Int. J. Anal. Appl. (2024), 22:202

International Journal of Analysis and Applications

A New Flexible Extension of the XLindley Distribution with Properties and Application on Income Tax and Carbon Fibers Data

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ABSTRACT. A new and enhanced extension of the XLindley distribution termed the sized-biased XLindley distribution (SBXLD), has been introduced. This new model explores two specific variants: the length-biased XLindley distribution and the area-biased XLindley distribution. Various crucial properties such as moments, moment generating function, quantile function, survival, and hazard functions, mean residual life function, and Rényi entropy have been derived and extensively investigated. For parameter estimation, five distinct methods have been employed to estimate the model parameters. Through a comprehensive simulation study, the most effective estimation method has been identified. The applicability and efficiency of the SBXLD model have been demonstrated using two datasets from different domains. It has been observed that the SBXLD model effectively analyzed these datasets and yielded superior results compared to other competitive distributions under consideration.

1. Introduction

Lifetime data modelling arises in various areas such as social sciences, biology, medicine, engineering, agronomy, agriculture, economics, survival analysis, and demography. However, numerous probability models can be utilized for these analytical datasets. The presumed distribution has a significant impact on the trustworthiness of the statistical analysis results. As a

Received Aug. 29, 2024

2020 Mathematics Subject Classification. 62E10.

Key words and phrases. weighted distribution; XLindley distribution; moments; estimation; data analysis.

https://doi.org/10.28924/2291-8639-22-2024-202

ISSN: 2291-8639

result, efforts are ongoing to discover distributions that closely fit the data to draw reliable conclusions. Many attempts have been made to broaden the classical distributions by employing various extended approaches. Some most commonly used parameter induction or mixture techniques are; Azzalini's approach [1], transmuted family [2], exponentiated-G [3], Beta-G [4], Kumaraswamy-G [5], Transformed-Transformer (T-X) family [6], Weibull-G [7], new Weibull-G [8], Fréchet-G [9], Generalized Burr III-G [10], and inverted Nadarajah–Haghighi power series [11].

Weighted probability distributions, created by weighting or other parameters, provide greater litheness. These distributions involve assigning weights to real-life data based on a weight function, rather than randomly selecting data points. The notion of weighted distributions was pioneered by [12]. Over the last 25 years, various fitting models for scanned data have used weighted distribution approaches. For a random variable X with a density function, the probability density function (pdf) of the weighted random variable X_w may be calculated using the following formula:

$$f_w(x) = \frac{w(x)f(x)}{E[w(X)]},$$

where $E[w(X)]$ is the normalizing factor and the term $w(x)$ makes the overall probability equal to 1. Cox [13] invented the notion of length-biased sampling. Further, Rao [14] introduced the theory of size-biased distribution, and if the weighted function has the following form $w(x) = x^c$, then the density function of the new model can be derived as

$$f_c(x) = \frac{x^c f(x)}{E_c[X]} \tag{1}$$

The probability models obtained for $c = 1$ and $c = 2$ are referred to as length-biased and area-biased distributions, respectively.

Owing to the importance of weighted distributions, many scholars have studied them in many settings. Examples of such studies include: Dara and Ahmad [15] investigated the length-biased exponential distribution, emphasizing its importance for modeling time-to-failure data in reliability investigations. Their findings underscored the distribution's simplicity and usefulness in real-world circumstances. Ahmad, Ahmad, and Ahmed [16] introduced the length-biased weighted Lomax distribution. Ayesha [17] investigated sized-biased Lindley distribution. Sharma, Dey, Singh, and Manzoor [18] focused on the length-biased weighted Maxwell distribution. Al-Omari, Al-Nasser, and Ciavolino [19] examined the length-biased weighted

Maxwell distribution. Alsmairan and Al-Omari [20] studied the weighted Suja distribution. Chouia, Zeghdoudi, Raman, and Beghriche [21] worked on size-biased Zeghdoudi distribution. Khogeer [22] derived and studied sized-biased Haq distribution.

Chouia and Zeghdoudi [23] introduced a new one-parameter flexible XLindley (XL) distribution by mixing exponential and Lindley distributions. The probability density function (pdf) and cumulative distribution (cdf) of XL distribution are defined in equation (1) and equation (2)

$$f(x;\delta) = \frac{\delta^2(2+\delta+x)e^{-\delta x}}{(1+\delta)^2}, \qquad x,\delta > 0, \tag{2}$$

and

$$F(x;\delta) = 1-\left(1+\frac{\delta x}{(1+\delta)^2}\right)e^{-\delta x} \ ; x > 0, \delta > 0. \tag{3}$$

A new flexible extension of the XLindley distribution called the sized-biased XLindley distribution, has been introduced using the concept proposed by Fisher in 1934. The key objectives of this study are as follows:

- The primary aim was to extend the XLindley distribution, and this was achieved by employing a weighted approach, allowing for the derivation and analysis of its key mathematical characteristics. Additionally, two sub-models, the length-biased and area-biased distributions, were also examined.
- The second major goal was to estimate the parameters of the model using five different estimation methods. A comprehensive simulation study was conducted to identify the most efficient estimation method.
- To demonstrate the applicability and flexibility of the new distribution, two datasets from different fields were analyzed.

The rest of the study is structured as follows: Section 2 is based on the derivation of the new distribution and its shapes. Some of the main mathematical properties are derived and discussed in Section 3. Section 4 discusses the estimation of model parameters and includes a simulation study. In Section 5, the practical utility of the proposed distribution is evaluated using two real-world datasets. The concluding remarks are provided in Section 7.

2. New Weighted Model

The new probability distribution is named sized-biased XLindley (SBXL) distribution using equation (1). The pdf of SBXL distribution is given by

$$f_{SBXL}(x) = \frac{\delta^{c+2}}{c!\,(\delta^2 + 2\delta + c + 1)} x^c (2 + \delta + x) e^{-\delta x};\ x > 0, \tag{4}$$

The cdf of SBXL distribution

$$F_{SBXL}(x) = \frac{(c + (1+\delta)^2)\Gamma(1+c) - \delta(2+\delta)\Gamma(1+c, x\delta) - \Gamma(2+c, x\delta)}{(c + (1+\delta)^2)c!}, \tag{5}$$

where $\Gamma(a,b) = \int_a^\infty x^{a-1} e^{-x} dx$ and $\Gamma(a) = \int_0^\infty x^{p-1} e^{-x} dx$ are the incomplete and complete gamma functions.

We plot some density curves based on some choices of parameters presented in Figure 1. It is interesting to note the density curves are positively skewed, as well as nearly symmetrical. The SBXL distribution is an unimodal distribution for a combination of parameters. These various forms provide the distribution additional flexibility in describing certain data types.

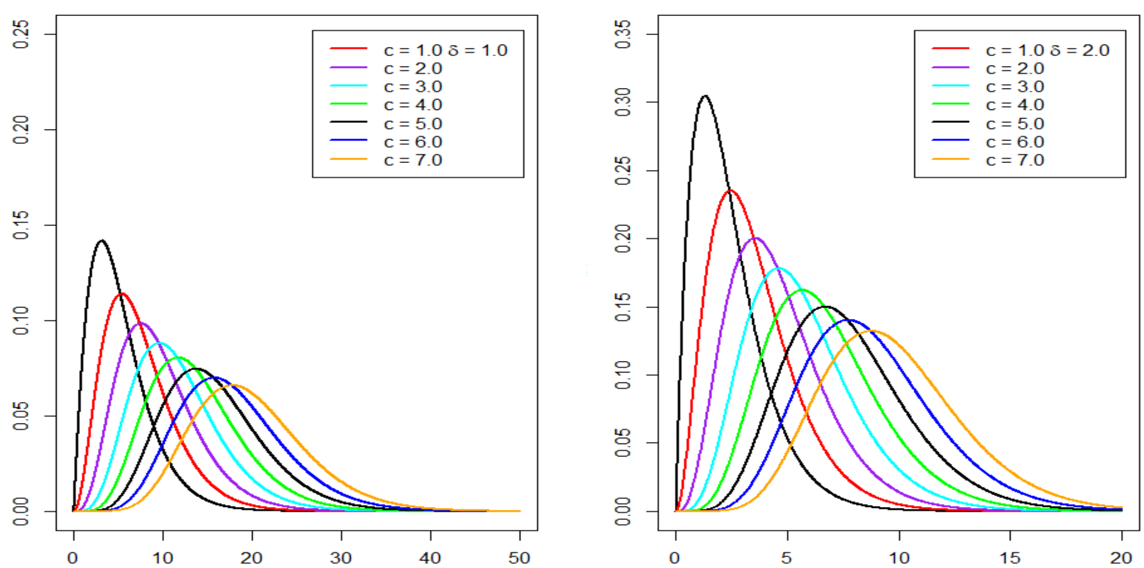

Fig. 1: Probability density curves of SBXL distribution for a combination of parameters

3. Statistical Properties of SBXL Distribution

The statistical characteristics of the SBXL distribution, such as moments, the moment generating function, Rényi entropy, and the mean residual life function, are derived in this section.

