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Generalised Ulam-Hyers Stability Analysis for System of Additive Functional Equation in Fuzzy and Random Normed Spaces: Direct and Fixed Point Approach

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Abstract. In this article, a new system of Functional Equations is proposed. The Ulam-Hyers stability of this class of equations is investigated using the product, sum, and mixed product-sum of powers of norms, as well as the general control function. The stability analysis is carried out in random and fuzzy normed spaces using fixed point and direct methods. One of the unique and interesting aspects of this study is that, three new and different kinds of FEs have been introduced and the stability analysis is derived for all three equations simultaneously.

1. Introduction

A fundamental field of mathematics called FEs offers a wide range of solutions to algebraic, analytical, applied, theoretical, and topological problems. Several mathematicians, including Abel, D' Alembert, Babbage, Cauchy, Euler, Gauss, Legendre, and Schroder have made significant contributions to the advancement of this domain. FEs is one of the key areas of modern research that is gaining traction among researchers worldwide. Due to their numerous applications, FEs are drawing the attention of more and more mathematical and empirical researchers. FEs are studied in several branches of mathematics such as number theory, abstract algebra, queueing theory, probability theory, differential geometry, and differential equations [1–3].

If any function from a given set of functions that approximates the equation is comparable to an exact solution of the equation, the equation is said to be stable in that set of functions. The study

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of stability is essential to FEs because it serves as an efficient and comprehensive tool for assessing the error that occurs when substituting functions that only approximately satisfy some equations with the optimal solutions to those equations. In modern parlance, an equation is said to be stable within a specific type of function if every function in that classification that considerably satisfies the equation is comparable to an exactsolution of the equation. In the recent years, mathematicians have explored a wide range of stability problems using various FEs (algebraic, logarithmic,radical, andreciprocal). [4–8, 23].

In 1940, Ulam brought up a key research problem on the stability of group homomorphisms [9]. In the following year, Hyers [10] identified an answer to Ulam's question for the Cauchy additive FE. A couple of decades later, Rassias [11] generalised Hyers' result, and Gavruta [12] then continued to expand on Rassias' findings by incorporating unbounded control functions. Presently,the stability concept established by Rassias and Gavruta is typically referred to as the "generalised Hyers-Ulam stability" of FEs. The Ulam stability can be analysed using a multitude of methods, the most common of which is the fixed-point technique, which is predicated on a basic outcome from fixed-point theory [13–17].

Mihet et al. [18] explored the Ulam stability of the following Cauchy FE in random normed spaces.

$$Z(\mathfrak{P}_1 + \mathfrak{P}_2) = Z(\mathfrak{P}_1) + Z(\mathfrak{P}_2)$$

Kim et al. [19] proposed a generalised version of Cauchy additive FE

$$Z\left(\frac{\mathfrak{P}_1 - \mathfrak{P}_2}{n} + \mathfrak{P}_3\right) + Z\left(\frac{\mathfrak{P}_2 - \mathfrak{P}_3}{n} + \mathfrak{P}_1\right) + Z\left(\frac{\mathfrak{P}_3 - \mathfrak{P}_1}{n} + \mathfrak{P}_2\right) = Z(\mathfrak{P}_1 + \mathfrak{P}_2 + \mathfrak{P}_3)$$

Furthermore, the authors deduced the Ulam stability of the above equation in fuzzy Banach spaces for any non-zero fixed integer n. It should go without saying that an equation such as the one- above can only be satisfied if a function Z is additive. As a natural outcome, the equation is commonly referred to as the Cauchy additive FE, and the Cauchy additive function isits general solution.Ghaffari et al. [20] documented the stability of cubic mappings in fuzzy normed spaces, while Baktash et al. [21] discussed the stability of cubic and quartic mappings in random normed spaces. Ravi et al. [22] employed the fixed-point technique to estimate the fuzzy stability of the generalised square root FE in multiple variables. Recently Agilan et.al exploring the stability results in various additive functional equation through various normed spaces such as [23–28]

Quite recently, Al-Ali et al. [29] studied the generalised Ulam-Hyers stability (GUHS) of a generalised p-radical FE

$$Z\left(\sqrt[p]{\sum_{i=1}^{k} \mathfrak{P}_i^p}\right) = \sum_{i=1}^{k} Z(\mathfrak{P}_i)$$

and inhomogeneous $p-$radical FE

$$Z\left(\sqrt[p]{\sum_{i=1}^{k} \mathfrak{P}_i^p}\right) = \sum_{i=1}^{k} Z(\mathfrak{P}_i) + G(\mathfrak{P}_1, \mathfrak{P}_2, \cdots, \mathfrak{P}_n)$$

in 2-Banach spaces.

Inspired by previous studies on FEs and their stability analyses, this article introduces a novel system of FEs and establishes the GUHS of various general control functions of the below equationsusing direct and fixed point approaches in fuzzy and random normed spaces.

$$Z\left(\frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}+\frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right)+Z\left(\frac{p}{q}\sqrt{\mathfrak{P}_3\mathfrak{P}_2}+\frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_1}\right) = \left(\frac{p}{q}+\frac{r}{s}\right)[Z(\sqrt{\mathfrak{P}_1\mathfrak{P}_2})+Z(\sqrt{\mathfrak{P}_2\mathfrak{P}_3})] \tag{1.1}$$

$$Z\left(\frac{p}{\sqrt{2}q}\sqrt{\mathfrak{P}_1^2+\mathfrak{P}_2^2}+\frac{r}{\sqrt{2}s}\sqrt{\mathfrak{P}_2^2+\mathfrak{P}_3^2}\right)+Z\left(\frac{p}{\sqrt{2}q}\sqrt{\mathfrak{P}_3^2+\mathfrak{P}_2^2}+\frac{r}{\sqrt{2}s}\sqrt{\mathfrak{P}_2^2+\mathfrak{P}_1^2}\right) = \left(\frac{p}{q}+\frac{r}{s}\right)\left[Z\left(\sqrt{\frac{\mathfrak{P}_1^2+\mathfrak{P}_2^2}{2}}\right)+Z\left(\sqrt{\frac{\mathfrak{P}_2^2+\mathfrak{P}_3^2}{2}}\right)\right] \tag{1.2}$$

$$Z\left(\frac{p}{\sqrt{3}q}\sqrt{\mathfrak{P}_1^2+\mathfrak{P}_1\mathfrak{P}_2+\mathfrak{P}_2^2}+\frac{r}{\sqrt{3}s}\sqrt{\mathfrak{P}_2^2+\mathfrak{P}_2\mathfrak{P}_3+\mathfrak{P}_3^2}\right) + Z\left(\frac{p}{\sqrt{3}q}\sqrt{\mathfrak{P}_3^2+\mathfrak{P}_3\mathfrak{P}_2+\mathfrak{P}_2^2}+\frac{r}{\sqrt{3}s}\sqrt{\mathfrak{P}_2^2+\mathfrak{P}_2\mathfrak{P}_1+\mathfrak{P}_1^2}\right) = \left(\frac{p}{q}+\frac{r}{s}\right)\left[Z\left(\sqrt{\frac{\mathfrak{P}_1^2+\mathfrak{P}_1\mathfrak{P}_2+\mathfrak{P}_2^2}{3}}\right)+Z\left(\sqrt{\frac{\mathfrak{P}_2^2+\mathfrak{P}_2\mathfrak{P}_3+\mathfrak{P}_3^2}{3}}\right)\right] \tag{1.3}$$

This is the first study in the literature that introduces a system of FEs and investigates the stability of these equations simultaneously in fuzzy and random normed spaces. This kind of research has not been performed on FEs before, which adds a significant weightage to this article. Hence, this study is unique and will be a substantial contribution to the available literature on the study of FEs.

2. General Solution

In this section, let us consider $\mathfrak{F}^*$ and $\mathfrak{I}^*$ to be real vector spaces.

Lemma 2.1. *If an odd mapping $Z:\mathfrak{F}^*\to\mathfrak{I}^*$ satisfies the FE*

$$Z(\mathfrak{P}_1+\mathfrak{P}_2)=Z(\mathfrak{P}_1)+Z(\mathfrak{P}_2) \tag{2.1}$$

then $Z:\mathfrak{F}^*\to\mathfrak{I}^*$ satisfies the FE (1.1) for all $\mathfrak{P}_1,\mathfrak{P}_2,\mathcal{U}_3\in\mathfrak{F}^*$.

Proof. If $\mathfrak{P}_1=\mathfrak{P}_2=0$ in (2.1), then $Z(0)=0$. Replacing $\mathfrak{P}_1$ by $-\mathfrak{P}_2$ in (2.1), the following result is obtained: $Z(-\mathfrak{P}_2)=-Z(\mathfrak{P}_2)$ for all $\mathfrak{P}_2\in\mathfrak{F}^*$. Replacing $\mathfrak{P}_2$ by $\mathfrak{P}_1$ in (2.1),

$$Z(2\mathfrak{P}_1)=2Z(\mathfrak{P}_1) \tag{2.2}$$

for all $\mathfrak{P}_1\in\mathfrak{F}^*$. By induction of n,

$$Z(n\mathfrak{P}_1)=n\,Z(\mathfrak{P}_1) \tag{2.3}$$

Taking $\mathfrak{P}_1 = \frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}$ and $\mathfrak{P}_2 = \frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}$ in (2.1) and using (2.3),

$$Z\left(\frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2} + \frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right) = \frac{p}{q}Z\left(\sqrt{\mathfrak{P}_1\mathfrak{P}_2}\right) + \frac{r}{s}\left(\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right) \tag{2.4}$$

for all $\mathfrak{P}_1, \mathfrak{P}_2 \in \mathfrak{F}^*$. Taking $\mathfrak{P}_1 = \frac{p}{q}\sqrt{\mathfrak{P}_3\mathfrak{P}_2}$ and $\mathfrak{P}_2 = \frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_1}$ in (2.1) and using (2.3),

$$Z\left(\frac{p}{q}\sqrt{\mathfrak{P}_3\mathfrak{P}_2} + \frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_1}\right) = \frac{p}{q}Z\left(\sqrt{\mathfrak{P}_3\mathfrak{P}_2}\right) + \frac{r}{s}\left(\sqrt{\mathfrak{P}_2\mathfrak{P}_1}\right) \tag{2.5}$$

for all $\mathfrak{P}_1, \mathfrak{P}_2 \in \mathfrak{F}^*$. Adding (2.4), (2.5), we reach Equation (1.1).

$$\begin{aligned} &Z\left(\frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2} + \frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right) + Z\left(\frac{p}{q}\sqrt{\mathfrak{P}_3\mathfrak{P}_2} + \frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_1}\right) \\ &\qquad = \left(\frac{p}{q} + \frac{r}{s}\right)[Z(\sqrt{\mathfrak{P}_1\mathfrak{P}_2}) + Z(\sqrt{\mathfrak{P}_2\mathfrak{P}_3})] \end{aligned} \tag{2.6}$$

for all $\mathfrak{P}_1, \mathfrak{P}_2, \mathcal{U}_3 \in \mathfrak{F}^*$. □

3. Fuzzy Stability Results:Direct Method

3.1. **Basics of Fuzzy Normed Spaces.** Basics of Fuzzy normed spaces one can see [30–34].

