Int. J. Anal. Appl. (2024), 22:195

International Journal of Analysis and Applications

Bipolar Fuzzy Magnified Translation of a Lattice

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Abstract. In this article, we introduce the bipolar fuzzy (BF) level subsets of a lattice, and we prove the characterisation BF level subset $\mathfrak{B}$ in terms of $\mathbb{L}$ forms a bipolar fuzzy lattice (BFL) and a bipolar fuzzy ideal (BFI). We show that if $\mathfrak{B}$ forms a BFL of $\mathbb{L}$, then the support set $\mathrm{Supp}(\mathfrak{B})$ is a crisp sublattice of $\mathbb{L}$. Also, we show that the converse necessarily does not hold in general, and we also proved the results for BFI. Moreover, we introduce and explore the concept of bipolar fuzzy magnified translation (BFMT) of a BFS. Also, we characterize a BFL and a BFI in terms of a BFMT. We show that the homomorphic image and pre-image of a BFMT of a BFL is also a BFL, and the BFMT of a BFI is also a BFI.

1. Introduction

In fuzzy set (FS) theory, Zadeh [21] defined a fuzzy set μ as a class of objects F along with a grade of M_{Sh} function. This M_{Sh} function $\mu(\hbar), \hbar \in F$, allocates to each object a grade of M_{Sh} ranging between 0 and one. FS allocates a single value to each object. This single value combines the evidence for $\hbar \in F$ and the evidence against $\hbar \in F$ without indicating how much there is for each. The single numbers in FS do not tell us completely about its accuracy. Several decision-makers and researchers felt that in proper decision-making, the evidence of $\hbar$ belonging to μ and evidence not belonging to μ are both necessary.

To counter this problem, Gau and Buehrer [7] popularized the vague sets. The Atanassov's [3] intuitionistic fuzzy sets and vague sets are mathematically equivalent by Bustince and Burillo [5].

Received: Sep. 6, 2024.

2020 *Mathematics Subject Classification.* 03G10, 03E72.

Key words and phrases. bipolar fuzzy set; bipolar fuzzy sublattice; bipolar fuzzy level subset; bipolar fuzzy magnified translation.

https://doi.org/10.28924/2291-8639-22-2024-195
ISSN: 2291-8639

According to Gau and Buehrer, a vague set $'\kappa'$ of a certain universe F is characterized by using a pair of functions (t_κ, f_κ) where t_κ and f_κ are functions from F into [0,1] such that $t_\kappa(\hbar) + f_\kappa(\hbar) \leq 1$ for all $\hbar \in F$. The t_κ is called the M_{Sh} function and $t_\kappa(\hbar)$ gives evidence of how much an element $\hbar$ belongs to κ and f_κ is called non-M_{Sh} function and $f_\kappa(\hbar)$ gives evidence of how much an element $\hbar$ does not belong to κ. These concepts are applied in decision-making, fuzzy control systems, knowledge discovery, fault diagnosis, etc. In fact, if F is a set, a function $'\kappa'$ from F to [0,1] is said to be a fuzzy subset of F. A fuzzy set of a set F is a mapping from F into $[0,1]$. A vague set κ of set F is a pair (t_κ, f_κ) where t_κ from F to $[0,1]$ is a M_{Sh} function and f_κ from F to $[0,1]$ is a non-M_{Sh} function satisfying the condition $t_\kappa(\hbar) + f_\kappa(\hbar) \leq 1$ for all $\hbar \in F$. Thus, the theory of vague sets is a generalization of the theory of fuzzy sets.

In particular, Ajmal and Thomas [1] both explored the theory of fuzzy sublattice (FL) and introduced the idea of fuzzy sets to lattice theory. After that, in 2011, Thomas and Nair [19] presented the idea of intuitionistic fuzzy lattices (IFLs). In 2017, Milles [12] investigated the characterization of IFIs and IFFs based on lattice operations. Rao [15] later researched rough vague lattices in 2019. Nageswara Rao et al. [14] introduced vague lattices (VL) in 2020. The principal IFI and IFF on a lattice were the subject of Boudaoud et al. [4] study in 2020. Milles [13] as well as studied the lattice of intuitionistic fuzzy topologies (IFT) produced by intuitionistic fuzzy relations in 2020. On residual lattices, Zhang and Qingguo [22] researched the intuitionistic fuzzy filter (IFF) theory.