3.1. Moments

The ordinary moment of SBXL distribution is defined as follows

$$E(X^r) = \int_0^\infty x^r f(x)dx,$$

$$= \frac{\delta^{c+2}}{c!\,(\delta^2 + 2\delta + c + 1)} \int_0^\infty x^{r+c} (2 + \delta + x)e^{-\delta x} dx,$$

$$= \frac{\delta^{c+2}}{c!\,(\delta^2 + 2\delta + c + 1)} \left((2 + \delta) \int_0^\infty x^{r+c} e^{-\delta x} dx + \int_0^\infty x^{r+c+1} e^{-\delta x} dx \right),$$

$$= \frac{\delta^{c+2}}{c!\,(\delta^2 + 2\delta + c + 1)} \left(\frac{(2 + \delta)\Gamma(r + c + 1)}{\delta^{r+c+1}} + \frac{\Gamma(r + c + 2)}{\delta^{r+c+2}} \right),$$

$$= \frac{1}{c!\,(\delta^2 + 2\delta + c + 1)} \left(\frac{(2 + \delta)\delta\Gamma(r + c + 1) + \Gamma(r + c + 2)}{\delta^r} \right),$$

$$= \frac{(r + c)!}{c!\,(\delta^2 + 2\delta + c + 1)} \left(\frac{(2 + \delta)\delta + (r + c + 1)}{\delta^r} \right),$$

$$= \frac{(r + c)!\,((1 + \delta)^2 + r + c)}{c!\,((1 + \delta)^2 + c)\delta^r}. \tag{6}$$

Based on equation (6) the first four moments of origin can be deduced as follows

$$E(X) = \frac{(1 + c)!\,((1 + \delta)^2 + 1 + c)}{c!\,((1 + \delta)^2 + c)\delta},$$

$$E(X^2) = \frac{(2 + c)!\,((1 + \delta)^2 + 2 + c)}{c!\,((1 + \delta)^2 + c)\delta^2},$$

$$E(X^3) = \frac{(3 + c)!\,((1 + \delta)^2 + 3 + c)}{c!\,((1 + \delta)^2 + c)\delta^3},$$

and

$$E(X^4) = \frac{(4 + c)!\,((1 + \delta)^2 + 4 + c)}{c!\,((1 + \delta)^2 + c)\delta^4}.$$

3.2. Moments generating function

The moment-generating function can be derived as

$$M_X(t) = \int_0^\infty e^{tx} f(x)dx,$$

$$= \frac{\delta^{c+2}}{c!\,(\delta^2 + 2\delta + c + 1)} \int_0^\infty e^{tx} x^c (2 + \delta + x)e^{-\delta x} dx,$$

$$= \frac{\delta^{c+2}}{c!\,(\delta^2+2\delta+c+1)}\left((2+\delta)\int_0^\infty e^{tx}\,x^c e^{-\delta x}dx + \int_0^\infty e^{tx}\,x^{c+1}e^{-\delta x}dx\right),$$

$$M_X(t) = \frac{\delta^{2+c}\big(c+(1+\delta)^2 - t(2+\delta)\big)}{(\delta-t)^{2+c}(c+(1+\delta)^2)}. \quad (7)$$

3.3. Rényi Entropy

The Rényi entropy of SBXL distribution can be derived as

$$H_\eta(X) = \frac{1}{1-\eta}\log\left\{\int_0^\infty \big(f(x)\big)^\eta\,dx\right\},$$

$$= \frac{1}{1-\eta}\log\left\{\int_0^\infty \left(\frac{\delta^{c+2}}{c!\,(\delta^2+2\delta+c+1)}x^c(2+\delta+x)e^{-\delta x}\right)^\eta dx\right\},$$

$$= \frac{1}{1-\eta}\log\left\{\left(\frac{\delta^{c+2}}{c!\,(\delta^2+2\delta+c+1)}\right)^\eta \int_0^\infty x^{\eta p}(2+\delta+x)^\eta e^{-\delta\eta x}\,dx\right\},$$

$$= \frac{1}{1-\eta}\log\left\{\left(\frac{\delta^{c+2}(2+\delta)}{c!\,(\delta^2+2\delta+c+1)}\right)^\eta \int_0^\infty \sum_{i=0}^\infty \binom{\eta}{i}\frac{x^{\eta c+i}}{(2+\delta)^i}\,e^{-\delta\eta x}\,dx\right\},$$

$$= \frac{1}{1-\eta}\log\left\{\left(\frac{\delta^{c+2}(2+\delta)}{c!\,(\delta^2+2\delta+c+1)}\right)^\eta \sum_{i=0}^\infty \binom{\eta}{i}\frac{1}{(2+\delta)^i}\int_0^\infty x^{\eta c+i}e^{-\delta\eta x}\,dx\right\},$$

$$H_\alpha(Y) = \frac{1}{1-\eta}\log\left\{\left(\frac{\delta^{c+2}(2+\delta)}{c!\,(\delta^2+2\delta+c+1)}\right)^\eta \sum_{i=0}^\infty \binom{\eta}{i}\frac{\eta^i\Gamma(1+i+\eta c)}{(2+\delta)^i(\eta\delta)^{1+i+\eta c}}\right\}. \quad (8)$$

3.4. Mean Residual Life

The mean residual life (MRL) function, often known as $m(t)$, is a notion used in survival analysis and reliability theory. It reflects a subject's projected remaining lifetime assuming it has lived up to time t. Mathematically, it is defined as:

$$m(t) = \mathbb{E}(X-t|X>t) = \frac{1}{S(t)}\left\{\int_t^\infty y f(y)\,dy - t\right\}. \quad (9)$$

Now consider the integral term

$$I = \int_t^\infty y f(y)\,dy,$$

$$= \frac{\delta^{c+2}}{c!\,(\delta^2 + 2\delta + c + 1)} \int_t^{\infty} x^{r+c}\,(2 + \delta + x) e^{-\delta x} dx,$$

$$= \frac{\delta^{c+2}}{c!\,(\delta^2 + 2\delta + c + 1)} \left((2 + \delta) \int_t^{\infty} x^{r+c}\, e^{-\delta x} dx + \int_t^{\infty} x^{r+c+1}\, e^{-\delta x} dx \right),$$

$$= \frac{\delta^{c+2}}{c!\,(\delta^2 + 2\delta + c + 1)} \left(\frac{(2 + \delta)\left(\Gamma(1 + r) + \left(\Gamma(1 + r, t\delta) - r\Gamma(r)\right)\right)}{\delta^{1+r}} + \left(\frac{\Gamma(2 + r) + (\Gamma(2 + r, t\delta) - \Gamma(2 + r))}{\delta^{2+r}} \right) \right),$$

Now the equation (9) takes the form

$$m(t) = \frac{1}{S(t)} \left\{ \left[\frac{\delta^{c+2}}{c!\,(\delta^2 + 2\delta + c + 1)} \left(\frac{(2 + \delta)\left(\Gamma(1 + r) + \left(\Gamma(1 + r, t\delta) - r\Gamma(r)\right)\right)}{\delta^{1+r}} + \left(\frac{\Gamma(2 + r) + (\Gamma(2 + r, t\delta) - \Gamma(2 + r))}{\delta^{2+r}} \right) \right) \right] - t \right\}, \tag{10}$$

4. Sub-models of the SBXL Distribution

In this section, we further explore two sub-models of SBXL distribution. We derived the density function of length-biased XLindley (LBXL) distribution by setting $c = 1$ and similarly area-biased XLindley (ABXL) distribution by assigning $c = 2$, respectively.

$$f_{SBXL}(x) = \frac{\delta^3}{(\delta^2 + 2\delta + 2)} x(2 + \delta + x)e^{-\delta x};\ x > 0, \delta > 0, \tag{11}$$

and

$$f_{ABXL}(x) = \frac{\delta^4}{2(\delta^2 + 2\delta + 3)} x^2(2 + \delta + x)e^{-\delta x}, \qquad x > 0, \delta > 0, \tag{12}$$

