Int. J. Anal. Appl. (2024), 22:200

International Journal of Analysis and Applications

$\ddot{\eta}$-Ricci Soliton and Its Applications on ϕ-Recurrent LP Sasakian Manifold

Pankaj Pandey[1], Kamakshi Sharma[1], B.B. Chaturvedi[2], Mohammad Nazrul Islam Khan[3,*]

[1] *School of Chemical Engineering and Physical Sciences, Lovely Professional University, Phagwara-144411, India*

[2] *Department of Pure & Applied Mathematics, Guru Ghasidas Vishwavidyalaya, Bilaspur (C.G.)-495009, India*

[3] *Department of Computer Engineering, College of Computer, Qassim University, Buraydah 51452, Saudi Arabia*

[*] *Corresponding author:* m.nazrul@qu.edu.sa

Abstract. The objective of this paper is to study the ϕ-recurrent nature and application of LP-Sasakian manifold associated with $\ddot{\eta}$-Ricci soliton. Initially, we introduce an example to show the existance of LP Sasakian manifold equipped with $\ddot{\eta}$-Ricci soliton. The idea of Ricci soliton recognized as a generalization of an Einstein metric and governing as the solution of partial differential equations representing Ricci flow. Some conditions have been obtained representing the nature of soliton (expanding, unchanged and shrinkingness) in pseudo projective, Weyl projective and semi generlized curvature with ϕ-recurrent condition. The application of $\ddot{\eta}$-Ricci Soliton on spacetime also discussed.

1. Introduction

The notion of symmetry in mathematical structures is commonly considered a fundamental aspect of mathematics, especially in the areas of Ricci soliton and paracontact geometry. In order to analyse compact three-dimensional manifold with positive Ricci curvature, Hamilton [13] explored the concept of Ricci flow before four decades. The idea of Ricci soliton recognized as a generalization of an Einstein metric and governing as the solution of partial differential equations representing Ricci flow. In case of Ricci flow, the Riemannian metric is proportional to $\frac{1}{2}Lg + Ric$ and this proportionality constant is called the soliton constant. The Ricci flow in terms of soliton

Received: Aug. 28, 2024.

2020 *Mathematics Subject Classification.* 53C25, 53C50, 53D10, 53C44.

Key words and phrases. Ricci soliton; LP-Sasakian manifold; ϕ-recurrent; pseudo projective; Weyl projective; differential equations; partial differential equations; semi-generalized.

https://doi.org/10.28924/2291-8639-22-2024-200
© 2024 the author(s).
ISSN: 2291-8639

constant λ (say) is defined by the following equation [13]:

$$\frac{1}{2}(L_v g)(G,K) + S(G,K) + \lambda g(G,K) = 0. \tag{1.1}$$

In (1.1), L is considered as Lie derivative along the complete vector field v and the Ricci tensor is denoted by S for all G, K vector fields in $\chi(M)$ (an algebra of tangent vector fields). Here, if the λ has positive (negative) value then we say expanding (compacting) soliton and is called unchanged in nature (steady) for $\lambda = 0$.

The concept of Ricci soliton on $f-$Kenmotsu manifold consisting almost conformal tensor has been generalized by Hui, Yadav and Patra [17]. They elaboarated ϕ-Ricci symmetric and cyclic Ricci parallel $f-$Kenmotsu manifold. Results on torse forming vector field have been obtained on Yamabe soliton by Mandal and Hui [19]. They apply semi-symmetric and projective semi-symmetric connection to discuss the existence conditions on compacting, steady and expanding. In [5] Chaubey and De investigated Riemannian manifolds equipped with Ricci soliton. In 2022 De, Sardar and De [11] extended the solitonic approch and defined Ricci-Yamabe soliton [11] by considering a special vector field. By taking into account the pointwise collinear property with characteristic vector field, Santu Dey [8] (2023) investigated the pointwise conformal Ricci soliton. Chaturvedi, Bhagat and Khan [4] (2023) elaborated perfect dark fluids (as a dust, radiation and stiff matter) associated with Bochner flat Lorentzian Kahler space-time including Ricci-Yamabe soliton. Hui et al. [16] (2017) studied Ricci solitons on Ricci pseudo-symmetric $(LCS)_n$-manifold. In [10] (2023) U.C. De, M. N. I. Khan and A. Sardar elaborated h-Almost Ricci-Yamabe solitons in paracontact geometry. Poddae, Balasubramanian and Sharma [27] (2024) extended the study of Yamabe soliton in contact geometry. Haseeb and Chaubey [15] explained *-Ricci soliton on LP Sasakian manifold. Pandey and Sharma ([25, 26]) (2023-24) explored the Ricci soliton and introduced the idea of generalized Z-Ricci soliton with generalized Z-tensor under the assumption of Sasakian metric. Cho and Kimura [6] presented and defined the idea of $\ddot{\eta}$-Ricci soliton with following system:

$$\frac{1}{2}(L_v g)(G,K) + S(G,K) + \lambda g(G,K) + \mu\ddot{\eta}(G)\ddot{\eta}(K) = 0. \tag{1.2}$$