Many different human decisions are founded on dual or bipolar-judgmental thinking, which has both a positive (+ve) and a negative (-ve) side. Keeping in view of this importance, in 2000, Lee [10] introduced the concept of bipolar fuzzy sets (BFSs). A BFS is a pair (μ^+, μ^-), where $\mu^+ : F \to [0,1]$ and $\mu^- : F \to [-1,0]$ are mappings. The BFSs are an extension of fuzzy sets (FSs) whose M_{Sh} degree range is $[-1,1]$. In a BFS, the M_{Sh} degree 0 of an element expresses that the element is irrelevant to the corresponding property, the M_{Sh} degree $(0,1]$ of an element expresses that the element somewhat satisfies the property. The M_{Sh} degree $[-1,0)$ of an element signifies that the element somewhat satisfies the implicit counter-property. The idea behind such a description is connected with the existence of bipolar information (e.g., positive information and negative information) about the given set. Positive (+ve) data represents what is presumed possible, while negative (-ve) data represents what is presumed impossible.

Let F be a universal set, and κ be a set over F that is defined by a positive M_{Sh} function μ_κ^+ and a negative M_{Sh} function μ_κ^-, where $\mu_\kappa^+ : F \to [0,1]$ and $\mu_\kappa^- : F \to [-1,0]$.Then κ is called a bipolar valued fuzzy set over F and can be written in form $\kappa = \{< \hbar, \mu_\kappa^+(\hbar), \mu_\kappa^-(\hbar) >| \hbar \in F\}$. Anitha et al. [2] have studied the BF subgroups of a group. Majumder and Sardar [11] studied fuzzy magnified translation on groups in 2008. Jun et al. [8] introduced BF translations in BCI/BCK algebras. Sharma [18] studied intuitionistic fuzzy magnified translation on groups in 2012. Majumder and Sardar [17] studied bipolar-valued fuzzy translation in semigroups in 2012. Udten et al. [20] explored the study of translation and density of a BFS in UP-algebras. Ria [16] studied BF translation, extension, and multiplication on bipolar anti-fuzzy ideals of K-algebras. Kalyani et al. [9] defined a BFMT of a bipolar fuzzy subgroup of a group.

2. Preliminaries

In this section, we will revisit and elaborate on the essential definitions that form the foundation of this study. To ensure a comprehensive understanding, we will carefully examine key terms and concepts, which are crucial for the progression and clarity of our research. This review will not only reinforce the theoretical framework but also highlight the relevance of these definitions in the context of our investigation.

Definition 2.1. *[6] A poset $(\mathbb{L}, \leq)$ is called a lattice if* $\sup\{p, q\}$ *(also denoted by $p \vee q$) and* $\inf\{p, q\}$ *(also denoted by $p \wedge q$) exist for every pair of elements p, q in $\mathbb{L}$.*

Definition 2.2. *[23] Let F be any non-empty set. A mapping $\psi : F \to [0,1]$ is called a fuzzy subset of F.*

Definition 2.3. *[6] Let $\psi : F \to [0,1]$ be any FS. Then the set $\{\psi(p) \mid p \in F\}$ is called the image of ψ and is denoted by $Im(\psi)$. For $t \in [0,1], \psi_t = \{p \in F \mid \psi(p) \geq t\}$ is called a level subset of ψ.*

Definition 2.4. *[7] A vague set κ in the universe of discourse F is characterized by two M_{Sh} functions given by*

(i) a truth M_{Sh} function $t_\kappa : F \to [0,1]$ and

(ii) a false M_{Sh} function $f_\kappa : F \to [0,1]$,

where $t_\kappa(p)$ is a lower bound of the grade of M_{Sh} of p derived from the evidence for p and $f_\kappa(p)$ is a lower bound on the negation of p derived from the evidence against p, with $t_\kappa(p) + f_\kappa(p) \leq 1$.