Definition 3.1. *Let $\mathfrak{F}^*$ be a real linear space. A function $\mathfrak{N} : \mathfrak{F}^* \times \mathbb{R} \to [0,1]$(the so-called fuzzy subset) is said to be a fuzzy norm on $\mathfrak{F}^*$ if for all $\mathfrak{P}_1, \mathfrak{P}_2 \in \mathfrak{F}^*$ and all $s, \mathfrak{O} \in \mathbb{R}$,*

$(F1)$ $\mathfrak{N}(\mathfrak{P}_1, c) = 0$ *for* $c \leq 0$;

$(F2)$ $\mathfrak{P}_1 = 0$ *if and only if* $\mathfrak{N}(\mathfrak{P}_1, c) = 1$ *for all* $c > 0$;

$(F3)$ $\mathfrak{N}(c\mathfrak{P}_1, \mathcal{T}) = \mathfrak{N}\left(\mathfrak{P}_1, \frac{\mathcal{T}}{|c|}\right)$ *if* $c \neq 0$;

$(F4)$ $\mathfrak{N}(\mathfrak{P}_1 + \mathfrak{P}_2, s + \mathcal{T}) \geq min\{\mathfrak{N}(\mathfrak{P}_1, s), \mathfrak{N}(\mathfrak{P}_2, \mathcal{T})\}$;

$(F5)$ $\mathfrak{N}(\mathfrak{P}_1, \cdot)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $lim_{\mathcal{T}\to\infty}\mathfrak{N}(\mathfrak{P}_1, \mathcal{T}) = 1$;

$(F6)$ *for* $\mathfrak{P}_1 \neq 0, \mathfrak{N}(\mathfrak{P}_1, \cdot)$ *is continuous on* $\mathbb{R}$.

The pair $(\mathfrak{F}^*, \mathfrak{N})$ is called a fuzzy normed linear space. $\mathfrak{N}(\mathfrak{F}^*, \mathcal{T})$ can be regarded as the truth-value of the statement the norm of $\mathfrak{P}_1$ is less than or equal to the real number $\mathcal{T}'$.

Here, $\mathfrak{F}^*$-linear space, $(\mathfrak{I}^*, \mathfrak{N}')$-fuzzy normed space and $(Y, \mathfrak{N}')$-fuzzy Banach space.
$Z : \mathfrak{F}^* \to \mathfrak{I}^*$

$$\begin{aligned} \mathbb{G}Z(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3) = &Z\left(\frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2} + \frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right) + Z\left(\frac{p}{q}\sqrt{\mathfrak{P}_3\mathfrak{P}_2} + \frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_1}\right) \\ &- \left(\frac{p}{q} + \frac{r}{s}\right)[Z(\sqrt{\mathfrak{P}_1\mathfrak{P}_2}) - Z(\sqrt{\mathfrak{P}_2\mathfrak{P}_3})] \end{aligned}$$

Theorem 3.1. *let the mapping $\mathfrak{K} : \mathfrak{F}^{*3} \to \mathfrak{I}^*$such that for some $\mathcal{M}$ with $0 < \left(\frac{\mathcal{M}}{2}\right)^{\mathscr{E}} < 1$*

$$\mathfrak{N}'\left(\mathfrak{K}\left(\mathscr{D}^{\mathscr{E}}\mathfrak{P}_1, \mathscr{D}^{\mathscr{E}}\mathfrak{P}_1, \mathscr{D}^{\mathscr{E}}\mathfrak{P}_1,\right), \mathfrak{O}\right) \geq \mathfrak{N}'\left(\mathcal{M}^{\mathscr{E}}\mathfrak{K}\left(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1\right), \mathfrak{O}\right) \tag{3.1}$$

for all $\mathfrak{P}_1 \in \mathfrak{F}^*$, then

$$\lim_{k\to\infty}\mathfrak{N}'\left(\mathfrak{K}\left(\mathscr{D}^{\mathscr{E}k}\mathfrak{P}_1, \mathscr{D}^{\mathscr{E}k}\mathfrak{P}_2, \mathscr{D}^{\mathscr{E}k}\mathfrak{P}_3\right), \mathscr{D}^{\mathscr{E}k}\mathfrak{O}\right) = 1. \tag{3.2}$$

Suppose that a function $Z : \mathfrak{F}^* \to \mathfrak{I}^*$ *satisfies the inequality*

$$\mathfrak{N}\left(\mathbb{G}Z(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3), \mathfrak{O}\right) \geq \mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3), \mathfrak{O}\right) \tag{3.3}$$

Then the limit

$$\mathscr{A}(\mathfrak{P}_1) = \mathfrak{N} - \lim_{k\to\infty} \frac{Z(\mathscr{D}^{\mathscr{E}k}\mathfrak{P}_1)}{\mathscr{D}^{\mathscr{E}k}} \tag{3.4}$$

exists for all $\mathfrak{P}_1 \in \mathfrak{F}^*$ *and the mapping* $\mathscr{A} : \mathfrak{F}^* \to \mathfrak{I}^*$ *such that*

$$\mathfrak{N}\left(Z(\mathfrak{P}_1) - \mathscr{A}(\mathfrak{P}_1), \mathfrak{O}\right) \geq \mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1), 2\mathfrak{O}|\mathscr{D} - \mathcal{M}|\right) \tag{3.5}$$

with $\mathscr{E} \in \{-1, 1\}$ *be fixed and* $\mathscr{D} = \left(\frac{p}{q} + \frac{r}{s}\right)$.

Proof. Assuming $\mathscr{E} = 1$ and replacing $(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)$ by $(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1)$ in (3.3), the result is obtained as

$$\mathfrak{N}\left(2Z(\mathscr{D}\mathfrak{P}_1) - 2\mathscr{D}Z(\mathfrak{P}_1), \mathfrak{O}\right) \geq \mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1), \mathfrak{O}\right) \tag{3.6}$$

Replacing $\mathfrak{P}_1$ by $\mathscr{D}^k\mathfrak{P}_1$ in (3.6), then

$$\mathfrak{N}\left(\frac{Z(\mathscr{D}^{k+1}\mathfrak{P}_1)}{\mathscr{D}} - Z(\mathscr{D}^k\mathfrak{P}_1), \frac{\mathfrak{O}}{2\mathscr{D}}\right) \geq \mathfrak{N}'\left(\mathfrak{K}(\mathscr{D}^k\mathfrak{P}_1, \mathscr{D}^k\mathfrak{P}_1, \mathscr{D}^k\mathfrak{P}_1), \mathfrak{O}\right) \tag{3.7}$$

Using (3.1), $(F3)$ in (3.7), the equation becomes

$$\mathfrak{N}\left(\frac{Z(\mathscr{D}^{k+1}\mathfrak{P}_1)}{\mathscr{D}} - Z(\mathscr{D}^k\mathfrak{P}_1), \frac{\mathfrak{O}}{2\mathscr{D}}\right) \geq \mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1), \frac{\mathfrak{O}}{\mathcal{M}^k}\right) \tag{3.8}$$

We know that it is easy to verify from (3.8), that

$$\mathfrak{N}\left(\frac{Z(\mathscr{D}^{k+1}\mathfrak{P}_1)}{\mathscr{D}^{(k+1)}} - \frac{Z(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k}, \frac{\mathfrak{O}}{2\mathscr{D}\cdot\mathscr{D}^k}\right) \geq \mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1), \frac{\mathfrak{O}}{\mathcal{M}^k}\right) \tag{3.9}$$

Replacing $\mathfrak{O}$ by $\mathcal{M}^k\mathfrak{O}$ in (3.9), the result is obtained as

$$\mathfrak{N}\left(\frac{Z(\mathscr{D}^{k+1}\mathfrak{P}_1)}{\mathscr{D}^{(k+1)}} - \frac{Z(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k}, \frac{\mathcal{M}^k\,\mathfrak{O}}{2\mathscr{D}\cdot\mathscr{D}^k}\right) \geq \mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1), \mathfrak{O}\right) \tag{3.10}$$

and

$$\frac{Z(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k} - Z(\mathfrak{P}_1) = \sum_{i=0}^{k-1}\left[\frac{Z(\mathscr{D}^{i+1}\mathfrak{P}_1)}{\mathscr{D}^{(i+1)}} - \frac{Z(\mathscr{D}^i\mathfrak{P}_1)}{\mathscr{D}^i}\right] \tag{3.11}$$

for all $\mathfrak{P}_1 \in \mathfrak{F}^*$. From equations (3.10) and (3.11),

$$\begin{aligned}
\mathfrak{N}&\left(\frac{Z(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k} - Z(\mathfrak{P}_1), \sum_{i=0}^{k-1}\frac{\mathcal{M}^i\,\mathfrak{O}}{2\mathscr{D}\cdot\mathscr{D}^i}\right)\\
&\geq min\bigcup_{i=0}^{k-1}\left\{\frac{Z(\mathscr{D}^{i+1}\mathfrak{P}_1)}{\mathscr{D}^{(i+1)}} - \frac{Z(\mathscr{D}^i\mathfrak{P}_1)}{\mathscr{D}^i}, \frac{\mathcal{M}^i\,\mathfrak{O}}{2\mathscr{D}\cdot\mathscr{D}^i}\right\}\\
&\geq min\bigcup_{i=0}^{k-1}\left\{\mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1), \mathfrak{O}\right)\right\}\\
&\geq \mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1), \mathfrak{O}\right)
\end{aligned} \tag{3.12}$$

Replacing $\mathfrak{P}_1$ by $\mathscr{D}^m\mathfrak{P}_1$ in (3.12) and using (3.1), $(F3)$,

$$\mathfrak{N}\left(\frac{Z(\mathscr{D}^{k+m}\mathfrak{P}_1)}{\mathscr{D}^{(k+m)}}-\frac{Z(\mathscr{D}^{m}\mathfrak{P}_1)}{\mathscr{D}^{m}},\sum_{i=0}^{k-1}\frac{\mathcal{M}^i\,\mathfrak{O}}{2\mathscr{D}\cdot\mathscr{D}^{(i+m)}}\right)\geq\mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1),\frac{\mathfrak{O}}{\mathcal{M}^m}\right) \tag{3.13}$$

Replacing $\mathfrak{O}$ by $\mathcal{M}^m\mathfrak{O}$ in (3.13),

$$\mathfrak{N}\left(\frac{Z(\mathscr{D}^{k+m}\mathfrak{P}_1)}{\mathscr{D}^{(k+m)}}-\frac{Z(\mathscr{D}^{m}\mathfrak{P}_1)}{\mathscr{D}^{m}},\sum_{i=m}^{m+k-1}\frac{\mathcal{M}^i\,\mathfrak{O}}{2\mathscr{D}\cdot\mathscr{D}^{i}}\right)\geq\mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1),\mathfrak{O}\right) \tag{3.14}$$