More specifically, the theory of (g, V, λ, μ) ($\ddot{\eta}$-Ricci soliton) simplifies to the Ricci soliton with $\mu = 0$. Majhi and Debabrata [18] (2019) extended η-Ricci soliton on LP-Sasakian manifold with some interesting results. The para-Kenmotsu manifold with $\ddot{\eta} - *-$Ricci soliton was described by Sardar, De, and Gezer [29] (2023). Additionally, it has been explained the para-Kenmotsu and para-Sasakian manifold admiting a gradient $\ddot{\eta} - *-$Ricci soliton in dimension three. An Einstein metric with $*-$conformal $\ddot{\eta}$-Ricci soliton is demonstrated by Sarkar and De [30] (2023). The $*-$conformal $\ddot{\eta}$-Ricci soliton admitting (κ, μ)-almost Kenmotsu structure studied in [30] and proved that $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ and $\ddot{\eta}$-Ricci soliton are locally isometric. Siddiqui, Oguzhan [31] (2020) considered η-Ricci soliton on Kenmotsu manifold with generalized symmetric metric connection. Shivaprasanna, Haque and Somashekhara [35] (2020) elaborated η-Ricci soliton on f-Kenmotsu manifolds. Almia and Upreti [1] (2023) discussed certain properties of η-Ricci soliton on η-Einstein para-Kenmotsu manifold. Mert and Atceken [21] (2023) explored almost η-Ricci solitons on pseudosymmetric Lorentzian generalized Sasakian space form. Haseeb, Bilal, Chaubey and Khan [14] studied geometry of indefinite Kenmotsu manifold as η-Ricci-Yamabe solitons.

On the other hand, the Sasakian manifold, under ϕ-recurrentness, was investigated by De et al. [12]. They demonstrated that, a Sasakian structure with a non-vanishing sectional curvature are similar to locally ϕ-symmetric manifolds, that implies the co-directionality of the characteristic and associated vector fields. By discribing the non-existence of generalized projectively ϕ-recurrent condition, Shaikh, Prakasha, and Ahmad [32] developed the concept on generalized ϕ-recurrent on LP Sasakian structure. Matsumoto [20] (1989) illustrated LP-Sasakian manifold and examined by Venkatesha, Bagewadi and Pradeep [38] (2011). Pandey and Chaturvedi continued this investigation by examining the Lorentzian metric and semi-symmetric constraints [23], [24]. In 2019, Vardhana, Venkatesha and Hui [39] examined semi generlized ϕ-recurrent conformal Ricci soliton on $(LCS)_n$ manifold.

Motivated by the above study, we are interested to investigate some new results on $\ddot{\eta}$-Ricci soliton under LP-Sasakian structure consisting special vector field. In this paper, covariant derivative operator is taken as ∇, the metric g assumes as a Lorentzian metric, the Riemann and Ricci curvature consider by R and S respectively and the scalar tensor by r.

We organize our paper in the following way: After the preliminaries (section 2), an example of $\ddot{\eta}$-Ricci soliton established to prove the existence of the considered manifold (section 3). In the next section 4, ϕ-recurrent properties have been discussed with LP-Sasakian structure on $\ddot{\eta}$-Ricci soliton. The next three subsequent sections (section: 5, 6 and 7) deal with pseudo projective, Weyl projective and semi generalized ϕ recurrent conditions. In the last section, application of $\ddot{\eta}$-Ricci soliton has been illustrated.

2. Preliminaries

An n-dimensional differentiable manifold (M^n, g) permitting a $(1,1)$ tensor field ϕ, a contravariant vector field ξ, a 1-form $\ddot{\eta}$, and a Lorentzian metric g, is called an LP-Sasakian manifold, if the following conditions hold [33]:

$$\begin{aligned} &\ddot{\eta}(\xi) = -1, \\ &\phi^2 G = G + \ddot{\eta}(G)\xi, \\ &g(\phi G, \phi K) = g(G, K) + \ddot{\eta}(G)\ddot{\eta}(K), \\ &g(G, \xi) = \ddot{\eta}(G), \end{aligned} \tag{2.1}$$

$$(\nabla_G \phi)K = g(G, K)\xi + \ddot{\eta}(K)G + 2\ddot{\eta}(G)\ddot{\eta}(K)\xi. \tag{2.2}$$

$$\phi\xi = 0, \ \ddot{\eta}(\phi G) = 0. \tag{2.3}$$

If we consider

$$\omega(G,K) = g(\phi G,K), \tag{2.4}$$

then closed 1-form $\ddot{\eta}$ provides the following in LP-Sasakian manifold

$$(\nabla_G \ddot{\eta})(K) = \omega(G,K),\ \omega(G,\xi) = 0, \tag{2.5}$$

for all the vector fields G, K, where ω denotes the symmetric (0,2) tensor.
Throughout the paper, we assume that LP-Sasakian manifold admits a special vector field defined by ([11], [2], [9])