We give below a formation of the definition of vague set in the following way, which makes Atanassov intuitionistic fuzzy sets [3] and Gau and Buehrer [7] vague sets in a mathematically equivalent form.

Definition 2.5. *[7] Let κ be a vague set of a universe F with true M_{Sh} function t_κ and false M_{Sh} function f_κ. For $\alpha, \Upsilon \in$ [0,1] with $\alpha \leq \Upsilon$, the (α, Υ)-cut or vague cut of a vague set κ is the crisp subset of F, given by $\kappa_{(\alpha,\Upsilon)} = \{p \in F \mid V_\kappa(p) \geq [\alpha, \Upsilon]\}$, i.e., $\kappa_{(\alpha,\Upsilon)} = \{p \in F \mid t_\kappa(p) \geq \alpha$ and $1 - f_\kappa(p) \geq \Upsilon\}$.*

Definition 2.6. *[7] The α-cut, κ_α of the vague set κ is the (α, α)-cut of κ and hence given by $\kappa_\alpha = \{p \in F \mid t_\kappa(p) \geq \alpha\}$.*

Definition 2.7. *[10] Suppose F is a universal set. A bipolar fuzzy set (BFS) $\mathbb{B}$ in F is an object having the form $\mathbb{B} = \{< \hbar, \mathbb{B}^P(\hbar), \mathbb{B}^N(\hbar) > \mid \hbar \in F\}$ where $\mathbb{B}^P : F \to [0,1]$ and $\mathbb{B}^N : F \to [-1,0]$ are a positive and negative M_{Sh} functions, respectively.*

Definition 2.8. *[2,10] Let F be a nonempty set, and let $\mathbb{B}_\vartheta, \mathbb{B}_\omega \in BPFS(F)$.*

(i) $\mathbb{B}_\vartheta$ is a subset of $\mathbb{B}_\omega$, denoted by $\mathbb{B}_\vartheta \subseteq \mathbb{B}_\omega$, if for each $\hbar \in F$, $\mathbb{B}^P_\vartheta(\hbar) \leq \mathbb{B}^P_\omega(\hbar)$ and $\mathbb{B}^N_\vartheta(\hbar) \geq \mathbb{B}^N_\omega(\hbar)$.

(ii) The complement of $\mathbb{B}_\vartheta$, denoted by $\mathbb{B}^c_\vartheta = ((\mathbb{B}^c_\vartheta)^N, (\mathbb{B}^c_\vartheta)^P)$, is a BFS in F defined as: for each $\hbar \in F$, $\mathbb{B}^c_\vartheta(\hbar) = (-1 - \mathbb{B}^N_\vartheta(\hbar), 1 - \mathbb{B}^P_\vartheta(\hbar))$, i.e.,$(\mathbb{B}^c_\vartheta)^P(\hbar) = 1 - \mathbb{B}^N_\vartheta(\hbar)$,$(\mathbb{B}^c_\vartheta)^N(\hbar) = -1 - \mathbb{B}^N_\vartheta(\hbar)$.

(iii) The intersection of $\mathbb{B}_\vartheta$ *and* $\mathbb{B}_\omega$*, denoted by* $\mathbb{B}_\vartheta \cap \mathbb{B}_\omega$*, is a BFS in* F *defined as: for each* $\hbar \in F$*,* $(\mathbb{B}_\vartheta \cap \mathbb{B}_\omega)(\hbar) = (\mathbb{B}^N_\vartheta(\hbar) \vee \mathbb{B}^N_\omega(\hbar), \mathbb{B}^P_\vartheta(\hbar) \wedge \mathbb{B}^P_\omega(\hbar))$.