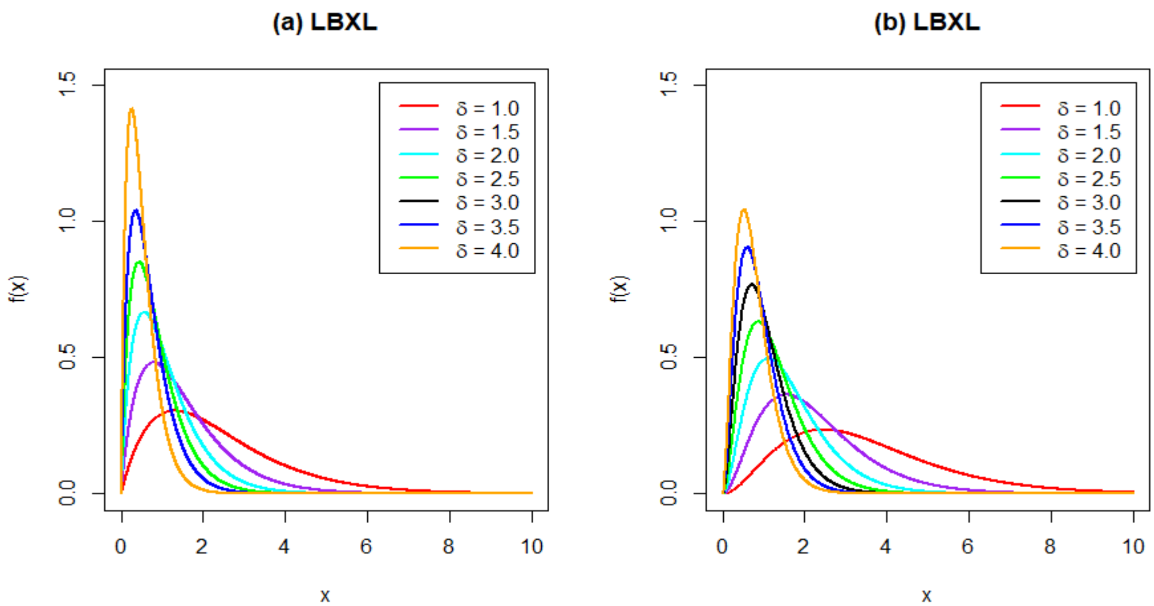

Fig. 2: The density function plots of the LBXL and ABXL models for some choices of δ.

The cdfs of the LBXL and ABXL distributions are

$$F_{LBXL}(x)=\frac{2(1+(1+\delta)^2)-\delta(2+\delta)\Gamma(2,x\delta)-\Gamma(3,x\delta)}{(1+(1+\delta)^2)}, \tag{13}$$

$$F_{ABXL}(x)=\frac{3(2+(1+\delta)^2)-\delta(2+\delta)\Gamma(3,x\delta)-\Gamma(4,x\delta)}{2(2+(1+\delta)^2)}, \tag{14}$$

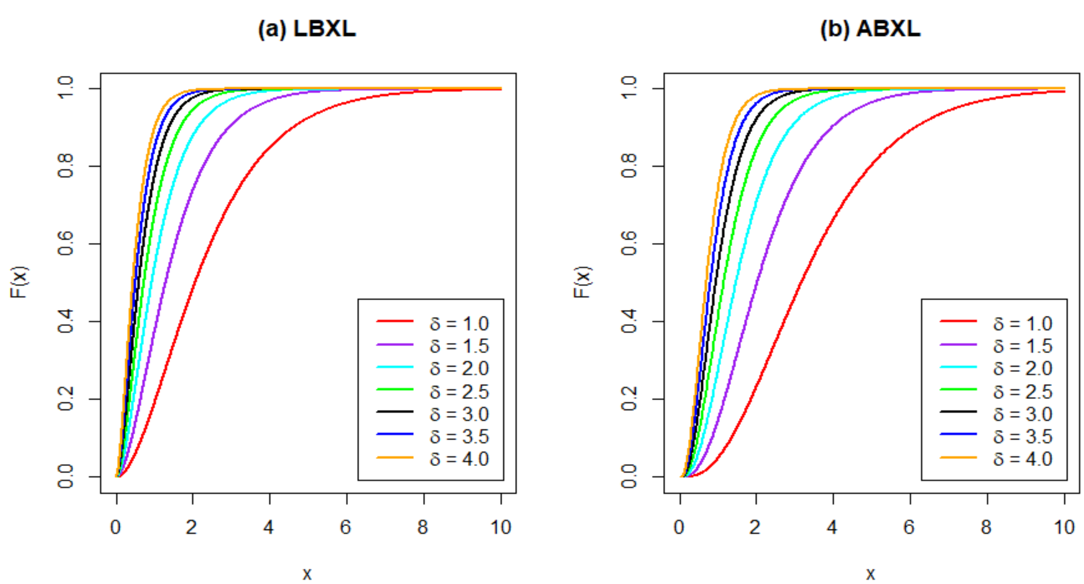

Fig. 3: The distribution function plots of the LBXL and ABXL models for some choices of δ.

The hazard functions of the LBXL and AXL distributions are

$$h_{LBXL}(x) = \frac{\delta^3 x(2+x+\delta)e^{-x\delta}}{\delta(2+\delta)\Gamma(2,x\delta)+\Gamma(3,x\delta)}, \tag{15}$$

$$h_{ABXL}(x) = \frac{\delta^4 x^2(2+x+\delta)e^{-x\delta}}{\delta(2+\delta)\Gamma(3,x\delta)+\Gamma(4,x\delta)}, \tag{16}$$

The hazard plots for each model are depicted in Figure 4.

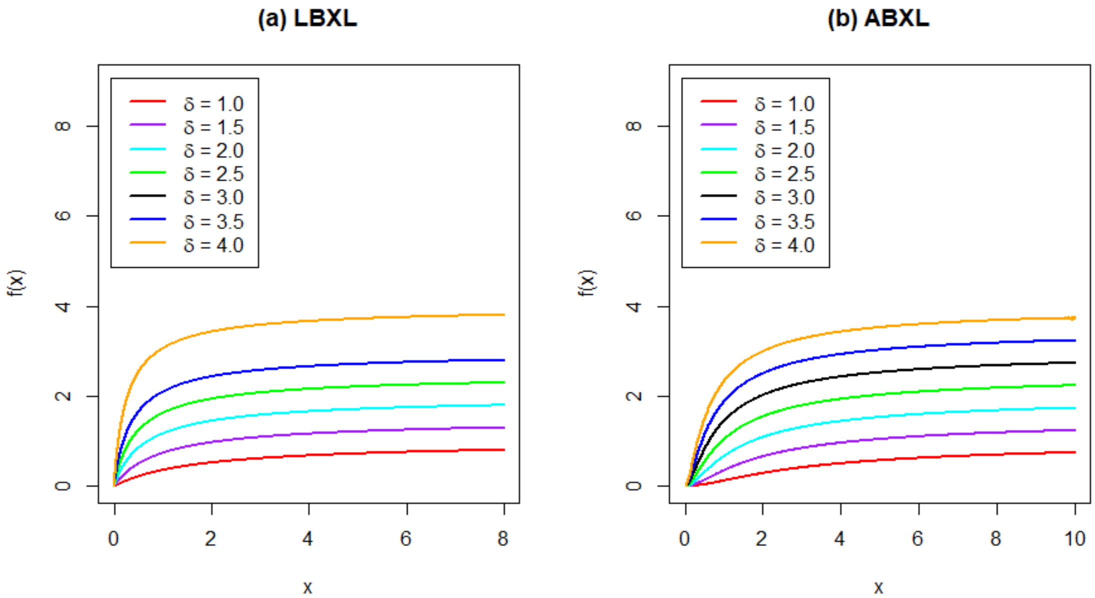

Fig. 4: The hazard plots of the LBXL and ABXL models for some choices of δ.

4.1. Statistical Characteristics of LBXL and ABXL Models

This subsection is devoted to the derivation of some important statistical properties. We derived moments, moment-generating function, and Shannon entropy.

The moments can be obtained by putting $p = 1,2$ in equation (6), respectively.

$$E(X_{LBXL}^r) = \frac{(r+1)!\,((1+\delta)^2+r+1)}{((1+\delta)^2+1)\delta^r},$$

and

$$E(X_{ABXL}^r) = \frac{(r+2)!\,((1+\delta)^2+r+2)}{2((1+\delta)^2+2)\delta^r}.$$

The moment-generating function for both models is presented below.

$$M_X(t) = \frac{\delta^3\big(1+(1+\delta)^2-t(2+\delta)\big)}{(\delta-t)^3(1+(1+\delta)^2)},$$

and

$$M_X(t)=\frac{\delta^2\left(2+(1+\delta)^2-t(2+\delta)\right)}{(\delta-t)^4\left(2+(1+\delta)^2\right)}.$$

Now we determine the numerical values of some renowned descriptive statistics such as average (Avg.), standard deviation (St.D.), coefficient of skewness, kurtosis, and variation. Table 1 presents a summary of these metrics.