$$\nabla_G \xi = \ddot{\eta}(G)\xi - G. \tag{2.6}$$

If (M^n, g) has to be supposed an LP-Sasakian structure $(\phi,\ \xi,\ \ddot{\eta},\ g)$ then the following relation obtained trivially:

$$R(\xi,G)K = \ddot{\eta}(K)G - g(G,K)\xi, \tag{2.7}$$

$$R(G,K)\xi = \ddot{\eta}(G)K - \ddot{\eta}(K)G, \tag{2.8}$$

$$S(G,\xi) = -2\ddot{\eta}(G), \tag{2.9}$$

$$S(\phi G,\phi K) = S(G,K) + 2\ddot{\eta}(G)\ddot{\eta}(K), \tag{2.10}$$

$$(\nabla_G \ddot{\eta})K = \ddot{\eta}(G)\ddot{\eta}(K) - g(G,K), \tag{2.11}$$

for all the the vector fields G, K.
By taking $V = \xi$ in (1.2) and using (2.6), we get

$$S(G,K) = (1-\lambda)g(G,K) - (\mu+1)\ddot{\eta}(G)\ddot{\eta}(K). \tag{2.12}$$

The above equation yields

$$QG = (1-\lambda)G - (\mu+1)\ddot{\eta}(G)\xi, \tag{2.13}$$

$$S(G,\xi) = (\mu-\lambda+2)\ddot{\eta}(G), \tag{2.14}$$

$$r = n(1-\lambda) + (\mu+1). \tag{2.15}$$

As a consequence, the covariant differentiation of Ricci curvature yields

$$(\nabla_H S)(K,\xi) = \nabla_H S(K,\xi) - S(\nabla_H K,\xi) - S(K,\nabla_H \xi). \tag{2.16}$$

The following lemma will be used in the subsequent sections:

Lemma 2.1. *[12] Let us consider the tuples (M^n, g) of ϕ-recurrent LP-Sasakian structure then ξ and ρ (characteristic and one-form vector fields respectively) seems co-directional and given by*

$$A(H) = -\ddot{\eta}(H)\ddot{\eta}(\rho). \tag{2.17}$$

The replacement by ξ to the field H in (2.17) gives

$$A(\xi) = \ddot{\eta}(\rho). \tag{2.18}$$

3. Existence of $\ddot{\eta}$-Ricci Soliton on LP Sasakian Manifold

Let us consider a 3-dimensional manifold $M^3 = \{(G, K, U) : U \neq 0\}$. Let the set $\{b_1, b_2, b_3\}$, denotes the basis on M^3, in the term of partial differential equations is given by

$$b_1 = b^u \frac{\partial}{\partial k},\ b_2 = b^u(\frac{\partial}{\partial g} + \frac{\partial}{\partial k}),\ b_3 = \frac{\partial}{\partial u}. \tag{3.1}$$

Let g be a Lorentzian metric defined by: $g(b_1, b_1) = g(b_2, b_2) = 1$, $g(b_3, b_3) = -1$, $g(b_1, b_2) = g(b_1, b_3) = g(b_2, b_3) = 0$, then Koszul formula yields

$$\begin{aligned} &\nabla_{b_1} b_1 = -b_3,\ \nabla_{b_1} b_2 = 0,\ \nabla_{b_1} b_3 = -b_1, \\ &\nabla_{b_2} b_1 = 0,\ \nabla_{b_2} b_2 = -b_3,\ \nabla_{b_2} b_3 = -b_2, \\ &\nabla_{b_3} b_1 = 0,\ \nabla_{b_1} b_3 = 0,\ \nabla_{b_3} b_3 = 0. \end{aligned} \tag{3.2}$$

By considering the 1-form $\ddot{\eta}$ and the $(1,1)$ tensor field ϕ as:

$$\ddot{\eta}(U) = g(U, b_3),$$

$$\phi(b_1) = -b_1,\ \phi(b_2) = -b_2,\ \phi(b_3) = 0,$$

the Riemann and the Ricci curvature coefficients obtain as

$$\begin{aligned} &R(b_1, b_2)b_3 = 0,\ R(b_2, b_3)b_3 = -b_2,\ R(b_1, b_3)b_3 = -b_1, \\ &R(b_1, b_2)b_2 = b_1,\ R(b_2, b_3)b_2 = -b_3,\ R(b_1, b_3)b_2 = 0, \\ &R(b_1, b_2)b_1 = -b_2,\ R(b_2, b_3)b_1 = 0,\ R(b_1, b_3)b_1 = -b_3. \end{aligned} \tag{3.3}$$

$$S(b_1, b_1) = S(b_2, b_2) = 2,\ S(b_3, b_3) = -2. \tag{3.4}$$

From (2.12), the Ricci curvature coefficients can be calculated

$$S(b_1, b_1) = S(b_2, b_2) = (1 - \lambda) \text{ and } S(b_3, b_3) = (\lambda - \mu - 2). \tag{3.5}$$

Since, the data $(\lambda, \mu) = (-1, -1)$ verifies the equation of $\ddot{\eta}$-Ricci soliton (1.2), therefore the metric defined by (3.1) represents the $\ddot{\eta}$-Ricci soliton satisfying LP Sasakian manifold.

