left(\eta\Pi_3\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\int_0^{f_2}\bar{\Pi}_2(\ell)U_{\mathcal{P}_\ell}d\ell$$
$$+\frac{\eta\Pi_3\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\iota_{\mathcal{E}}}U_{\mathcal{K}}-\frac{\eta\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\iota_{\mathcal{E}}}\int_0^{f_3}\bar{\Pi}_3(\ell)U_{\mathcal{K}_\ell}d\ell$$
$$=\left(1-\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\right)\left(\tau-\iota_{\mathcal{J}}U_{\mathcal{J}}\right)+\left(\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{U_{\mathcal{E}}}-\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)\right)U_{\mathcal{E}}$$
$$+\left(\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{U_{\mathcal{P}}}-\frac{\alpha+\iota_{\mathcal{P}}}{\Pi_1}+\frac{\alpha\Pi_2\left(\eta\Pi_3\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\right)U_{\mathcal{P}}$$
$$+\left(\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{U_{\mathcal{K}}}-\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)\right)U_{\mathcal{K}}-\frac{\lambda\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\delta\iota_{\mathcal{E}}}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)U_{\mathcal{L}}^2. \tag{3.3}$$

Using inequality (2.16), we obtain

$$\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{U_{\mathcal{E}}}-\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)\le\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)-\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)=0.$$

Further, Hypotheses **H4** and **H5** in case of $U_{\mathcal{J}}^0=\dfrac{\tau}{\iota_{\mathcal{J}}}$ imply that

$$\begin{aligned}
&\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{U_{\mathcal{P}}}-\frac{\alpha+\iota_{\mathcal{P}}}{\Pi_1}+\frac{\alpha\Pi_2\left(\eta\Pi_3\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\\
&\le\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)\wp_{\mathcal{P}}\left(U_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}-\frac{\alpha+\iota_{\mathcal{P}}}{\Pi_1}+\frac{\alpha\Pi_2\left(\eta\Pi_3\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\\
&\le\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)\wp_{\mathcal{P}}\left(U_{\mathcal{J}}^0\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)}-\frac{\alpha+\iota_{\mathcal{P}}}{\Pi_1}+\frac{\alpha\Pi_2\left(\eta\Pi_3\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\\
&=\wp_{\mathcal{P}}\left(U_{\mathcal{J}}^0\right)-\frac{\alpha+\iota_{\mathcal{P}}}{\Pi_1}+\frac{\alpha\Pi_2\left(\eta\Pi_3\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}\\
&=\frac{\alpha\Pi_2\left(\eta\Pi_3\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^0\right)+\iota_{\mathcal{E}}\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^0\right)\right)+\iota_{\mathcal{K}}\iota_{\mathcal{E}}\wp_{\mathcal{P}}\left(U_{\mathcal{J}}^0\right)}{\iota_{\mathcal{K}}\iota_{\mathcal{E}}}-\frac{\alpha+\iota_{\mathcal{P}}}{\Pi_1}\\
&=\frac{\alpha+\iota_{\mathcal{P}}}{\Pi_1}\left(\tilde{\mathfrak{R}}_0-1\right).
\end{aligned}$$

Furthermore

$$\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^{0}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{U_{\mathcal{K}}}-\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^{0}\right)\leq\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^{0}\right)\wp_{\mathcal{K}}\left(U_{\mathcal{J}}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}-\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^{0}\right)\leq\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^{0}\right)\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^{0}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^{0}\right)}-\wp_{\mathcal{K}}\left(U_{\mathcal{J}}^{0}\right)=0.$$

Upon conducting a straightforward computation and incorporating the value $U_{\mathcal{J}}^{0}=\tau/\iota_{\mathcal{J}}$, Eq. (3.3) will manifest in the subsequent manner:

$$\frac{d\Xi_2}{dt}\leq\tau\left(1-\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^{0}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{U_{\mathcal{J}}^{0}}\right)+\frac{\alpha+\iota_{\mathcal{P}}}{\Pi_1}\left(\tilde{\mathfrak{K}}_0-1\right)U_{\mathcal{P}}-\frac{\lambda\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^{0}\right)}{\delta\iota_{\mathcal{E}}}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)U_{\mathcal{L}}^{2}.$$

From Hypothesis **H3**, we have

$$\left(1-\frac{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}^{0}\right)}{\wp_{\mathcal{E}}\left(U_{\mathcal{J}}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{U_{\mathcal{J}}^{0}}\right)\leq 0.$$

Evidently, for $\tilde{\mathfrak{K}}_0\leq 1$, the rate $\frac{d\Xi_2}{dt}$ remains non-positive. Furthermore, $\frac{d\Xi_2}{dt}$ equals zero at $U_{\mathcal{J}}=U_{\mathcal{J}}^{0}$ with $U_{\mathcal{P}}=U_{\mathcal{E}}=U_{\mathcal{L}}=0$. It follows that all solutions convey to M_2'. Within M_2', every element satisfy $U_{\mathcal{J}}(t)=U_{\mathcal{J}}^{0}$ and $U_{\mathcal{P}}(t)=U_{\mathcal{E}}(t)=U_{\mathcal{L}}(t)=0$ for all t. Following this, the first equation of model (3.1) along with Hypothesis **H1** results in:

$$0=\dot{U}_{\mathcal{J}}=\tau-\iota_{\mathcal{J}}U_{\mathcal{J}}^{0}-\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{0},U_{\mathcal{K}}(t)\right)\Longrightarrow\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{0},U_{\mathcal{K}}(t)\right)=0\Longrightarrow U_{\mathcal{K}}(t)=0,\ \text{for all } t.$$

Then, $M_2'=\left\{O^0\right\}$. Consequently, according to the L.I.P, the inference can be drawn that O^0 exhibits global asymptotic stability whenever $\tilde{\mathfrak{K}}_0\leq 1$ [53], [54]. □

Theorem 3.2. *Given $\tilde{\mathfrak{K}}_0>1$ the fulfillment of Hypotheses **H1**-**H4** and **H6**, it can be concluded that the equilibrium O^1 for system (3.1) is guaranteed to be G.A.S in $\hat{\Delta}\backslash\Upsilon$.*

Proof. Building up $\Xi_3(U_{\mathcal{J}},U_{\mathcal{P}},U_{\mathcal{K}},U_{\mathcal{E}},U_{\mathcal{L}})$ according to Eq. (3.4) as follows:

$$\begin{aligned}\Xi_3=&\,U_{\mathcal{J}}-U_{\mathcal{J}}^{1}-\int_{U_{\mathcal{J}}^{1}}^{U_{\mathcal{J}}}\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(\gamma,U_{\mathcal{E}}^{1}\right)}d\gamma+\frac{U_{\mathcal{P}}^{1}}{\Pi_1}F\left(\frac{U_{\mathcal{P}}}{U_{\mathcal{P}}^{1}}\right)\\&+\frac{\eta\Pi_3U_{\mathcal{K}}^{1}\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)+U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}{\iota_{\mathcal{K}}U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)}F\left(\frac{U_{\mathcal{K}}}{U_{\mathcal{K}}^{1}}\right)\\&+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}}F\left(\frac{U_{\mathcal{E}}}{U_{\mathcal{E}}^{1}}\right)+\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{2U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)\left(\delta-\theta U_{\mathcal{L}}^{1}\right)}\left(U_{\mathcal{L}}-U_{\mathcal{L}}^{1}\right)^{2}\\&+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\int_{t-\ell}^{t}F\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}(\gamma),U_{\mathcal{E}}(\gamma)\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}\right)d\gamma d\ell\\&+\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\int_{t-\ell}^{t}F\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}(\gamma),U_{\mathcal{P}}(\gamma)\right)}{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)}\right)d\gamma d\ell\end{aligned}$$

$$+\frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\int_{t-\ell}^{t}F\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}(\gamma),U_{\mathcal{K}}(\gamma)\right)}{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}\right)d\gamma d\ell$$
$$+\frac{\alpha U^1_{\mathcal{P}}\left(\eta\Pi_3U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)}{\iota_{\mathcal{K}}U^1_{\mathcal{K}}U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\int_0^{f_2}\bar{\Pi}_2(\ell)\int_{t-\ell}^{t}F\left(\frac{U_{\mathcal{P}}(\gamma)}{U^1_{\mathcal{P}}}\right)d\gamma d\ell$$
$$+\frac{\eta U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\int_0^{f_3}\bar{\Pi}_3(\ell)\int_{t-\ell}^{t}F\left(\frac{U_{\mathcal{K}}(\gamma)}{U^1_{\mathcal{K}}}\right)d\gamma d\ell. \tag{3.4}$$

We can deduce from the equilibrium condition derived from the last equation of system (3.1) that $\delta-\theta U^1_{\mathcal{L}}=\frac{\iota_{\mathcal{L}}U^1_{\mathcal{L}}}{U^1_{\mathcal{E}}}>0$. The positivity definiteness of Ξ_3 becomes evident. When calculating $\frac{d\Xi_3}{dt}$ along the trajectories of the model described in (3.1), we obtain

$$\frac{d\Xi_3}{dt}=\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\dot{U}_{\mathcal{J}}+\frac{1}{\Pi_1}\left(1-\frac{U^1_{\mathcal{P}}}{U_{\mathcal{P}}}\right)\dot{U}_{\mathcal{P}}$$
$$+\frac{\eta\Pi_3U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\iota_{\mathcal{K}}U^1_{\mathcal{K}}U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\left(1-\frac{U^1_{\mathcal{K}}}{U_{\mathcal{K}}}\right)\dot{U}_{\mathcal{K}}$$
$$+\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\left(1-\frac{U^1_{\mathcal{E}}}{U_{\mathcal{E}}}\right)\dot{U}_{\mathcal{E}}+\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\left(\delta-\theta U^1_{\mathcal{L}}\right)}\left(U_{\mathcal{L}}-U^1_{\mathcal{L}}\right)\dot{U}_{\mathcal{L}}$$
$$+\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}\right)\right)d\ell$$
$$+\frac{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}-\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)}{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}+\ln\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)}{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}\right)\right)d\ell$$
$$+\frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}-\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)}{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}+\ln\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)}{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}\right)\right)d\ell$$
$$+\frac{\alpha U^1_{\mathcal{P}}\left(\eta\Pi_3U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)}{\iota_{\mathcal{K}}U^1_{\mathcal{K}}U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\int_0^{f_2}\bar{\Pi}_2(\ell)\left(\frac{U_{\mathcal{P}}}{U^1_{\mathcal{P}}}-\frac{U_{\mathcal{P}_\ell}}{U^1_{\mathcal{P}}}\right.$$
$$\left.+\ln\left(\frac{U_{\mathcal{P}_\ell}}{U_{\mathcal{P}}}\right)\right)d\ell+\frac{\eta U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\int_0^{f_3}\bar{\Pi}_3(\ell)\left(\frac{U_{\mathcal{K}}}{U^1_{\mathcal{K}}}-\frac{U_{\mathcal{K}_\ell}}{U^1_{\mathcal{K}}}+\ln\left(\frac{U_{\mathcal{K}_\ell}}{U_{\mathcal{K}}}\right)\right)d\ell$$
$$=\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\left(\tau-\iota_{\mathcal{J}}U_{\mathcal{J}}-\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)-\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)-\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\right)$$
$$+\frac{1}{\Pi_1}\left(1-\frac{U^1_{\mathcal{P}}}{U_{\mathcal{P}}}\right)\left(\int_0^{f_1}\bar{\Pi}_1(\ell)\left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)+\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)\right.\right.$$
$$\left.\left.+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)\right)d\ell-\left(\alpha+\iota_{\mathcal{P}}\right)U_{\mathcal{P}}\right)$$
$$+\frac{\eta\Pi_3U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\iota_{\mathcal{K}}U^1_{\mathcal{K}}U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\left(1-\frac{U^1_{\mathcal{K}}}{U_{\mathcal{K}}}\right)\left(\alpha\int_0^{f_2}\bar{\Pi}_2(\ell)U_{\mathcal{P}_\ell}d\ell-\iota_{\mathcal{K}}U_{\mathcal{K}}\right)$$

$$
\begin{aligned}
&+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)}\left(1-\frac{U_{\mathcal{E}}^{1}}{U_{\mathcal{E}}}\right)\left(\eta\int_{0}^{f_3}\bar{\Pi}_3\left(\ell\right)U_{\mathcal{K}_\ell}d\ell-\iota_{\mathcal{E}}U_{\mathcal{E}}-\lambda U_{\mathcal{L}}U_{\mathcal{E}}\right)\\
&+\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)\left(\delta-\theta U_{\mathcal{L}}^{1}\right)}\left(U_{\mathcal{L}}-U_{\mathcal{L}}^{1}\right)\left(\delta U_{\mathcal{E}}-\iota_{\mathcal{L}}U_{\mathcal{L}}-\theta U_{\mathcal{L}}U_{\mathcal{E}}\right)\\
&+\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)-\frac{1}{\Pi_1}\int_{0}^{f_1}\bar{\Pi}_1\left(\ell\right)\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)d\ell\\
&+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\Pi_1}\int_{0}^{f_1}\bar{\Pi}_1\left(\ell\right)\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}\right)d\ell\\
&+\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)-\frac{1}{\Pi_1}\int_{0}^{f_1}\bar{\Pi}_1\left(\ell\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)d\ell\\
&+\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)}{\Pi_1}\int_{0}^{f_1}\bar{\Pi}_1\left(\ell\right)\ln\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)}{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}\right)d\ell\\
&+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)-\frac{1}{\Pi_1}\int_{0}^{f_1}\bar{\Pi}_1\left(\ell\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)d\ell\\
&+\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}{\Pi_1}\int_{0}^{f_1}\bar{\Pi}_1\left(\ell\right)\ln\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)}{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}\right)d\ell\\
&+\frac{\alpha\Pi_2U_{\mathcal{P}}^{1}\left(\eta\Pi_3U_{\mathcal{K}}^{1}\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)+U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)\right)}{\iota_{\mathcal{K}}U_{\mathcal{K}}^{1}U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)}\frac{U_{\mathcal{P}}}{U_{\mathcal{P}}^{1}}\\
&-\frac{\alpha\left(\eta\Pi_3U_{\mathcal{K}}^{1}\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)+U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)\right)}{\iota_{\mathcal{K}}U_{\mathcal{K}}^{1}U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)}\int_{0}^{f_2}\bar{\Pi}_2\left(\ell\right)U_{\mathcal{P}_\ell}d\ell\\
&+\frac{\alpha U_{\mathcal{P}}^{1}\left(\eta\Pi_3U_{\mathcal{K}}^{1}\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)+U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)\right)}{\iota_{\mathcal{K}}U_{\mathcal{K}}^{1}U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)}\int_{0}^{f_2}\bar{\Pi}_2\left(\ell\right)\ln\left(\frac{U_{\mathcal{P}_\ell}}{U_{\mathcal{P}}}\right)d\ell\\
&+\frac{\eta\Pi_3U_{\mathcal{K}}^{1}\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)}\frac{U_{\mathcal{K}}}{U_{\mathcal{K}}^{1}}-\frac{\eta\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)}\int_{0}^{f_3}\bar{\Pi}_3\left(\ell\right)U_{\mathcal{K}_\ell}d\ell\\
&+\frac{\eta U_{\mathcal{K}}^{1}\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)}\int_{0}^{f_3}\bar{\Pi}_3\left(\ell\right)\ln\left(\frac{U_{\mathcal{K}_\ell}}{U_{\mathcal{K}}}\right)d\ell.
\end{aligned}
$$

Thus

$$
\begin{aligned}
\frac{d\Xi_3}{dt}=&\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)}\right)\left(\tau-\iota_{\mathcal{J}}U_{\mathcal{J}}\right)+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)}\left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)\right)\\
&-\frac{1}{\Pi_1}\int_{0}^{f_1}\bar{\Pi}_1\left(\ell\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)U_{\mathcal{P}}^{1}}{U_{\mathcal{P}}}d\ell-\frac{1}{\Pi_1}\int_{0}^{f_1}\bar{\Pi}_1\left(\ell\right)\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)U_{\mathcal{P}}^{1}}{U_{\mathcal{P}}}d\ell\\
&-\frac{1}{\Pi_1}\int_{0}^{f_1}\bar{\Pi}_1\left(\ell\right)\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)U_{\mathcal{P}}^{1}}{U_{\mathcal{P}}}d\ell-\frac{\alpha+\iota_{\mathcal{P}}}{\Pi_1}U_{\mathcal{P}}+\frac{\alpha+\iota_{\mathcal{P}}}{\Pi_1}U_{\mathcal{P}}^{1}
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\alpha\left(\eta\Pi_3 U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)}{\iota_{\mathcal{K}}U^1_{\mathcal{K}}U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\int_0^{f_2}\bar{\Pi}_2(\ell)\frac{U_{\mathcal{P}_\ell}U^1_{\mathcal{K}}}{U_{\mathcal{K}}}d\ell\\
&-\frac{\eta\Pi_3 U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{U^1_{\mathcal{K}}U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\left(U_{\mathcal{K}}-U^1_{\mathcal{K}}\right)\\
&-\frac{\eta\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\int_0^{f_3}\bar{\Pi}_3(\ell)\frac{U_{\mathcal{K}_\ell}U^1_{\mathcal{E}}}{U_{\mathcal{E}}}d\ell-\frac{\iota_{\mathcal{E}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\left(U_{\mathcal{E}}-U^1_{\mathcal{E}}\right)\\
&-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}U_{\mathcal{L}}\left(U_{\mathcal{E}}-U^1_{\mathcal{E}}\right)+\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\left(\delta-\theta U^1_{\mathcal{L}}\right)}\left(U_{\mathcal{L}}-U^1_{\mathcal{L}}\right)\left(\delta U_{\mathcal{E}}-\iota_{\mathcal{L}}U_{\mathcal{L}}-\theta U_{\mathcal{L}}U_{\mathcal{E}}\right)\\
&+\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}\right)d\ell\\
&+\frac{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\ln\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)}{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}\right)d\ell\\
&+\frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\ln\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)}{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}\right)d\ell\\
&+\frac{\alpha\Pi_2 U^1_{\mathcal{P}}\left(\eta\Pi_3 U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)}{\iota_{\mathcal{K}}U^1_{\mathcal{K}}U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\frac{U_{\mathcal{P}}}{U^1_{\mathcal{P}}}\\
&+\frac{\alpha U^1_{\mathcal{P}}\left(\eta\Pi_3 U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)}{\iota_{\mathcal{K}}U^1_{\mathcal{K}}U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\int_0^{f_2}\bar{\Pi}_2(\ell)\ln\left(\frac{U_{\mathcal{P}_\ell}}{U_{\mathcal{P}}}\right)d\ell\\
&+\frac{\eta\Pi_3 U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\frac{U_{\mathcal{K}}}{U^1_{\mathcal{K}}}+\frac{\eta U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\int_0^{f_3}\bar{\Pi}_3(\ell)\ln\left(\frac{U_{\mathcal{K}_\ell}}{U_{\mathcal{K}}}\right)d\ell.
\end{aligned}
\tag{3.5}
$$