Using $(F3)$ in (3.14),then

$$\mathfrak{N}\left(\frac{Z(\mathscr{D}^{k+m}\mathfrak{P}_1)}{\mathscr{D}^{(k+m)}}-\frac{Z(\mathscr{D}^{m}\mathfrak{P}_1)}{\mathscr{D}^{m}},\mathfrak{O}\right)\geq\mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1),\frac{r}{\sum_{i=m}^{m+k-1}\frac{\mathcal{M}^i}{2\mathscr{D}\cdot\mathscr{D}^i}}\right) \tag{3.15}$$

for all $\mathfrak{P}_1\in\mathfrak{F}^*$ and all $\mathfrak{O}>0$ and all $m,k\geq 0$. Since $0<\mathcal{M}<\mathscr{D}$ and $\sum_{i=0}^{k}\left(\frac{d}{\mathscr{D}}\right)^i<\infty$, the Cauchy criterion for convergence and $(F5)$ implies that $\left\{\frac{Z(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k}\right\}$ is a Cauchy sequence in $(\mathfrak{I}^*,N)$. Since $(\mathfrak{I}^*,N)$ is a fuzzy Banach space, this sequence converges to some point $\mathscr{A}(\mathfrak{P}_1)\in\mathfrak{I}^*$. So one can define the mapping $\mathscr{A}:\mathfrak{F}^*\to\mathfrak{I}^*$ by

$$\mathscr{A}(\mathfrak{P}_1)=\mathfrak{N}-\lim_{k\to\infty}\frac{Z(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k}$$

for all $\mathfrak{P}_1\in\mathfrak{F}^*$. Substituting $m=0$ in (3.15),

$$\mathfrak{N}\left(\frac{Z(\mathscr{D}^{k}\mathfrak{P}_1)}{\mathscr{D}^{k}}-Z(\mathfrak{P}_1),\mathfrak{O}\right)\geq\mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1),\frac{\mathfrak{O}}{\sum_{i=0}^{k-1}\frac{\mathcal{M}^i}{2\mathscr{D}\cdot\mathscr{D}^i}}\right) \tag{3.16}$$

for all $\mathfrak{P}_1\in\mathfrak{F}^*$ and all $\mathfrak{O}>0$. Letting $\mathcal{L}\to\infty$ in (3.16) and using $(F6)$,

$$\mathfrak{N}\left(Z(\mathfrak{P}_1)-\mathscr{A}(\mathfrak{P}_1),\mathfrak{O}\right)\geq\mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1),2\mathfrak{O}(\mathscr{D}-\mathcal{M})\right)$$

for all $\mathfrak{P}_1\in\mathfrak{F}^*$ and all $\mathfrak{O}>0$.

To prove $\mathscr{A}$ satisfies the equation(1.1),

Replacing $(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3)$ by $(\mathscr{D}^k\mathfrak{P}_1,\mathscr{D}^k\mathfrak{P}_2,\mathscr{D}^k\mathfrak{P}_3)$ in (3.3), respectively, the result is obtained as

$$\mathfrak{N}\left(\frac{1}{\mathscr{D}^k}DZ(\mathscr{D}^k\mathfrak{P}_1,\mathscr{D}^k\mathfrak{P}_2,\mathscr{D}^k\mathfrak{P}_3),\mathfrak{O}\right)\geq\mathfrak{N}'\left(\mathfrak{K}(\mathscr{D}^k\mathfrak{P}_1,\mathscr{D}^k\mathfrak{P}_2,\mathscr{D}^k\mathfrak{P}_3),\mathscr{D}^k\mathfrak{O}\right) \tag{3.17}$$

Now,

$$\begin{aligned}&\mathfrak{N}\left(Z\left(\frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}+\frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right)+Z\left(\frac{p}{q}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}+\frac{r}{s}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}\right)\right.\\&\qquad\left.-\left(\frac{p}{q}+\frac{r}{s}\right)\left[Z(\sqrt{\mathfrak{P}_1\mathfrak{P}_2})-Z(\sqrt{\mathfrak{P}_2\mathfrak{P}_3})\right],\mathfrak{O}\right)\\&\geq min\left\{\mathfrak{N}\left(\mathscr{A}\left(\frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}+\frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right)-\frac{1}{\mathscr{D}^k}Z\left(\mathscr{D}^k\left(\frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}+\frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right)\right),\frac{\mathfrak{O}}{5}\right),\right.\\&\qquad\mathfrak{N}\left(\mathscr{A}\left(\frac{p}{q}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}+\frac{r}{s}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}\right)-\frac{1}{\mathscr{D}^k}Z\left(\mathscr{D}^k\left(\frac{p}{q}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}+\frac{r}{s}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}\right)\right),\frac{\mathfrak{O}}{5}\right),\end{aligned}$$

$$\mathfrak{N}\Bigg(-\left(\frac{p}{q}+\frac{r}{s}\right)\Big[\mathscr{A}(\sqrt{\mathfrak{P}_1\mathfrak{P}_2})\Big]+\frac{1}{\mathscr{D}^k}\left(\mathscr{D}^k\left(\frac{p}{q}+\frac{r}{s}\right)\Big[Z(\sqrt{\mathfrak{P}_1\mathfrak{P}_2})\Big]\right),\frac{\mathfrak{O}}{5}\Bigg),$$
$$\mathfrak{N}\Bigg(-\left(\frac{p}{q}+\frac{r}{s}\right)\Big[\mathscr{A}(\sqrt{\mathfrak{P}_2\mathfrak{P}_3})\Big]+\frac{1}{\mathscr{D}^k}\left(\mathscr{D}^k\left(\frac{p}{q}+\frac{r}{s}\right)\Big[Z(\sqrt{\mathfrak{P}_2\mathfrak{P}_3})\Big]\right),\frac{\mathfrak{O}}{5}\Bigg),$$
$$\mathfrak{N}\Bigg(\frac{1}{\mathscr{D}^k}Z\left(\mathscr{D}^k\left(\frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}+\frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right)\right)+Z\left(\mathscr{D}^k\left(\frac{p}{q}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}+\frac{r}{s}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}\right)\right)$$
$$-\frac{1}{\mathscr{D}^k}\left(\mathscr{D}^k\left(\frac{p}{q}+\frac{r}{s}\right)\Big[Z(\sqrt{\mathfrak{P}_1\mathfrak{P}_2})-Z(\sqrt{\mathfrak{P}_2\mathfrak{P}_3})\Big]\right),\frac{\mathfrak{O}}{5}\Bigg)\Bigg\} \tag{3.18}$$

Using (3.17) and $(F5)$ in (3.18),

$$\mathfrak{N}\Bigg(\mathscr{A}\left(\frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}+\frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right)+\mathscr{A}\left(\frac{p}{q}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}+\frac{r}{s}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}\right)$$
$$-\left(\frac{p}{q}+\frac{r}{s}\right)[\mathscr{A}(\sqrt{\mathfrak{P}_1\mathfrak{P}_2})-\mathscr{A}(\sqrt{\mathfrak{P}_2\mathfrak{P}_3})],\mathfrak{O}\Bigg)$$
$$\geq min\left\{1,1,1,1,\mathfrak{N}'\left(\mathfrak{K}(\mathscr{D}^k\mathfrak{P}_1,\mathscr{D}^k\mathfrak{P}_2,\mathscr{D}^k\mathfrak{P}_3),\mathscr{D}^k\mathfrak{O}\right)\right\}$$
$$\geq \mathfrak{N}'\left(\mathfrak{K}(\mathscr{D}^k\mathfrak{P}_1,\mathscr{D}^k\mathfrak{P}_2,\mathscr{D}^k\mathfrak{P}_3),\mathscr{D}^k\mathfrak{O}\right) \tag{3.19}$$

Letting $k\to\infty$ in (3.19) and using (3.2),then

$$\mathfrak{N}\Bigg(\mathscr{A}\left(\frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}+\frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right)+\mathscr{A}\left(\frac{p}{q}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}+\frac{r}{s}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}\right)$$
$$-\left(\frac{p}{q}+\frac{r}{s}\right)\Big[\mathscr{A}(\sqrt{\mathfrak{P}_1\mathfrak{P}_2})-\mathscr{A}(\sqrt{\mathfrak{P}_2\mathfrak{P}_3})\Big],\mathfrak{O}\Bigg)=1 \tag{3.20}$$

Using $(F2)$, we arrive

$$\mathscr{A}\left(\frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}+\frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right)+\mathscr{A}\left(\frac{p}{q}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}+\frac{r}{s}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}\right)$$
$$=\left(\frac{p}{q}+\frac{r}{s}\right)\Big[\mathscr{A}(\sqrt{\mathfrak{P}_1\mathfrak{P}_2})+\mathscr{A}(\sqrt{\mathfrak{P}_2\mathfrak{P}_3})\Big]$$

Finally $\mathscr{A}$ satisfies the additive functional equation (1.1).
In order to prove $\mathscr{A}(\mathfrak{P}_1)$ is unique.

$$\mathfrak{N}(\mathscr{A}(\mathfrak{P}_1)-\mathscr{A}'(\mathfrak{P}_1),\mathfrak{O})=\mathfrak{N}\left(\frac{\mathscr{A}(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k}-\frac{\mathscr{A}'(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k},\mathfrak{O}\right)$$
$$\geq min\left\{\mathfrak{N}\left(\frac{\mathscr{A}(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k}-\frac{Z(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k},\frac{\mathfrak{O}}{2}\right),\mathfrak{N}\left(\frac{Z(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k}-\frac{\mathscr{A}'(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k},\frac{\mathfrak{O}}{2}\right)\right\}$$
$$\geq \mathfrak{N}'\left(\mathfrak{K}(\mathscr{D}^k\mathfrak{P}_1,\mathscr{D}^k\mathfrak{P}_1,\mathscr{D}^k\mathfrak{P}_1),\frac{2\mathfrak{O}\,\mathscr{D}^k(\mathscr{D}-\mathcal{M})}{2}\right)$$
$$\geq \mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1),\frac{2\mathfrak{O}\,\mathscr{D}^k(\mathscr{D}-\mathcal{M})}{2\mathcal{M}^k}\right)$$

Since

$$\lim_{k\to\infty} \frac{2\mathfrak{O}\,\mathscr{D}^k(\mathscr{D}-\mathcal{M})}{2\mathcal{M}^k} = \infty,$$

the result is obtained as

$$\lim_{k\to\infty} \mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1), \frac{2\mathfrak{O}\,\mathscr{D}^k(\mathscr{D}-\mathcal{M})}{2\mathcal{M}^k}\right) = 1.$$

Finally,

$$\mathfrak{N}(\mathscr{A}(\mathfrak{P}_1) - \mathscr{A}'(\mathfrak{P}_1), \mathfrak{O}) = 1$$

for all $\mathfrak{P}_1 \in \mathfrak{F}^*$ and all $\mathfrak{O} > 0$, hence $\mathscr{A}(\mathfrak{P}_1) = \mathscr{A}'(\mathfrak{P}_1)$. Therefore $\mathscr{A}(\mathfrak{P}_1)$ is unique. □