4. ϕ-Recurrentness on $\ddot{\eta}$-Ricci Soliton

Definition 4.1. *An LP-Sasakian manifold is said to be ϕ-recurrent if there exists a non-zero 1-form A such that ([22], [7], [28]*

$$\phi^2(\nabla_H R)(G,K)U = A(H)R(G,K)U, \tag{4.1}$$

for arbitrary vector fields G, K, U, H.

Applicability of (2.1) on (4.1) yields

$$(\nabla_H R)(G,K)U + \ddot{\eta}((\nabla_H R)(G,K)U)\xi = A(H)R(G,K)U. \tag{4.2}$$

Contracting (4.2), we obtain

$$(\nabla_H S)(K,U) = A(H)S(K,U). \tag{4.3}$$

Replacing $U = \xi$ in (4.3) and together with (2.9) produces

$$(\nabla_H S)(K,\xi) = -2A(H)\ddot{\eta}(K). \tag{4.4}$$

On the other hand (2.16) yields the following with the help of (2.6), (2.9) and (2.11)

$$(\nabla_H S)(K,\xi) = -2g(H,K) + S(K,H). \tag{4.5}$$

Using (4.4) in (4.5), we get

$$S(H,K) = 2g(H,K) - 2A(H)\ddot{\eta}(K). \tag{4.6}$$

Using lemma (2.1), equation (4.6) reduces to

$$S(H,\xi) = 2\ddot{\eta}(H)[1 - \ddot{\eta}(\rho)]. \tag{4.7}$$

Using (2.14) and (2.18) in (4.7), we get

$$\lambda = \mu + A(\xi). \tag{4.8}$$

Therefore, we have

Theorem 4.1. *Let (M^n, g) be an LP Sasakian ϕ recurrent space in the context of $\ddot{\eta}$-Ricci soliton then solitonic behaviour depends on 1-form A and explained by:*

- $A(\xi) \in (-\mu,\ \infty)$ *provides expanding nature,*
- $A(\xi) \in (-\infty, -\mu)$ *gives shrinking nature,*
- $A(\xi) = -\mu$ *reflects that neither expand nor compact (steady).*

5. Pseudo Projective ϕ-Recurrentness on $\ddot{\eta}$-Ricci Soliton

Definition 5.1. *In an LP-Sasakian manifold (M^n, g), the pseudo-projective curvature tensor $\bar{P}$ is defined by ([37], [3], [36])*

$$\begin{aligned}\bar{P}(G,K)U = aR(G,K)U + b\,[S(K,U)G - S(G,U)K] - \\ \frac{r}{n}(\frac{a}{n-1} + b)\,[g(K,U)G - g(G,U)K].\end{aligned} \tag{5.1}$$

Definition 5.2. *An LP-Sasakian manifold is said to be pseudo-projective ϕ-recurrent if there exists a non-zero 1-form A such that [22]*

$$\phi^2(\nabla_H \bar{P})(G,K)U = A(H)\bar{P}(G,K)U, \tag{5.2}$$

for arbitrary vector fields G, K, U, H.

Equation (2.1) and (5.2) give

$$(\nabla_H \bar{P})(G,K)U + \ddot{\eta}((\nabla_H \bar{P})(G,K)U)\xi = A(H)\bar{P}(G,K)U. \tag{5.3}$$

Contracting, we have

$$(\nabla_H S)(K,U) = A(H)[S(K,U) - \frac{r}{n}g(K,U)]. \tag{5.4}$$

Replacing $U = \xi$ in (5.4) and using (2.9) and (2.1) result out

$$(\nabla_H S)(K,\xi) = -\frac{dr(H)}{n}\ddot{\eta}(K) - A(H)[(2 + \frac{r}{n}]\ddot{\eta}(K). \tag{5.5}$$

Since r is constant in (2.15), we get from (5.5)

$$(\nabla_H S)(K,\xi) = -A(H)\ddot{\eta}(K)[2 + \frac{r}{n}]. \tag{5.6}$$

In view of (5.6) and (4.5), we have

$$S(H,K) = 2g(H,K) - A(H)\ddot{\eta}(K)[(2 + \frac{r}{n}]. \tag{5.7}$$

Using lemma (2.1) in above, we get

$$S(H,\xi) = \ddot{\eta}(H)[2 - \ddot{\eta}(\rho)\frac{r}{n}]. \tag{5.8}$$

With the help of (2.14), (2.15) and (2.18), equation (5.8) releases

$$\lambda = \frac{\mu + A(\xi)\frac{(3n-\mu-1)}{n}}{1 + nA(\xi)}. \tag{5.9}$$

Therefore, we can state

Theorem 5.1. *Let (M^n, g) be a pseudo projective LP Sasakian ϕ recurrent space in the context of $\ddot{\eta}$-Ricci soliton then solitonic behaviour is dependent on one-form A and given by*