Implementing the specified conditions for equilibrium O^1

$$
\begin{aligned}
&\tau=\iota_{\mathcal{J}}U^1_{\mathcal{J}}+\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right),\\
&\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)=\frac{\left(\alpha+\iota_{\mathcal{P}}\right)U^1_{\mathcal{P}}}{\Pi_1},\\
&U^1_{\mathcal{K}}=\frac{\alpha\Pi_2U^1_{\mathcal{P}}}{\iota_{\mathcal{K}}},\qquad U^1_{\mathcal{E}}=\frac{\eta\Pi_3U^1_{\mathcal{K}}}{\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}},\qquad \delta U^1_{\mathcal{E}}=\iota_{\mathcal{L}}U^1_{\mathcal{L}}+\theta U^1_{\mathcal{L}}U^1_{\mathcal{E}},
\end{aligned}
$$

we derive

$$
\begin{aligned}
\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)&=\frac{\eta\Pi_3 U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)},\\
&=\frac{\alpha\Pi_2 U^1_{\mathcal{P}}\left(\eta\Pi_3 U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)}{\iota_{\mathcal{K}}U^1_{\mathcal{K}}U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}.
\end{aligned}
$$

Therefore, Eq. (3.5) will be expressed as follows:

$$
\begin{aligned}
\frac{d\Xi_3}{dt} =& \left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\left(\iota_{\mathcal{J}}U^1_{\mathcal{J}}-\iota_{\mathcal{J}}U_{\mathcal{J}}\right)+\left(2-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\left(\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\right.\\
&\left.+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)+\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\\
&+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{P}}}d\ell\\
&-\frac{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)U_{\mathcal{P}}}d\ell\\
&-\frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{P}}}d\ell-\frac{\left(\alpha+\iota_{\mathcal{P}}\right)U^1_{\mathcal{P}}}{\Pi_1}\frac{U_{\mathcal{P}}}{U^1_{\mathcal{P}}}\\
&-\frac{\alpha U^1_{\mathcal{P}}\left(\eta\Pi_3U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)}{\iota_{\mathcal{K}}U^1_{\mathcal{K}}U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\int_0^{f_2}\bar{\Pi}_2(\ell)\frac{U_{\mathcal{P}_\ell}U^1_{\mathcal{K}}}{U^1_{\mathcal{P}}U_{\mathcal{K}}}d\ell\\
&-\frac{\eta\Pi_3U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\left(\frac{U_{\mathcal{K}}}{U^1_{\mathcal{K}}}-1\right)\\
&-\frac{\eta U^1_{\mathcal{K}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\int_0^{f_3}\bar{\Pi}_3(\ell)\frac{U_{\mathcal{K}_\ell}U^1_{\mathcal{E}}}{U^1_{\mathcal{K}}U_{\mathcal{E}}}d\ell-\frac{\iota_{\mathcal{E}}\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\left(U_{\mathcal{E}}-U^1_{\mathcal{E}}\right)\\
&-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}U_{\mathcal{L}}\left(U_{\mathcal{E}}-U^1_{\mathcal{E}}\right)+\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}U^1_{\mathcal{L}}\left(U_{\mathcal{E}}-U^1_{\mathcal{E}}\right)-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}U^1_{\mathcal{L}}\left(U_{\mathcal{E}}-U^1_{\mathcal{E}}\right)\\
&+\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\left(\delta-\theta U^1_{\mathcal{L}}\right)}\left(U_{\mathcal{L}}-U^1_{\mathcal{L}}\right)\left(\delta U_{\mathcal{E}}-\iota_{\mathcal{L}}U_{\mathcal{L}}-\theta U_{\mathcal{L}}U_{\mathcal{E}}-\delta U^1_{\mathcal{E}}+\iota_{\mathcal{L}}U^1_{\mathcal{L}}\right.\\
&\left.+\theta U^1_{\mathcal{L}}U^1_{\mathcal{E}}+\theta U^1_{\mathcal{L}}U_{\mathcal{E}}-\theta U^1_{\mathcal{L}}U_{\mathcal{E}}\right)+\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}\right)d\ell\\
&+\frac{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\ln\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)}{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}\right)d\ell\\
&+\frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\ln\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)}{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}\right)d\ell\\
&+\left(\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)\frac{U_{\mathcal{P}}}{U^1_{\mathcal{P}}}\\
&+\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_2}\int_0^{f_2}\bar{\Pi}_2(\ell)\ln\left(\frac{U_{\mathcal{P}_\ell}}{U_{\mathcal{P}}}\right)d\ell
\end{aligned}
$$

$$+\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\frac{U_{\mathcal{K}}}{U^1_{\mathcal{K}}}+\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_3}\int_0^{f_3}\bar{\Pi}_3(\ell)\ln\left(\frac{U_{\mathcal{K}_\ell}}{U_{\mathcal{K}}}\right)d\ell.$$

Consequently, the form of the expression becomes as follows:

$$\begin{aligned}
\frac{d\Xi_3}{dt}=&\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\left(\iota_{\mathcal{J}}U^1_{\mathcal{J}}-\iota_{\mathcal{J}}U_{\mathcal{J}}\right)+\left(2-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\Big(\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\\
&+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\Big)+\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\\
&+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{P}}}d\ell\\
&-\frac{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)U_{\mathcal{P}}}d\ell\\
&-\frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{P}}}d\ell\\
&-\left(\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)\frac{U_{\mathcal{P}}}{U^1_{\mathcal{P}}}\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_2}\int_0^{f_2}\bar{\Pi}_2(\ell)\frac{U_{\mathcal{P}_\ell}U^1_{\mathcal{K}}}{U^1_{\mathcal{P}}U_{\mathcal{K}}}d\ell\\
&-\left(\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)\left(\frac{U_{\mathcal{K}}}{U^1_{\mathcal{K}}}-1\right)\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_3}\int_0^{f_3}\bar{\Pi}_3(\ell)\frac{U_{\mathcal{K}_\ell}U^1_{\mathcal{E}}}{U^1_{\mathcal{K}}U_{\mathcal{E}}}d\ell-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\left(U_{\mathcal{E}}-U^1_{\mathcal{E}}\right)\\
&-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)}\left(U_{\mathcal{L}}-U^1_{\mathcal{L}}\right)\left(U_{\mathcal{E}}-U^1_{\mathcal{E}}\right)+\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\left(\delta-\theta U^1_{\mathcal{L}}\right)}\left(\delta-\theta U^1_{\mathcal{L}}\right)\left(U_{\mathcal{L}}-U^1_{\mathcal{L}}\right)\\
&\times\left(U_{\mathcal{E}}-U^1_{\mathcal{E}}\right)-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\left(\delta-\theta U^1_{\mathcal{L}}\right)}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)\left(U_{\mathcal{L}}-U^1_{\mathcal{L}}\right)^2\\
&+\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}\right)d\ell\\
&+\frac{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\ln\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)}{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}\right)d\ell
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}{\Pi_{1}}\int_{0}^{f_{1}}\bar{\Pi}_{1}\left(\ell\right)\ln\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_{\ell}},U_{\mathcal{K}_{\ell}}\right)}{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}\right)d\ell+\left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\right.\\
&\left.+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)\right)\frac{U_{\mathcal{P}}}{U_{\mathcal{P}}^{1}}+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}{\Pi_{2}}\int_{0}^{f_{2}}\bar{\Pi}_{2}\left(\ell\right)\ln\left(\frac{U_{\mathcal{P}_{\ell}}}{U_{\mathcal{P}}}\right)d\ell\\
&+\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\frac{U_{\mathcal{K}}}{U_{\mathcal{K}}^{1}}+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\Pi_{3}}\int_{0}^{f_{3}}\bar{\Pi}_{3}\left(\ell\right)\ln\left(\frac{U_{\mathcal{K}_{\ell}}}{U_{\mathcal{K}}}\right)d\ell\\
=&\,\iota_{\mathcal{J}}U_{\mathcal{J}}^{1}\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{U_{\mathcal{J}}^{1}}\right)+\left(2-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)}\right)\left(\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)+\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)\right.\\
&\left.+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)\right)+\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)}+\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)}\\
&+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\Pi_{1}}\int_{0}^{f_{1}}\bar{\Pi}_{1}\left(\ell\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_{\ell}},U_{\mathcal{E}_{\ell}}\right)U_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)U_{\mathcal{P}}}d\ell\\
&-\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)}{\Pi_{1}}\int_{0}^{f_{1}}\bar{\Pi}_{1}\left(\ell\right)\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_{\ell}},U_{\mathcal{P}_{\ell}}\right)U_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}d\ell\\
&-\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}{\Pi_{1}}\int_{0}^{f_{1}}\bar{\Pi}_{1}\left(\ell\right)\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_{\ell}},U_{\mathcal{K}_{\ell}}\right)U_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)U_{\mathcal{P}}}d\ell-\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)\frac{U_{\mathcal{P}}}{U_{\mathcal{P}}^{1}}\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}{\Pi_{2}}\int_{0}^{f_{2}}\bar{\Pi}_{2}\left(\ell\right)\frac{U_{\mathcal{P}_{\ell}}U_{\mathcal{K}}^{1}}{U_{\mathcal{P}}^{1}U_{\mathcal{K}}}d\ell-\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)\frac{U_{\mathcal{K}}}{U_{\mathcal{K}}^{1}}+2\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\\
&+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\Pi_{3}}\int_{0}^{f_{3}}\bar{\Pi}_{3}\left(\ell\right)\frac{U_{\mathcal{K}_{\ell}}U_{\mathcal{E}}^{1}}{U_{\mathcal{K}}^{1}U_{\mathcal{E}}}d\ell-\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\frac{U_{\mathcal{E}}}{U_{\mathcal{E}}^{1}}\\
&-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)\left(\delta-\theta U_{\mathcal{L}}^{1}\right)}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)\left(U_{\mathcal{L}}-U_{\mathcal{L}}^{1}\right)^{2}\\
&+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\Pi_{1}}\int_{0}^{f_{1}}\bar{\Pi}_{1}\left(\ell\right)\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_{\ell}},U_{\mathcal{E}_{\ell}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}\right)d\ell\\
&+\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)}{\Pi_{1}}\int_{0}^{f_{1}}\bar{\Pi}_{1}\left(\ell\right)\ln\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_{\ell}},U_{\mathcal{P}_{\ell}}\right)}{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}\right)d\ell\\
&+\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}{\Pi_{1}}\int_{0}^{f_{1}}\bar{\Pi}_{1}\left(\ell\right)\ln\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_{\ell}},U_{\mathcal{K}_{\ell}}\right)}{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}\right)d\ell\\
&+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}{\Pi_{2}}\int_{0}^{f_{2}}\bar{\Pi}_{2}\left(\ell\right)\ln\left(\frac{U_{\mathcal{P}_{\ell}}}{U_{\mathcal{P}}}\right)d\ell
\end{aligned}
$$

$$+\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_3}\int_0^{f_3}\bar{\Pi}_3\left(\ell\right)\ln\left(\frac{U_{\mathcal{K}_\ell}}{U_{\mathcal{K}}}\right)d\ell.$$

Additionally, we have the following equalities:

$$\begin{aligned}
\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}\right)&=\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{P}}}\right)+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\\
&\quad+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{E}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)U^1_{\mathcal{E}}}\right)+\ln\left(\frac{U_{\mathcal{P}}U^1_{\mathcal{K}}}{U^1_{\mathcal{P}}U_{\mathcal{K}}}\right)+\ln\left(\frac{U_{\mathcal{K}}U^1_{\mathcal{E}}}{U^1_{\mathcal{K}}U_{\mathcal{E}}}\right),\\
\ln\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)}{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}\right)&=\ln\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)U_{\mathcal{P}}}\right)+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\\
&\quad+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)U_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)U^1_{\mathcal{P}}}\right),\\
\ln\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)}{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}\right)&=\ln\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{P}}}\right)+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\\
&\quad+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{K}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)U^1_{\mathcal{K}}}\right)+\ln\left(\frac{U_{\mathcal{P}}U^1_{\mathcal{K}}}{U^1_{\mathcal{P}}U_{\mathcal{K}}}\right),\\
\ln\left(\frac{U_{\mathcal{P}_\ell}}{U_{\mathcal{P}}}\right)&=\ln\left(\frac{U_{\mathcal{P}_\ell}U^1_{\mathcal{K}}}{U^1_{\mathcal{P}}U_{\mathcal{K}}}\right)+\ln\left(\frac{U^1_{\mathcal{P}}U_{\mathcal{K}}}{U_{\mathcal{P}}U^1_{\mathcal{K}}}\right),\\
\ln\left(\frac{U_{\mathcal{K}_\ell}}{U_{\mathcal{K}}}\right)&=\ln\left(\frac{U_{\mathcal{K}_\ell}U^1_{\mathcal{E}}}{U^1_{\mathcal{K}}U_{\mathcal{E}}}\right)+\ln\left(\frac{U^1_{\mathcal{K}}U_{\mathcal{E}}}{U_{\mathcal{K}}U^1_{\mathcal{E}}}\right).
\end{aligned}$$

Therefore, $\frac{d\Xi_3}{dt}$ will be

$$\begin{aligned}
\frac{d\Xi_3}{dt}&=\iota_{\mathcal{J}}U^1_{\mathcal{J}}\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{U^1_{\mathcal{J}}}\right)+\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}-\frac{U_{\mathcal{E}}}{U^1_{\mathcal{E}}}\right)\\
&\quad+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}-\frac{U_{\mathcal{P}}}{U^1_{\mathcal{P}}}\right)\\
&\quad+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}-\frac{U_{\mathcal{K}}}{U^1_{\mathcal{K}}}\right)\\
&\quad+\left(\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)\left(2-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\\
&\quad-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1\left(\ell\right)\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{P}}}d\ell
\end{aligned}$$

$$
\begin{aligned}
&-\frac{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)U_{\mathcal{P}}}d\ell\\
&-\frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{P}}}d\ell\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_2}\int_0^{f_2}\bar{\Pi}_2(\ell)\frac{U_{\mathcal{P}_\ell}U^1_{\mathcal{K}}}{U^1_{\mathcal{P}}U_{\mathcal{K}}}d\ell+2\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_3}\int_0^{f_3}\bar{\Pi}_3(\ell)\frac{U_{\mathcal{K}_\ell}U^1_{\mathcal{E}}}{U^1_{\mathcal{K}}U_{\mathcal{E}}}d\ell-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\left(\delta-\theta U^1_{\mathcal{L}}\right)}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)\left(U_{\mathcal{L}}-U^1_{\mathcal{L}}\right)^2\\
&+\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left[\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{P}}}\right)+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\right.\\
&\left.+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{E}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)U^1_{\mathcal{E}}}\right)+\ln\left(\frac{U_{\mathcal{P}}U^1_{\mathcal{K}}}{U^1_{\mathcal{P}}U_{\mathcal{K}}}\right)+\ln\left(\frac{U_{\mathcal{K}}U^1_{\mathcal{E}}}{U^1_{\mathcal{K}}U_{\mathcal{E}}}\right)\right]d\ell\\
&+\frac{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left[\ln\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)U_{\mathcal{P}}}\right)\right.\\
&\left.+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)U_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)U^1_{\mathcal{P}}}\right)\right]d\ell\\
&+\frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left[\ln\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{P}}}\right)+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\right.\\
&\left.+\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{K}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)U^1_{\mathcal{K}}}\right)+\ln\left(\frac{U_{\mathcal{P}}U^1_{\mathcal{K}}}{U^1_{\mathcal{P}}U_{\mathcal{K}}}\right)\right]d\ell\\
&+\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_2}\int_0^{f_2}\bar{\Pi}_2(\ell)\left[\ln\left(\frac{U_{\mathcal{P}_\ell}U^1_{\mathcal{K}}}{U^1_{\mathcal{P}}U_{\mathcal{K}}}\right)+\ln\left(\frac{U^1_{\mathcal{P}}U_{\mathcal{K}}}{U_{\mathcal{P}}U^1_{\mathcal{K}}}\right)\right]d\ell\\
&+\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_3}\int_0^{f_3}\bar{\Pi}_3(\ell)\left[\ln\left(\frac{U_{\mathcal{K}_\ell}U^1_{\mathcal{E}}}{U^1_{\mathcal{K}}U_{\mathcal{E}}}\right)+\ln\left(\frac{U^1_{\mathcal{K}}U_{\mathcal{E}}}{U_{\mathcal{K}}U^1_{\mathcal{E}}}\right)\right]d\ell\\
=&\,\iota_{\mathcal{J}}U^1_{\mathcal{J}}\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{U^1_{\mathcal{J}}}\right)+\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}-\frac{U_{\mathcal{E}}}{U^1_{\mathcal{E}}}\right)\\
&+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}-\frac{U_{\mathcal{P}}}{U^1_{\mathcal{P}}}\right)\\
&+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}-\frac{U_{\mathcal{K}}}{U^1_{\mathcal{K}}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\left(\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}-1-\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\right)\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left[\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{P}}}-1-\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{P}}}\right)\right]d\ell\\
&-\frac{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left[\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)U_{\mathcal{P}}}-1-\ln\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)U_{\mathcal{P}}}\right)\right]d\ell\\
&-\frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left[\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{P}}}-1-\ln\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{P}}}\right)\right]d\ell\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_2}\int_0^{f_2}\bar{\Pi}_2(\ell)\left[\frac{U_{\mathcal{P}_\ell}U^1_{\mathcal{K}}}{U^1_{\mathcal{P}}U_{\mathcal{K}}}-1-\ln\left(\frac{U_{\mathcal{P}_\ell}U^1_{\mathcal{K}}}{U^1_{\mathcal{P}}U_{\mathcal{K}}}\right)\right]d\ell\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_3}\int_0^{f_3}\bar{\Pi}_3(\ell)\left[\frac{U_{\mathcal{K}_\ell}U^1_{\mathcal{E}}}{U^1_{\mathcal{K}}U_{\mathcal{E}}}-1-\ln\left(\frac{U_{\mathcal{K}_\ell}U^1_{\mathcal{E}}}{U^1_{\mathcal{K}}U_{\mathcal{E}}}\right)\right]d\ell\\
&+\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{E}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)U^1_{\mathcal{E}}}\right)+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)U_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)U^1_{\mathcal{P}}}\right)\\
&+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{K}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)U^1_{\mathcal{K}}}\right)\\
&-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\left(\delta-\theta U^1_{\mathcal{L}}\right)}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)\left(U_{\mathcal{L}}-U^1_{\mathcal{L}}\right)^2.
\end{aligned}
$$