Table 1: Certain computational metrics are generated from the LBXL and ABXL distributions.

	LBXL					**ABXL**				
δ	**Avg.**	**St.D.**	**CV**	**Skewness**	**Kurtosis**	**Avg.**	**St.D.**	**CV**	**Skewness**	**Kurtosis**
0.5	5.231	3.378	0.646	1.222	5.197	7.412	3.957	0.534	1.027	4.564
1.0	2.400	1.625	0.677	1.298	5.471	3.500	1.937	0.553	1.067	4.680
1.5	1.517	1.049	0.691	1.345	5.668	2.242	1.264	0.564	1.098	4.781
2.0	1.100	0.768	0.698	1.373	5.791	1.636	0.932	0.569	1.118	4.853
2.5	0.860	0.604	0.702	1.388	5.865	1.284	0.735	0.572	1.131	4.900
3.0	0.706	0.497	0.704	1.397	5.911	1.056	0.606	0.574	1.139	4.932
3.5	0.598	0.422	0.705	1.403	5.939	0.896	0.515	0.575	1.144	4.952
4.0	0.519	0.366	0.706	1.406	5.957	0.778	0.448	0.576	1.147	4.966

The Shannon entropy for ABXL and LBXL distributions can be calculated as

$$H(X)=E\left[-\log\left(f(x)\right)\right]$$

Now using the log on the density function of the LBXL distribution,

$$\log\left(f_{LBXL}(x)\right)=\log\left(\frac{\delta^3}{(\delta^2+2\delta+2)}\right)+\log(x)+\log(2+\delta+x)-\delta x.$$

and

$$H_{LBXL}(X)=-\frac{\delta^3}{(\delta^2+2\delta+2)}+\frac{\delta^3}{(\delta^2+2\delta+2)}\int_0^\infty(\log(x))\,x(2+\delta+x)e^{-\delta x}dx$$
$$-\frac{\delta^3}{(\delta^2+2\delta+2)}\int_0^\infty\log(2+\delta+x)\,x(2+\delta+x)e^{-\delta x}dx$$
$$+\frac{\delta^4}{(\delta^2+2\delta+2)}\int_0^\infty x^2\,(2+\delta+x)e^{-\delta x}dx,$$

$$H_{LBXL}(X) = -\frac{\delta^3}{(\delta^2+2\delta+2)} + \frac{\delta^3\big(3+\delta(2+\delta) - \text{EulerGamma}\big(2+\delta(2+\delta)\big) - \big(2+\delta(2+\delta)\big)\log(\delta)\big)}{(2+2\delta+\delta^2)^2} + \frac{6\delta\big(4+\delta(2+\delta)\big)}{(2+2\delta+\delta^2)^2} - \frac{\delta^3}{(\delta^2+2\delta+2)}\int_0^\infty \log(2+\delta+x)\, x(2+\delta+x)e^{-\delta x}dx.$$

Similarly, the Shannon entropy of the ABXL distribution can be determined. Table 2 displays the numerical entropy values for both the LBXL and ABXL models, as calculated for various δ values.

Table 2: Entropy values for the models ABXL and LBXL distributions

$\boldsymbol{\delta}$	$\boldsymbol{H_{LBXL}(X)}$	$\boldsymbol{H_{ABXL}(X)}$	$\boldsymbol{\delta}$	$\boldsymbol{H_{LBXL}(X)}$	$\boldsymbol{H_{ABXL}(X)}$
0.3	-3.02843	-3.21979	1.7	-1.15352	-1.40808
0.5	-2.49256	-2.69868	1.9	-1.03094	-1.28858
0.8	-1.98548	-2.21055	2.0	-0.97458	-1.23349
0.9	-1.85642	-2.08662	2.2	-0.87018	-1.13122
1.1	-1.63523	-1.87400	2.4	-0.77529	-1.03801
1.3	-1.45035	-1.69578	2.6	-0.68841	-0.95244
1.5	-1.29188	-1.54247	3.0	-0.53408	-0.80003

5. Estimation and Simulation Study

This section is based on the parameter estimation of both sub-models LBXL and ABXL using four renowned classical estimation methods. The behavior of these estimation methods is assessed using a detailed simulation study with small and large sample sizes.

5.1. Maximum Likelihood Estimation

A random sample $x_1, x_2, \dots, x_n$ of size n is taken from both sub-models. The log-likelihood functions of LBXL and ABXL distributions are given below, respectively

$$l(\delta)_{LBXL} = n\log\left(\frac{\delta^3}{\delta^2+2\delta+2}\right) + \sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log(2+\delta+x_i) - \delta\sum_{i=1}^{n}x_i,$$

$$l(\delta)_{ABXL} = n\log\left(\frac{\delta^4}{2(\delta^2+2\delta+3)}\right) + \sum_{i=1}^{n}\log({x_i}^2) + \sum_{i=1}^{n}\log(2+\delta+x_i) - \delta\sum_{i=1}^{n}x_i.$$

We differentiate the above equations for parameter δ and the non-linear equations are

$$\frac{\partial l(\delta)_{LBXL}}{\partial\delta}=\frac{n(6+4\delta+\delta^2)}{\delta(2+2\delta+\delta^2)}+\sum_{i=1}^{n}\frac{1}{2+\delta+x_i}-\sum_{i=1}^{n}x_i,$$

and

$$\frac{\partial l(\delta)_{ABXL}}{\partial\delta}=\frac{2n(6+3\delta+\delta^2)}{\delta(3+2\delta+\delta^2)}+\sum_{i=1}^{n}\frac{1}{2+\delta+x_i}-\sum_{i=1}^{n}x_i.$$

Because finding an exact solution to the above-derived equations can be difficult, one approach to optimizing them is to utilize an approach, for example, the Newton-Raphson algorithm. We used the ***optim*** function using R-software.

5.2. Anderson Darling Estimation

In this subsection, we consider another renowned estimation approach named Anderson Darling (AD) estimation which was also previously utilized by [24–26]. Let $x_1, x_2, \dots, x_n$ be a random sample drawn from distribution and $X_{1:n}, X_{2:n}, \dots, X_{n:n}$ are ordered observations. The Anderson Darling estimates can be obtained by minimizing the below distance for parameter δ.

$$\mathrm{AD}(\delta)_{LBXL}=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(F\left(x_{(i:n)}\middle|\delta\right)_{LBXL}\right)+\log\left(1-F\left(x_{(n+1-i)}\middle|\delta\right)_{LBXL}\right)\right],$$

and

$$\mathrm{AD}(\delta)_{ABXL}=-n-\frac{1}{n}\sum_{i=1}^{n}(2i-1)\left[\log\left(F\left(x_{(i:n)}\middle|\delta\right)_{ABXL}\right)+\log\left(1-F\left(x_{(n+1-i)}\middle|\delta\right)_{ABXL}\right)\right],$$

Where $F(.)$ is the cdf of under-study distributions.

5.3. Ordinary Least Squares Estimation

The Ordinary Least Squares (OLS) estimators for LBXL and ABXL models can be attained by minimizing the below-given distance

$$\mathrm{OLS}(\delta)_{LBXL}=\sum_{i=1}^{n}\left[F\left(x_{(i:n)}\middle|\delta\right)_{LBXL}-\frac{i}{n+1}\right]^2,$$

and

$$\mathrm{OLS}(\delta)_{ABXL}=\sum_{i=1}^{n}\left[F\left(x_{(i:n)}\middle|\delta\right)_{ABXL}-\frac{i}{n+1}\right]^2.$$

5.4. Cramer Von-Misses Estimation

The Cramer von Misses (CVM) is also a distance-based estimation approach. Various authors utilized this estimation method to estimate the model parameters [27]. The estimates can be obtained by minimizing the following distance

$$\mathrm{CVM}(\delta)_{LBXL} = \frac{1}{12n} + \sum_{i=1}^{n} \left[\log\left(F\left(x_{(i:n)} \middle| \delta\right)_{LBXL} \right) - \frac{2i-1}{2n} \right]^2,$$

and

$$\mathrm{CVM}(\delta)_{ABXL} = \frac{1}{12n} + \sum_{i=1}^{n} \left[\log\left(F\left(x_{(i:n)} \middle| \delta\right)_{ABXL} \right) - \frac{2i-1}{2n} \right]^2.$$

5.5. Simulation Study

This subsection is devoted to a comprehensive simulation study that is utilized to assess the effectiveness of the estimators discussed earlier. The random samples are generated from LBXL and ABXL distributions based on different sample sizes such as $n = 25, 50, 100, 150, 200$, and 250. We replicate this process 10,000 times for each sample size. We computed average estimates (AE), absolute bias (AB), mean relative errors (MRE), and mean square errors (MSE) for different values of parameter δ. The formulas of AS, MRE, and MSE are given below.

$$\mathrm{ABs} = \frac{\sum_{i=1}^{N} \left|\hat{\delta} - \delta\right|}{N},$$

$$\mathrm{MRE} = \frac{\sum_{i=1}^{N} \left|\hat{\delta} - \delta\right| / \delta}{N},$$

and

$$\mathrm{MSE} = \frac{\sum_{i=1}^{N} \left(\hat{\delta} - \delta\right)^2}{N}.$$

The findings of the detailed simulation study are listed in Tables 3–8. From these tables, it is observed that estimation methods efficiently estimate the model parameters. The AB, MRE, and MSE are showing a decreasing pattern with increasing sample size.