- $A(\xi) > -\frac{\mu n}{3n-\mu-1}$*, and* $A(\xi) > -\frac{1}{n}$ *implies expansion in nature,*
- $A(\xi) < -\frac{\mu n}{3n-\mu-1}$ *and* $A(\xi) < -\frac{1}{n}$ *implies compact in nature,*
- $A(\xi) = -\frac{\mu n}{3n-\mu-1}$ *found no variation in nature.*

6. Weyl Projective ϕ-Recurrentness on $\ddot{\eta}$-Ricci Soliton

Definition 6.1. *The Weyl projective curvature tensor of (M^n, g) is given by*

$$P(G,K)U = R(G,K)U - \frac{1}{n-1}[S(K,U)G - S(G,U)K]. \tag{6.1}$$

Definition 6.2. *An LP-Sasakian manifold is said to be Weyl-projective ϕ-recurrent manifold if there exists a non-zero 1-form A such that*

$$\phi^2(\nabla_H P)(G,K)U = A(H)P(G,K)U, \tag{6.2}$$

for arbitrary vector fields G, K, U, H.

Equation (2.1) and (6.2) deliver

$$(\nabla_H P)(G,K)U + \ddot{\eta}((\nabla_H P)(G,K)U)\xi = A(H)P(G,K)U. \tag{6.3}$$

Contracting (6.3) yields

$$(\nabla_H S)(K,U) = \frac{1}{(n-1)}A(H)[nS(K,U) - rg(K,U)]. \tag{6.4}$$

Replacing $U = \xi$ in (6.4) and using (2.9) and (2.1) reduce to

$$(\nabla_H S)(K,\xi) = \frac{1}{(n-1)}A(H)\ddot{\eta}(K)[2n + r]. \tag{6.5}$$

With the help of (4.5), equation (6.5) yields

$$S(H,K) = 2g(H,K) - \frac{1}{(n-1)}A(H)\ddot{\eta}(K)[2n + r]. \tag{6.6}$$

The lemma (2.1) and (6.6) resulting

$$S(K,\xi) = [2 + \frac{1}{(n-1)}A(\xi)(2n + r)]\ddot{\eta}(K). \tag{6.7}$$

Using (2.14), (2.15) and (2.18) in (6.7), we have

$$\lambda = \frac{(n-1)\mu - A(\xi)(3n - \mu - 1)}{n(1 - A(\xi)) - 1}. \tag{6.8}$$

Therefore, we conclude that

Theorem 6.1. *Let (M^n, g) be a Weyl projective LP Sasakian ϕ recurrent space in the context of $\ddot{\eta}$-Ricci soliton then solitonic behaviour of the flow is given by*

- *expanding flow obtain for $A(\xi) < \frac{(n-1)\mu}{3n-\mu-1}$ and $A(\xi) < \frac{(n-1)}{n}$,*
- *shrinking flow obtain for $A(\xi) > \frac{(n-1)\mu}{3n-\mu-1}$ and $A(\xi) > \frac{(n-1)}{n}$,*
- *flow maintains the same behaviour for $A(\xi) = \frac{(n-1)\mu}{3n-\mu-1}$.*

7. Semi Generalized ϕ-Recurrentness on $\ddot{\eta}$-Ricci Soliton

Definition 7.1. *[39] An LP-Sasakian manifold satisfying the condition*

$$\phi^2((\nabla_Z R)(G,K)U) = A(Z)R(G,K)U + B(Z)g(K,U)G, \tag{7.1}$$

is called a semi-generalized ϕ-recurrent manifold.

Using (2.1) in (7.1), we obtain

$$g((\nabla_H R)(G,K)U,V) + \ddot{\eta}((\nabla_H R)(G,K)U)\ddot{\eta}(V) = \tag{7.2}$$
$$A(H)g(R(G,K)U,V) + B(H)g(K,U)g(G,V).$$

Contracting above yields

$$(\nabla_H S)(K,U) = A(H)S(K,U) + nB(H)g(K,U). \tag{7.3}$$

Put $U = \xi$ in (7.3) and using (2.1), (2.9) and (4.5), we get

$$S(H,K) = 2g(H,K) + \ddot{\eta}(K)[nB(H) - 2A(H)]. \tag{7.4}$$

Equation (7.4) implies

$$S(\phi H,\phi K) = 2g(\phi H,\phi K) + \ddot{\eta}(\phi K)[nB(\phi H) - 2A(\phi H)]. \tag{7.5}$$

$$S(\phi H,\phi K) = 2g(\phi H,\phi K). \tag{7.6}$$

On using (2.1) and (2.10), (7.6) reduces to

$$S(H,K) = 2g(H,K). \tag{7.7}$$

Using (2.14), above equation provides

$$\lambda = \mu \tag{7.8}$$

If the soliton becomes steady, we have $\lambda = \mu = 0$ which implies non-existance of $\ddot{\eta}$-Ricci soliton and therefore following statement can be stated