By simplifying the previous result, we arrive at

$$
\begin{aligned}
\frac{d\Xi_3}{dt}=&\ \iota_{\mathcal{J}}U^1_{\mathcal{J}}\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{U^1_{\mathcal{J}}}\right)\\
&+\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}-\frac{U_{\mathcal{E}}}{U^1_{\mathcal{E}}}-1+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{E}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)U^1_{\mathcal{E}}}\right)\\
&+\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}-\frac{U_{\mathcal{P}}}{U^1_{\mathcal{P}}}-1+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)U_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)U^1_{\mathcal{P}}}\right)\\
&+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}-\frac{U_{\mathcal{K}}}{U^1_{\mathcal{K}}}-1+\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{K}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)U^1_{\mathcal{K}}}\right)\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left[F\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{P}}}\right)+F\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\right]d\ell
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left[F\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)U_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}\right)+F\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)}\right)\right]d\ell\\
&-\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left[F\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)U_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)U_{\mathcal{P}}}\right)+F\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)}\right)\right]d\ell\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}{\Pi_2}\int_0^{f_2}\bar{\Pi}_2(\ell)F\left(\frac{U_{\mathcal{P}_\ell}U_{\mathcal{K}}^{1}}{U_{\mathcal{P}}^{1}U_{\mathcal{K}}}\right)d\ell\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\Pi_3}\int_0^{f_3}\bar{\Pi}_3(\ell)F\left(\frac{U_{\mathcal{K}_\ell}U_{\mathcal{E}}^{1}}{U_{\mathcal{K}}^{1}U_{\mathcal{E}}}\right)d\ell\\
&-\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\left[\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)U_{\mathcal{E}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)U_{\mathcal{E}}^{1}}-1-\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)U_{\mathcal{E}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)U_{\mathcal{E}}^{1}}\right)\right]\\
&-\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)\left[\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)U_{\mathcal{P}}^{1}}-1-\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)U_{\mathcal{P}}^{1}}\right)\right]\\
&-\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)\left[\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)U_{\mathcal{K}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)U_{\mathcal{K}}^{1}}-1-\ln\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)U_{\mathcal{K}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)U_{\mathcal{K}}^{1}}\right)\right]\\
&-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{U_{\mathcal{E}}^{1}\left(\iota_{\mathcal{E}}+\lambda U_{\mathcal{L}}^{1}\right)\left(\delta-\theta U_{\mathcal{L}}^{1}\right)}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)\left(U_{\mathcal{L}}-U_{\mathcal{L}}^{1}\right)^2\\
=\;&\iota_{\mathcal{J}}U_{\mathcal{J}}^{1}\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{U_{\mathcal{J}}^{1}}\right)\\
&+\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)}-\frac{U_{\mathcal{E}}}{U_{\mathcal{E}}^{1}}\right)\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}\right)\\
&+\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)}-\frac{U_{\mathcal{P}}}{U_{\mathcal{P}}^{1}}\right)\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}\right)\\
&+\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}-\frac{U_{\mathcal{K}}}{U_{\mathcal{K}}^{1}}\right)\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{K}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}\right)\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left[F\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{E}_\ell}\right)U_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)U_{\mathcal{P}}}\right)+F\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)}\right)\right.\\
&\left.+F\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)U_{\mathcal{E}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)U_{\mathcal{E}}^{1}}\right)\right]d\ell-\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left[F\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{P}_\ell}\right)U_{\mathcal{P}}^{1}}{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}\right)\right.\\
&\left.+F\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)}\right)+F\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{P}}^{1}\right)U_{\mathcal{P}}}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}^{1},U_{\mathcal{E}}^{1}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)U_{\mathcal{P}}^{1}}\right)\right]d\ell
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_1}\int_0^{f_1}\bar{\Pi}_1(\ell)\left[F\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}_\ell},U_{\mathcal{K}_\ell}\right)U^1_{\mathcal{P}}}{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{P}}}\right)\right.\\
&\left.+F\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)+F\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)U_{\mathcal{K}}}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)U^1_{\mathcal{K}}}\right)\right]d\ell\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)+\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\Pi_2}\int_0^{f_2}\bar{\Pi}_2(\ell)F\left(\frac{U_{\mathcal{P}_\ell}U^1_{\mathcal{K}}}{U^1_{\mathcal{P}}U_{\mathcal{K}}}\right)d\ell\\
&-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\Pi_3}\int_0^{f_3}\bar{\Pi}_3(\ell)F\left(\frac{U_{\mathcal{K}_\ell}U^1_{\mathcal{E}}}{U^1_{\mathcal{K}}U_{\mathcal{E}}}\right)d\ell\\
&-\frac{\lambda\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{U^1_{\mathcal{E}}\left(\iota_{\mathcal{E}}+\lambda U^1_{\mathcal{L}}\right)\left(\delta-\theta U^1_{\mathcal{L}}\right)}\left(\iota_{\mathcal{L}}+\theta U_{\mathcal{E}}\right)\left(U_{\mathcal{L}}-U^1_{\mathcal{L}}\right)^2.
\end{aligned}
$$

From Hypothesis **H2**, we have

$$
\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\left(1-\frac{U_{\mathcal{J}}}{U^1_{\mathcal{J}}}\right)\leq 0.
$$

In addition, from Eqs. (2.17)-(2.19), we conclude

$$
\begin{aligned}
&\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}-\frac{U_{\mathcal{E}}}{U^1_{\mathcal{E}}}\right)\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U_{\mathcal{E}}\right)}\right)\leq 0,\\
&\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}-\frac{U_{\mathcal{P}}}{U^1_{\mathcal{P}}}\right)\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}\right)\leq 0,\\
&\left(\frac{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}-\frac{U_{\mathcal{K}}}{U^1_{\mathcal{K}}}\right)\left(1-\frac{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}\right)\leq 0.
\end{aligned}
$$

Consequently, when $\tilde{\mathfrak{R}}_0 > 1$, it is established that $\frac{d\Xi_3}{dt} \leq 0$ holds true for all positive values of $U_{\mathcal{J}}, U_{\mathcal{P}}, U_{\mathcal{K}}, U_{\mathcal{E}}$ and $U_{\mathcal{L}}$. Additionally, $\frac{d\Xi_3}{dt} = 0$ is realized when $U_{\mathcal{J}} = U^1_{\mathcal{J}}, U_{\mathcal{P}} = U^1_{\mathcal{P}}, U_{\mathcal{K}} = U^1_{\mathcal{K}}, U_{\mathcal{E}} = U^1_{\mathcal{E}}$ and $U_{\mathcal{L}} = U^1_{\mathcal{L}}$. As a result, with $M'_3 = \left\{O^1\right\}$, we can affirm that the equilibrium O^1 is G.A.S under the condition $\tilde{\mathfrak{R}}_0 > 1$ in accordance of L.I.P [53], [54]. □

4. Illustrative examples and numerical simulations

Dive into the real world! This section brings our theoretical predictions to life through numerical simulations. We use MATLAB solvers' ode45 and dde23 to explore the dynamics of systems (2.1) and (3.1). We utilize Crowley-Martin functions for the incidence rates in system (2.1), while choosing saturation incidence for system (3.1). Additionally, within the framework of model (2.1), we investigate the dynamic impact of impaired antibody immunity and the effect of the Crowley-Martin functions parameters on the dynamics. Furthermore, our analysis extends to exploring the influences of time delays and the saturation effect on the dynamics of system (3.1). We utilize sensitivity analysis to reveal how basic reproduction ratios respond to variations in parameter values.

4.1. **Numerical simulation for model (2.1).** The Crowley-Martin functional response was pioneered by P. H. Crowley and E. K. Martin [58]. The Crowley-Martin functional response is considered predator-dependent, whereas predator interference involves both prey and predator populations. This type of response is influenced by the predator presence. It is assumed in Crowley-Martin that the predator-feeding rate will decline with higher predator density, even when prey density is high. The influence of predator interference on feeding rate remains significant regardless of whether an individual predator is handling or searching for prey at any given moment [59–62]. The Crowley-Martin functional response is distinguished by its cabability to explain the saturation effects inherent in viral replication, which are influenced by the availability of target cells. From a biological point of view, it is well known that viral replication reaches an upper limit due to the finite availability of target cells [63]. The subsequent Crowley-Martin incidence rates will be selected for system (2.1):

$$\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}},U_{\mathcal{E}})=\frac{\mu_{\mathcal{E}}U_{\mathcal{J}}U_{\mathcal{E}}}{(1+\nu U_{\mathcal{J}})(1+\varphi_{\mathcal{E}}U_{\mathcal{E}})},$$
$$\mathcal{F}_{\mathcal{P}}(U_{\mathcal{J}},U_{\mathcal{P}})=\frac{\mu_{\mathcal{P}}U_{\mathcal{J}}U_{\mathcal{P}}}{(1+\nu U_{\mathcal{J}})(1+\varphi_{\mathcal{P}}U_{\mathcal{P}})},$$
$$\mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}},U_{\mathcal{K}})=\frac{\mu_{\mathcal{K}}U_{\mathcal{J}}U_{\mathcal{K}}}{(1+\nu U_{\mathcal{J}})(1+\varphi_{\mathcal{K}}U_{\mathcal{K}})},$$

where $\mu_{\mathcal{X}}>0,\mathcal{X}\in\{\mathcal{E},\mathcal{P},\mathcal{K}\}$, account for the infection rate constants. Parameters $\nu>0$ and $\varphi_{\mathcal{X}}>0$, $\mathcal{X}\in\{\mathcal{E},\mathcal{P},\mathcal{K}\}$, represent the Crowley-Martin infection rate constants. Concerning the feeding rate, ν represents the handling time, and $\varphi_{\mathcal{X}}$ indicates the level of interference among predators (HIV-1 particles and infected cells). This functional response is useful for modeling infections like HIV-1, as the rate of infection experiences a saturation effect because of the intracellular replication [63]. Hence, the structure of system (2.1) will be as follows:

$$\begin{cases}\dot{U}_{\mathcal{J}} &= \tau-\iota_{\mathcal{J}}U_{\mathcal{J}}-\dfrac{\mu_{\mathcal{E}}U_{\mathcal{J}}U_{\mathcal{E}}}{(1+\nu U_{\mathcal{J}})(1+\varphi_{\mathcal{E}}U_{\mathcal{E}})}-\dfrac{\mu_{\mathcal{P}}U_{\mathcal{J}}U_{\mathcal{P}}}{(1+\nu U_{\mathcal{J}})(1+\varphi_{\mathcal{P}}U_{\mathcal{P}})}\\ &\quad-\dfrac{\mu_{\mathcal{K}}U_{\mathcal{J}}U_{\mathcal{K}}}{(1+\nu U_{\mathcal{J}})(1+\varphi_{\mathcal{K}}U_{\mathcal{K}})},\\ \dot{U}_{\mathcal{P}} &= \dfrac{\mu_{\mathcal{E}}U_{\mathcal{J}}U_{\mathcal{E}}}{(1+\nu U_{\mathcal{J}})(1+\varphi_{\mathcal{E}}U_{\mathcal{E}})}+\dfrac{\mu_{\mathcal{P}}U_{\mathcal{J}}U_{\mathcal{P}}}{(1+\nu U_{\mathcal{J}})(1+\varphi_{\mathcal{P}}U_{\mathcal{P}})}\\ &\quad+\dfrac{\mu_{\mathcal{K}}U_{\mathcal{J}}U_{\mathcal{K}}}{(1+\nu U_{\mathcal{J}})(1+\varphi_{\mathcal{K}}U_{\mathcal{K}})}-(\alpha+\iota_{\mathcal{P}})U_{\mathcal{P}},\\ \dot{U}_{\mathcal{K}} &= \alpha U_{\mathcal{P}}-\iota_{\mathcal{K}}U_{\mathcal{K}},\\ \dot{U}_{\mathcal{E}} &= \eta U_{\mathcal{K}}-\iota_{\mathcal{E}}U_{\mathcal{E}}-\lambda U_{\mathcal{L}}U_{\mathcal{E}},\\ \dot{U}_{\mathcal{L}} &= \delta U_{\mathcal{E}}-\iota_{\mathcal{L}}U_{\mathcal{L}}-\theta U_{\mathcal{L}}U_{\mathcal{E}}.\end{cases} \tag{4.1}$$

Clearly, $\mathcal{F}_{\mathcal{X}}(U_{\mathcal{J}},U_{\mathcal{X}}),\mathcal{X}\in\{\mathcal{E},\mathcal{P},\mathcal{K}\}$ are continuously differentiable functions. In addition, $\mathcal{F}_{\mathcal{X}}(U_{\mathcal{J}},U_{\mathcal{X}})$ for $\mathcal{X}\in\{\mathcal{E},\mathcal{P},\mathcal{K}\}$ satisfying the following: $\mathcal{F}_{\mathcal{X}}(U_{\mathcal{J}},U_{\mathcal{X}})>0$ and $\mathcal{F}_{\mathcal{X}}(0,U_{\mathcal{X}})=\mathcal{F}_{\mathcal{X}}(U_{\mathcal{J}},0)=0$ for all $U_{\mathcal{J}} > 0$ and $U_{\mathcal{X}} > 0$. Hence, Hypothesis **H1** is fulfilled. Moreover, we have

$$\frac{\partial \mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}}, U_{\mathcal{X}}\right)}{\partial U_{\mathcal{J}}} = \frac{\mu_{\mathcal{X}} U_{\mathcal{X}}}{\left(1+\nu U_{\mathcal{J}}\right)^2\left(1+\varphi_{\mathcal{X}} U_{\mathcal{X}}\right)} > 0, \quad \text{for all } U_{\mathcal{J}}, U_{\mathcal{X}} > 0,$$
$$\frac{\partial \mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}}, U_{\mathcal{X}}\right)}{\partial U_{\mathcal{X}}} = \frac{\mu_{\mathcal{X}} U_{\mathcal{J}}}{\left(1+\nu U_{\mathcal{J}}\right)\left(1+\varphi_{\mathcal{X}} U_{\mathcal{X}}\right)^2} > 0, \quad \text{for all } U_{\mathcal{J}}, U_{\mathcal{X}} > 0,$$

where $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$. This means that Hypothesis **H2** is valid. It is obvious that

$$\wp_{\mathcal{X}}\left(U_{\mathcal{J}}\right) = \frac{\partial \mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{X}}} = \frac{\mu_{\mathcal{X}} U_{\mathcal{J}}}{1+\nu U_{\mathcal{J}}} > 0, \quad \text{for all } U_{\mathcal{J}} > 0,\ \mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}.$$

Furthermore

$$\wp'_{\mathcal{X}}\left(U_{\mathcal{J}}\right) = \frac{d}{dU_{\mathcal{J}}}\left(\frac{\partial \mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{X}}}\right) = \frac{\mu_{\mathcal{X}}}{\left(1+\nu U_{\mathcal{J}}\right)^2} > 0, \quad \text{for all } U_{\mathcal{J}} > 0,\ \mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\},$$

which confirms that Hypothesis **H3** is also met. Moreover, we have

$$\frac{\partial}{\partial U_{\mathcal{X}}}\left(\frac{\mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}}, U_{\mathcal{X}}\right)}{U_{\mathcal{X}}}\right) = -\frac{\mu_{\mathcal{X}} \varphi_{\mathcal{X}} U_{\mathcal{J}}}{\left(1+\nu U_{\mathcal{J}}\right)\left(1+\varphi_{\mathcal{X}} U_{\mathcal{X}}\right)^2} < 0, \quad \text{for all } U_{\mathcal{J}}, U_{\mathcal{X}} > 0,\ \mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}.$$

Then, Hypothesis **H4** is verified. It is observable that $\frac{\wp_{\mathcal{X}}(U_{\mathcal{J}})}{\wp_{\mathcal{E}}(U_{\mathcal{J}})} = \frac{\mu_{\mathcal{X}}}{\mu_{\mathcal{E}}}, \mathcal{X} \in \{\mathcal{P}, \mathcal{K}\}$. Hence, Hypothesis **H5** is satisfied. Furthermore, we have

$$\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}_{\mathcal{E}}^1\right)} = \frac{\mu_{\mathcal{P}} U_{\mathcal{P}}\left(1+\varphi_{\mathcal{E}} \bar{U}_{\mathcal{E}}^1\right)}{\mu_{\mathcal{E}} \bar{U}_{\mathcal{E}}^1\left(1+\varphi_{\mathcal{P}} U_{\mathcal{P}}\right)}, \qquad \frac{\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{P}}^1\right)}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{E}}^1\right)} = \frac{\mu_{\mathcal{P}} \bar{U}_{\mathcal{P}}^1\left(1+\varphi_{\mathcal{E}} \bar{U}_{\mathcal{E}}^1\right)}{\mu_{\mathcal{E}} \bar{U}_{\mathcal{E}}^1\left(1+\varphi_{\mathcal{P}} \bar{U}_{\mathcal{P}}^1\right)},$$
$$\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}_{\mathcal{E}}^1\right)} = \frac{\mu_{\mathcal{K}} U_{\mathcal{K}}\left(1+\varphi_{\mathcal{E}} \bar{U}_{\mathcal{E}}^1\right)}{\mu_{\mathcal{E}} \bar{U}_{\mathcal{E}}^1\left(1+\varphi_{\mathcal{K}} U_{\mathcal{K}}\right)}, \qquad \frac{\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{K}}^1\right)}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{E}}^1\right)} = \frac{\mu_{\mathcal{K}} \bar{U}_{\mathcal{K}}^1\left(1+\varphi_{\mathcal{E}} \bar{U}_{\mathcal{E}}^1\right)}{\mu_{\mathcal{E}} \bar{U}_{\mathcal{E}}^1\left(1+\varphi_{\mathcal{K}} \bar{U}_{\mathcal{K}}^1\right)},$$

and

$$\begin{aligned}
&\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}_{\mathcal{E}}^1\right) U_{\mathcal{P}}} - \frac{\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{P}}^1\right)}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{E}}^1\right) \bar{U}_{\mathcal{P}}^1}\right)\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}}, U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}_{\mathcal{E}}^1\right)} - \frac{\mathcal{F}_{\mathcal{P}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{P}}^1\right)}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{E}}^1\right)}\right) \\
&\quad = -\frac{\varphi_{\mathcal{P}} \mu_{\mathcal{P}}^2\left(1+\varphi_{\mathcal{E}} \bar{U}_{\mathcal{E}}^1\right)^2\left(U_{\mathcal{P}}-\bar{U}_{\mathcal{P}}^1\right)^2}{\mu_{\mathcal{E}}^2\left(\bar{U}_{\mathcal{E}}^1\right)^2\left(1+\varphi_{\mathcal{P}} U_{\mathcal{P}}\right)^2\left(1+\varphi_{\mathcal{P}} \bar{U}_{\mathcal{P}}^1\right)^2} \leq 0, \\
&\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}_{\mathcal{E}}^1\right) U_{\mathcal{K}}} - \frac{\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{K}}^1\right)}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{E}}^1\right) \bar{U}_{\mathcal{K}}^1}\right)\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}}, U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}}, \bar{U}_{\mathcal{E}}^1\right)} - \frac{\mathcal{F}_{\mathcal{K}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{K}}^1\right)}{\mathcal{F}_{\mathcal{E}}\left(\bar{U}_{\mathcal{J}}^1, \bar{U}_{\mathcal{E}}^1\right)}\right) \\
&\quad = -\frac{\varphi_{\mathcal{K}} \mu_{\mathcal{K}}^2\left(1+\varphi_{\mathcal{E}} \bar{U}_{\mathcal{E}}^1\right)^2\left(U_{\mathcal{K}}-\bar{U}_{\mathcal{K}}^1\right)^2}{\mu_{\mathcal{E}}^2\left(\bar{U}_{\mathcal{E}}^1\right)^2\left(1+\varphi_{\mathcal{K}} U_{\mathcal{K}}\right)^2\left(1+\varphi_{\mathcal{K}} \bar{U}_{\mathcal{K}}^1\right)^2} \leq 0,
\end{aligned}$$

for all $U_{\mathcal{P}}, U_{\mathcal{K}} > 0, U_{\mathcal{J}} \in \left(0, \bar{U}_{\mathcal{J}}^0\right)$. Consequently, Hypothesis **H6** is also satisfied. Given that Hypotheses **H1**-**H6** are fulfilled, the conclusions regarding global stability, as outlined in Theorems

2.1 and 2.2 persist. System (4.1) is associated with the basic reproduction ratio, provided as

$$\Re_{0(4.1)} = \frac{\bar{U}^0_{\mathcal{J}}\left(\eta\alpha\mu_{\mathcal{E}} + \iota_{\mathcal{E}}\iota_{\mathcal{K}}\mu_{\mathcal{P}} + \alpha\iota_{\mathcal{E}}\mu_{\mathcal{K}}\right)}{\iota_{\mathcal{E}}\iota_{\mathcal{K}}\left(\alpha+\iota_{\mathcal{P}}\right)\left(1+\nu\bar{U}^0_{\mathcal{J}}\right)} = \Re_{0\mathcal{E}(4.1)} + \Re_{0\mathcal{P}(4.1)} + \Re_{0\mathcal{K}(4.1)}, \tag{4.2}$$

where

$$\Re_{0\mathcal{E}(4.1)} = \frac{\eta\alpha\bar{U}^0_{\mathcal{J}}\mu_{\mathcal{E}}}{\iota_{\mathcal{E}}\iota_{\mathcal{K}}\left(\alpha+\iota_{\mathcal{P}}\right)\left(1+\nu\bar{U}^0_{\mathcal{J}}\right)},$$
$$\Re_{0\mathcal{P}(4.1)} = \frac{\bar{U}^0_{\mathcal{J}}\mu_{\mathcal{P}}}{\left(\alpha+\iota_{\mathcal{P}}\right)\left(1+\nu\bar{U}^0_{\mathcal{J}}\right)},$$
$$\Re_{0\mathcal{K}(4.1)} = \frac{\alpha\bar{U}^0_{\mathcal{J}}\mu_{\mathcal{K}}}{\iota_{\mathcal{K}}\left(\alpha+\iota_{\mathcal{P}}\right)\left(1+\nu\bar{U}^0_{\mathcal{J}}\right)}.$$

4.1.1. *Stability of equilibria.* We conduct numerical simulations for the system described in (4.1), employing the parameter with values outlined in Table 1. Several parameters have been drawn from previous research, complemented by specific choices ($\mu_{\mathcal{X}}$, θ, ν and $\varphi = \varphi_{\mathcal{X}}$ for $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$) made for this study. For the assessment of equilibrium stability in system (4.1), we commence simulations using three varied sets of initial states:

I.S.1: $(U_{\mathcal{J}}(0), U_{\mathcal{P}}(0), U_{\mathcal{K}}(0), U_{\mathcal{E}}(0), U_{\mathcal{L}}(0)) = (720, 10, 2.3, 2.6, 0.08)$,
I.S.2: $(U_{\mathcal{J}}(0), U_{\mathcal{P}}(0), U_{\mathcal{K}}(0), U_{\mathcal{E}}(0), U_{\mathcal{L}}(0)) = (690, 8, 2, 2, 0.07)$,
I.S.3: $(U_{\mathcal{J}}(0), U_{\mathcal{P}}(0), U_{\mathcal{K}}(0), U_{\mathcal{E}}(0), U_{\mathcal{L}}(0)) = (660, 6, 1.6, 1.5, 0.06)$.