Corollary 3.1. *Let the mapping $Z : \mathfrak{F}^* \to \mathfrak{I}^*$ satisfies*

$$N\left(\mathbb{G}Z(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3),\mathfrak{O}\right) \geq \begin{cases} \mathfrak{N}'\left(\Lambda,\mathfrak{O}\right), & \\ \mathfrak{N}'\left(\Lambda \sum_{i=1}^{3} \|\mathcal{H}_i\|^s,\mathfrak{O}\right), & s \neq 1; \\ \mathfrak{N}'\left(\Lambda \prod_{i=1}^{3} \|\mathcal{H}_i\|^s,\mathfrak{O}\right), & s \neq \frac{1}{3}; \\ \mathfrak{N}'\left(\Lambda\left(\prod_{i=1}^{3} \|\mathcal{H}_i\|^s + \sum_{i=1}^{3} \|\mathcal{H}_i\|^{3s}\right),\mathfrak{O}\right), & s \neq \frac{1}{3}; \end{cases} \tag{3.21}$$

Then there exists a unique additive mapping $\mathscr{A} : \mathfrak{F}^* \to \mathfrak{I}^*$ such that

$$N\left(Z(\mathfrak{P}_1) - \mathscr{A}(\mathfrak{P}_1),\mathfrak{O}\right) \geq \begin{cases} \mathfrak{N}'\left(\Lambda, 2|\mathscr{D}-1|\mathfrak{O}\right), \\ \mathfrak{N}'\left(3\Lambda\|\mathfrak{P}_1\|^s, 2|\mathscr{D}-\mathscr{D}^s|\mathfrak{O}\right), \\ \mathfrak{N}'\left(\Lambda\|\mathfrak{P}_1\|^{3s}, 2|\mathscr{D}-\mathscr{D}^{3s}|\mathfrak{O}\right), \\ \mathfrak{N}'\left(4\Lambda\|\mathfrak{P}_1\|^{3s}, 2|\mathscr{D}-\mathscr{D}^{3s}|\mathfrak{O}\right) \end{cases} \tag{3.22}$$

3.2. **Fuzzy Stability Results:Fixed Point Method.** The following part of the section details some fundamental concepts of fixed point theory see [35]. To prove the stability result the following are defined:

δ_i is a constant such that

$$\delta_i = \begin{cases} \mathscr{D} & if \quad i = 0, \\ \frac{1}{\mathscr{D}} & if \quad i = 1 \end{cases}$$

and Z is the set such that

$$Z = \{\alpha \mid \alpha : \mathfrak{F}^* \to \mathfrak{I}^*, \alpha(0) = 0\}.$$

Theorem 3.2. *Let the mapping $Z : \mathfrak{F}^* \to \mathfrak{I}^*$ for which there exists a function $\mathfrak{K} : \mathfrak{F}^{*3} \to \mathfrak{I}^*$ with the condition*

$$\lim_{k\to\infty} \mathfrak{N}'\left(\mathfrak{K}\left(\delta_i^k\mathfrak{P}_1, \delta_i^k\mathfrak{P}_2, \delta_i^k\mathfrak{P}_3\right), \delta_i^k\mathfrak{O}\right) = 1 \tag{3.23}$$

and satisfying the inequality

$$\mathfrak{N}\left(\mathbb{G}\,Z(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3),\mathfrak{O}\right) \geq \mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3),\mathfrak{O}\right) \tag{3.24}$$

If there exists $L = L(i)$ such that the function

$$\mathfrak{P}_1 \to \mathscr{E}(\mathfrak{P}_1) = \frac{1}{2}\mathfrak{K}\left(\frac{\mathfrak{P}_1}{\mathscr{D}}, \frac{\mathfrak{P}_1}{\mathscr{D}}, \frac{\mathfrak{P}_1}{\mathscr{D}}\right),$$

has the property

$$\mathfrak{N}'\left(L\frac{1}{\delta_i}\mathscr{E}(\delta_i\mathfrak{P}_1),\mathfrak{O}\right)=\mathfrak{N}'\left(\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right),\ \forall\mathfrak{P}_1\in\mathfrak{F}^*,\mathfrak{O}>0. \tag{3.25}$$

Then there exists a additive mapping $\mathscr{A}:\mathfrak{F}^*\to\mathfrak{I}^*$ *such that*

$$\mathfrak{N}\left(Z(\mathfrak{P}_1)-Q(\mathfrak{P}_1),\mathfrak{O}\right)\geq\mathfrak{N}'\left(\frac{L^{1-i}}{1-L}\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right),\ \forall\mathfrak{P}_1\in\mathfrak{F}^*,\mathfrak{O}>0, \tag{3.26}$$

where $\mathscr{D}=\left(\frac{p}{q}+\frac{r}{s}\right)$.

Proof. Let the general metric d on Z, such that

$$d(\alpha,\beta)=inf\left\{K\text{ß}\mathfrak{N}(0,\infty)|\mathfrak{N}\left(\alpha(\mathfrak{P}_1)-\beta(\mathfrak{P}_1),\mathfrak{O}\right)\geq\mathfrak{N}'\left(K\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right),\mathfrak{P}_1\in\mathfrak{F}^*,\mathfrak{O}>0\right\}.$$

It is easy to see that (Z,d) is complete. $T:Z\to Z$ is defined as $T\alpha(\mathfrak{P}_1)=\dfrac{1}{\delta_i}\alpha(\delta_i\mathfrak{P}_1)$, for all $\mathfrak{P}_1\in\mathfrak{F}^*$. If $\alpha,\beta\in Z$, then $d(\alpha,\beta)\leq K$

$$\begin{aligned}
\Rightarrow\quad & \mathfrak{N}\left(\alpha(\mathfrak{P}_1)-\beta(\mathfrak{P}_1),\mathfrak{O}\right)\geq\mathfrak{N}'\left(K\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right)\\
\Rightarrow\quad & \mathfrak{N}\left(\frac{\alpha(\delta_i\mathfrak{P}_1)}{\delta_i}-\frac{\beta(\delta_i\mathfrak{P}_1)}{\delta_i},\mathfrak{O}\right)\geq\mathfrak{N}'\left(\frac{K}{\delta_i}\mathscr{E}(\delta_i\mathfrak{P}_1),\mathfrak{O}\right)\\
\Rightarrow\quad & \mathfrak{N}\left(T\alpha(\mathfrak{P}_1)-T\beta(\mathfrak{P}_1),\mathfrak{O}\right)\geq\mathfrak{N}'\left(KL\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right)\\
\Rightarrow\quad & d\left(T\alpha(\mathfrak{P}_1),T\beta(\mathfrak{P}_1)\right)\leq KL\\
\Rightarrow\quad & d\left(T\alpha,T\beta\right)\leq Ld(\alpha,\beta)
\end{aligned} \tag{3.27}$$

for all $\alpha,\beta\in Z$. Therefore, T is a strictly contractive mapping on Z with Lipschitz constant L. Replacing $(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3)$ by $(\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1)$ in (3.24),

$$\mathfrak{N}\left(2Z(\mathscr{D}\mathfrak{P}_1)-2\mathscr{D}Z(\mathfrak{P}_1),\mathfrak{O}\right)\geq\mathfrak{N}'\left(\mathfrak{K}(\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1),\mathfrak{O}\right). \tag{3.28}$$

for all $\mathfrak{P}_1\in\mathfrak{F}^*$, $\mathfrak{O}>0$. Using (F3) in (3.28),

$$\mathfrak{N}\left(\frac{Z(\mathfrak{P}_1)}{\mathscr{D}}-Z(\mathfrak{P}_1),\mathfrak{O}\right)\geq\mathfrak{N}'\left(\frac{1}{2\mathscr{D}}\mathfrak{K}(\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1),\mathfrak{O}\right) \tag{3.29}$$

for all $\mathfrak{P}_1\in\mathfrak{F}^*$, $\mathfrak{O}>0$ with the help of (3.25) when $i=0$, it follows from (3.29) that

$$\begin{aligned}
\Rightarrow\quad & \mathfrak{N}\left(\frac{Z(\mathfrak{P}_1)}{\mathscr{D}}-Z(\mathfrak{P}_1),\mathfrak{O}\right)\geq\mathfrak{N}'\left(L\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right)\\
\Rightarrow\quad & d(TZ,Z)\leq L=L^1=L^{1-i}
\end{aligned} \tag{3.30}$$

Replacing $\mathfrak{P}_1$ by $\frac{\mathfrak{P}_1}{\mathscr{D}}$ in (3.28), the result is obtained as

$$\mathfrak{N}\left(Z(\mathfrak{P}_1)-\mathscr{D}Z\left(\frac{\mathfrak{P}_1}{\mathscr{D}}\right),\mathfrak{O}\right)\geq\mathfrak{N}'\left(\frac{1}{2}\mathfrak{K}\left(\frac{\mathfrak{P}_1}{\mathscr{D}},\frac{\mathfrak{P}_1}{\mathscr{D}},\frac{\mathfrak{P}_1}{\mathscr{D}}\right),\mathfrak{O}\right) \tag{3.31}$$

With the help of (3.25) when $i=1$, it follows from (3.31)

$$\begin{aligned}
\Rightarrow\quad & \mathfrak{N}\left(Z(\mathfrak{P}_1)-\mathscr{D}Z\left(\frac{\mathfrak{P}_1}{\mathscr{D}}\right),\mathfrak{O}\right)\geq\mathfrak{N}'\left(\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right)\\
\Rightarrow\quad & d(Z,TZ)\leq 1=L^0=L^{1-i}
\end{aligned} \tag{3.32}$$

Then from (3.30) and (3.32),

$$d(Z, TZ) \leq L^{1-i} < \infty$$

It follows that a fixed point $\mathscr{A}$ of T in Z exists such that

$$\mathscr{A}(\mathfrak{P}_1) = N - \lim_{k\to\infty} \frac{Z(\mathscr{D}^k \mathfrak{P}_1)}{\mathscr{D}^k}, \qquad \forall \mathfrak{P}_1 \in \mathfrak{F}^*, \mathfrak{O} > 0. \tag{3.33}$$

Replacing $(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)$ by $(\delta_i \mathfrak{P}_1, \delta_i \mathfrak{P}_2, \delta_i \mathfrak{P}_3)$ in (3.24),

$$\mathfrak{N}\left(\frac{1}{\delta_i^k} DZ(\delta_i \mathfrak{P}_1, \delta_i \mathfrak{P}_2, \delta_i \mathfrak{P}_3), \mathfrak{O}\right) \geq \mathfrak{N}'\left(\mathfrak{K}(\delta_i \mathfrak{P}_1, \delta_i \mathfrak{P}_2, \delta_i \mathfrak{P}_3), \delta_i^k \mathfrak{O}\right) \tag{3.34}$$

for all $\mathfrak{O} > 0$ and all $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3 \in \mathfrak{F}^*$. Using the same technique as in the Theorem 3.1, the function $\mathscr{A} : \mathfrak{F}^* \to \mathfrak{I}^*$ can be proved to satisfy the functional equation (1.1).