Theorem 7.1. *There does not exist any proper steady type $\ddot{\eta}$-Ricci soliton on semi-generalized ϕ-recurrent LP Sasakian space.*

Corollary 7.1. *Let (M^n, g) be a semi-generalized LP Sasakian ϕ recurrent space equipped with $\ddot{\eta}$-Ricci soliton then is an Einstein manifold.*

Corollary 7.2. *Let (M^n, g) be a semi-generalized LP Sasakian ϕ recurrent space equipped with $\ddot{\eta}$-Ricci soliton then it is either expanding or shrinking accordingly $\mu > 0$ and $\mu < 0$ respectively.*

8. Space-time on $\ddot{\eta}$-Ricci Soliton

The Lorentzian para-Sasakian spacetime of dimension four is that we will study in this section. With the cosmological constant κ, the Einstein's field equation is given by

$$S(G,K) - \frac{r}{2}g(G,K) + \kappa g(G,K) = \tau T(G,K). \tag{8.1}$$

T is the energy momentum $(0,2)$ type tensor for all vector fields G, K and τ indicates the gravitational constant. A perfect fluid is described by energy momentum tensor T given by

$$T(G,K) = (\rho - p)A(G)A(K) + pg(G,K), \tag{8.2}$$

where ρ: the density function, p: the pressure function and A: the *one*-form examined by $g(G,V) = A(G)$ and V denotes the flow vector. In terms of the fact that if the density function ρ and the pressure p vanish in the same manner, then fluid substance is not pure. If the flow vector field in LP-Sasakian spacetime is considered as characteristic vector field ξ then energy momentum tensor consequently takes the form

$$T(G,K) = (\rho - p)\ddot{\eta}(G)\ddot{\eta}(K) + pg(G,K). \tag{8.3}$$

Substituting (2.12) in (8.1), we get

$$T(G,K) = -\frac{1}{\tau}\Big[(1 - \lambda - \frac{r}{2} + \kappa)g(G,K) - (\mu + 1)\ddot{\eta}(G)\ddot{\eta}(K)\Big]. \tag{8.4}$$

Therefore, we have

Theorem 8.1. *The energy tensor of the perfect fluid spacetime on LP-Sasakian space in the context of $\ddot{\eta}$-Ricci soliton, is given by*

$$T(G,K) = \frac{1}{\tau}\Big[(1 - \lambda - \frac{r}{2} + \kappa)g(G,K) - (\mu + 1)\ddot{\eta}(G)\ddot{\eta}(K)\Big].$$

compairing (2.9) and (2.14), we get

$$\mu = \lambda - 4. \tag{8.5}$$

Equation (8.4) and (8.5) provide the scalar curvature of energy tensor

$$T_{sca} = \frac{1}{\tau}[5\lambda + 4\kappa - 7]. \tag{8.6}$$

Therefore, the statement can be given

Theorem 8.2. *The scalar curvature of energy tensor of perfect fluid LP-Sasakian spacetime in the context of $\ddot{\eta}$-Ricci soliton is given by*

$$T_{sca} = \frac{1}{\tau}[5\lambda + 4\kappa - 7].$$

From (8.3) and (8.4), we obtain

$$\begin{aligned} &\frac{1}{\tau}\Big[(1 - \lambda - \frac{r}{2} + \kappa)g(G,K) - (\mu + 1)\ddot{\eta}(G)\ddot{\eta}(K)\Big] = \\ &(\rho - p)\ddot{\eta}(G)\ddot{\eta}(K) + pg(G,K), \end{aligned} \tag{8.7}$$

Put $G = K = \xi$ in (8.7), yields

$$\lambda = -[2\tau(\rho - 2p) + 1 + 2\kappa]. \tag{8.8}$$

Theorem 8.3. *If a perfect fluid LP-Sasakian spacetime on $\ddot{\eta}$-Ricci soliton satisfies the field equation, then soliton constant λ is given by*

$$\lambda = -[2\tau(\rho - 2p) + 1 + 2\kappa].$$

Corollary 8.1. *If a perfect fluid LP-Sasakian spacetime equipped a $\ddot{\eta}$-Ricci soliton then cosmological constant κ and soliton constant λ are related by*

$$4\kappa + 5\lambda = \tau T_{sca} + 7.$$

Conclusion

In this paper, ϕ-recurrent characteristics have been studied on an n-dimensional LP Sasakian structure equipped with $\ddot{\eta}$-Ricci soliton associated with special vector field. The properties such as expanding, shrinking and steady have been discussed for pseudo-projective, Weyl-projective and semi generalized ϕ-recurrent LP-Sasakian manifold associated with $\ddot{\eta}$-Ricci soliton. The energy momentum and its scalar curvature have been illustrated in LP-Sasakian spacetime in the context of $\ddot{\eta}$-Ricci soliton.