Table 3: The parameter estimates based on considered estimation methods for parameter $\delta = 0.5$

Measure	n	LBXL				ABXL			
		AD	ML	OLS	CVM	AD	ML	OLS	CVM
AEs	25	0.50719	0.50566	0.50538	0.50949	0.50291	0.50449	0.50131	0.50453
	50	0.50050	0.50636	0.50211	0.50368	0.50013	0.50377	0.49990	0.49916
	100	0.50041	0.50191	0.50277	0.50061	0.50048	0.50111	0.50188	0.50195
	150	0.50120	0.50219	0.50202	0.50065	0.50035	0.50047	0.50005	0.49994
	200	0.50090	0.50045	0.50127	0.50165	0.50064	0.50142	0.50123	0.49940
	250	0.50112	0.50072	0.49986	0.50031	0.50008	0.50102	0.50089	0.50182
ABs	25	0.00719	0.00566	0.00538	0.00949	0.00291	0.00449	0.00131	0.00453
	50	0.00050	0.00636	0.00211	0.00368	0.00013	0.00377	0.00010	0.00084
	100	0.00041	0.00191	0.00277	0.00061	0.00048	0.00111	0.00188	0.00195
	150	0.00120	0.00219	0.00202	0.00065	0.00035	0.00047	0.00005	0.00006
	200	0.00090	0.00045	0.00127	0.00165	0.00064	0.00142	0.00123	0.00060
	250	0.00112	0.00072	0.00014	0.00031	0.00008	0.00102	0.00089	0.00182
MREs	25	0.01437	0.01132	0.00437	0.01899	0.00583	0.00898	0.00314	0.00905
	50	0.00099	0.01272	0.00209	0.00736	0.00025	0.00753	0.00152	0.00167
	100	0.00081	0.00381	0.00106	0.00122	0.00095	0.00222	0.00071	0.00391
	150	0.00239	0.00438	0.00072	0.00131	0.00069	0.00093	0.00049	0.00012
	200	0.00179	0.00090	0.00055	0.00329	0.00128	0.00285	0.00037	0.00119
	250	0.00223	0.00143	0.00040	0.00062	0.00016	0.00203	0.00030	0.00363
MSEs	25	0.00435	0.00351	0.01075	0.00429	0.00307	0.00269	0.00262	0.00292
	50	0.00180	0.00179	0.00422	0.00217	0.00131	0.00126	0.00020	0.00144
	100	0.00099	0.00073	0.00555	0.00097	0.00063	0.00068	0.00376	0.00075
	150	0.00061	0.00062	0.00403	0.00067	0.00045	0.00043	0.00009	0.00051
	200	0.00045	0.00045	0.00254	0.00057	0.00035	0.00033	0.00246	0.00035
	250	0.00037	0.00035	0.00029	0.00039	0.00026	0.00024	0.00178	0.00027

Table 4: The parameter estimates based on considered estimation methods for parameter $\delta = 1.0$

Measure	n	LBXL				ABXL			
		AD	ML	OLS	CVM	AD	ML	OLS	CVM
AEs	25	1.00768	1.01479	1.01049	1.01308	1.00268	1.00896	1.00337	1.00754
	50	1.00133	1.00828	1.00608	1.00496	1.00368	1.00400	1.00328	1.00310
	100	1.00369	1.00170	1.00557	1.00183	1.00046	1.00082	1.00374	1.00363
	150	1.00329	1.00035	1.00250	1.00007	1.00379	0.99961	1.00197	1.00179
	200	1.00070	1.00257	1.00018	1.00197	0.99986	1.00264	1.00007	1.00246
	250	1.00124	1.00320	1.00088	1.00266	1.00124	1.00060	1.00145	1.00152
ABs	25	0.00768	0.01479	0.01049	0.01308	0.00268	0.00896	0.00337	0.00754
	50	0.00133	0.00828	0.00608	0.00496	0.00368	0.00400	0.00328	0.00310
	100	0.00369	0.00170	0.00557	0.00183	0.00046	0.00082	0.00374	0.00363
	150	0.00329	0.00035	0.00250	0.00007	0.00379	0.00039	0.00197	0.00179
	200	0.00070	0.00257	0.00018	0.00197	0.00014	0.00264	0.00007	0.00246
	250	0.00124	0.00320	0.00088	0.00266	0.00124	0.00060	0.00145	0.00152
MREs	25	0.00768	0.01479	0.01791	0.01308	0.00268	0.00896	0.01270	0.00754
	50	0.00133	0.00828	0.00872	0.00496	0.00368	0.00400	0.00658	0.00310
	100	0.00369	0.00170	0.00476	0.00183	0.00046	0.00082	0.00284	0.00363
	150	0.00329	0.00035	0.00292	0.00007	0.00379	0.00039	0.00211	0.00179
	200	0.00070	0.00257	0.00191	0.00197	0.00014	0.00264	0.00155	0.00246
	250	0.00124	0.00320	0.00164	0.00266	0.00124	0.00060	0.00125	0.00152
MSEs	25	0.01668	0.01598	0.01049	0.01857	0.01109	0.00998	0.00337	0.01206
	50	0.00802	0.00750	0.00608	0.00954	0.00587	0.00574	0.00328	0.00637
	100	0.00415	0.00366	0.00557	0.00456	0.00262	0.00285	0.00374	0.00308
	150	0.00272	0.00233	0.00250	0.00270	0.00175	0.00164	0.00197	0.00203
	200	0.00201	0.00183	0.00018	0.00209	0.00147	0.00126	0.00007	0.00162
	250	0.00157	0.00154	0.00088	0.00174	0.00105	0.00107	0.00145	0.00110

Table 5: The parameter estimates based on considered estimation methods for parameter $\delta = 1.5$

Measure	n	LBXL				ABXL			
		AD	ML	OLS	CVM	AD	ML	OLS	CVM
AEs	25	1.51760	1.52874	1.52200	1.52990	1.50887	1.51100	1.50568	1.50974
	50	1.51158	1.50918	1.50827	1.50612	1.50589	1.51377	1.49846	1.50838
	100	1.50541	1.50219	1.50222	1.50552	1.50394	1.50318	1.50160	1.50346
	150	1.50264	1.50340	1.50526	1.50558	1.50226	1.50251	1.49975	1.49892
	200	1.50423	1.50492	1.50252	1.50278	1.50183	1.50052	1.49953	1.50172
	250	1.50370	1.50608	1.49941	1.50136	1.49854	1.50148	1.50126	1.50423
ABs	25	0.01760	0.02874	0.02200	0.02990	0.00887	0.01100	0.00568	0.00974
	50	0.01158	0.00918	0.00827	0.00612	0.00589	0.01377	0.00154	0.00838
	100	0.00541	0.00219	0.00222	0.00552	0.00394	0.00318	0.00160	0.00346
	150	0.00264	0.00340	0.00526	0.00558	0.00226	0.00251	0.00025	0.00108
	200	0.00423	0.00492	0.00252	0.00278	0.00183	0.00052	0.00047	0.00172
	250	0.00370	0.00608	0.00059	0.00136	0.00146	0.00148	0.00126	0.00423
MREs	25	0.01173	0.01916	0.04707	0.01993	0.00591	0.00733	0.02887	0.00650
	50	0.00772	0.00612	0.02080	0.00408	0.00392	0.00918	0.01409	0.00559
	100	0.00361	0.00146	0.00986	0.00368	0.00263	0.00212	0.00699	0.00231
	150	0.00176	0.00226	0.00655	0.00372	0.00151	0.00168	0.00469	0.00072
	200	0.00282	0.00328	0.00502	0.00185	0.00122	0.00035	0.00350	0.00115
	250	0.00246	0.00406	0.00390	0.00091	0.00097	0.00099	0.00267	0.00282
MSEs	25	0.03827	0.03999	0.01467	0.04648	0.02619	0.02344	0.00379	0.02842
	50	0.02052	0.01714	0.00552	0.02016	0.01305	0.01242	0.00103	0.01414
	100	0.01012	0.00846	0.00148	0.01030	0.00619	0.00677	0.00107	0.00718
	150	0.00608	0.00612	0.00351	0.00719	0.00402	0.00403	0.00016	0.00428
	200	0.00449	0.00397	0.00168	0.00468	0.00303	0.00286	0.00031	0.00343
	250	0.00374	0.00344	0.00039	0.00397	0.00252	0.00242	0.00084	0.00288