Table 1. Summary of model parameters and references.

Parameter	Value	Reference	Parameter	Value	Reference
τ	10	[64–66]	η	2.6	[74], [75]
$\iota_{\mathcal{J}}$	0.01	[5], [64], [67]	$\iota_{\mathcal{E}}$	2.4	[74], [75]
$\mu_{\mathcal{E}}$	varied	-	δ	0.01	[64], [76]
$\mu_{\mathcal{P}}$	varied	-	$\iota_{\mathcal{L}}$	0.3	[64], [77]
$\mu_{\mathcal{K}}$	varied	-	θ	varied	-
α	0.2	[64], [68]	φ_1	varied	-
$\iota_{\mathcal{P}}$	0.17	[64], [68]	φ_2	varied	-
$\iota_{\mathcal{K}}$	0.8	[69], [70]	φ_3	varied	-
λ	0.06	[71–73]	ν	varied	-

Utilizing the infection rates $\mu_{\mathcal{X}}$ for $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$, the stability of equilibria is governed by $\Re_{0(4.1)}$, as defined in Eq. (4.2). To scrutinize the impact of variations in these parameters, we explore two distinct scenarios:

Stability of $\bar{O}^0_{(4.1)}$. For the parameters $\mu_{\mathcal{E}} = 0.02$, $\mu_{\mathcal{P}} = 0.00002$, $\mu_{\mathcal{K}} = 0.0001$, $\nu = \varphi = 0.02$ and $\theta = 0.001$, the computed value of $\Re_{0(4.1)} = 0.703$ is below 1. As illustrated in Figure

1, the trajectories originating from I.S.1-I.S.3 approach the equilibrium $\bar{O}^0_{(4.1)} = (1000, 0, 0, 0, 0)$. This observation reinforces the assertion that $\bar{O}^0_{(4.1)}$ is being G.A.S, aligning with the outcome established in Theorem 2.1. Biologically speaking, this situation implies the complete eradication of the infection, signifying the successful elimination of the pathogen by the human body.

Stability of $\bar{O}^1_{(4.1)}$. Choosing $\mu_{\mathcal{E}} = 0.03$, $\mu_{\mathcal{P}} = 0.0001$, $\mu_{\mathcal{K}} = 0.0004$, $\nu = \varphi = 0.02$ and $\theta = 0.001$, yields $\Re_{0(4.1)} = 1.071 > 1$. Consequently, the equilibrium $\bar{O}^1_{(4.1)}$ exists when $\Re_{0(4.1)} > 1$, with specific values given by $\bar{O}^1_{(4.1)} = (688.079, 8.43, 2.108, 2.279, 0.075)$. In Figure 2, the numerical results align with the findings of Theorem 2.2, indicating the convergence of solutions for system (4.1) to $\bar{O}^1_{(4.1)}$ when $\Re_{0(4.1)} > 1$ across all I.S.1-I.S.3. From a biological standpoint, this situation implies the coexistence of both HIV-1 particles and antibodies within the host organism.

4.1.2. *Impact of impaired antibody immunity.* In this given context, we introduce fluctuations in the antibody impairment rate constant θ value while maintaining specific values for $\mu_{\mathcal{E}} = 0.03$, $\mu_{\mathcal{P}} = 0.0001$, $\mu_{\mathcal{K}} = 0.0004$, $\nu = 0.02$, and $\varphi = \varphi_{\mathcal{X}} = 0.02$, where $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$. We aim to explore how antibody immune impairment impacts the dynamics of system (4.1), by obtaining numerical solutions with different θ values as provided in Table 2. Under these circumstances, we employ the subsequent initial state:

I.S.4: $(U_{\mathcal{J}}(0), U_{\mathcal{P}}(0), U_{\mathcal{K}}(0), U_{\mathcal{E}}(0), U_{\mathcal{L}}(0)) = (684, 8.5, 2.13, 2.3, 0.04)$.

Table 2. Effect of the antibody impairment rate constant, θ, on the model's equilibria.

Antibody impairment rate constant θ	Equilibrium $\bar{O}^1_{(4.1)}$
0	$(688.133, 8.429, 2.107, 2.278, 0.076)$
0.03	$(686.796, 8.465, 2.116, 2.289, 0.062)$
0.2	$(683.739, 8.548, 2.137, 2.313, 0.03)$
2	$(681.28, 8.614, 2.154, 2.333, 0.005)$

As θ increases, we notice a decline in the quantity of antibodies, as shown in Table 2. This decrease is accompanied by a greater quantity of infected cells, whether latent or active, along with a higher quantity of HIV-1 particles. Due to this, there is a reduction in the count of healthy cells. Notably, Figure 3 reveals that the stability criteria of the equilibria remain unaffected by antibody immune impairment. This is a consequence of the fact that the parameter $\Re_{0(4.1)}$ stays constant regardless of varying θ values.

4.1.3. *The effect of level of interference among predators (HIV-1 particles and infected cells), $\boldsymbol{\varphi}$, on the system dynamics.* In this case, we fix specific values for $\mu_{\mathcal{E}} = 0.03$, $\mu_{\mathcal{P}} = 0.0001$, $\mu_{\mathcal{K}} = 0.0004$, $\nu = 0.02$ and $\theta = 0.1$, We numerically solve system (4.1) with various values of $\varphi = \varphi_{\mathcal{X}}$ for $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$, accompanied by the following initial state:

I.S.5: $(U_{\mathcal{J}}(0), U_{\mathcal{P}}(0), U_{\mathcal{K}}(0), U_{\mathcal{E}}(0), U_{\mathcal{L}}(0)) = (680, 12, 3, 3.5, 0.05)$.

Upon examining the results in Table 3, a positive correlation is apparent between φ and healthy cells, meaning that when φ becomes higher, the concentration of healthy cells becomes higher too. In contrast, the counts of other compartments become lower. Notably, the parameter $\mathfrak{R}_{0(4.1)}$ remains independent of φ. Consequently, Figure 4 illustrates that altering φ does not have an impact on the stability of equilibria.

TABLE 3. Level of interference among HIV-1 particles and infected cells, φ, on the dynamic of system (4.1).

Level of interference φ	Equilibrium $\bar{O}^1_{(4.1)}$
0	$(405.258, 16.074, 4.019, 4.347, 0.059)$
0.018	$(665.613, 9.037, 2.259, 2.445, 0.045)$
0.06	$(861.13, 3.753, 0.938, 1.016, 0.025)$
2	$(995.266, 0.128, 0.032, 0.035, 0.001)$

4.1.4. *The effect of handling time among predators (HIV-1 particles and infected cells),* $\boldsymbol{\nu}$*, on the system dynamics.* In this scenario, setting specific values for $\mu_{\mathcal{E}} = 0.03$, $\mu_{\mathcal{P}} = 0.0001$, $\mu_{\mathcal{K}} = 0.0004$, $\theta = 0.1$, and $\varphi = \varphi_{\mathcal{X}} = 0.2$, for $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$, and considering various values of ν, we numerically solve system (4.1) together with the following initial state:

I.S.6: $(U_{\mathcal{J}}(0), U_{\mathcal{P}}(0), U_{\mathcal{K}}(0), U_{\mathcal{E}}(0), U_{\mathcal{L}}(0)) = (500, 30, 4, 4.3, 0.04)$.

Our analysis of Table 4 reveals a positive correlation between ν and healthy cells. This indicates that an increase in ν corresponds to a higher concentration of healthy cells, while the counts of other compartments exhibit a decrease. This is, in fact, due to the dependence of the parameter $\mathfrak{R}_{0(4.1)}$ on ν. Therefore, Figure 5 confirms that ν alters the stability properties of equilibria. Furthermore, we derive the subsequent observations:

(i) $\bar{O}^1_{(4.1)}$ is G.A.S when $0 < \nu < 0.0215$.
(ii) $\bar{O}^0_{(4.1)}$ is G.A.S when $\nu \geq 0.0215$.

TABLE 4. The effect of handling time among HIV-1 particles and infected cells, ν, on the dynamic of system (4.1).

Handling time ν	Equilibria	$\mathfrak{R}_{0(4.1)}$
0	$\bar{O}^1_{(4.1)} = (103.542, 24.229, 6.057, 6.551, 0.069)$	22.5
0.01	$\bar{O}^1_{(4.1)} = (439.635, 15.145, 3.786, 4.096, 0.058)$	2.045
0.014	$\bar{O}^1_{(4.1)} = (693.629, 8.28, 2.07, 2.24, 0.043)$	1.5
0.0215	$\bar{O}^0_{(4.1)} = (1000, 0, 0, 0, 0)$	1
0.1	$\bar{O}^0_{(4.1)} = (1000, 0, 0, 0, 0)$	0.223

4.2. **Numerical simulation for model (3.1).** In the context of numerical analysis, we adopt the following specific form regarding the probability distribution functions $\pi_i(\ell)$, for $i = 1, 2, 3$:

$$\pi_i(\ell) = \delta_*(\ell - \ell_i), \quad \ell_i \in [0, f_i], \quad i = 1, 2, 3,$$

where $\delta_*(.)$ is the Dirac delta function. When $f_i \to \infty$, we observe

$$\int_0^\infty \pi_i(\ell)\, d\ell = 1, \quad i = 1, 2, 3.$$

Furthermore, we obtain

$$\Pi_i = \int_0^\infty \delta_*(\ell - \ell_i)\, e^{-b_i \ell} d\ell = e^{-b_i \ell_i}, \qquad i = 1, 2, 3.$$

Additionally, we will select the saturation incidence rates of infection as follows:

$$\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}}, U_{\mathcal{E}}) = \frac{\mu_{\mathcal{E}} U_{\mathcal{J}} U_{\mathcal{E}}}{1 + \epsilon_{\mathcal{E}} U_{\mathcal{E}}}, \quad \mathcal{F}_{\mathcal{P}}(U_{\mathcal{J}}, U_{\mathcal{P}}) = \frac{\mu_{\mathcal{P}} U_{\mathcal{J}} U_{\mathcal{P}}}{1 + \epsilon_{\mathcal{P}} U_{\mathcal{P}}}, \quad \mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}}, U_{\mathcal{K}}) = \frac{\mu_{\mathcal{K}} U_{\mathcal{J}} U_{\mathcal{K}}}{1 + \epsilon_{\mathcal{K}} U_{\mathcal{K}}},$$

here, the positive constants $\mu_{\mathcal{X}}$ for $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$ denote the infection rate constants, while $\epsilon_{\mathcal{X}} > 0$, $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$ represents the saturation parameters. Consequently, Based on the aforementioned reasoning, system (3.1) was modified into a system with a discrete time delay, as illustrated below:

$$\begin{cases} \dot{U}_{\mathcal{J}} &= \tau - \iota_{\mathcal{J}} U_{\mathcal{J}} - \dfrac{\mu_{\mathcal{E}} U_{\mathcal{J}} U_{\mathcal{E}}}{1 + \epsilon_{\mathcal{E}} U_{\mathcal{E}}} - \dfrac{\mu_{\mathcal{P}} U_{\mathcal{J}} U_{\mathcal{P}}}{1 + \epsilon_{\mathcal{P}} U_{\mathcal{P}}} - \dfrac{\mu_{\mathcal{K}} U_{\mathcal{J}} U_{\mathcal{K}}}{1 + \epsilon_{\mathcal{K}} U_{\mathcal{K}}}, \\ \dot{U}_{\mathcal{P}} &= e^{-b_1 \ell_1} \left(\dfrac{\mu_{\mathcal{E}} U_{\mathcal{J}}(t - \ell_1)\, U_{\mathcal{E}} U_{\mathcal{J}}(t - \ell_1)}{1 + \epsilon_{\mathcal{E}} U_{\mathcal{E}}(t - \ell_1)} + \dfrac{\mu_{\mathcal{P}} U_{\mathcal{J}}(t - \ell_1)\, U_{\mathcal{P}}(t - \ell_1)}{1 + \epsilon_{\mathcal{P}} U_{\mathcal{P}}(t - \ell_1)} \right. \\ & \qquad \left. + \dfrac{\mu_{\mathcal{K}} U_{\mathcal{J}}(t - \ell_1)\, U_{\mathcal{K}}(t - \ell_1)}{1 + \epsilon_{\mathcal{K}} U_{\mathcal{K}}(t - \ell_1)} \right) - (\alpha + \iota_{\mathcal{P}})\, U_{\mathcal{P}}, \\ \dot{U}_{\mathcal{K}} &= \alpha e^{-b_2 \ell_2} U_{\mathcal{P}}(t - \ell_2) - \iota_{\mathcal{K}} U_{\mathcal{K}}, \\ \dot{U}_{\mathcal{E}} &= \eta e^{-b_3 \ell_3} U_{\mathcal{K}}(t - \ell_3) - \iota_{\mathcal{E}} U_{\mathcal{E}} - \lambda U_{\mathcal{L}} U_{\mathcal{E}}, \\ \dot{U}_{\mathcal{L}} &= \delta U_{\mathcal{E}} - \iota_{\mathcal{L}} U_{\mathcal{L}} - \theta U_{\mathcal{L}} U_{\mathcal{E}}. \end{cases} \tag{4.3}$$

The functions $\mathcal{F}_{\mathcal{X}}(U_{\mathcal{J}}, U_{\mathcal{X}})$, $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$ are evidently continuously differentiable. Additionally, for each $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$, these functions satisfy the Hypotheses: $\mathcal{F}_{\mathcal{X}}(U_{\mathcal{J}}, U_{\mathcal{X}}) > 0$ and $\mathcal{F}_{\mathcal{X}}(0, U_{\mathcal{X}}) = \mathcal{F}_{\mathcal{X}}(U_{\mathcal{J}}, 0) = 0$ for all $U_{\mathcal{J}} > 0$ and $U_{\mathcal{X}} > 0$. Thus, the fulfillment of Hypothesis **H1** is evident. Furthermore, we observe

$$\begin{aligned} \frac{\partial \mathcal{F}_{\mathcal{X}}(U_{\mathcal{J}}, U_{\mathcal{X}})}{\partial U_{\mathcal{J}}} &= \frac{\mu_{\mathcal{X}} U_{\mathcal{X}}}{1 + \epsilon_{\mathcal{X}} U_{\mathcal{X}}} > 0, \quad \text{for all } U_{\mathcal{X}} > 0, \\ \frac{\partial \mathcal{F}_{\mathcal{X}}(U_{\mathcal{J}}, U_{\mathcal{X}})}{\partial U_{\mathcal{X}}} &= \frac{\mu_{\mathcal{X}} U_{\mathcal{J}}}{(1 + \epsilon_{\mathcal{X}} U_{\mathcal{X}})^2} > 0, \quad \text{for all } U_{\mathcal{J}}, U_{\mathcal{X}} > 0, \end{aligned}$$

where $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$. This can be expressed by stating that Hypothesis **H2** is valid. It is obvious that

$$\wp_{\mathcal{X}}(U_{\mathcal{J}}) = \frac{\partial \mathcal{F}_{\mathcal{X}}(U_{\mathcal{J}}, 0)}{\partial U_{\mathcal{X}}} = \mu_{\mathcal{X}} U_{\mathcal{J}} > 0, \quad \text{for all } U_{\mathcal{J}} > 0,\ \mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}.$$

Furthermore

$$\wp'_{\mathcal{X}}(U_{\mathcal{J}}) = \frac{d}{dU_{\mathcal{J}}} \left(\frac{\partial \mathcal{F}_{\mathcal{X}}(U_{\mathcal{J}}, 0)}{\partial U_{\mathcal{X}}} \right) = \mu_{\mathcal{X}} > 0,\ \mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\},$$

which confirms that Hypothesis **H3** is also met. Moreover, we have

$$\frac{\partial}{\partial U_{\mathcal{X}}}\left(\frac{\mathcal{F}_{\mathcal{X}}\left(U_{\mathcal{J}},U_{\mathcal{X}}\right)}{U_{\mathcal{X}}}\right)=-\frac{\mu_{\mathcal{X}}\epsilon_{\mathcal{X}}U_{\mathcal{J}}}{\left(1+\epsilon_{\mathcal{X}}U_{\mathcal{X}}\right)^2}<0,\quad \text{for all } U_{\mathcal{J}},U_{\mathcal{X}}>0,\ \mathcal{X}\in\{\mathcal{E},\mathcal{P},\mathcal{K}\}.$$