since $\mathscr{A}$ is a unique fixed point of T in the set

$$\Delta = \{Z \in Z | d(Z, \mathscr{A}) < \infty\},$$

$\mathscr{A}$ is a unique function such that

$$\mathfrak{N}\left(Z(\mathfrak{P}_1) - \mathscr{A}(\mathfrak{P}_1), \mathfrak{O}\right) \geq \mathfrak{N}'\left(K\mathscr{E}(\mathfrak{P}_1), \mathfrak{O}\right) \tag{3.35}$$

Using the fixed point alternative again, we get

$$\begin{aligned}
& d(Z, \mathscr{A}) \leq \frac{1}{1-L} d(Z, TZ) \\
\Rightarrow \quad & d(Z, \mathscr{A}) \leq \frac{L^{1-i}}{1-L} \\
\Rightarrow \quad & \mathfrak{N}\left(Z(\mathfrak{P}_1) - \mathscr{A}(\mathfrak{P}_1), \mathfrak{O}\right) \geq \mathfrak{N}'\left(\frac{L^{1-i}}{1-L} \mathscr{E}(\mathfrak{P}_1), \mathfrak{O}\right),
\end{aligned} \tag{3.36}$$

for all $\mathfrak{P}_1 \in \mathfrak{F}^*$ and $\mathfrak{O} > 0$. Hence proved. □

Corollary 3.2. *Let the mapping $Z : \mathfrak{F}^* \to \mathfrak{I}^*$ satisfies*

$$N\left(\mathbb{G}Z(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3), \mathfrak{O}\right) \geq \begin{cases} \mathfrak{N}'\left(\Lambda, \mathfrak{O}\right), \\ \mathfrak{N}'\left(\Lambda \sum_{i=1}^{3} \|\mathcal{H}_i\|^s, \mathfrak{O}\right), & s \neq 1; \\ \mathfrak{N}'\left(\Lambda \prod_{i=1}^{3} \|\mathcal{H}_i\|^s, \mathfrak{O}\right), & s \neq \frac{1}{3}; \\ \mathfrak{N}'\left(\Lambda\left(\prod_{i=1}^{3} \|\mathcal{H}_i\|^s + \sum_{i=1}^{3} \|\mathcal{H}_i\|^{3s}\right), \mathfrak{O}\right), & s \neq \frac{1}{3}; \end{cases} \tag{3.37}$$

Then there exists a additive mapping $\mathscr{A} : \mathfrak{F}^* \to \mathfrak{I}^*$ such that

$$N\left(Z(\mathfrak{P}_1) - \mathscr{A}(\mathfrak{P}_1), \mathfrak{O}\right) \geq \begin{cases} \mathfrak{N}'\left(\Lambda, 2|\mathscr{D} - 1|\mathfrak{O}\right), \\ \mathfrak{N}'\left(3\Lambda\|\mathfrak{P}_1\|^s, 2|\mathscr{D} - \mathscr{D}^s|\mathfrak{O}\right), \\ \mathfrak{N}'\left(\Lambda\|\mathfrak{P}_1\|^{3s}, 2|\mathscr{D} - \mathscr{D}^{3s}|\mathfrak{O}\right), \\ \mathfrak{N}'\left(4\Lambda\|\mathfrak{P}_1\|^{3s}, 2|\mathscr{D} - \mathscr{D}^{3s}|\mathfrak{O}\right) \end{cases} \tag{3.38}$$

Proof. Setting

$$\mathfrak{K}(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3) = \begin{cases} \Lambda, \\ \Lambda \sum_{i=1}^{3} \|\mathcal{H}_i\|^s, \\ \Lambda \prod_{i=1}^{3} \|\mathcal{H}_i\|^s, \\ \Lambda \left(\prod_{i=1}^{3} \|\mathcal{H}_i\|^s + \sum_{i=1}^{3} \|\mathcal{H}_i\|^{3s} \right). \end{cases}$$

for all $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3 \in \mathfrak{F}^*$. Then,

$$\begin{aligned} &\mathfrak{N}'\left(\mathfrak{K}(\delta_i^k \mathfrak{P}_1, \delta_i^k \mathfrak{P}_2, \delta_i^k \mathfrak{P}_3), \delta_i^k \mathfrak{O}\right) \\ &= \begin{cases} \mathfrak{N}'\left(\Lambda, \delta_i^k \mathfrak{O}\right) \\ \mathfrak{N}'\left(\Lambda \sum_{i=1}^{3} \|\mathcal{H}_i\|^s, \delta_i^{(1-s)k} \mathfrak{O}\right) \\ \mathfrak{N}'\left(\Lambda \prod_{i=1}^{3} \|\mathcal{H}_i\|^s, \delta_i^{(1-3s)k} \mathfrak{O}\right) \\ \mathfrak{N}'\left(\Lambda \left(\prod_{i=1}^{3} \|\mathcal{H}_i\|^s + \sum_{i=1}^{3} \|\mathcal{H}_i\|^{3s} \right), \delta_i^{(1-3s)k} \mathfrak{O}\right) \end{cases} \\ &= \begin{cases} \to 1 \text{ as } k \to \infty, \\ \to 1 \text{ as } k \to \infty, \\ \to 1 \text{ as } k \to \infty, \\ \to 1 \text{ as } k \to \infty. \end{cases} \end{aligned}$$

Thus, (3.23) is holds. But $\mathscr{E}(\mathfrak{P}_1) = \frac{1}{2}\mathfrak{K}\left(\frac{\mathfrak{P}_1}{\mathscr{D}}, \frac{\mathfrak{P}_1}{\mathscr{D}}, \frac{\mathfrak{P}_1}{\mathscr{D}}\right)$ has the property

$$\mathfrak{N}'\left(L \frac{1}{\delta_i} \mathscr{E}(\delta_i \mathfrak{P}_1), \mathfrak{O}\right) \geq \mathfrak{N}'\left(\mathscr{E}(\mathfrak{P}_1), \mathfrak{O}\right) \ \forall \mathfrak{P}_1 \in \mathfrak{F}^*, \mathfrak{O} > 0.$$

Hence

$$\mathfrak{N}'\left(\mathscr{E}(\mathfrak{P}_1), \mathfrak{O}\right) = \mathfrak{N}'\left(\mathfrak{K}\left(\frac{\mathfrak{P}_1}{\mathscr{D}}, \frac{\mathfrak{P}_1}{\mathscr{D}}, \frac{\mathfrak{P}_1}{\mathscr{D}}\right), 2\mathfrak{O}\right) = \begin{cases} \mathfrak{N}'\left(\Lambda, 2\mathfrak{O}\right), \\ \mathfrak{N}'\left(\frac{3\Lambda}{\mathscr{D}^s} \|\mathfrak{P}_1\|^s, 2\mathfrak{O}\right), \\ \mathfrak{N}'\left(\frac{\Lambda}{\mathscr{D}^{3s}} \|\mathfrak{P}_1\|^{3s}, 2\mathfrak{O}\right), \\ \mathfrak{N}'\left(\frac{4\Lambda}{\mathscr{D}^{3s}} \|\mathfrak{P}_1\|^{3s}, 2\mathfrak{O}\right). \end{cases}$$

Now,

$$\mathfrak{N}'\left(\frac{1}{\delta_i}\mathscr{E}(\delta_i\mathfrak{P}_1),\mathfrak{O}\right)=\begin{cases}\mathfrak{N}'\left(\dfrac{\Lambda}{\delta_i},2\mathfrak{O}\right),\\ \mathfrak{N}'\left(\dfrac{\Lambda}{\delta_i}\left(\dfrac{n}{\mathscr{D}^s}\right)\|\delta_i\mathfrak{P}_1\|^s,2\mathfrak{O}\right),\\ \mathfrak{N}'\left(\dfrac{\Lambda}{\delta_i}\left(\dfrac{1}{\mathscr{D}^{3s}}\right)\|\delta_i\mathfrak{P}_1\|^{3s},2\mathfrak{O}\right),\\ \mathfrak{N}'\left(\dfrac{\Lambda}{\delta_i}\left(\dfrac{4}{\mathscr{D}^{3s}}\right)\|\delta_i\mathfrak{P}_1\|^{3s},2\mathfrak{O}\right)\end{cases}=\begin{cases}\mathfrak{N}'\left(\delta_i^{-1}\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right),\\ \mathfrak{N}'\left(\delta_i^{s-1}\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right),\\ \mathfrak{N}'\left(\delta_i^{3s-1}\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right),\\ \mathfrak{N}'\left(\delta_i^{3s-1}\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right).\end{cases}$$

Now from (3.26), the following cases for conditions (i) and (ii) are proved.

Type:1 $L=\mathscr{D}^{-1}$ for $s=0$ if $i=0$

$$\begin{aligned}\mathfrak{N}\left(Z(\mathfrak{P}_1)-\mathscr{A}(\mathfrak{P}_1),\mathfrak{O}\right)&\geq\mathfrak{N}'\left(\frac{\mathscr{D}^{-1}}{1-\mathscr{D}^{-1}}\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right)\\&=\mathfrak{N}'\left(\frac{\Lambda}{2(\mathscr{D}-1)}\|\mathfrak{P}_1\|^s,\mathfrak{O}\right)=\mathfrak{N}'\left(\Lambda\|\mathfrak{P}_1\|^s,2(\mathscr{D}-1)\mathfrak{O}\right).\end{aligned}$$

Type:2 $L=\mathscr{D}^{3}$ for $s=0$ if $i=1$

$$\begin{aligned}\mathfrak{N}\left(Z(\mathfrak{P}_1)-\mathscr{A}(\mathfrak{P}_1),\mathfrak{O}\right)&\geq\mathfrak{N}'\left(\frac{1}{1-\mathscr{D}^{3}}\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right)\\&=\mathfrak{N}'\left(\frac{\Lambda}{2(1-\mathscr{D})}\|\mathfrak{P}_1\|^s,\mathfrak{O}\right)=\mathfrak{N}'\left(\Lambda\|\mathfrak{P}_1\|^s,2(1-\mathscr{D})\mathfrak{O}\right).\end{aligned}$$

Type:3 $L=\mathscr{D}^{s-1}$ for $s>3$ if $i=0$

$$\begin{aligned}\mathfrak{N}\left(Z(\mathfrak{P}_1)-\mathscr{A}(\mathfrak{P}_1),\mathfrak{O}\right)&\geq\mathfrak{N}'\left(\frac{\mathscr{D}^{s-1}}{1-\mathscr{D}^{s-1}}\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right)\\&=\mathfrak{N}'\left(\frac{3\Lambda}{2(\mathscr{D}-\mathscr{D}^s)}\|\mathfrak{P}_1\|^s,\mathfrak{O}\right)=\mathfrak{N}'\left(3\Lambda\|\mathfrak{P}_1\|^s,2(\mathscr{D}-\mathscr{D}^s)\mathfrak{O}\right).\end{aligned}$$