(iv) The union of $\mathbb{B}_\vartheta$ *and* $\mathbb{B}_\omega$*, denoted by* $\mathbb{B}_\vartheta \cup \mathbb{B}_\omega$*, is a BFS in* F *defined as: for each* $\hbar \in F$*,* $(\mathbb{B}_\vartheta \cup \mathbb{B}_\omega)(\hbar) = (\mathbb{B}^N_\vartheta(\hbar) \wedge \mathbb{B}^N_\omega(\hbar), \mathbb{B}^P_\vartheta(\hbar) \vee \mathbb{B}^P_\omega(\hbar))$.

Definition 2.9. *[2,10] Let* $\mathbb{B}$ *be a bipolar fuzzy set of a universe* F*. For* $\alpha \in [0,1]$ *and* $\Upsilon \in [-1,0]$*, the* (α, Υ) *level subset cut of a bipolar fuzzy set* κ *is the crisp subset of* F *is given by* $\mathbb{B}_{(\alpha,\Upsilon)} = \{\hbar \in F \mid \mathbb{B}^P(\hbar) \geq \alpha$ *and* $\mathbb{B}^N(\hbar) \leq \Upsilon\}$.

Definition 2.10. *[2,10] The Support set of a bipolar fuzzy subset* $\mathbb{B} =< \mathbb{B}^P, \mathbb{B}^N >$ *denoted by* $\mathrm{Supp}(\mathbb{B})$ *and is defined by* $\mathrm{Supp}(\mathbb{B}) = \{\hbar \mid \mathbb{B}^P(\hbar) \neq 0$ *or* $\mathbb{B}^N(\hbar) \neq 0\}$.

Definition 2.11. *[20] For any BFS* $\mathbb{B} =< \mathbb{B}^N, \mathbb{B}^P >$ *in universe of discourse* $\mathbb{D}$*, we denote* $\triangledown = -1 - \inf\{\mathbb{B}^N(\mathcal{T}) \mid \mathcal{T} \in \mathbb{D}\}$ *and* $\triangle = 1 - \sup\{\mathbb{B}^P(\mathcal{T}) \mid \mathcal{T} \in \mathbb{D}\}$*. Let* $\mathbb{B} =< \mathbb{B}^N, \mathbb{B}^P >$ *be a BFS in* $\mathbb{D}$ *and* $(\theta, \vartheta) \in [\triangledown, 0] \times [0, \triangle]$*. By a bipolar fuzzy* (θ, ϑ)*-translation of* $\mathbb{B} =< \mathbb{B}^N, \mathbb{B}^P >$*, we mean a BFS* $\mathbb{B}^T_{(\theta,\vartheta)} =< \mathbb{B}^N_{(\theta,T)}, \mathbb{B}^P_{(\vartheta,T)} >$*, where* $\mathbb{B}^N_{(\theta,T)} : \mathbb{D} \to [-1,0]$ *defined by* $\mathbb{B}^N_{(\theta,T)}(\mathcal{T}) = \mathbb{B}^N(\mathcal{T}) + \theta$ *and* $\mathbb{B}^P_{(\vartheta,T)} : \mathbb{D} \to [0,1]$ *defined by* $\mathbb{B}^P_{(\vartheta,T)}(\mathcal{T}) = \mathbb{B}^P(\mathcal{T}) + \vartheta$ *for all* $\mathcal{T} \in \mathbb{D}$.