Hence, the verification of Hypothesis **H4** is straightforward. Additionally, it is evident that $\frac{\wp_{\mathcal{X}}(U_{\mathcal{J}})}{\wp_{\mathcal{E}}(U_{\mathcal{J}})}=\frac{\mu_{\mathcal{X}}}{\mu_{\mathcal{E}}}$, for $\mathcal{X}\in\{\mathcal{P},\mathcal{K}\}$. Consequently, Hypothesis **H5** is satisfied. Moreover, we have:

$$\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}=\frac{\mu_{\mathcal{P}}U_{\mathcal{P}}\left(1+\epsilon_{\mathcal{E}}U^1_{\mathcal{E}}\right)}{\mu_{\mathcal{E}}U^1_{\mathcal{E}}\left(1+\epsilon_{\mathcal{P}}U_{\mathcal{P}}\right)},\qquad \frac{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}=\frac{\mu_{\mathcal{P}}U^1_{\mathcal{P}}\left(1+\epsilon_{\mathcal{E}}U^1_{\mathcal{E}}\right)}{\mu_{\mathcal{E}}U^1_{\mathcal{E}}\left(1+\epsilon_{\mathcal{P}}U^1_{\mathcal{P}}\right)},$$
$$\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}=\frac{\mu_{\mathcal{K}}U_{\mathcal{K}}\left(1+\epsilon_{\mathcal{E}}U^1_{\mathcal{E}}\right)}{\mu_{\mathcal{E}}U^1_{\mathcal{E}}\left(1+\epsilon_{\mathcal{K}}U_{\mathcal{K}}\right)},\qquad \frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}=\frac{\mu_{\mathcal{K}}U^1_{\mathcal{K}}\left(1+\epsilon_{\mathcal{E}}U^1_{\mathcal{E}}\right)}{\mu_{\mathcal{E}}U^1_{\mathcal{E}}\left(1+\epsilon_{\mathcal{K}}U^1_{\mathcal{K}}\right)},$$

and

$$\begin{aligned}&\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{P}}}-\frac{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)U^1_{\mathcal{P}}}\right)\left(\frac{\mathcal{F}_{\mathcal{P}}\left(U_{\mathcal{J}},U_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}-\frac{\mathcal{F}_{\mathcal{P}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{P}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\\
&=-\frac{\epsilon_{\mathcal{P}}\mu^2_{\mathcal{P}}\left(1+\epsilon_{\mathcal{E}}U^1_{\mathcal{E}}\right)^2\left(U_{\mathcal{P}}-U^1_{\mathcal{P}}\right)^2}{\mu^2_{\mathcal{E}}\left(U^1_{\mathcal{E}}\right)^2\left(1+\epsilon_{\mathcal{P}}U_{\mathcal{P}}\right)^2\left(1+\epsilon_{\mathcal{P}}U^1_{\mathcal{P}}\right)^2}\le 0,\\
&\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)U_{\mathcal{K}}}-\frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)U^1_{\mathcal{K}}}\right)\left(\frac{\mathcal{F}_{\mathcal{K}}\left(U_{\mathcal{J}},U_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U_{\mathcal{J}},U^1_{\mathcal{E}}\right)}-\frac{\mathcal{F}_{\mathcal{K}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{K}}\right)}{\mathcal{F}_{\mathcal{E}}\left(U^1_{\mathcal{J}},U^1_{\mathcal{E}}\right)}\right)\\
&=-\frac{\epsilon_{\mathcal{K}}\mu^2_{\mathcal{K}}\left(1+\epsilon_{\mathcal{E}}U^1_{\mathcal{E}}\right)^2\left(U_{\mathcal{K}}-U^1_{\mathcal{K}}\right)^2}{\mu^2_{\mathcal{E}}\left(U^1_{\mathcal{E}}\right)^2\left(1+\epsilon_{\mathcal{K}}U_{\mathcal{K}}\right)^2\left(1+\epsilon_{\mathcal{K}}U^1_{\mathcal{K}}\right)^2}\le 0,\end{aligned}$$

for all $U_{\mathcal{P}},U_{\mathcal{K}}>0, U_{\mathcal{J}}\in\left(0,U^0_{\mathcal{J}}\right)$. Hence, Hypothesis **H6** is fulfilled as well. With the fulfillment of Hypotheses **H1**-**H6**, the global stability results stated in Theorems 3.1 and 3.2 persist. Accordingly, the determination of the basic reproduction ratio for system (4.3) is outlined below:

$$\begin{aligned}\tilde{\mathfrak{R}}_{0(4.3)}&=\frac{U^0_{\mathcal{J}}e^{-b_1\ell_1}\left(\alpha e^{-b_2\ell_2}\left(\mu_{\mathcal{E}}\eta e^{-b_3\ell_3}+\mu_{\mathcal{K}}\iota_{\mathcal{E}}\right)+\mu_{\mathcal{P}}\iota_{\mathcal{E}}\iota_{\mathcal{K}}\right)}{\iota_{\mathcal{E}}\iota_{\mathcal{K}}\left(\alpha+\iota_{\mathcal{P}}\right)}\\
&=\tilde{\mathfrak{R}}_{0\mathcal{E}(4.3)}+\tilde{\mathfrak{R}}_{0\mathcal{P}(4.3)}+\tilde{\mathfrak{R}}_{0\mathcal{K}(4.3)},\end{aligned}\tag{4.4}$$

where

$$\begin{aligned}\tilde{\mathfrak{R}}_{0\mathcal{E}(4.3)}&=\frac{\eta\alpha U^0_{\mathcal{J}}\mu_{\mathcal{E}}e^{-b_1\ell_1-b_2\ell_2-b_3\ell_3}}{\iota_{\mathcal{E}}\iota_{\mathcal{K}}\left(\alpha+\iota_{\mathcal{P}}\right)},\\
\tilde{\mathfrak{R}}_{0\mathcal{P}(4.3)}&=\frac{U^0_{\mathcal{J}}\mu_{\mathcal{P}}e^{-b_1\ell_1}}{\alpha+\iota_{\mathcal{P}}},\\
\tilde{\mathfrak{R}}_{0\mathcal{K}(4.3)}&=\frac{\alpha U^0_{\mathcal{J}}\mu_{\mathcal{K}}e^{-b_1\ell_1-b_2\ell_2}}{\iota_{\mathcal{K}}\left(\alpha+\iota_{\mathcal{P}}\right)}.\end{aligned}$$

4.2.1. *The stability implications of time delays on equilibria.* To assess how time delay parameters affect the solutions of system (4.3), we maintain constant values for parameters $\mu_{\mathcal{E}} = 0.003$, $\mu_{\mathcal{P}} = 0.0001$, $\mu_{\mathcal{K}} = 0.0004$, $\theta = 0.001$, $b_1 = 0.1$, $b_2 = 0.2$, $b_3 = 0.3$, and set $\epsilon = \epsilon_{\mathcal{X}} = 0.0001$, $\mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$. Furthermore, you can refer to Table 1 for the values of the other parameters. Next, we examine how the dynamics are affected by experimenting with different variations of the delay parameters ℓ_i, where i takes values from 1 to 3. The stability of equilibria is highly sensitive to changes in the parameters ℓ_i due to the dependence of $\tilde{\mathfrak{R}}_{0(4.3)}$ on them (as indicated in Eq. (4.4)). Let's delve into the following choices for the delay parameters:

D.P.1: $\ell_1 = 0.004$, $\ell_2 = 0.002$, $\ell_3 = 0.003$.

D.P.2: $\ell_1 = 0.02$, $\ell_2 = 0.03$, $\ell_3 = 0.01$.

D.P.3: $\ell_1 = 0.6$, $\ell_2 = 0.7$, $\ell_3 = 0.8$.

D.P.4: $\ell_1 = 1.287$, $\ell_2 = 1.7$, $\ell_3 = 2.8$.

D.P.5: $\ell_1 = 4$, $\ell_2 = 5$, $\ell_3 = 6$.

D.P.6: $\ell_1 = 7$, $\ell_2 = 8$, $\ell_3 = 9$.

We address the initial state for solving system (4.3) as follows:

I.S.7: $(U_{\mathcal{J}}(v), U_{\mathcal{P}}(v), U_{\mathcal{K}}(v), U_{\mathcal{E}}(v), U_{\mathcal{L}}(v)) = (600, 5, 2, 1, 0.2), v \in [-\ell, 0], \ell = \max\{\ell_1, \ell_2, \ell_3\}$.

Table 5 displays the values of $\tilde{\mathfrak{R}}_{0(4.3)}$ corresponding to different ℓ_i values, where $i = 1, 2, 3$. Significantly reducing $\tilde{\mathfrak{R}}_{0(4.3)}$ is noteworthy when the ℓ_i parameters increase. Figure 6 visually depicts the numerical solutions derived from the system, emphasizing the significant impact of the included time-delays. Specifically, a positive correlation exists between ℓ_i and the count of healthy cells, indicating that their values increase simultaneously, while a decrease is observed in the counts of other compartments. This suggests the potential creation of a new category of medication aimed at extending delay times and ultimately mitigating HIV-1 multiplication.

Table 5. Analyzing the influence of delay parameters, ℓ_i, on System (4.3).

Delay parameters ℓ_1, ℓ_2, ℓ_3	Equilibria	$\tilde{\mathfrak{R}}_{0(4.3)}$
0.004, 0.002, 0.003	$O^1_{(4.3)} = (367.297, 17.093, 4.272, 4.606, 0.151)$	2.732
0.02, 0.03, 0.01	$O^1_{(4.3)} = (370.35, 16.984, 4.22, 4.542, 0.149)$	2.71
0.6, 0.7, 0.8	$O^1_{(4.3)} = (529.992, 11.963, 2.6, 2.212, 0.073)$	1.89
1.287, 1.7, 2.8	$O^0_{(4.3)} = (1000, 0, 0, 0, 0)$	1
4, 5, 6	$O^0_{(4.3)} = (1000, 0, 0, 0, 0)$	0.337
7, 8, 9	$O^0_{(4.3)} = (1000, 0, 0, 0, 0)$	0.176

4.2.2. *The saturation infection rates effect on the system dynamics.* In this scenario, we set $\mu_{\mathcal{E}} = 0.003$, $\mu_{\mathcal{P}} = 0.0001$, $\mu_{\mathcal{K}} = 0.0004$, $\theta = 0.001$ and $\ell = \ell_i = 0.002$, $i = 1, 2, 3$. We consider different values of $\epsilon = \epsilon_{\mathcal{X}}, \mathcal{X} \in \{\mathcal{E}, \mathcal{P}, \mathcal{K}\}$ and numerically solve system (4.3) with the initial state I.S.7 to analyze the impact of saturation infection rates on the dynamics of system (4.3). Analyzing Table 6 reveals a correlation wherein an augmentation in ϵ leads to an increase in the count of healthy cells, while there is a reduction in the counts of other compartments. Clearly, the parameter $\tilde{\Re}_{0(4.3)}$ does not depends on ϵ. Therefore, Figure 7 confirms that the saturation infection does not alter the stability properties of equilibria.

Table 6. The dynamics of system (4.3) affected by saturation parameters.

Saturation parameter ϵ	Equilibrium $\tilde{Q}_{1(4.3)}$
0.002	$(371.136, 16.993, 4.247, 4.58, 0.15)$
0.08	$(492.947, 13.701, 3.424, 3.696, 0.122)$
0.3	$(666.725, 9.006, 2.251, 2.432, 0.08)$
1	$(839.906, 4.326, 1.081, 1.169, 0.039)$

4.3. **Analysis of parameter sensitivity.**

4.3.1. *Investigating the sensitivity of parameters for model (4.1).* The principal goal of analyzing parameter sensitivity is to determine the variable that has the most substantial impact on the basic reproduction ratio $\Re_0$, which should be the focus of control strategies. To achieve this, we will employ direct differentiation, a method that calculates sensitivity indices by considering changes in variables based on parameters. We can define the normalized forward sensitivity index of $\Re_0$, which depends on the differentiability with respect to a parameter ϱ, as follows:

$$S_{\varrho} = \frac{\varrho}{\Re_0}\frac{\partial \Re_0}{\partial \varrho}, \tag{4.5}$$

For each parameter belonging to $\Re_{0(4.1)}$, we utilized Eq. (4.5) to compute the sensitivity index values. For this purpose, we considered the parameters specified in Table 1, while also incorporating supplementary values for $\mu_{\mathcal{E}} = 0.03$, $\mu_{\mathcal{P}} = 0.0001$, $\mu_{\mathcal{K}} = 0.0004$ and $\nu = 0.02$. Sensitivity index values for $\Re_{0(4.1)}$ are showcased in Table 7 and Figure 8, revealing positive index values for τ, $\mu_{\mathcal{E}}$, $\mu_{\mathcal{P}}$, $\mu_{\mathcal{K}}$, η and α. In this context, higher values of these parameters correspond to a higher prevalence of HIV-1 infection within this population. In contrast, the remaining indices show negativity, implying that an increase in $\iota_{\mathcal{J}}$, $\iota_{\mathcal{P}}$, $\iota_{\mathcal{K}}$, $\iota_{\mathcal{E}}$ and ν corresponds to a reduction in the $\Re_{0(4.1)}$ value. Interestingly, the parameter δ, representing antibodies responsiveness, demonstrates no impact on $\Re_{0(4.1)}$.

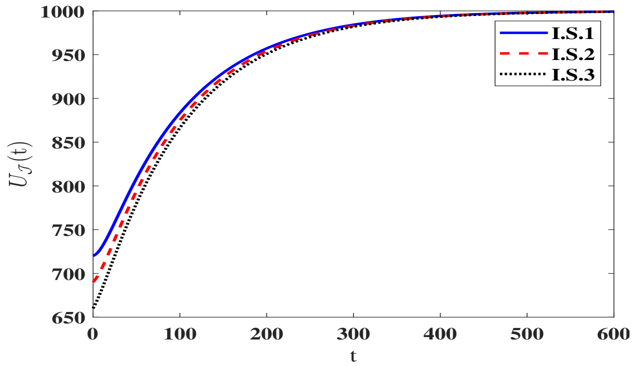
Table 7. Sensitivity analysis for the basic reproduction ratio for system (4.1), $\mathfrak{R}_{0(4.1)}$.

Parameter ϱ	S.Index	Value	Parameter ϱ	S.Index	Value
τ	S_τ	0.048	α	S_α	0.447
$\iota_{\mathcal{J}}$	$S_{\iota_{\mathcal{J}}}$	-0.048	$\iota_{\mathcal{E}}$	$S_{\iota_{\mathcal{E}}}$	-0.976
$\mu_{\mathcal{E}}$	$S_{\mu_{\mathcal{E}}}$	0.976	$\iota_{\mathcal{P}}$	$S_{\iota_{\mathcal{P}}}$	-0.459
$\mu_{\mathcal{P}}$	$S_{\mu_{\mathcal{P}}}$	0.012	$\iota_{\mathcal{K}}$	$S_{\iota_{\mathcal{K}}}$	-0.988
$\mu_{\mathcal{K}}$	$S_{\mu_{\mathcal{K}}}$	0.012	ν	S_ν	-0.952
η	S_η	0.976	δ	S_δ	0

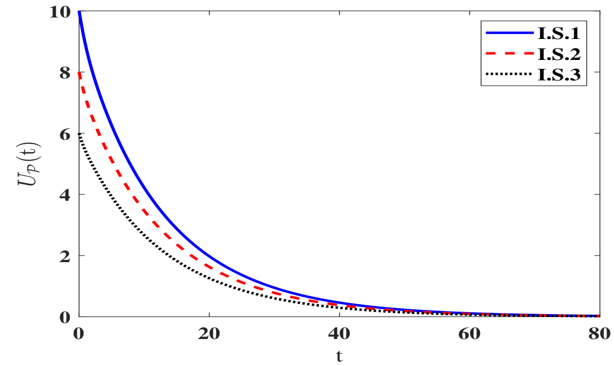
4.3.2. *Investigating the sensitivity of parameters for model (4.3).* Here, Eq. (4.5) will be applied to calculate sensitivity indices concerning the basic reproduction ratio for system (4.3), $\tilde{\mathfrak{R}}_{0(4.3)}$, considering each parameter contributing to the basic reproduction ratio. The calculations were performed with the parameters specified in Table 1, supplemented by the inclusion of the additional parameters: $\mu_{\mathcal{E}} = 0.003$, $\mu_{\mathcal{P}} = 0.0001$, $\mu_{\mathcal{K}} = 0.0004$, $b_1 = 0.1$, $b_2 = 0.2$, $b_3 = 0.3$, $\ell_1 = 0.6$, $\ell_2 = 0.7$, and $\ell_3 = 0.8$. The sensitivity index values for $\tilde{\mathfrak{R}}_{0(4.3)}$ are presented in Table 8, and Figure 9 provides a graphical representation. Positive indices for τ, $\mu_{\mathcal{E}}$, $\mu_{\mathcal{P}}$, $\mu_{\mathcal{K}}$, η and α suggest that an increase in these values will elevate the prevalence of HIV-1 disease. Conversely, negative indices, including $\iota_{\mathcal{J}}$, $\iota_{\mathcal{P}}$, $\iota_{\mathcal{K}}$, $\iota_{\mathcal{E}}$, b_1, b_2, b_3, ℓ_1, ℓ_2, and ℓ_3 indicate that an increase in these values will lower the parameter $\tilde{\mathfrak{R}}_{0(4.3)}$. Furthermore, the parameter representing antibodies responsiveness, δ, demonstrates no discernible effect on $\tilde{\mathfrak{R}}_{0(4.3)}$.

Table 8. Sensitivity analysis for the basic reproduction ratio for system (4.3) $\tilde{\mathfrak{R}}_{0(4.3)}$.

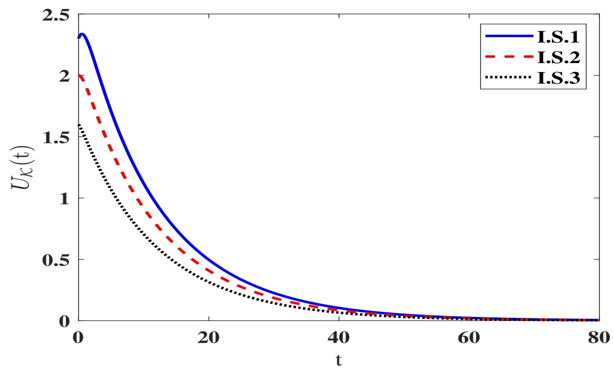
Parameter ϱ	S.Index	Value	Parameter ϱ	S.Index	Value
τ	S_τ	1	$\iota_{\mathcal{P}}$	$S_{\iota_{\mathcal{P}}}$	-0.459
$\iota_{\mathcal{J}}$	$S_{\iota_{\mathcal{J}}}$	-1	$\iota_{\mathcal{K}}$	$S_{\iota_{\mathcal{K}}}$	-0.865
$\mu_{\mathcal{E}}$	$S_{\mu_{\mathcal{E}}}$	0.748	b_1	S_{b_1}	-0.06
$\mu_{\mathcal{P}}$	$S_{\mu_{\mathcal{P}}}$	0.135	b_2	S_{b_2}	-0.121
$\mu_{\mathcal{K}}$	$S_{\mu_{\mathcal{K}}}$	0.117	b_3	S_{b_3}	-0.18
η	S_η	0.748	ℓ_1	S_{ℓ_1}	-0.06
α	S_α	0.325	ℓ_2	S_{ℓ_2}	-0.121
$\iota_{\mathcal{E}}$	$S_{\iota_{\mathcal{E}}}$	-0.748	ℓ_3	S_{ℓ_3}	-0.18
δ	S_δ	0			