Type:4 $L=\mathscr{D}^{1-s}$ for $s<1$ if $i=1$

$$\begin{aligned}\mathfrak{N}\left(Z(\mathfrak{P}_1)-\mathscr{A}(\mathfrak{P}_1),\mathfrak{O}\right)&\geq\mathfrak{N}'\left(\frac{1}{1-\mathscr{D}^{1-s}}\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right)\\&=\mathfrak{N}'\left(\frac{3\Lambda}{2(\mathscr{D}^s-\mathscr{D})}\|\mathfrak{P}_1\|^s,\mathfrak{O}\right)=\mathfrak{N}'\left(3\Lambda\|\mathfrak{P}_1\|^s,2(\mathscr{D}^s-\mathscr{D})\mathfrak{O}\right).\end{aligned}$$

Type:5 $L=\mathscr{D}^{3s-1}$ for $s>\frac{1}{3}$ if $i=0$

$$\begin{aligned}\mathfrak{N}\left(Z(\mathfrak{P}_1)-\mathscr{A}(\mathfrak{P}_1),\mathfrak{O}\right)&\geq\mathfrak{N}'\left(\frac{\mathscr{D}^{3s-1}}{1-\mathscr{D}^{3s-1}}\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right)\\&=\mathfrak{N}'\left(\frac{\Lambda}{2(\mathscr{D}-\mathscr{D}^{3s})}\|\mathfrak{P}_1\|^{3s},\mathfrak{O}\right)=\mathfrak{N}'\left(\Lambda\|\mathfrak{P}_1\|^{3s},2(\mathscr{D}-\mathscr{D}^{3s})\mathfrak{O}\right).\end{aligned}$$

Type:6 $L=\mathscr{D}^{1-3s}$ for $s<\frac{1}{3}$ if $i=1$

$$\begin{aligned}\mathfrak{N}\left(Z(\mathfrak{P}_1)-\mathscr{A}(\mathfrak{P}_1),\mathfrak{O}\right)&\geq\mathfrak{N}'\left(\frac{1}{1-\mathscr{D}^{1-3s}}\mathscr{E}(\mathfrak{P}_1),\mathfrak{O}\right)\\&=\mathfrak{N}'\left(\frac{\Lambda}{2(\mathscr{D}^{3s}-\mathscr{D})}\|\mathfrak{P}_1\|^{3s},\mathfrak{O}\right)=\mathfrak{N}'\left(\Lambda\|\mathfrak{P}_1\|^{3s},2(\mathscr{D}^{3s}-\mathscr{D})\mathfrak{O}\right).\end{aligned}$$

Type:7 $L = \mathscr{D}^{3s-1}$ for $s > \frac{1}{3}$ if $i = 0$

$$\begin{aligned}\mathfrak{N}\left(Z(\mathfrak{P}_1) - \mathscr{A}(\mathfrak{P}_1), \mathfrak{O}\right) &\geq \mathfrak{N}'\left(\frac{\mathscr{D}^{3s-1}}{1-\mathscr{D}^{3s-1}}\mathscr{E}(\mathfrak{P}_1), \mathfrak{O}\right)\\ &= \mathfrak{N}'\left(\frac{4\Lambda}{2(\mathscr{D}-\mathscr{D}^{3s})}\|\mathfrak{P}_1\|^{3s}, \mathfrak{O}\right) = \mathfrak{N}'\left(4\Lambda\|\mathfrak{P}_1\|^{3s}, 2(\mathscr{D}-\mathscr{D}^{3s})\mathfrak{O}\right).\end{aligned}$$

Type:8 $L = \mathscr{D}^{1-3s}$ for $s < \frac{1}{3}$ if $i = 1$

$$\begin{aligned}\mathfrak{N}\left(Z(\mathfrak{P}_1) - \mathscr{A}(\mathfrak{P}_1), \mathfrak{O}\right) &\geq \mathfrak{N}'\left(\frac{1}{1-\mathscr{D}^{1-3s}}\mathscr{E}(\mathfrak{P}_1), \mathfrak{O}\right)\\ &= \mathfrak{N}'\left(\frac{4\Lambda}{2(\mathscr{D}^{3s}-\mathscr{D})}\|\mathfrak{P}_1\|^{3s}, \mathfrak{O}\right) = \mathfrak{N}'\left(4\Lambda\|\mathfrak{P}_1\|^{3s}, 2(\mathscr{D}^{3s}-\mathscr{D})\mathfrak{O}\right).\end{aligned}$$

Hence the proof is complete □

4. Stability Results: Random Normed Space: Direct Method

4.1. **Basics of Random Normed Spaces.** Fundamentals of Random normed spaces one can see [36–40].

Definition 4.1. *A random normed space (briefly, RN-space) is a triple $(\mathfrak{F}^*, \mu, \mathcal{T})$, where X is a vector space, T is a continuous $\mathcal{T}-$norm and μ is a mapping from $\mathfrak{F}^*$ into D^+ satisfying the following conditions:*
(RN1) $\mu_{\mathfrak{P}_1}(\mathcal{T}) = \mathfrak{N}(\mathcal{T})$ *for all* $\mathcal{T} > 0$ *if and only if* $\mathfrak{P}_1 = 0$;
(RN2) $\mu_{\alpha\,\mathfrak{P}_1}(\mathcal{T}) = \mu_{\mathfrak{P}_1}(\mathcal{T}/|\alpha|)$ *for all* $\mathfrak{P}_1 \in \mathfrak{F}^*$, *and* $\alpha \in \mathbb{R}$ *with* $\alpha \neq 0$;
(RN3) $\mu_{\mathfrak{P}_1+\mathfrak{P}_2}(\mathcal{T}+s) \geq T\left(\mu_{\mathfrak{P}_1}(\mathcal{T}), \mu_{\mathfrak{P}_2}(s)\right)$ *for all* $\mathfrak{P}_1, \mathfrak{P}_2 \in \mathfrak{F}^*$ *and* $\mathcal{T}, s \geq 0$.

Let us take $\mathfrak{F}^*$- linear space , $(\mathfrak{I}^*, \mu, T)$ a complete RN-space.
A mapping $DZ : \mathfrak{F}^* \to \mathfrak{I}^*$ is defined by

$$\begin{aligned}\mathbb{G}Z(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3) = Z\left(\frac{p}{q}\sqrt{\mathfrak{P}_1\mathfrak{P}_2} + \frac{r}{s}\sqrt{\mathfrak{P}_2\mathfrak{P}_3}\right) + Z\left(\frac{p}{q}\sqrt{\mathfrak{P}_2\mathfrak{P}_3} + \frac{r}{s}\sqrt{\mathfrak{P}_1\mathfrak{P}_2}\right)\\ -\left(\frac{p}{q}+\frac{r}{s}\right)\left[Z(\sqrt{\mathfrak{P}_1\mathfrak{P}_2}) - Z(\sqrt{\mathfrak{P}_2\mathfrak{P}_3})\right]\end{aligned}$$

for all $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3 \in \mathfrak{F}^*$.

Theorem 4.1. *Let the odd mapping $Z : \mathfrak{F}^* \to \mathfrak{I}^*$ for which there exist a function $\eta : \mathfrak{F}^{*n} \to D^+$ with*

$$\begin{aligned}\lim_{k\to\infty} T_{i=0}^{\infty}\left(\eta_{\mathscr{D}^{(k+i)j}\mathfrak{P}_1, \mathscr{D}^{(k+i)j}\mathfrak{P}_2, \mathscr{D}^{(k+i)j}\mathfrak{P}_3}\left(\mathscr{D}^{(k+i+1)j}\,\mathcal{T}\right)\right) = 1\\ = \lim_{k\to\infty}\eta_{\mathscr{D}^{kj}\mathfrak{P}_1, \mathscr{D}^{kj}\mathfrak{P}_2, \mathscr{D}^{kj}\mathfrak{P}_3}\left(\mathscr{D}^{kj}\,\mathcal{T}\right)\end{aligned} \tag{4.1}$$

such that

$$\mu_{\mathbb{G}Z(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3)}(\mathcal{T}) \geq \eta_{\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3}(\mathcal{T}) \tag{4.2}$$

for all $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3 \in \mathfrak{F}^*$ and all $\mathcal{T} > 0$.
Then there exists a additive mapping $\mathscr{A} : \mathfrak{F}^* \to \mathfrak{I}^*$ satisfying

$$\mu_{\mathscr{A}(\mathfrak{P}_1)-Z(\mathfrak{P}_1)}(\mathcal{T}) \geq T_{i=0}^{\infty}\left(\eta_{\mathscr{D}^{(i+1)j}\mathfrak{P}_1, \mathscr{D}^{(i+1)j}\mathfrak{P}_1, \mathscr{D}^{(i+1)j}\mathfrak{P}_1}\left(\mathscr{D}^{(i+1)j}\,\mathcal{T}\right)\right) \tag{4.3}$$

The mapping $\mathscr{A}(\mathfrak{P}_1)$ *is defined as*

$$\mu_{\mathscr{A}(\mathfrak{P}_1)}(\mathcal{T}) = \lim_{k\to\infty} \mu_{\frac{Z(\mathscr{D}^{kj}\mathfrak{P}_1)}{\mathscr{D}^{kj}}}(\mathcal{T}) \tag{4.4}$$

for all $\mathfrak{P}_1 \in \mathfrak{F}^*$ *and all* $\mathcal{T} > 0$ *with* $j = \pm 1$ *and* $\mathscr{D} = \left(\frac{p}{q} + \frac{r}{s}\right)$.