Table 6: The parameter estimates based on considered estimation methods for parameter $\delta = 2.0$

Measure	n	LBXL				ABXL			
		AD	ML	OLS	CVM	AD	ML	OLS	CVM
AEs	25	2.02152	2.03009	2.02730	2.02145	2.01328	2.02572	2.02445	2.01691
	50	2.01531	2.02660	2.01351	2.02012	2.00632	2.01660	2.00073	2.00665
	100	1.99652	2.00788	2.00321	1.99877	2.00418	2.00595	1.99902	2.00802
	150	2.00196	2.00669	2.00230	2.00320	2.00385	2.00560	1.99874	2.00147
	200	2.00424	2.00339	1.99563	2.00334	2.00034	2.00342	2.00637	2.00350
	250	2.00475	2.00259	2.00678	2.00385	1.99865	2.00607	1.99964	2.00227
ABs	25	0.02152	0.03009	0.02730	0.02145	0.01328	0.02572	0.02445	0.01691
	50	0.01531	0.02660	0.01351	0.02012	0.00632	0.01660	0.00073	0.00665
	100	0.00348	0.00788	0.00321	0.00123	0.00418	0.00595	0.00098	0.00802
	150	0.00196	0.00669	0.00230	0.00320	0.00385	0.00560	0.00126	0.00147
	200	0.00424	0.00339	0.00437	0.00334	0.00034	0.00342	0.00637	0.00350
	250	0.00475	0.00259	0.00678	0.00385	0.00135	0.00607	0.00036	0.00227
MREs	25	0.01076	0.01505	0.08145	0.01072	0.00664	0.01286	0.05176	0.00846
	50	0.00765	0.01330	0.04419	0.01006	0.00316	0.00830	0.02666	0.00332
	100	0.00174	0.00394	0.01900	0.00061	0.00209	0.00298	0.01254	0.00401
	150	0.00098	0.00335	0.01285	0.00160	0.00193	0.00280	0.00808	0.00074
	200	0.00212	0.00169	0.00947	0.00167	0.00017	0.00171	0.00639	0.00175
	250	0.00238	0.00129	0.00758	0.00193	0.00067	0.00304	0.00491	0.00113
MSEs	25	0.07785	0.06911	0.01365	0.08344	0.05079	0.04639	0.01222	0.05029
	50	0.03508	0.03353	0.00676	0.04099	0.02431	0.02390	0.00037	0.02482
	100	0.01819	0.01573	0.00160	0.01851	0.01202	0.01033	0.00049	0.01296
	150	0.01144	0.00980	0.00115	0.01307	0.00790	0.00754	0.00063	0.00810
	200	0.00942	0.00818	0.00219	0.00940	0.00586	0.00551	0.00318	0.00662
	250	0.00685	0.00669	0.00339	0.00747	0.00499	0.00429	0.00018	0.00515

Table 7: The parameter estimates based on considered estimation methods for parameter $\delta = 2.5$

Measure	n	LBXL				ABXL			
		AD	ML	OLS	CVM	AD	ML	OLS	CVM
AEs	25	2.51117	2.54365	2.53409	2.55469	2.52049	2.55183	2.52585	2.52390
	50	2.52215	2.50698	2.50270	2.51424	2.51497	2.50875	2.51483	2.50974
	100	2.50235	2.50909	2.50806	2.51251	2.50314	2.51218	2.49990	2.49912
	150	2.50273	2.50056	2.50883	2.51497	2.49962	2.50821	2.49983	2.50064
	200	2.50477	2.50071	2.50444	2.50662	2.50389	2.50754	2.49767	2.49301
	250	2.50336	2.49734	2.50417	2.50397	2.50420	2.49617	2.50310	2.50690
ABs	25	0.01117	0.04365	0.03409	0.05469	0.02049	0.05183	0.02585	0.02390
	50	0.02215	0.00698	0.00270	0.01424	0.01497	0.00875	0.01483	0.00974
	100	0.00235	0.00909	0.00806	0.01251	0.00314	0.01218	0.00010	0.00088
	150	0.00273	0.00056	0.00883	0.01497	0.00038	0.00821	0.00017	0.00064
	200	0.00477	0.00071	0.00444	0.00662	0.00389	0.00754	0.00233	0.00699
	250	0.00336	0.00266	0.00417	0.00397	0.00420	0.00383	0.00310	0.00690
MREs	25	0.00447	0.01746	0.14398	0.02187	0.00820	0.02073	0.08701	0.00956
	50	0.00886	0.00279	0.05922	0.00570	0.00599	0.00350	0.04106	0.00390
	100	0.00094	0.00363	0.03121	0.00500	0.00125	0.00487	0.01956	0.00035
	150	0.00109	0.00023	0.02040	0.00599	0.00015	0.00328	0.01431	0.00025
	200	0.00191	0.00029	0.01634	0.00265	0.00156	0.00302	0.01115	0.00280
	250	0.00134	0.00107	0.01234	0.00159	0.00168	0.00153	0.00800	0.00276
MSEs	25	0.11920	0.08176	0.01364	0.13298	0.08768	0.12041	0.01034	0.08185
	50	0.05722	0.03234	0.00108	0.06654	0.03766	0.04932	0.00593	0.04240
	100	0.02722	0.01716	0.00322	0.03140	0.01671	0.02636	0.00004	0.01932
	150	0.01958	0.01196	0.00353	0.02033	0.01178	0.01774	0.00007	0.01399
	200	0.01494	0.00948	0.00178	0.01617	0.01006	0.01307	0.00093	0.00989
	250	0.01221	0.00695	0.00167	0.01243	0.00753	0.01023	0.00124	0.00878

Table 8: The parameter estimates based on considered estimation methods for parameter $\delta = 3.0$

Measure	n	LBXL				ABXL			
		AD	ML	OLS	CVM	AD	ML	OLS	CVM
AEs	25	3.03338	3.06495	3.06076	3.03777	3.03548	3.03812	3.01057	3.01985
	50	3.03011	3.01060	3.01060	3.03202	3.01712	3.02051	3.01895	3.02083
	100	3.01408	3.01694	3.01522	3.00190	3.00285	3.01131	3.01095	3.00920
	150	3.00956	3.00092	3.00091	3.00411	2.99918	3.00555	3.00255	2.99933
	200	3.00175	3.00469	3.00651	3.00571	3.00367	3.00730	3.00276	3.00776
	250	3.00152	3.00453	3.01107	3.01149	2.99876	3.00445	2.99634	3.00572
ABs	25	0.03338	0.06495	0.06076	0.03777	0.03548	0.03812	0.01057	0.01985
	50	0.03011	0.01060	0.01060	0.03202	0.01712	0.02051	0.01895	0.02083
	100	0.01408	0.01694	0.01522	0.00190	0.00285	0.01131	0.01095	0.00920
	150	0.00956	0.00092	0.00091	0.00411	0.00082	0.00555	0.00255	0.00067
	200	0.00175	0.00469	0.00651	0.00571	0.00367	0.00730	0.00276	0.00776
	250	0.00152	0.00453	0.01107	0.01149	0.00124	0.00445	0.00366	0.00572
MREs	25	0.01113	0.02165	0.21265	0.01259	0.01183	0.01271	0.12658	0.00662
	50	0.01004	0.00353	0.09210	0.01067	0.00571	0.00684	0.05938	0.00694
	100	0.00469	0.00565	0.04789	0.00063	0.00095	0.00377	0.03170	0.00307
	150	0.00319	0.00031	0.03068	0.00137	0.00027	0.00185	0.02099	0.00022
	200	0.00058	0.00156	0.02645	0.00190	0.00122	0.00243	0.01467	0.00259
	250	0.00051	0.00151	0.01792	0.00383	0.00041	0.00148	0.01265	0.00191
MSEs	25	0.17077	0.19030	0.02025	0.19797	0.10681	0.10706	0.00352	0.13480
	50	0.08394	0.08277	0.00353	0.09394	0.05671	0.05265	0.00632	0.05919
	100	0.04313	0.04146	0.00507	0.04445	0.02899	0.02854	0.00365	0.03182
	150	0.02842	0.02631	0.00030	0.03002	0.01837	0.01621	0.00085	0.02117
	200	0.02087	0.01907	0.00217	0.02243	0.01405	0.01388	0.00092	0.01574
	250	0.01654	0.01592	0.00369	0.01770	0.01070	0.01075	0.00122	0.01293