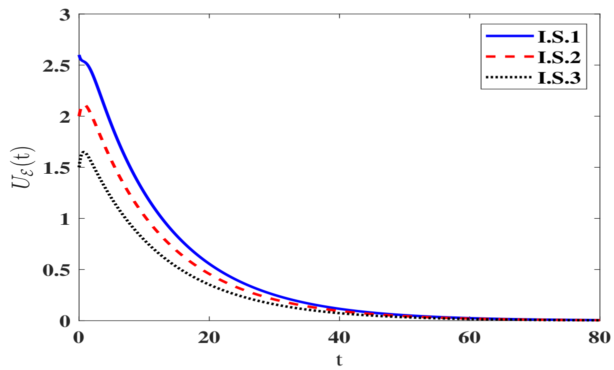
(A) Healthy CD4$^+$T cells



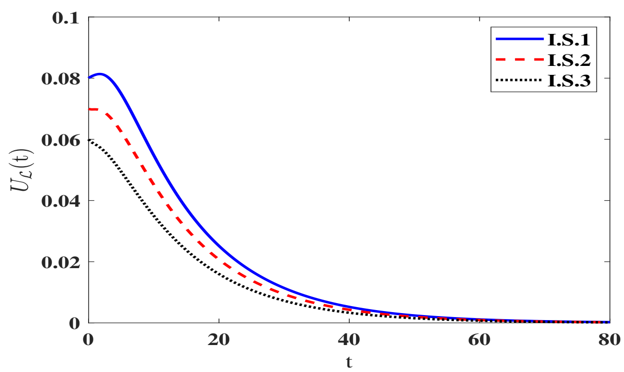
(B) Latently infected cells



(C) Actively infected cells

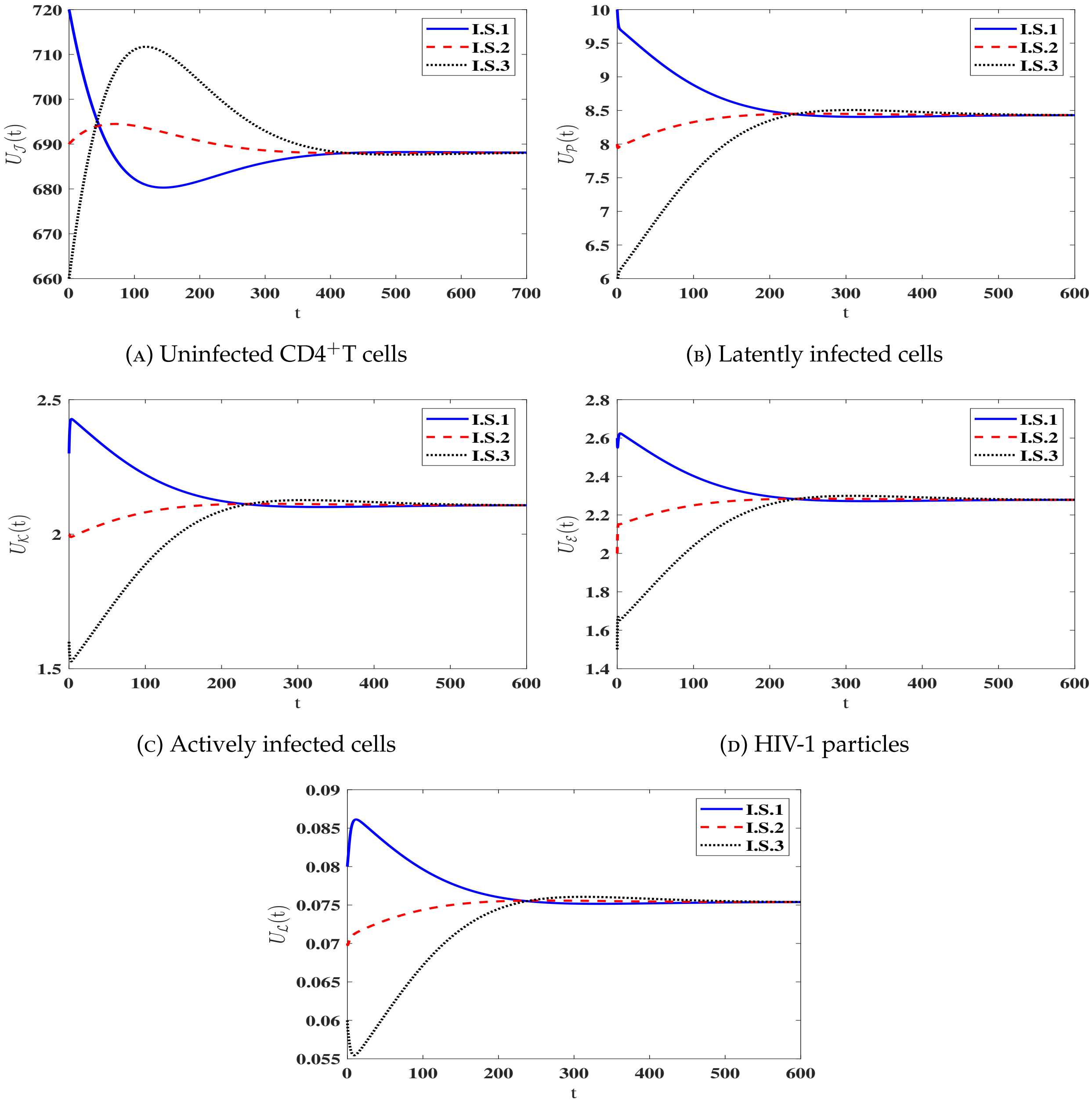


(D) HIV-1 particles



(E) Antibodies

FIGURE 1. Solution patterns of the dynamical system (4.1) in the state of $\Re_{0(4.1)} \leq 1$.



(A) Uninfected CD4^{+}T cells

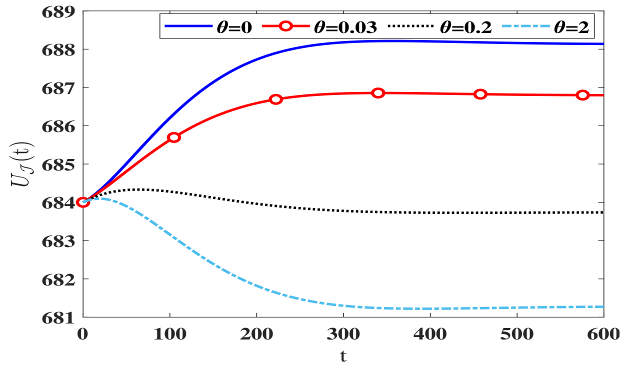
(B) Latently infected cells

(C) Actively infected cells

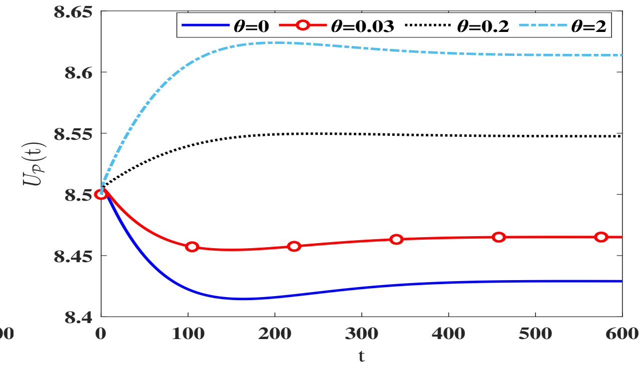
(D) HIV-1 particles

(E) Antibodies

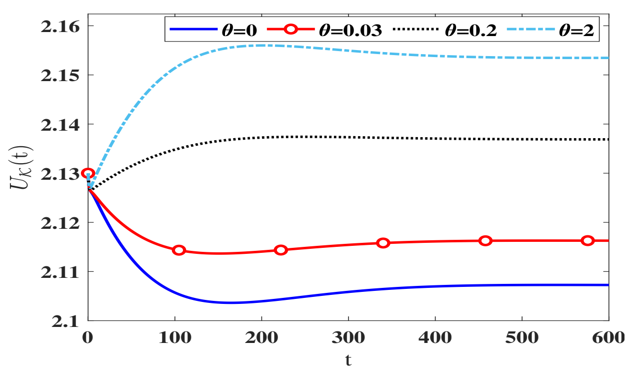
FIGURE 2. Solution patterns of the dynamical system (4.1) in the state of $\mathfrak{R}_{0(4.1)} > 1$.



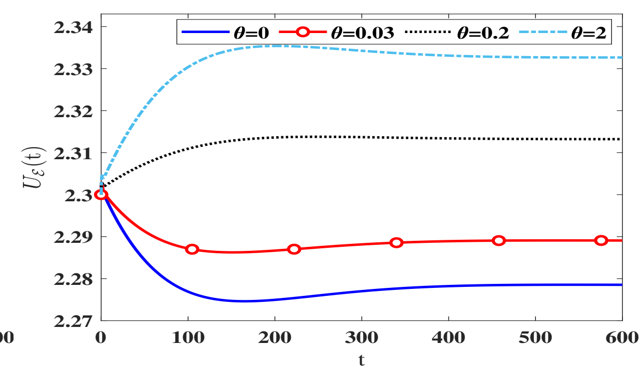
(A) Uninfected CD4$^+$T cells



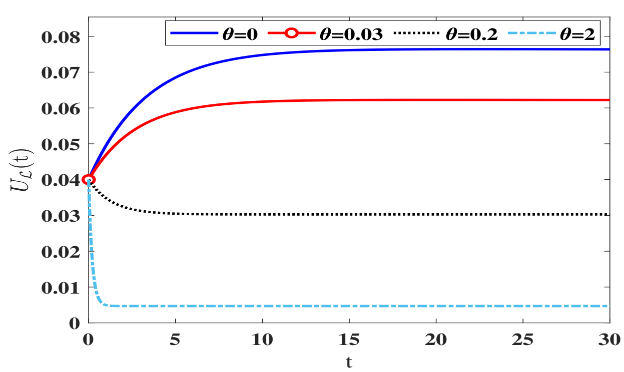
(B) Latently infected cells



(C) Actively infected cells

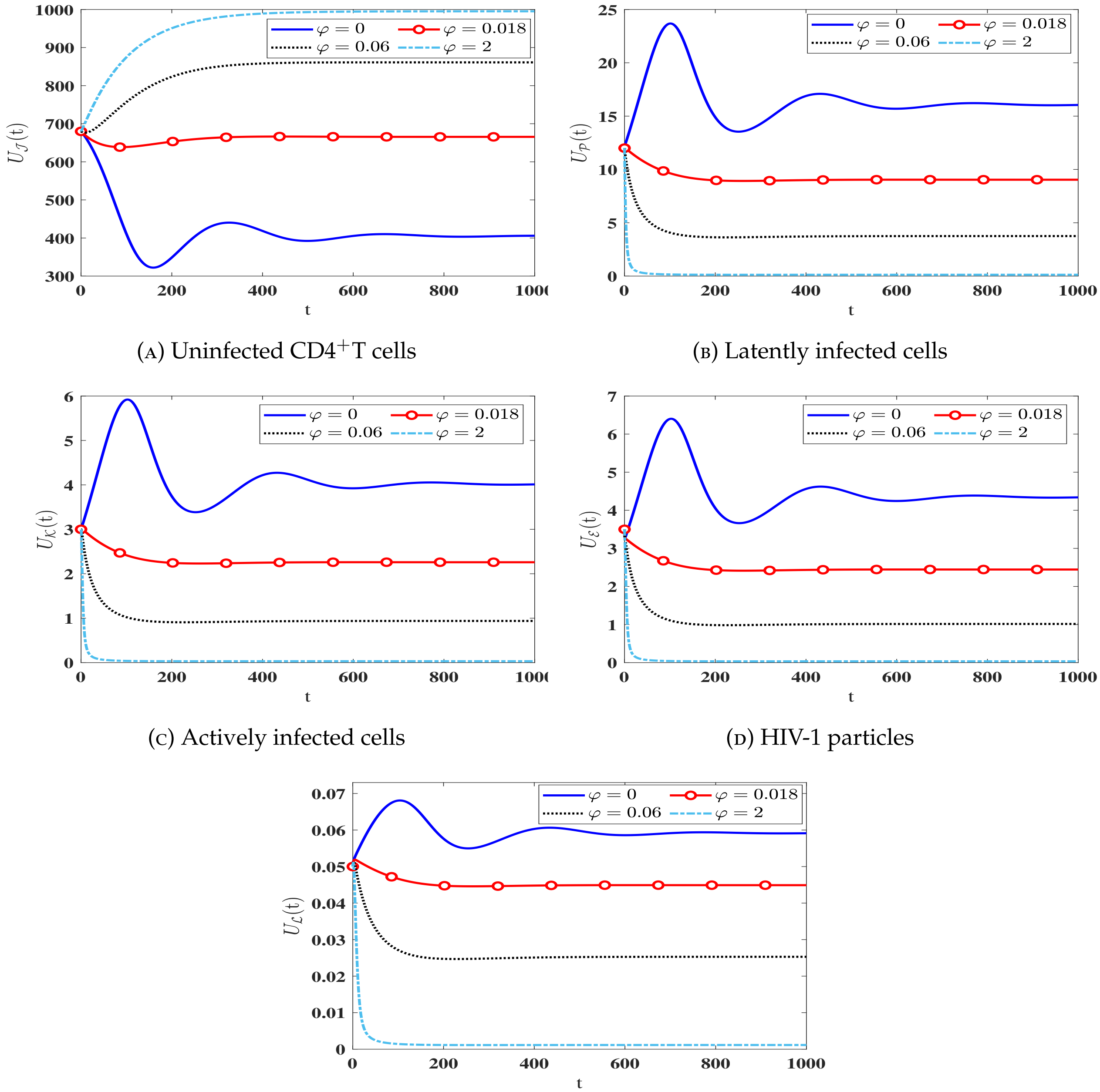


(D) HIV-1 particles



(E) Antibodies

FIGURE 3. Solution patterns of the dynamical system (4.1) along different θ values.



(A) Uninfected CD4$^+$T cells

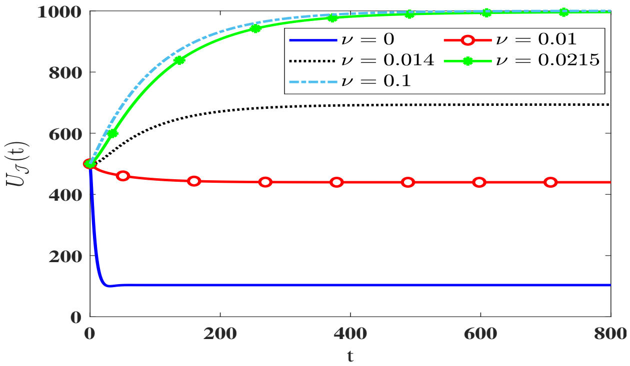
(B) Latently infected cells

(C) Actively infected cells

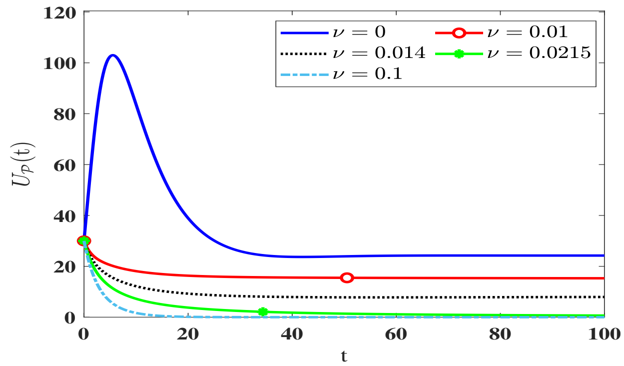
(D) HIV-1 particles

(E) Antibodies

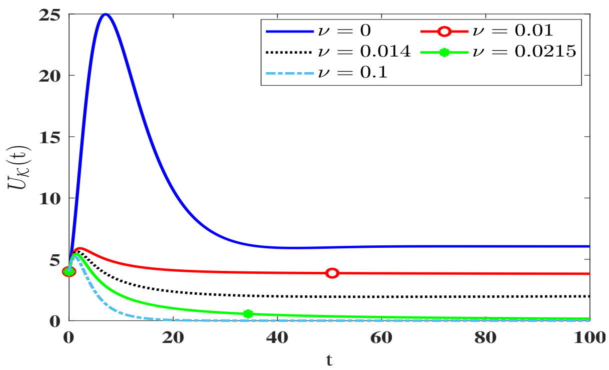
FIGURE 4. The effect of φ on the solution patterns in system (4.1).



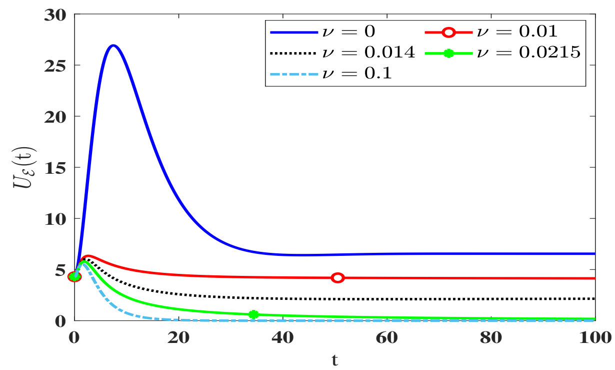
(A) Uninfected CD4$^+$T cells



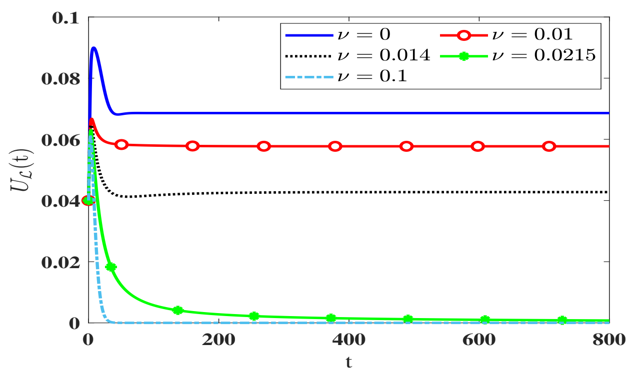
(B) Latently infected cells



(C) Actively infected cells

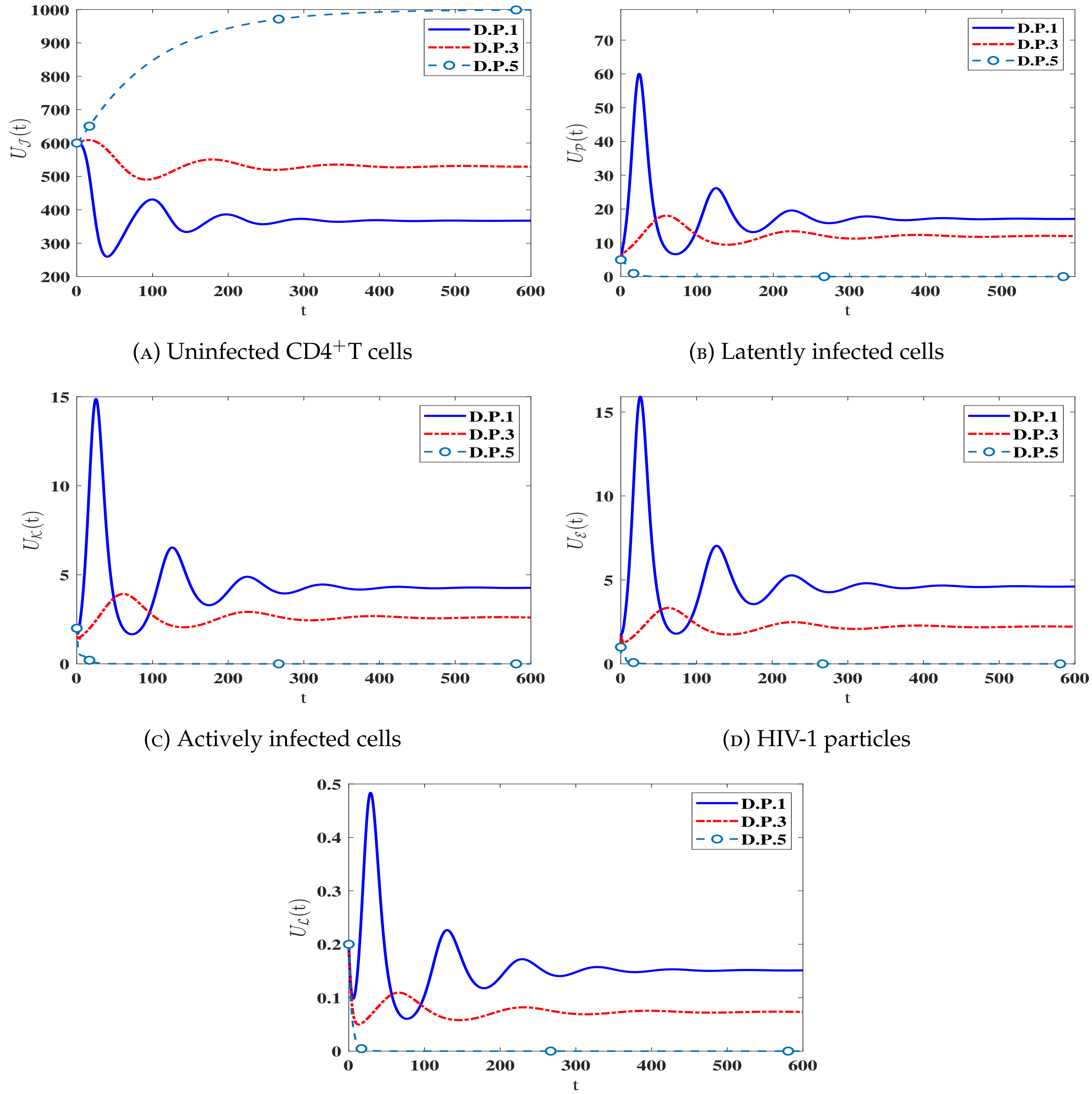


(D) HIV-1 particles



(E) Antibodies

FIGURE 5. The effect of ν on the solution patterns in system (4.1).