Proof. Assuming $j = 1$ and applying $(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)$ by $(\mathfrak{P}_1, \mathfrak{P}_1, \mathfrak{P}_1)$in (4.2),

$$\mu_{2Z(\mathfrak{P}_1)-2\mathscr{D}Z(\mathfrak{P}_1)}(\mathcal{T}) \geq \eta_{\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1}(\mathcal{T}) \tag{4.5}$$

It follows from (4.5) and $(RN2)$ that

$$\mu_{\frac{Z(\mathfrak{P}_1)}{\mathscr{D}}-Z(\mathfrak{P}_1)}(\mathcal{T}) \geq \eta_{\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1}(2\mathscr{D}\mathcal{T}) \tag{4.6}$$

Replacing $\mathfrak{P}_1$ by $\mathscr{D}^k\mathfrak{P}_1$ in (4.6),

$$\mu_{\frac{Z(\mathscr{D}^{k+1}\mathfrak{P}_1)}{\mathscr{D}^{(k+1)}}-\frac{Z(\mathscr{D}^{k}\mathfrak{P}_1)}{\mathscr{D}^{k}}}(\mathcal{T}) \geq \eta_{\mathscr{D}^k\mathfrak{P}_1,\mathscr{D}^k\mathfrak{P}_1,\mathscr{D}^k\mathfrak{P}_1}(2\mathscr{D}^{(k+1)}t) \tag{4.7}$$

We know that, it is easy to see that

$$\frac{Z(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k} - Z(\mathfrak{P}_1) = \sum_{i=0}^{k-1} \frac{Z(\mathscr{D}^{i+1}\mathfrak{P}_1)}{\mathscr{D}^{(i+1)}} - \frac{Z(\mathscr{D}^{i}\mathfrak{P}_1)}{\mathscr{D}^{i}} \tag{4.8}$$

for all $\mathfrak{P}_1 \in \mathfrak{F}^*$. From equations (4.7) and (4.8),

$$\begin{aligned}
\mu_{\frac{Z(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k}-Z(\mathfrak{P}_1)}(\mathcal{T}) &= \mu_{\sum_{i=0}^{k-1} \frac{Z(\mathscr{D}^{i+1}\mathfrak{P}_1)}{\mathscr{D}^{(i+1)}}-\frac{Z(\mathscr{D}^{i}\mathfrak{P}_1)}{\mathscr{D}^{i}}}(\mathcal{T}) \\
&\geq T_{i=0}^{k-1}\mu_{\frac{Z(\mathscr{D}^{i+1}\mathfrak{P}_1)}{\mathscr{D}^{(i+1)}}-\frac{Z(\mathscr{D}^{i}\mathfrak{P}_1)}{\mathscr{D}^{i}}}(\mathcal{T}) \\
&\geq T_{i=0}^{k-1}\eta_{\mathscr{D}^i\mathfrak{P}_1,\mathscr{D}^i\mathfrak{P}_1,\mathscr{D}^i\mathfrak{P}_1}(2\mathscr{D}^{(i+1)}\mathcal{T})
\end{aligned} \tag{4.9}$$

In order to prove the convergence of the sequence

$$\left\{\frac{Z(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k}\right\},$$

$\mathfrak{P}_1$ is replaced by $\mathscr{D}^m\mathfrak{P}_1$ in (4.9). The result is obtained as follows.

$$\begin{aligned}
\mu_{\frac{Z(\mathscr{D}^{k+m}\mathfrak{P}_1)}{\mathscr{D}^{(k+m)}}-Z(\mathfrak{P}_1)}(\mathcal{T}) &\geq T_{i=0}^{k-1}\eta_{\mathscr{D}^{i+m}\mathfrak{P}_1,\mathscr{D}^{i+m}\mathfrak{P}_1,\mathscr{D}^{i+m}\mathfrak{P}_1}(2\mathscr{D}^{(i+m+1)}\mathcal{T}) \\
&= T_{i=m}^{m+n-1}\eta_{\mathscr{D}^i\mathfrak{P}_1,\mathscr{D}^i\mathfrak{P}_1,\mathscr{D}^i\mathfrak{P}_1}(2\mathscr{D}^{(i+1)}\mathfrak{P}_1) \\
&\to 0 \text{ as } m \to \infty
\end{aligned} \tag{4.10}$$

Thus $\left\{\frac{Z(\mathscr{D}^k\mathfrak{P}_1)}{\mathscr{D}^k}\right\}$ is a Cauchy sequence. Since $\mathfrak{F}^*$ is complete there exists a mapping $\mathscr{A} : \mathfrak{F}^* \to \mathfrak{I}^*$,

$$\mu_{\mathscr{A}(\mathfrak{P}_1)}(\mathcal{T}) = \lim_{k\to\infty} \mu_{\frac{Z(\mathscr{D}^{kj}\mathfrak{P}_1)}{\mathscr{D}^{kj}}}(\mathcal{T})$$

Letting $m = 0$ and $\mathcal{L} \to \infty$ in (4.10), the result is (4.3) for all $\mathfrak{P}_1 \in \mathfrak{F}^*$ and all $\mathcal{T} > 0$. Replacing $(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)$ by $(\mathscr{D}^k\mathfrak{P}_1, \mathscr{D}^k\mathfrak{P}_2, \mathscr{D}^k\mathfrak{P}_3)$,

$$\begin{aligned}\mu_{\frac{1}{\mathscr{D}^k}DZ(\mathscr{D}^k\mathfrak{P}_1,\mathscr{D}^k\mathfrak{P}_2,\mathscr{D}^k\mathfrak{P}_3)}(\mathcal{T}) &\geq \eta_{\mathscr{D}^k\mathfrak{P}_1,\mathscr{D}^k\mathfrak{P}_2,\mathscr{D}^k\mathfrak{P}_3}(\mathscr{D}^k\ t)\\ &= T_{i=m}^{m+k-1}\Big(\eta_{\mathscr{D}^{i+1}\mathfrak{P}_1,\mathscr{D}^{i+1}\mathfrak{P}_1,\mathscr{D}^{i+1}\mathfrak{P}_1}\Big)(\mathscr{D}^{(i+m+1)}t) \end{aligned} \tag{4.11}$$

Taking $\mathcal{L} \to \infty$ on both sides, it can be seen that $\mathscr{A}$ satisfies (1.1) for all $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3 \in \mathfrak{F}^*$.
Therefore the mapping $\mathscr{A} : \mathfrak{F}^* \to \mathfrak{I}^*$ is additive.

Finally, to prove the uniqueness of the additive function $\mathscr{A}$ subject to (4.4), let us assume that there exist a additive function $\mathscr{A}'$ which satisfies (4.3) and (4.4). Since $\mathscr{A}(\mathscr{D}^k\mathfrak{P}_1) = \mathscr{D}^k\mathscr{A}(\mathfrak{P}_1)$ and $\mathscr{A}'(\mathscr{D}^k\mathfrak{P}_1) = \mathscr{D}^k\mathscr{A}'(\mathfrak{P}_1)$ for all $\mathfrak{P}_1 \in \mathfrak{F}^*$ and all $\mathcal{L} \in \mathbb{N}$, it follows from (4.4)

$$\begin{aligned}\mu_{\mathscr{A}(\mathfrak{P}_1)-\mathscr{A}'(\mathfrak{P}_1)}(2\mathcal{T}) &= \mu_{\mathscr{A}(\mathscr{D}^k\mathfrak{P}_1)-\mathscr{A}'(\mathscr{D}^k\mathfrak{P}_1)}(2\mathscr{D}^k\mathcal{T})\\ &= \mu_{\mathscr{A}(\mathscr{D}^k\mathfrak{P}_1)-Z(\mathscr{D}^k\mathfrak{P}_1)+Z(\mathscr{D}^k\mathfrak{P}_1)-\mathscr{A}'(\mathscr{D}^k\mathfrak{P}_1)}(2\mathscr{D}^k\mathcal{T})\\ &\geq T\Big(\mu_{\mathscr{A}(\mathscr{D}^k\mathfrak{P}_1)-Z(\mathscr{D}^k\mathfrak{P}_1)}(\mathscr{D}^k\mathcal{T}), \mu_{Z(\mathscr{D}^k\mathfrak{P}_1)-\mathscr{A}'(\mathscr{D}^k\mathfrak{P}_1)}(\mathscr{D}^k\mathcal{T})\Big)\\ &= T\Big(T_{i=0}^{\infty}\Big(\eta_{\mathscr{D}^{(i+k+1)}\mathfrak{P}_1,\mathscr{D}^{(i+k+1)}\mathfrak{P}_1,\mathscr{D}^{(i+k+1)}\mathfrak{P}_1}\Big)(2\mathscr{D}^{(i+k+1)}\mathcal{T}),\\ &\qquad T_{i=0}^{\infty}\Big(\eta_{\mathscr{D}^{(i+k+1)}\mathfrak{P}_1,\mathscr{D}^{(i+k+1)}\mathfrak{P}_1,\mathscr{D}^{(i+k+1)}\mathfrak{P}_1}\Big)(2\mathscr{D}^{(i+k+1)}\mathcal{T})\Big)\\ &\to\ 0\ as\ k\ \to\ \infty\end{aligned}$$

Hence $\mathscr{A}$ is unique. □

Corollary 4.1. *Let the mapping $Z : \mathfrak{F}^* \to \mathfrak{I}^*$ satisfies the inequality*

$$\mu_{GZ(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3)}(\mathcal{T}) \geq \begin{cases} \eta_{\Lambda}(\mathcal{T}), & \\ \eta_{\Lambda\sum_{i=1}^{3}\|\mathcal{H}_i\|^s}(\mathcal{T}), & s \neq 3;\\ \eta_{\Lambda\prod_{i=1}^{3}\|\mathcal{H}_i\|^s}(\mathcal{T}), & s \neq \frac{1}{3};\\ \eta_{\Lambda\left(\prod_{i=1}^{3}\|\mathcal{H}_i\|^s+\sum_{i=1}^{3}\|\mathcal{H}_i\|^{3s}\right)}(\mathcal{T}), & s \neq \frac{1}{3}; \end{cases} \tag{4.12}$$

Then there exists a unique additive function $\mathscr{A} : \mathfrak{F}^* \to \mathfrak{I}^*$ then

$$\mu_{Z(\mathfrak{P}_1)-\mathscr{A}(\mathfrak{P}_1)}(\mathcal{T}) \leq \begin{cases} \eta_{\frac{\Lambda}{2|\mathscr{D}-1|}}(\mathcal{T}),\\ \eta_{\frac{n\Lambda\|\mathfrak{P}_1\|^s}{2|\mathscr{D}-\mathscr{D}^s|}}(\mathcal{T}),\\ \eta_{\frac{\Lambda\|\mathfrak{P}_1\|^{3s}}{2|\mathscr{D}-\mathscr{D}^{3s}|}}(\mathcal{T}),\\ \eta_{\frac{\Lambda\|\mathfrak{P}_1\|^{3s}}{2}\left(\frac{4}{|\mathscr{D}-\mathscr{D}^{3s}|}\right)}(\mathcal{T}), \end{cases} \tag{4.13}$$

for all $\mathfrak{P}_1 \in \mathfrak{F}^*$ and all $\mathcal{T} > 0$.

4.2. **Stability Results: Random Normed Space: Fixed Point Method.** Let $\mathfrak{F}^*$ be a vector space and (Y,μ,T) be a complete RN-space.

Theorem 4.2. *Let the mapping* $Z:\mathfrak{F}^*\to\mathfrak{I}^*$ *for which there exist a function* $\eta:\mathfrak{F}^{*3}\to D^+$ *with the condition*

$$\lim_{k\to\infty}\eta_{\delta_i^k\mathfrak{P}_1,\delta_i^k\mathfrak{P}_2,\delta_i^k\mathfrak{P}_3}\left(\delta_i^k t\right)=1,\ \ \forall\ \mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3\in\mathfrak{P}_1,\mathcal{T}>0 \tag{4.14}$$

and satisfying

$$\mu_{GZ(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3)}(\mathcal{T})\geq\eta_{\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3}(\mathcal{T}),\ \forall\ \mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3\in\mathfrak{P}_1,\mathcal{T}>0. \tag{4.15}$$

There exists $L=L(i)$ *such that the function*

$$\mathfrak{P}_1\to\mathscr{E}(\mathfrak{P}_1,\mathcal{T})=\eta_{\frac{\mathfrak{P}_1}{\mathscr{D}},\frac{\mathfrak{P}_1}{\mathscr{D}},\frac{\mathfrak{P}_1}{\mathscr{D}}}(2\mathcal{T}),$$

has the property

$$\mathscr{E}(\mathfrak{P}_1,\mathcal{T})\leq L\frac{1}{\delta_i}\mathscr{E}\left(\delta_i\mathfrak{P}_1,\mathcal{T}\right),\ \forall\mathfrak{P}_1\in\mathfrak{F}^*,\mathcal{T}>0. \tag{4.16}$$

There exists a additive function $\mathscr{A}:\mathfrak{F}^*\to\mathfrak{I}^*$ *satisfying*

$$\mu_{\mathscr{A}(\mathfrak{P}_1)-Z(\mathfrak{P}_1)}\left(\frac{L^{1-i}}{1-L}\mathcal{T}\right)\geq\mathscr{E}(\mathfrak{P}_1,\mathcal{T}),\ \forall\mathfrak{P}_1\in\mathfrak{F}^*,\mathcal{T}>0, \tag{4.17}$$

here $\mathscr{D}=\left(\frac{p}{q}+\frac{r}{s}\right)$.