6. Application SBXL Distribution

Now we utilize the newly proposed distribution on two datasets from different disciplines to demonstrate its usefulness. Since the proposed distributions in our study involve two sub-models, we conducted comparisons with other different distributions to ensure a fair evaluation. The performance of the new evaluated against renowned lifetime distributions such as inverted Kumaraswamy distribution [28], Burr III distribution[29], Xgamma distribution [30], Haq distribution ([31]), XLindley distribution [23], and Lindley distribution [32].

The model parameters for all the distributions under consideration were estimated using the maximum likelihood method. The fitted distributions were compared based on the maximized log-likelihood value, Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Cramer von Mises test (W), and Kolmogorov-Smirnov test.

Data Set I:

The first dataset concerns the tensile strength of approximately 100 carbon fiber observations, originally discussed by [33]. The data points include values such as 3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, 2.00, 1.17, 2.17, 0.39, 2.79, 1.08, 2.88, 2.73, 2.87, 3.19, 1.87, 2.95, 2.67, 4.20, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 1.47, 3.22, 3.15, 2.97, 1.61, 2.05, 3.60, 3.11, 1.69, 4.90, 3.39, 3.22, 2.55, 3.56, 2.38, 1.92, 0.98, 1.59, 1.73, 1.71, 1.18, 4.38, 0.85, 1.80, 2.12, and 3.65.

Visual representations of this dataset, including the TTT plot, box plots, and violin plot are shown in Figure 5.

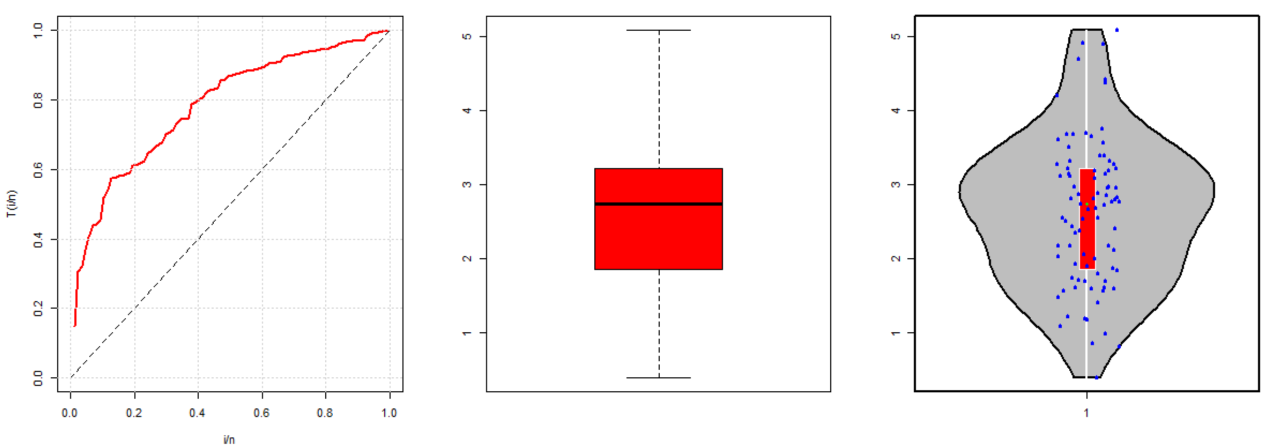

Fig. 5: TTT, box, and violin plots for the first dataset

Table 9 shows the parameter estimates, standard errors (SEs), and model selection criteria for the first dataset. Figure 6 also shows a visual comparison between the fitted PDF, CDF, P-P, contour, and profile log-likelihood plots.

Table 9: MLEs, SEs, and model selection criteria obtained from dataset I

Distribution	Log-L	AIC	BIC	CVM	KS	Est. Parameters (SEs)	
SBXL	- 124.23	252.46	257.4	0.1600	0.1083	2.3853 (0.3513)	4.9121 (0.9011)
QXL	-130.39	264.79	269.72	0.8389	0.1701	0.9878 (0.0799)	-0.6851 (0.0286)
BIII	-141.44	286.89	291.82	27.790	0.9866	5.1466 (0.6281)	2.2837 (0.1568)
IK	-139.93	283.87	288.80	0.5230	0.1331	3.0140 (0.2359)	26.194 (6.5304)
LBXL	-142.22	286.43	288.90	1.1088	0.2031	0.9194 (0.0586)	-
ABXL	-132.17	266.34	268.81	0.5752	0.1418	1.2955 (0.0708)	-
Haq	-171.70	345.39	347.86	3.1041	0.3386	0.6756 (0.0508)	-
Xgamma	-160.67	323.33	325.80	2.4690	0.3047	0.8493 (0.0600)	-
XLindley	-164.58	331.16	333.63	2.4400	0.3003	0.5361 (0.0432)	-
Lindley	-158.28	318.56	321.03	2.0613	0.2771	0.6156 (0.0484)	-

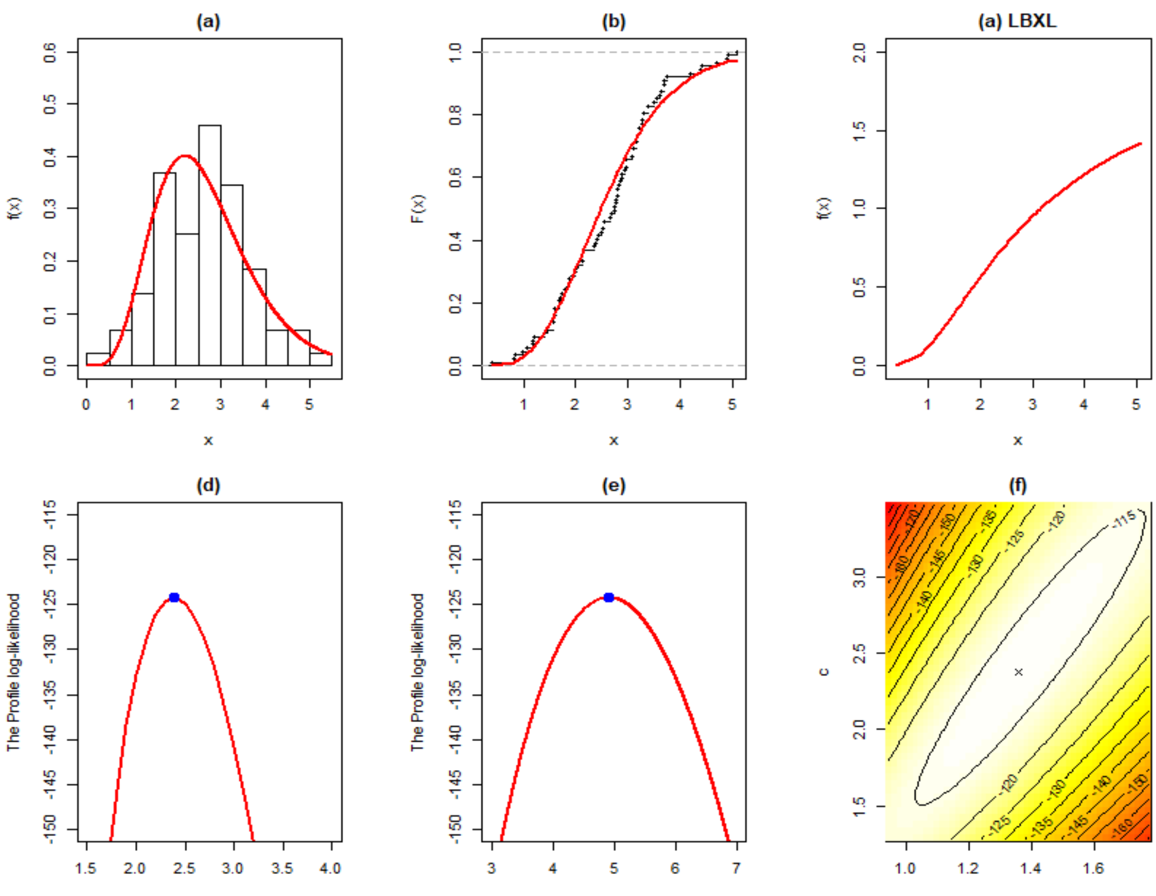

Fig. 6: Fitted density function, distribution function, hazard function, contour, and profile log-likelihood plots based new model for dataset I.