(A) Uninfected CD4^{+}T cells

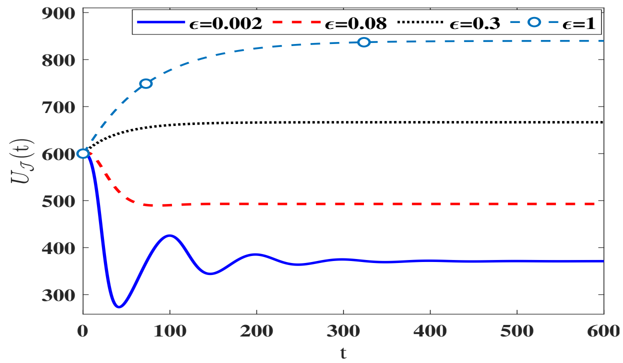
(B) Latently infected cells

(C) Actively infected cells

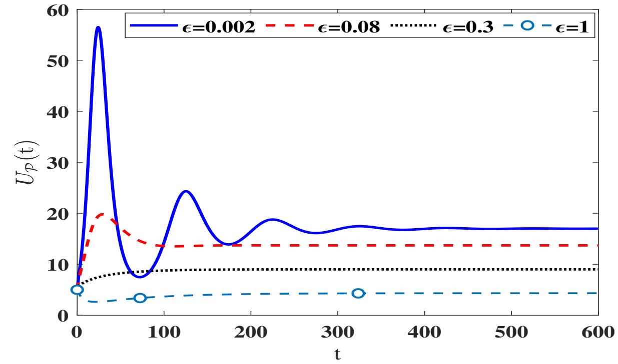
(D) HIV-1 particles

(E) Antibodies

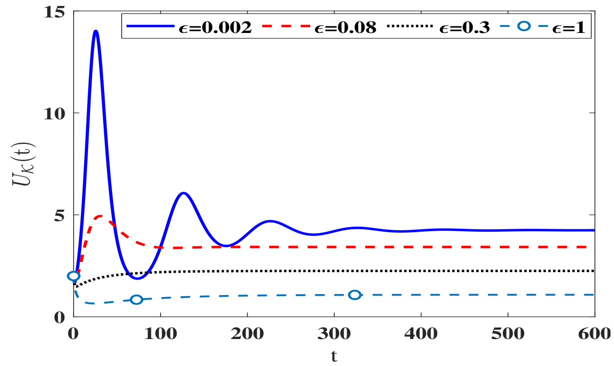
FIGURE 6. The influence of ℓ_i on the solution patterns in system (4.3).



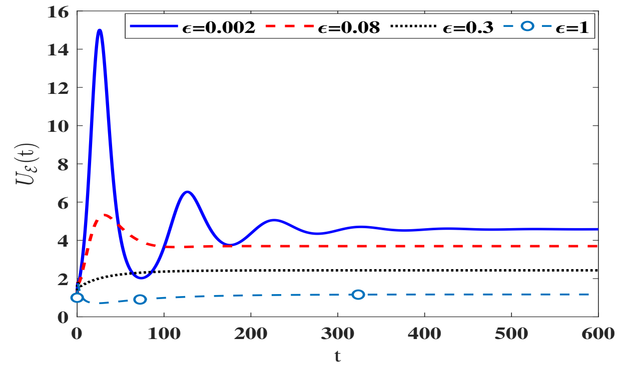
(A) Uninfected CD4$^+$T cells



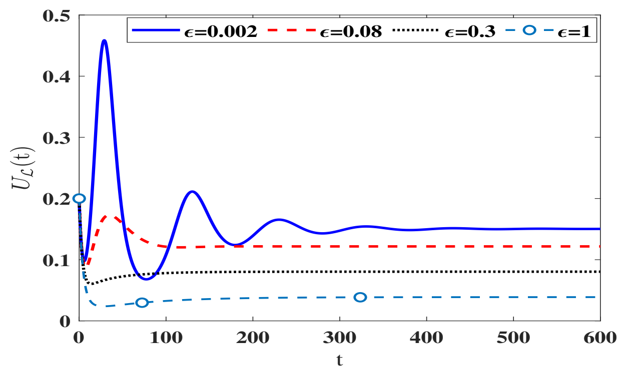
(B) Latently infected cells



(C) Actively infected cells



(D) HIV-1 particles



(E) Antibodies

FIGURE 7. The influence of saturation infection rates on the solution patterns in system (4.3).

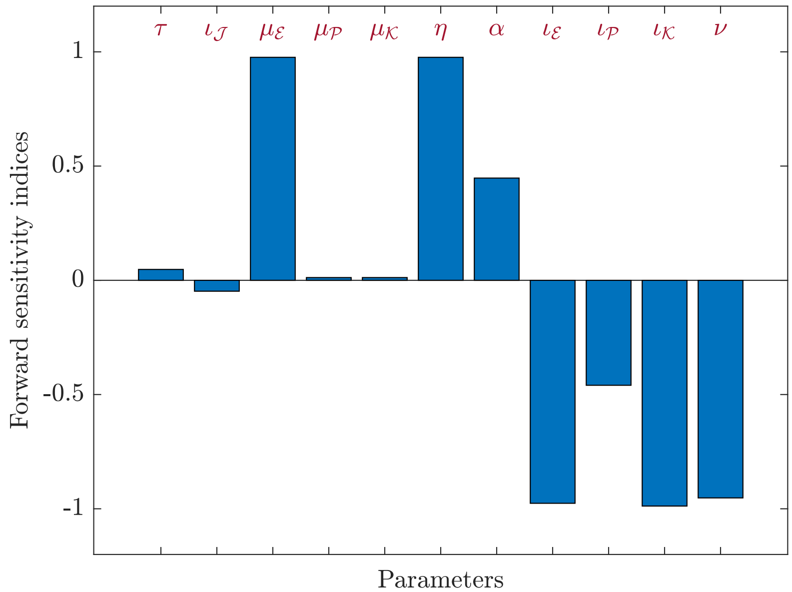


FIGURE 8. Analyzing sensitivity of parameters for $\mathfrak{R}_{0(4.1)}$ in model (4.1).

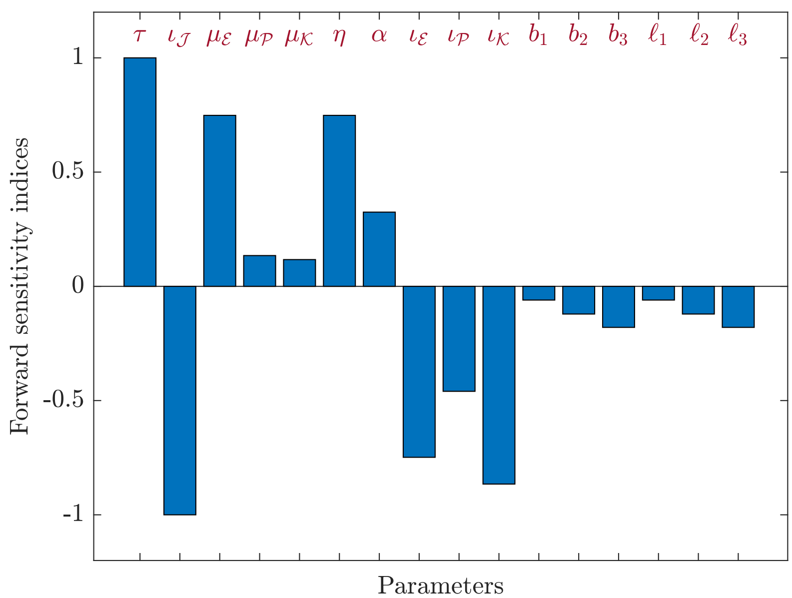


FIGURE 9. Analyzing sensitivity of parameters for $\tilde{\mathfrak{R}}_{0(4.3)}$ in model (4.3).

5. Discussion

To emphasize the importance of integrating the latent CIM spread in our investigated models, we analyze model (2.1) in the context of various antiviral interventions, outlined as follows:

$$\begin{cases} \dot{U}_{\mathcal{J}} &= \tau - \iota_{\mathcal{J}} U_{\mathcal{J}} - (1-\varsigma_1)\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}}, U_{\mathcal{E}}) - (1-\varsigma_2)\mathcal{F}_{\mathcal{P}}(U_{\mathcal{J}}, U_{\mathcal{P}}) \\ &\quad - (1-\varsigma_3)\mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}}, U_{\mathcal{K}}), \\ \dot{U}_{\mathcal{P}} &= (1-\varsigma_1)\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}}, U_{\mathcal{E}}) + (1-\varsigma_2)\mathcal{F}_{\mathcal{P}}(U_{\mathcal{J}}, U_{\mathcal{P}}) \\ &\quad + (1-\varsigma_3)\mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}}, U_{\mathcal{K}}) - (\alpha + \iota_{\mathcal{P}}) U_{\mathcal{P}}, \\ \dot{U}_{\mathcal{K}} &= \alpha U_{\mathcal{P}} - \iota_{\mathcal{K}} U_{\mathcal{K}}, \\ \dot{U}_{\mathcal{E}} &= \eta U_{\mathcal{K}} - \iota_{\mathcal{E}} U_{\mathcal{E}} - \lambda U_{\mathcal{L}} U_{\mathcal{E}}, \\ \dot{U}_{\mathcal{L}} &= \delta U_{\mathcal{E}} - \iota_{\mathcal{L}} U_{\mathcal{L}} - \theta U_{\mathcal{L}} U_{\mathcal{E}}. \end{cases} \tag{5.1}$$

The parameter $0 \le \varsigma_1 \le 1$ represents the effectiveness of antiviral therapy in blocking VIM transmission, while $0 \le \varsigma_2, \varsigma_3 \le 1$ denote the therapeutic efficacies in blocking latent and active CIM transmissions, respectively [78].

For system (5.1), the following establishes the basic reproduction ratio:

$$\mathfrak{R}_0 = \frac{(1-\varsigma_1)\eta\alpha}{\iota_{\mathcal{E}}\iota_{\mathcal{K}}(\alpha+\iota_{\mathcal{P}})} \frac{\partial \mathcal{F}_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{E}}} + \frac{(1-\varsigma_2)}{\alpha+\iota_{\mathcal{P}}} \frac{\partial \mathcal{F}_{\mathcal{P}}\left(\bar{U}^0_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{P}}} + \frac{(1-\varsigma_3)\alpha}{\iota_{\mathcal{K}}(\alpha+\iota_{\mathcal{P}})} \frac{\partial \mathcal{F}_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{K}}}.$$

Assuming ς to be equal to ς_1, ς_2, and ς_3, we obtain:

$$\begin{aligned} \mathfrak{R}^{\varsigma}_0 &= (1-\varsigma)\left[\frac{\eta\alpha}{\iota_{\mathcal{E}}\iota_{\mathcal{K}}(\alpha+\iota_{\mathcal{P}})} \frac{\partial \mathcal{F}_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{E}}} + \frac{1}{\alpha+\iota_{\mathcal{P}}} \frac{\partial \mathcal{F}_{\mathcal{P}}\left(\bar{U}^0_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{P}}} + \frac{\alpha}{\iota_{\mathcal{K}}(\alpha+\iota_{\mathcal{P}})} \frac{\partial \mathcal{F}_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{K}}}\right] \\ &= (1-\varsigma)\mathfrak{R}_0. \end{aligned}$$

Currently, we compute the drug efficacy, ς, which results in $\mathfrak{R}^{\varsigma}_0 < 1$ and stabilizes $\bar{O}^0$ in system (5.1) as follows:

$$1 \ge \varsigma > \tilde{\varsigma}_{\min} = \max\left\{0, 1 - \frac{1}{\mathfrak{R}_0}\right\}. \tag{5.2}$$

Neglecting the spread of latent CIM in model (5.1) yields

$$\begin{cases} \dot{U}_{\mathcal{J}} &= \tau - \iota_{\mathcal{J}} U_{\mathcal{J}} - (1-\varsigma)\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}}, U_{\mathcal{E}}) - (1-\varsigma)\mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}}, U_{\mathcal{K}}), \\ \dot{U}_{\mathcal{P}} &= (1-\varsigma)\mathcal{F}_{\mathcal{E}}(U_{\mathcal{J}}, U_{\mathcal{E}}) + + (1-\varsigma)\mathcal{F}_{\mathcal{K}}(U_{\mathcal{J}}, U_{\mathcal{K}}) - (\alpha + \iota_{\mathcal{P}}) U_{\mathcal{P}}, \\ \dot{U}_{\mathcal{K}} &= \alpha U_{\mathcal{P}} - \iota_{\mathcal{K}} U_{\mathcal{K}}, \\ \dot{U}_{\mathcal{E}} &= \eta U_{\mathcal{K}} - \iota_{\mathcal{E}} U_{\mathcal{E}} - \lambda U_{\mathcal{L}} U_{\mathcal{E}}, \\ \dot{U}_{\mathcal{L}} &= \delta U_{\mathcal{E}} - \iota_{\mathcal{L}} U_{\mathcal{L}} - \theta U_{\mathcal{L}} U_{\mathcal{E}}, \end{cases} \tag{5.3}$$

and the definition of the basic reproduction ratio of model (5.3) will be

$$\hat{\mathfrak{R}}^{\varsigma}_0 = (1-\varsigma)\left[\frac{\eta\alpha}{\iota_{\mathcal{E}}\iota_{\mathcal{K}}(\alpha+\iota_{\mathcal{P}})} \frac{\partial \mathcal{F}_{\mathcal{E}}\left(\bar{U}^0_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{E}}} + \frac{\alpha}{\iota_{\mathcal{K}}(\alpha+\iota_{\mathcal{P}})} \frac{\partial \mathcal{F}_{\mathcal{K}}\left(\bar{U}^0_{\mathcal{J}}, 0\right)}{\partial U_{\mathcal{K}}}\right] = (1-\varsigma)\hat{\mathfrak{R}}_0.$$

We identify the drug efficacy, denoted as ς, that ensures $\hat{\Re}_0^{\varsigma} < 1$, thereby stabilizing the equilibrium $\bar{O}^0$ of system (5.3). The condition is expressed as:

$$1 \geq \varsigma > \hat{\varsigma}_{\min} = \max\left\{0, 1 - \frac{1}{\hat{\Re}_0}\right\}. \tag{5.4}$$

It is evident that $\hat{\Re}_0 < \Re_0$. Interestingly, comparing Eqs. (5.2) and (5.4) reveals that $\hat{\varsigma}_{\min}$ is always smaller than $\varsigma_{\min}$. Therefore, applying drugs with efficacy ς in case of $\hat{\varsigma}_{\min} \leq \varsigma < \varsigma_{\min}$, will guarantee that $\hat{\Re}_0^{\varsigma} < 1$, ensuring the global asymptotic stability of $\bar{O}^0$ in system (5.3). However, $\Re_0^{\varsigma} > 1$, indicating the instability of $\bar{O}^0$ in system (5.1) . As a result, drug therapies designed without considering the spread of the latent CIM may be inaccurate or insufficient for achieving virus eradication from the body.

6. Conclusion

This work formulates two novel models providing insights into HIV-1 dynamics while considering antibody immune impairment. These models comprise five compartments: circulating HIV-1 particles, infected cells harboring latent and active types, immune cells of the antibodies type, and healthy cells from $CD4^{+}T$ lymphocytes. To enhance realism, we introduced a scenario in which healthy cells acquire susceptibility to infection upon exposure to infectious compartments (HIV-1 particles, and infected cells). The second model takes the first model a step further by incorporating DDEs. Both models incorporate general functions to represent the incidence rates. Notably, the solutions produced by these models display properties of nonnegativity and boundedness. Within this framework, we identified two crucial equilibria: the virus-free equilibrium, $\bar{O}^0$ (O^0), and the virus-persistence equilibrium, $\bar{O}^1$ (O^1). We calculated the basic reproduction ratios, referred to as $\Re_0$ ($\tilde{\Re}_0$), which are essential for determining whether the equilibria mentioned earlier exist and remain stable over time. It is noteworthy that $\Re_0$ ($\tilde{\Re}_0$) consists of three separate components, representing the contributions from VIM, latent CIM, and active CIM. The Lyapunov function technique and L.I.P. are employed to comprehend the system's behavior over time by studying the global asymptotic stability of the equilibria. Two key findings emerged from our investigation: firstly, the virus-free equilibrium $\bar{O}^0$ (O^0) is being G.A.S whenever $\Re_0 < 1$ ($\tilde{\Re}_0 < 1$), ultimately leading to the disappearance of the infection. In contrast, the equilibrium $\bar{O}^0$ (O^0) loses its stability and the virus-persistence equilibrium $\bar{O}^1$ (O^1) is being G.A.S whenever $\Re_0 > 1$ ($\tilde{\Re}_0 > 1$), indicating a long-term infection. For experimental verification of our theoretical derivations, we employed numerical simulations, which yielded results consistent with our analytical solutions. Our exploration encompassed the influence of antibody immune impairment, time delays, and latent CIM on HIV-1 dynamics, revealing reduced immune function as a key factor contributing significantly to disease progression. Moreover, time delays significantly reduced the basic reproduction ratio ($\tilde{\Re}_0$), leading to suppressed HIV-1 reproduction. Therefore, eliminating HIV-1 requires prioritizing strategies that keep $\tilde{\Re}_0$ below 1. Our study also highlights the critical role of latent CIM spread in HIV-1 dynamics. Additionally, a sensitivity analysis revealed key factors influencing the system's behavior, deepening our understanding of HIV-1 dynamics.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S.Y. Zhang, M.A. Taye, The Efficacy of Antiviral Drug, HIV Viral Load and the Immune Response, arXiv:2101.10413 [physics.bio-ph] (2021). http://arxiv.org/abs/2101.10413.

[2] D. Wodarz, D.N. Levy, Human Immunodeficiency Virus Evolution Towards Reduced Replicative Fitnessin Vivoand the Development of AIDS, Proc. R. Soc. B. 274 (2007), 2481–2491. https://doi.org/10.1098/rspb.2007.0413.

[3] S.L. Swain, K.K. McKinstry, T.M. Strutt, Expanding Roles for $CD4^+$ T Cells in Immunity to Viruses, Nat. Rev. Immunol. 12 (2012), 136–148. https://doi.org/10.1038/nri3152.

[4] UNAIDS, 2023 UNAIDS Global AIDS Update, https://thepath.unaids.org.