Proof. Let the general metric d on Z, such that

$$d(\alpha,\beta)=inf\left\{K\text{ß}\mathfrak{R}(0,\infty)|\mu_{\alpha(\mathfrak{P}_1)-\beta(\mathfrak{P}_1)}(K\mathcal{T})\geq\mathscr{E}(\mathfrak{P}_1,\mathcal{T}),\mathfrak{P}_1\in\mathfrak{F}^*,\mathcal{T}>0\right\}.$$

It is evident that (Z,d) is complete. $T:Z\to Z$ can be defined as $T\alpha(\mathfrak{P}_1)=\dfrac{1}{\delta_i}\alpha(\delta_i\mathfrak{P}_1)$, for all $\mathfrak{P}_1\in\mathfrak{F}^*$. Now for $\alpha,\beta\in Z$, $d(\alpha,\beta)\leq K$

$$\begin{aligned}
\Rightarrow&\quad \mu_{\alpha(\mathfrak{P}_1)-h(\mathfrak{P}_1)}(K\mathcal{T})\geq\mathscr{E}(\mathfrak{P}_1,\mathcal{T})\\
\Rightarrow&\quad \mu_{\frac{\alpha(\delta_i\mathfrak{P}_1)}{\delta_i}-\frac{\beta(\delta_i\mathfrak{P}_1)}{\delta_i}}\left(\frac{K\mathcal{T}}{\delta_i}\right)\geq\mathscr{E}(\delta_i\mathfrak{P}_1,\mathcal{T})\\
\Rightarrow&\quad \mu_{T\alpha(\mathfrak{P}_1)-T\beta(\mathfrak{P}_1)}\left(\frac{Kt}{\delta_i}\right)\geq\mathscr{E}(\mathfrak{P}_1,\mathcal{T})\\
\Rightarrow&\quad d\left(T\alpha(\mathfrak{P}_1),T\beta(\mathfrak{P}_1)\right)\leq KL\\
\Rightarrow&\quad d\left(T\alpha,T\beta\right)\leq Ld(\alpha,\beta)
\end{aligned} \tag{4.18}$$

for all $\alpha,\beta\in Z$. Therefore, T is strictly contractive mapping on Z with Lipschitz constant L. Replacing $(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3)$ by $(\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1)$ in (4.15),

$$\mu_{\frac{Z(\mathfrak{P}_1)}{\mathscr{D}}-Z(\mathfrak{P}_1)}\left(\frac{t}{\mathscr{D}}\right)\geq\eta_{\mathfrak{P}_1,\mathfrak{P}_1,\mathfrak{P}_1}(2\mathcal{T}) \tag{4.19}$$

With the help of (4.16) when $i=0$, it follows from (4.19), then

$$\begin{aligned}
\Rightarrow&\quad \mu_{\frac{Z(\mathfrak{P}_1)}{\mathscr{D}}-Z(\mathfrak{P}_1)}\left(\frac{t}{\mathscr{D}^3}\right)\geq\mathscr{E}(\mathfrak{P}_1,t)\\
\Rightarrow&\quad d(TZ,Z)\leq L=L^{1-0}<\infty.
\end{aligned} \tag{4.20}$$

Again replacing $\mathfrak{P}_1$ by $\frac{\mathfrak{P}_1}{\mathscr{D}}$ in (4.19), the result is

$$\mu_{Z(\mathfrak{P}_1)-\mathscr{D}Z\left(\frac{\mathfrak{P}_1}{\mathscr{D}}\right)}(\mathcal{T}) \geq \eta_{\frac{\mathfrak{P}_1}{\mathscr{D}},\frac{\mathfrak{P}_1}{\mathscr{D}},\frac{\mathfrak{P}_1}{\mathscr{D}}}(2\mathcal{T}) \tag{4.21}$$

With the help of (4.16) when $i = 1$, it follows from (4.21), then

$$\Rightarrow \quad \mu_{Z(\mathfrak{P}_1)-\mathscr{D}Z\left(\frac{\mathfrak{P}_1}{\mathscr{D}}\right)}(\mathcal{T}) \geq \mathscr{E}(\mathfrak{P}_1, \mathcal{T})$$

$$\Rightarrow \quad d(Z, TZ) \leq 1 = L^0 = L^{1-i} \tag{4.22}$$

Then from (4.20) and (4.22), it can be concluded that

$$d(Z, TZ) \leq L^0 < \infty$$

It follows that there exists a fixed point $\mathscr{A}$ of T in Z such that

$$\mu_{\mathscr{A}(\mathfrak{P}_1)}(\mathcal{T}) = \lim_{k\to\infty} \frac{\mu_{Z(\delta_i^k \mathfrak{P}_1)}}{\delta_i^k}(\mathcal{T}), \qquad \forall \mathfrak{P}_1 \in \mathfrak{F}^*, \mathcal{T} > 0. \tag{4.23}$$

Replacing $(\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3)$ by $(\delta_i^k \mathfrak{P}_1, \delta_i^k \mathfrak{P}_2, \delta_i^k \mathfrak{P}_3)$ in (4.15),

$$\mu_{\frac{1}{\delta_i^k} DZ(\delta_i^k \mathfrak{P}_1, \delta_i^k \mathfrak{P}_2, \delta_i^k \mathfrak{P}_3)}(\mathcal{T}) \geq \eta_{\delta_i^k \mathfrak{P}_1, \delta_i^k \mathfrak{P}_2, \delta_i^k \mathfrak{P}_3}\left(\delta_i^k \mathcal{T}\right) \tag{4.24}$$

for all $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3 \in \mathfrak{F}^*$ and all $\mathcal{T} > 0$.

By fixed point alternative, since $\mathscr{A}$ is unique fixed point of T in the set

$$\Delta = \{Z \in Z | d(Z, \mathscr{A}) < \infty\},$$

therefore $\mathscr{A}$ is a uniqe function such that

$$\mu_{Z(\mathfrak{P}_1)-\mathscr{A}(\mathfrak{P}_1)}(K\mathcal{T}) \geq \mathscr{E}(\mathfrak{P}_1, \mathcal{T}) \tag{4.25}$$

and

$$d(Z, \mathscr{A}) \leq \frac{1}{1-L} d(Z, TZ)$$

$$\Rightarrow \quad d(Z, \mathscr{A}) \leq \frac{L^{1-i}}{1-L}$$

$$\Rightarrow \quad \mu_{Z(\mathfrak{P}_1)-\mathscr{A}(\mathfrak{P}_1)}\left(\frac{L^{1-i}}{1-L}\mathcal{T}\right) \geq \mathscr{E}(\mathfrak{P}_1, \mathcal{T}) \tag{4.26}$$

□

Corollary 4.2. *Let the function $Z : \mathfrak{F}^* \to \mathfrak{I}^*$ satisfies the inequality*

$$\mu_{GZ(\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3)}(\mathcal{T}) \quad \geq \begin{cases} \eta_\Lambda(\mathcal{T}), & \\ \eta_{\Lambda \sum_{i=1}^3 \|\mathcal{H}_i\|^s}(\mathcal{T}), & s \neq 3; \\ \eta_{\Lambda \prod_{i=1}^3 \|\mathcal{H}_i\|^s}(\mathcal{T}), & s \neq \frac{1}{3}; \\ \eta_{\Lambda\left(\prod_{i=1}^3 \|\mathcal{H}_i\|^s + \sum_{i=1}^3 \|\mathcal{H}_i\|^{3s}\right)}(\mathcal{T}), & s \neq \frac{1}{3}; \end{cases} \tag{4.27}$$

Then there exists a unique additive function $\mathscr{A} : \mathfrak{F}^* \to \mathfrak{I}^*$ *then*

$$\mu_{Z(\mathfrak{P}_1)-\mathscr{A}(\mathfrak{P}_1)}(\mathcal{T}) \leq \begin{cases} \eta_{\frac{\Lambda}{2|\mathscr{D}-1|}}(\mathcal{T}), \\ \eta_{\frac{n\Lambda\|\mathfrak{P}_1\|^s}{2|\mathscr{D}-\mathscr{D}^s|}}(\mathcal{T}), \\ \eta_{\frac{\Lambda\|\mathfrak{P}_1\|^{3s}}{2|\mathscr{D}-\mathscr{D}^{3s}|}}(\mathcal{T}), \\ \eta_{\frac{\Lambda\|\mathfrak{P}_1\|^{3s}}{2}\left(\frac{4}{|\mathscr{D}-\mathscr{D}^{3s}|}\right)}(\mathcal{T}), \end{cases} \tag{4.28}$$

for all $\mathfrak{P}_1 \in \mathfrak{F}^*$ *and all* $\mathcal{T} > 0$.

Proof. Let

$$\eta_{\mathfrak{P}_1,\mathfrak{P}_2,\mathfrak{P}_3}(\mathcal{T}) = \begin{cases} \eta_{\Lambda}(\mathcal{T}), \\ \eta_{\Lambda\sum_{i=1}^{3}\|\mathcal{H}_i\|^s}(\mathcal{T}), \\ \eta_{\Lambda\prod_{i=1}^{3}\|\mathcal{H}_i\|^s}(\mathcal{T}), \\ \eta_{\Lambda\left(\prod_{i=1}^{3}\|\mathcal{H}_i\|^s+\sum_{i=1}^{3}\|\mathcal{H}_i\|^{3s}\right)}(\mathcal{T}), \end{cases}$$

for all $\mathfrak{P}_1, \mathfrak{P}_2, \mathfrak{P}_3 \in \mathfrak{F}^*$ and all $\mathcal{T} > 0$. □

5. Conclusion

In this research, a novel system of additive FE (1.1) has been proposed. The generalised Ulam-Hyers stability for these equations are then investigated in fuzzy and random normed spaces using direct and fixed point techniques. This kind of effective stability analysis for a novel system of equations has not been attempted before, which makes the results of the study quite unique and important to the study of FEs. Some applications of the results derived for the newly proposed FEs have also been explored to introduce the readers to the practical applications of the results. The Hyers-Ulam stability for these equations (1.1) can also be determined in the future in various normed spaces.

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