Acknowledgement: The Researchers would like to thank the Deanship of Graduate Studies and Scientific Research at Qassim University for financial support (QU-APC-2024-9/1).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] P. Almia, J. Upreti, Certain Properties of η-Ricci Soliton on η-Einstein Para-Kenmotsu Manifolds, Filomat 37 (2023), 9575–9585. https://doi.org/10.2298/FIL2328575A.

[2] M.B. Ayed, K.E. Mehdi, Multiplicity Results for the Scalar Curvature Problem on Half Spheres, Discr. Contin. Dyn. Syst. 44 (2024), 1878–1900. https://doi.org/10.3934/dcds.2024013.

[3] S. Elsaeed, O. Moaaz, K.S. Nisar, M. Zakarya, E.M. Elabbasy, Sufficient Criteria for Oscillation of Even-Order Neutral Differential Equations with Distributed Deviating Arguments, AIMS Math. 9 (2024), 15996–16014. https://doi.org/10.3934/math.2024775.

[4] B.B. Chaturvedi, P. Bhagat, M.N.I. Khan, Novel Theorems for a Bochner Flat Lorentzian Kahler Space-Time Manifold with η-Ricci-Yamabe Solitons, Chaos Solitons Fractals: X 11 (2023), 100097. https://doi.org/10.1016/j.csfx.2023.100097.

[5] S.K. Chaubey, U.C. De, Three-Dimensional Riemannian Manifolds and Ricci Solitons, Quaest. Math. 45 (2022), 765–778. https://doi.org/10.2989/16073606.2021.1895352.

[6] J.T. Cho, M. Kimura, Ricci Solitons and Real Hypersurfaces in a Complex Space Form, Tohoku Math. J. 61 (2009), 205–212. https://doi.org/10.2748/tmj/1245849443.

[7] N. Omar, S. Serra-Capizzano, B. Qaraad, F. Alharbi, O. Moaaz, E.M. Elabbasy, More Effective Criteria for Testing the Oscillation of Solutions of Third-Order Differential Equations, Axioms 13 (2024), 139. https://doi.org/10.3390/axioms13030139.

[8] S. Dey, Conformal Ricci Soliton and Almost Conformal Ricci Soliton in Paracontact Geometry, Int. J. Geom. Methods Modern Phys. 20 (2023), 2350041. https://doi.org/10.1142/S021988782350041X.

[9] K. De, M.N.I. Khan, U.C. De, Characterizations of Generalized Robertson-Walker Spacetimes Concerning Gradient Solitons, Heliyon 10 (2024), e25702. https://doi.org/10.1016/j.heliyon.2024.e25702.

[10] U.C. De, M.N.I. Khan, A. Sardar, h-Almost Ricci-Yamabe Solitons in Paracontact Geometry, Mathematics 10 (2022), 3388. https://doi.org/10.3390/math10183388.

[11] U.C. De, A. Sardar, K. De, Ricci-Yamabe Solitons and 3-Dimensional Riemannian Manifolds, Turk. J. Math. 46 (2022), 1078–1088. https://doi.org/10.55730/1300-0098.3143.

[12] U.C. De, A.A. Shaikh, S. Biswas, On ϕ–Recurrent Sasakian Manifolds, Novi Sad J. Math. 33 (2003), 43–48.

[13] R.S. Hamilton, Three-Manifolds with Positive Ricci Curvature, J. Diff. Geom. 17 (1982), 255–306.

[14] A. Haseeb, M. Bilal, S.K. Chaubey, M.N.I. Khan, Geometry of Indefinite Kenmotsu Manifolds as $*\eta$-Ricci-Yamabe Solitons, Axioms 11 (2022), 461. https://doi.org/10.3390/axioms11090461.

[15] A. Haseeb, S.K. Chaubey, Lorentzian Para-Sasakian Manifolds and *-Ricci Solitons, Kragujevac J. Math. 48 (2024), 167–179.

[16] S.K. Hui, R.S. Lemence, D. Chakraborty, Ricci Solitons on Ricci Pseudosymmetric $(LCS)_n$-Manifolds, arXiv:1707.03618 [math.DG], (2017). http://arxiv.org/abs/1707.03618.

[17] S. Hui, S. Yadav, A. Patra, Almost Conformal Ricci Solitons on f-Kenmotsu Manifolds, Khayyam J. Math. 5 (2019), 89–104. https://doi.org/10.22034/kjm.2019.81221.

[18] P. Majhi, D. Kar, Eta-Ricci Solitons on LP-Sasakian Manifolds, Rev. Union Mat. Argentina 60 (2019), 391–405. https://doi.org/10.33044/revuma.v60n2a07.