Data Set II:

The second data set illustrated below is also utilized by [34] which represents the breaking stress of carbon fibers of 50mm in length (measured in GPa). The recorded data points are as follows: 0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12,2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85,2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31,3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90.

The descriptive behavior is illustrated visually using different plots such as box, violine, and TTT plots presented in Figure 7. The parameter estimates and goodness-of-fit measures for the second dataset are presented in Table 10.

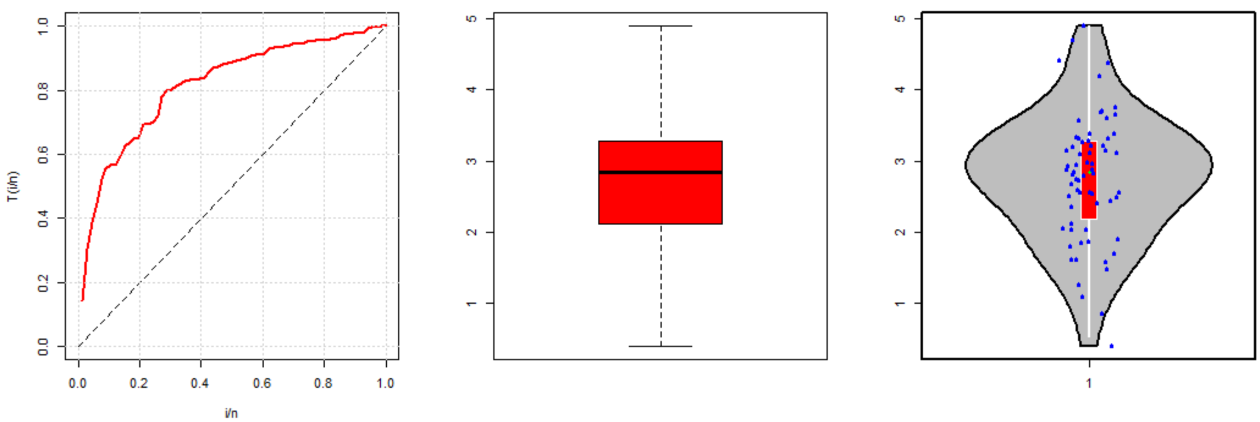

Fig. 7: Box, Violine, and TTT q-q plot for the dataset II

Table 10: MLEs, SEs, and model selection criteria obtained from dataset II

Distribution	Log-L	AIC	BIC	CVM	KS	Est. Parameters (SEs)	
SBXL	-91.069	186.14	190.52	0.2442	0.1324	2.7946 (0.4752)	6.3581 (1.2874)
QXL	-102.23	208.46	212.84	1.1667	0.2459	0.9113 (0.0860)	-0.6546 (0.0346)
BIII	-110.06	224.13	228.51	21.417	0.9912	5.8410 (0.8334)	2.2413 (0.1688)
IK	-108.76	221.52	225.90	0.7466	0.1895	2.9331 (0.2503)	27.794 (7.6780)
LBXL	-109.56	221.12	223.31	1.3498	0.2358	0.8807 (0.0643)	-
ABXL	-100.92	203.84	206.03	0.8766	0.2053	1.2394 (0.0776)	-
Haq	-133.08	268.16	270.35	2.8911	0.3629	0.6544 (0.0561)	-
Xgamma	-123.72	249.44	251.63	2.4150	0.3279	0.8210 (0.0661)	-
XLindley	-127.46	256.93	259.12	2.3715	0.3217	0.5149 (0.0475)	-
Lindley	-122.38	246.77	248.96	2.0914	0.2977	0.5902 (0.0532)	-

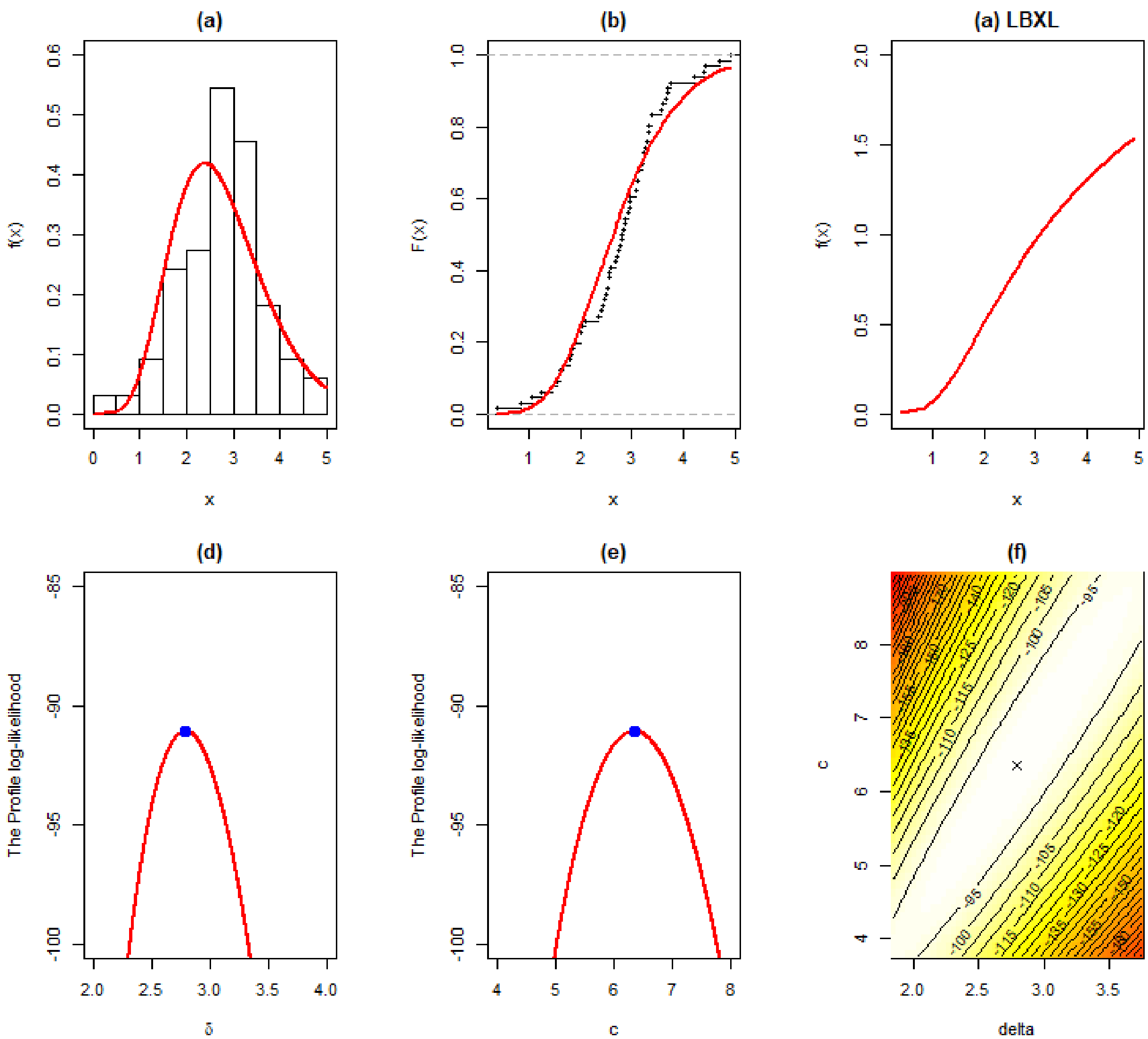

Fig.8: Fitted density function, distribution function, hazard function, contour, and profile log-likelihood plots based new model for dataset II.

7. Conclusion

This paper introduces and investigates a novel one-parameter weighted distribution known as the size-biased XLindley distribution. The study dives into its mathematical features and investigates two sub-models: length-biased and area-biased distributions. The model's parameters are calculated using five distinct ways, and the most efficient method is determined through extensive simulation. To show the new distribution's relevance and adaptability, two datasets from various domains are examined. The findings show that this novel model produces more efficient results than the previous one-parameter probability distributions examined in the study.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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