[5] D. Callaway, HIV-1 Infection and Low Steady State Viral Loads, Bull. Math. Biol. 64 (2002), 29–64. https://doi.org/10.1006/bulm.2001.0266.

[6] A.S. Perelson, P.W. Nelson, Mathematical Analysis of HIV-1 Dynamics in Vivo, SIAM Rev. 41 (1999), 3–44. https://doi.org/10.1137/s0036144598335107.

[7] M.A. Nowak, R.M. May, Virus Dynamics, Oxford University Press, Oxford, 2000.

[8] M.A. Nowak, C.R.M. Bangham, Population Dynamics of Immune Responses to Persistent Viruses, Science 272 (1996), 74–79. https://doi.org/10.1126/science.272.5258.74.

[9] E.A. Hernandez-Vargas, R.H. Middleton, Modeling the Three Stages in HIV Infection, J. Theor. Biol. 320 (2013), 33–40. https://doi.org/10.1016/j.jtbi.2012.11.028.

[10] M.A. Stafford, L. Corey, Y. Cao, E.S. Daar, D.D. Ho, A.S. Perelson, Modeling Plasma Virus Concentration during Primary HIV Infection, J. Theor. Biol. 203 (2000), 285–301. https://doi.org/10.1006/jtbi.2000.1076.

[11] D. Wodarz, R.M. May, M.A. Nowak, The Role of Antigen-Independent Persistence of Memory Cytotoxic T Lymphocytes, Int. Immunol. 12 (2000), 467–477. https://doi.org/10.1093/intimm/12.4.467.

[12] C. Jolly, Q.J. Sattentau, Retroviral Spread by Induction of Virological Synapses, Traffic 5 (2004), 643–650. https://doi.org/10.1111/j.1600-0854.2004.00209.x.

[13] C. Vargas-Garcia, R. Zurakowski, A. Singh, Synaptic Transmission May Provide an Evolutionary Benefit to Hiv Through Modulation of Latency, J. Theor. Biol. 455 (2018), 261–268. https://doi.org/10.1016/j.jtbi.2018.07.030.

[14] M. Sourisseau, N. Sol-Foulon, F. Porrot, F. Blanchet, O. Schwartz, Inefficient Human Immunodeficiency Virus Replication in Mobile Lymphocytes, J. Virol. 81 (2007), 1000–1012. https://doi.org/10.1128/jvi.01629-06.

[15] H. Sato, J. Orensteint, D. Dimitrov, M. Martin, Cell-to-Cell Spread of HIV-1 Occurs Within Minutes and May Not Involve the Participation of Virus Particles, Virology 186 (1992), 712–724. https://doi.org/10.1016/0042-6822(92)90038-q.

[16] S. Iwami, J.S. Takeuchi, S. Nakaoka, F. Mammano, F. Clavel, H. Inaba, T. Kobayashi, N. Misawa, K. Aihara, Y. Koyanagi, K. Sato, Cell-to-Cell Infection by HIV Contributes Over Half of Virus Infection, eLife 4 (2015), e08150. https://doi.org/10.7554/elife.08150.

[17] N.L. Komarova, D. Wodarz, Virus Dynamics in the Presence of Synaptic Transmission, Math. Biosci. 242 (2013), 161–171. https://doi.org/10.1016/j.mbs.2013.01.003.

[18] N. Martin, Q. Sattentau, Cell-to-Cell HIV-1 Spread and Its Implications for Immune Evasion, Curr. Opin. HIV AIDS 4 (2009), 143–149. https://doi.org/10.1097/coh.0b013e328322f94a.

[19] A. Sigal, J.T. Kim, A.B. Balazs, E. Dekel, A. Mayo, R. Milo, D. Baltimore, Cell-to-Cell Spread of HIV Permits Ongoing Replication Despite Antiretroviral Therapy, Nature 477 (2011), 95–98. https://doi.org/10.1038/nature10347.

[20] Y. Gao, J. Wang, Threshold Dynamics of a Delayed Nonlocal Reaction-Diffusion Hiv Infection Model With Both Cell-Free and Cell-to-Cell Transmissions, J. Math. Anal. Appl. 488 (2020), 124047. https://doi.org/10.1016/j.jmaa.2020.124047.

[21] J. Lin, R. Xu, X. Tian, Threshold Dynamics of an HIV-1 Virus Model With Both Virus-to-Cell and Cell-to-Cell Transmissions, Intracellular Delay, and Humoral Immunity, Appl. Math. Comp. 315 (2017), 516–530. https://doi.org/10.1016/j.amc.2017.08.004.

[22] T. Zheng, Y. Luo, Z. Teng, Spatial dynamics of a viral infection model with immune response and nonlinear incidence, Z. Angew. Math. Phys. 74 (2023), 124. https://doi.org/10.1007/s00033-023-02015-8.

[23] Y. Luo, L. Zhang, T. Zheng, Z. Teng, Analysis of a Diffusive Virus Infection Model With Humoral Immunity, Cell-to-Cell Transmission and Nonlinear Incidence, Physica A: Stat. Mech. Appl. 535 (2019), 122415. https://doi.org/10.1016/j.physa.2019.122415.

[24] R. Zhang, S. Liu, Global Dynamics of an Age-Structured Within-Host Viral Infection Model With Cell-to-Cell Transmission and General Humoral Immunity Response, Math. Biosci. Eng. 17 (2020), 1450–1478. https://doi.org/10.3934/mbe.2020075.

[25] Z. She, X. Jiang, Threshold Dynamics of a General Delayed Within-Host Viral Infection Model With Humoral Immunity and Two Modes of Virus Transmission, Discr. Contin. Dyn. Syst.-Ser. B 26 (2021), 3835–3861. https://doi.org/10.3934/dcdsb.2020259.

[26] S. Pan, S.P. Chakrabarty, Threshold Dynamics of HCV Model With Cell-to-Cell Transmission and a Non-Cytolytic Cure in the Presence of Humoral Immunity, Comm. Nonlinear Sci. Numer. Simul. 61 (2018), 180–197. https://doi.org/10.1016/j.cnsns.2018.02.010.

[27] M. Dhar, S. Samaddar, P. Bhattacharya, Modeling the Cell-to-Cell Transmission Dynamics of Viral Infection Under the Exposure of Non-Cytolytic Cure, J. Appl. Math. Comp. 65 (2020), 885–911. https://doi.org/10.1007/s12190-020-01420-w.

[28] A. De Milito, B Lymphocyte Dysfunctions in HIV Infection, Curr. HIV Res. 2 (2004), 11–21. https://doi.org/10.2174/1570162043485068.

[29] S. Amu, N. Ruffin, B. Rethi, F. Chiodi, Impairment of B-cell Functions during HIV-1 Infection, AIDS 27 (2013), 2323–2334. https://doi.org/10.1097/qad.0b013e328361a427.

[30] F. Chiodi, G. Scarlatti, Editorial: HIV-Induced Damage of B Cells and Production of HIV Neutralizing Antibodies, Front. Immunol. 9 (2018), 297. https://doi.org/10.3389/fimmu.2018.00297.

[31] P. Lydyard, A. Whelan, M. Fanger, BIOS Instant Notes in Immunology, Taylor & Francis, 2005. https://doi.org/10.4324/9780203808306.

[32] H. Miao, X. Abdurahman, Z. Teng, L. Zhang, Dynamical Analysis of a Delayed Reaction-Diffusion Virus Infection Model With Logistic Growth and Humoral Immune Impairment, Chaos Solitons Fractals 110 (2018), 280–291. https://doi.org/10.1016/j.chaos.2018.03.006.

[33] A.M. Elaiw, S.F. Alshehaiween, A.D. Hobiny, Global Properties of a Delay-Distributed HIV Dynamics Model Including Impairment of B-Cell Functions, Mathematics 7 (2019), 837. https://doi.org/10.3390/math7090837.

[34] A.M. Elaiw, S.F. Alshehaiween, A.D. Hobiny, Global Properties of HIV Dynamics Models Including Impairment of B-Cell Functions, J. Biol. Syst. 28 (2020), 1–25. https://doi.org/10.1142/s0218339020500011.

[35] P. Wu, H. Zhao, Mathematical Analysis of an Age-Structured HIV/AIDS Epidemic Model With Haart and Spatial Diffusion, Nonlinear Anal.: Real World Appl. 60 (2021), 103289. https://doi.org/10.1016/j.nonrwa.2021.103289.

[36] P. Wu, Z. Feng, X. Zhang, Global Dynamics of an Age–space Structured HIV/AIDS Model With Viral Load-Dependent Infection and Conversion Rates, J. Comp. Appl. Math. 412 (2022), 114309. https://doi.org/10.1016/j.cam.2022.114309.

[37] P. Wu, R. Zhang, A. Din, Mathematical Analysis of an Age-Since Infection and Diffusion HIV/AIDS Model With Treatment Adherence and Dirichlet Boundary Condition, Math. Comp. Simul. 214 (2023), 1–27. https://doi.org/10.1016/j.matcom.2023.06.018.

[38] L. Rong, A.S. Perelson, Modeling HIV Persistence, the Latent Reservoir, and Viral Blips, J. Theor. Biol. 260 (2009), 308–331. https://doi.org/10.1016/j.jtbi.2009.06.011.

[39] B.J. Nath, K. Sadri, H.K. Sarmah, K. Hosseini, An Optimal Combination of Antiretroviral Treatment and Immunotherapy for Controlling HIV Infection, Math. Comp. Simul. 217 (2024), 226–243. https://doi.org/10.1016/j.matcom.2023.10.012.

[40] T.W. Chun, L. Stuyver, S.B. Mizell, L.A. Ehler, J.A.M. Mican, M. Baseler, A.L. Lloyd, M.A. Nowak, A.S. Fauci, Presence of an Inducible HIV-1 Latent Reservoir During Highly Active Antiretroviral Therapy, Proc. Natl. Acad. Sci. U.S.A. 94 (1997), 13193–13197. https://doi.org/10.1073/pnas.94.24.13193.

[41] J.K. Wong, M. Hezareh, H.F. Gunthard, D.V. Havlir, C.C. Ignacio, C.A. Spina, D.D. Richman, Recovery of Replication-Competent HIV Despite Prolonged Suppression of Plasma Viremia, Science 278 (1997), 1291–1295. https://doi.org/10.1126/science.278.5341.1291.

[42] L.M. Agosto, M.B. Herring, W. Mothes, A.J. Henderson, HIV-1-Infected CD4+ T Cells Facilitate Latent Infection of Resting CD4+ T Cells through Cell-Cell Contact, Cell Rep. 24 (2018), 2088–2100. https://doi.org/10.1016/j.celrep.2018.07.079.

[43] W. Wang, X. Wang, K. Guo, W. Ma, Global Analysis of a Diffusive Viral Model With Cell-to-cell Infection and Incubation Period, Math. Methods Appl. Sci. 43 (2020), 5963–5978. https://doi.org/10.1002/mma.6339.

[44] K. Hattaf, H. Dutta, Modeling the Dynamics of Viral Infections in Presence of Latently Infected Cells, Chaos Solitons Fractals 136 (2020), 109916. https://doi.org/10.1016/j.chaos.2020.109916.

[45] A.M. Elaiw, N.H. AlShamrani, Stability of a General CTL-Mediated Immunity HIV Infection Model With Silent Infected Cell-to-Cell Spread, Adv. Diff. Equ. 2020 (2020), 355. https://doi.org/10.1186/s13662-020-02818-3.

[46] N.H. AlShamrani, R.H. Halawani, W. Shammakh, A.M. Elaiw, Stability of Impaired Humoral Immunity HIV-1 Models with Active and Latent Cellular Infections, Computation 11 (2023), 207. https://doi.org/10.3390/computation11100207.

[47] A.M. Elaiw, S.F. Alshehaiween, Global Stability of Delay-distributed Viral Infection Model With Two Modes of Viral Transmission and B-cell Impairment, Math. Methods Appl. Sci. 43 (2020), 6677–6701. https://doi.org/10.1002/mma.6408.

[48] A.M. Elaiw, S.F. Alshehaiween, A.D. Hobiny, Impact of B-Cell Impairment on Virus Dynamics With Time Delay and Two Modes of Transmission, Chaos Solitons Fractals 130 (2020), 109455. https://doi.org/10.1016/j.chaos.2019.109455.

[49] A.M. Elaiw, N.H. AlShamrani, Global Stability of Humoral Immunity Virus Dynamics Models With Nonlinear Infection Rate and Removal, Nonlinear Anal.: Real World Appl. 26 (2015), 161–190. https://doi.org/10.1016/j.nonrwa.2015.05.007.

[50] H. Shu, Y. Chen, L. Wang, Impacts of the Cell-Free and Cell-to-Cell Infection Modes on Viral Dynamics, J. Dyn. Diff. Equ. 30 (2017), 1817–1836. https://doi.org/10.1007/s10884-017-9622-2.

[51] P. van den Driessche, J. Watmough, Reproduction Numbers and Sub-Threshold Endemic Equilibria for Compartmental Models of Disease Transmission, Math. Biosci. 180 (2002), 29–48. https://doi.org/10.1016/s0025-5564(02)00108-6.

[52] A. Korobeinikov, Global Properties of Basic Virus Dynamics Models, Bull. Math. Biol. 66 (2004), 879–883. https://doi.org/10.1016/j.bulm.2004.02.001.

[53] J.K. Hale, S. M. V. Lunel, Introduction to Functional Differential Equations, Springer, New York, 1993.

[54] H.K. Khalil, Nonlinear Systems, 3rd Edition, Prentice Hall, Upper Saddle River, 2002.

[55] G. Huang, Y. Takeuchi, W. Ma, Lyapunov Functionals for Delay Differential Equations Model of Viral Infections, SIAM J. Appl. Math. 70 (2010), 2693–2708. https://doi.org/10.1137/090780821.

[56] J. Xu, G. Huang, Global Stability and Bifurcation Analysis of a Virus Infection Model with Nonlinear Incidence and Multiple Delays, Fractal Fract. 7 (2023), 583. https://doi.org/10.3390/fractalfract7080583.

[57] Y. Kuang, Delay Differential Equations With Applications in Population Dynamics, Academic Press, San Diego, 1993.

[58] P.H. Crowley, E.K. Martin, Functional Responses and Interference within and between Year Classes of a Dragonfly Population, J. N. Amer. Benthol. Soc. 8 (1989), 211–221. https://doi.org/10.2307/1467324.

[59] R.K. Upadhyay, R.K. Naji, Dynamics of a Three Species Food Chain Model With Crowley–martin Type Functional Response, Chaos Solitons Fractals 42 (2009), 1337–1346. https://doi.org/10.1016/j.chaos.2009.03.020.

[60] F. Gazori, M. Hesaaraki, Global Stability of a Within-Host Dynamics of Malaria and the Immune System With Crowley-Martin Functional Response, Asian J. Math. Comp. Res. 3 (2015), 201–219.

[61] J.P. Tripathi, S. Tyagi, S. Abbas, Global Analysis of a Delayed Density Dependent Predator–prey Model With Crowley-martin Functional Response, Comm. Nonlinear Sci. Numer. Simul. 30 (2016), 45–69. https://doi.org/10.1016/j.cnsns.2015.06.008.

[62] R.K.K. Upadhyay, S.N.N. Raw, V. Rai, Dynamical Complexities in a Tri-Trophic Hybrid Food Chain Model With Holling Type II and Crowley–martin Functional Responses, Nonlinear Anal.: Model. Control 15 (2010), 361–375. https://doi.org/10.15388/na.15.3.14331.

[63] I. Nali, A. Dénes, Global Dynamics of a Within-Host Model for Usutu Virus, Computation 11 (2023), 226. https://doi.org/10.3390/computation11110226.

[64] S.K. Sahani, Yashi, Effects of Eclipse Phase and Delay on the Dynamics of HIV Infection, J. Biol. Syst. 26 (2018), 421–454. https://doi.org/10.1142/s0218339018500195.

[65] F. Li, W. Ma, Dynamics Analysis of an HTLV-1 Infection Model With Mitotic Division of Actively Infected Cells and Delayed CTL Immune Response, Math Methods Appl. Sci. 41 (2018), 3000–3017. https://doi.org/10.1002/mma.4797.

[66] A.S. Perelson, D.E. Kirschner, R. De Boer, Dynamics of HIV Infection of CD4+ T Cells, Math. Biosci. 114 (1993), 81–125. https://doi.org/10.1016/0025-5564(93)90043-a.

[67] H. Mohri, S. Bonhoeffer, S. Monard, A.S. Perelson, D.D. Ho, Rapid Turnover of T Lymphocytes in SIV-Infected Rhesus Macaques, Science 279 (1998), 1223–1227. https://doi.org/10.1126/science.279.5354.1223.

[68] Y. Wang, J. Liu, L. Liu, Viral Dynamics of an HIV Model With Latent Infection Incorporating Antiretroviral Therapy, Adv. Diff. Equ. 2016 (2016), 225. https://doi.org/10.1186/s13662-016-0952-x.

[69] S. Wang, X. Song, Z. Ge, Dynamics Analysis of a Delayed Viral Infection Model With Immune Impairment, Appl. Math. Model. 35 (2011), 4877–4885. https://doi.org/10.1016/j.apm.2011.03.043.

[70] C. Yan, W. Wang, Modeling HIV Dynamics Under Combination Therapy with Inducers and Antibodies, Bull. Math. Biol. 81 (2019), 2625–2648. https://doi.org/10.1007/s11538-019-00621-0.

[71] C. Sun, L. Li, J. Jia, Hopf Bifurcation of an Hiv-1 Virus Model With Two Delays and Logistic Growth, Math. Model. Nat. Phenom. 15 (2020), 16. https://doi.org/10.1051/mmnp/2019038.

[72] S. Wang, D. Zou, Global Stability of In-Host Viral Models With Humoral Immunity and Intracellular Delays, Appl. Math. Model. 36 (2012), 1313–1322. https://doi.org/10.1016/j.apm.2011.07.086.

[73] T. Wang, Z. Hu, F. Liao, W. Ma, Global Stability Analysis for Delayed Virus Infection Model With General Incidence Rate and Humoral Immunity, Math. Comp. Simul. 89 (2013), 13–22. https://doi.org/10.1016/j.matcom.2013.03.004.

[74] K. Allali, J. Danane, Y. Kuang, Global Analysis for an HIV Infection Model with CTL Immune Response and Infected Cells in Eclipse Phase, Appl. Sci. 7 (2017), 861. https://doi.org/10.3390/app7080861.

[75] Q. Sun, L. Min, Y. Kuang, Global Stability of Infection-free State and Endemic Infection State of a Modified Human Immunodeficiency Virus Infection Model, IET Syst. Biol. 9 (2015), 95–103. https://doi.org/10.1049/iet-syb.2014.0046.

[76] A.M. Elaiw, Global Threshold Dynamics in Humoral Immunity Viral Infection Models Including an Eclipse Stage of Infected Cells, J. Korean Soc. Ind. Appl. Math. 19 (2015), 137–170. https://doi.org/10.12941/JKSIAM.2015.19.137.

[77] K. Hattaf, Global Stability and Hopf Bifurcation of a Generalized Viral Infection Model With Multi-Delays and Humoral Immunity, Physica A: Stat. Mech. Appl. 545 (2020), 123689. https://doi.org/10.1016/j.physa.2019.123689.

[78] X. Wang, L. Rong, HIV Low Viral Load Persistence Under Treatment: Insights From a Model of Cell-to-Cell Viral Transmission, Appl. Math. Lett. 94 (2019), 44–51. https://doi.org/10.1016/j.aml.2019.02.019.