[19] Y. ChandraMandal, S. Kumar Hui, Yamabe Solitons with Potential Vector Field as Torse Forming, Cubo (Temuco) 20 (2018), 37–47. https://doi.org/10.4067/S0719-06462018000300037.

[20] K. Matsumoto, On Lorentzian Paracontact Manifolds, Bull. Yamagata Univ. Nat. Sci. 12 (1989), 151–156.

[21] T. Mert, M. Atçeken, On the Almost η-Ricci Solitons on Pseudosymmetric Lorentz Generalized Sasakian Space Forms, Univ. J. Math. Appl. 6 (2023), 43–52. https://doi.org/10.32323/ujma.1236596.

[22] H.G. Nagaraja, K. Venu, Ricci Solitons in Kenmotsu Manifold, J. Inf. Math. Sci. 8 (2016), 29–36.

[23] P. Pandey, On Weakly Cyclic Generalized Z-Symmetric Manifolds, Nat. Acad. Sci. Lett. 43 (2020), 347–350. https://doi.org/10.1007/s40009-020-00875-6.

[24] P. Pandey, B.B. Chaturvedi, On a Lorentzian Complex Space Form, Nat. Acad. Sci. Lett. 43 (2020), 351–353. https://doi.org/10.1007/s40009-020-00874-7.

[25] P. Pandey, K. Sharma, Some Results on Ricci Soliton on Contact Metric Manifolds, Facta Univ. Ser: Math. Inf. 38 (2023), 473–486.

[26] P. Pandey, K. Sharma, Generalized Z-Ricci Soliton on Sasakian 3-Manifold with Generalized Z-Tensor, AIP Conf. Proc. 3087 (2024), 060001. https://doi.org/10.1063/5.0199390.

[27] R. Poddar, S. Balasubramanian, R. Sharma, Yamabe Solitons in Contact Geometry, N. Z. J. Math. 54 (2023), 49–55. https://doi.org/10.53733/286.

[28] A. Sardar, M.N.I. Khan, U.C. De, η-$*$-Ricci Solitons and Almost Co-Kähler Manifolds, Mathematics 9 (2021), 3200. https://doi.org/10.3390/math9243200.

[29] A. Sardar, U.C. De, A. Gezer, η-$*$-Ricci Solitons and Paracontact Geometry, J. Anal. 31 (2023), 2861–2876. https://doi.org/10.1007/s41478-023-00620-4.

[30] S. Sarkar, S. Dey, $*$-Conformal η-Ricci Soliton within the Framework of Kenmotsu Manifolds, Ricerche Mat. (2023). https://doi.org/10.1007/s11587-023-00781-1.

[31] M.D. Siddiqi, O. Bahadır, Eta-Ricci Solitons on Kenmotsu Manifold With Generalized Symmetric Metric Connection, Facta Univ. Ser.: Math. Inf. 35 (2020), 295–310. https://doi.org/10.22190/FUMI2002295S.

[32] A.A. Shaikh, D.G. Prakasha, H. Ahmad, On Generalized ϕ-Recurrent LP-Sasakian Manifolds, J. Egypt. Math. Soc. 23 (2015), 161–166. https://doi.org/10.1016/j.joems.2013.12.019.

[33] A.A. Shaikh, S. Biswas, On LP-Sasakian Manifolds, Bull. Malays. Math. Sci. Soc. 27 (2004), 17–26.

[34] B. Shanmukha, V. Venkatesha, Some Ricci Solitons on Kenmotsu Manifold, J. Anal. 28 (2020), 1155–1164. https://doi.org/10.1007/s41478-020-00243-z.

[35] G.S. Shivaprasanna, M.S. Haque, G. Somashekhara, η-Ricci Soliton on f-Kenmotsu Manifolds, J. Phys.: Conf. Ser. 1543 (2020), 012007. https://doi.org/10.1088/1742-6596/1543/1/012007.

[36] M.D. Siddiqi, S.K. Chaubey, M.N.I. Khan, $f(R,T)$-Gravity Model with Perfect Fluid Admitting Einstein Solitons, Mathematics 10 (2021), 82. https://doi.org/10.3390/math10010082.

[37] A. Singh, R.K. Pandey, A. Prakash, K. Sachin, On a Pseudo Projective ϕ-Recurrent Sasakian Manifolds, J. Math. Computer Sci. 14 (2015), 309–314.

[38] Venkatesha, C.S. Bagewadi, K.T. Pradeep Kumar, Some Results on Lorentzian Para-Sasakian Manifolds, ISRN Geom. 2011 (2011), 161523. https://doi.org/10.5402/2011/161523.

[39] S.V. Vishnuvardhana, Venkatesha, S.K. Hui, Certain Results on (LCS)n-Manifolds, arXiv:1901.06873 [math.DG], (2019). https://doi.org/10.48550/ARXIV.1901.06873.