Definition 2.12. *[16] Let* $\mathbb{B} =< \mathbb{B}^N, \mathbb{B}^P >$ *be a BFS in universe of discourse* $\mathbb{D}$ *and* $(\alpha, \beta) \in [0,1]$*,* $(\theta, \vartheta) \in [\triangledown, 0] \times [0, \triangle]$*. By a BFMT of* $\mathbb{B} =< \mathbb{B}^N, \mathbb{B}^P >$*, we mean a BFS* $M = \{< r, \mathbb{B}^N_{(\alpha,\theta)}(r), \mathbb{B}^P_{(\beta,\vartheta)}(r) >\mid r \in \mathbb{D}\}$ *or simply as* $M = \{< r, \mathbb{B}^N_M(r), \mathbb{B}^P_M(r) >\mid r \in \mathbb{D}\}$*, where* $\mathbb{B}^N_M = \mathbb{B}^N_{(\alpha,\theta)} : \mathbb{D} \to [-1,0]$ *and* $\mathbb{B}^P_M(r) = \mathbb{B}^P_{(\beta,\vartheta)} : \mathbb{D} \to [0,1]$ *defined by* $\mathbb{B}^N_M(r) = \mathbb{B}^N_{(\alpha,\theta)}(r) = \alpha\mathbb{B}^N(r) + \theta$ *and* $\mathbb{B}^P_M(r) = \mathbb{B}^P_{(\beta,\vartheta)}(r) = \beta\mathbb{B}^P(r) + \vartheta$ *for all* $r \in \mathbb{D}$.

3. Bipolar fuzzy level subsets of a lattice

Theorem 3.1. *Suppose* $\mathfrak{B} =< \mathfrak{B}^P, \mathfrak{B}^N >$ *where* $\mathfrak{B}^P : \mathbb{L} \to [0,1]$ *and* $\mathfrak{B}^N : \mathbb{L} \to [-1,0]$*, is a BFS of* $\mathbb{L}$*. Then* $\mathfrak{B}$ *is a BFL of* $\mathbb{L}$ *if and only if* $\mathfrak{B}_{(\alpha,\beta)}$*,the level subset which is non-empty is a sublattice of* $\mathbb{L}$ *for every* $\alpha \in [0,1]$ *and* $\beta \in [-1,0]$.

Proof. Suppose that $\mathfrak{B} =< \mathfrak{B}^P, \mathfrak{B}^N >\in BFL(\mathbb{L})$. Let $\hbar, s \in B_{(\alpha,\beta)}$. Then $\mathfrak{B}^P(\hbar) \geq \alpha, \mathfrak{B}^P(s) \geq \alpha$ and $\mathfrak{B}^N(\hbar) \leq \beta, \mathfrak{B}^N(s) \leq \beta$. Now, $\mathfrak{B}^P(\hbar \vee s) \geq \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s)\} \geq \alpha$ and $\mathfrak{B}^P(\hbar \vee s) \leq \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(s)\} \leq \beta$. Thus, $\hbar \vee s \in B_{(\alpha,\beta)}$. Now, $\mathfrak{B}^P(\hbar \wedge s) \geq \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s)\} \geq \alpha$ and $\mathfrak{B}^N(\hbar \wedge s) \leq \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(s)\} \leq \beta$. Thus, $\hbar \wedge s \in \mathfrak{B}_{(\alpha,\beta)}$. Hence, $\mathfrak{B}_{(\alpha,\beta)}$ is a sublattice of $\mathbb{L}$.

Conversely, assume that $\mathfrak{B}_{(\alpha,\beta)}$ is a sublattice of $\mathbb{L}$. Suppose $\hbar, s \in \mathbb{L}$. Then $\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s) \in [0,1]$ and $\mathfrak{B}^N(\hbar), \mathfrak{B}^N(s) \in [-1,0]$. Choose $\alpha = \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s)\}$ and $\beta = \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(s)\}$. Then $\mathfrak{B}^P(\hbar) \geq \alpha, \mathfrak{B}^P(s) \geq \alpha, \mathfrak{B}^N(\hbar) \leq \beta, \mathfrak{B}^N(s) \leq \beta$, so $\hbar, s \in \mathfrak{B}_{(\alpha,\beta)}$. As $\mathfrak{B}_{(\alpha,\beta)}$ is a sublattice of $\mathbb{L}$, we have $\hbar \vee s \in \mathfrak{B}_{(\alpha,\beta)}$ and $\hbar \wedge s \in \mathfrak{B}_{(\alpha,\beta)}$. Hence, $\mathfrak{B}^P(\hbar \vee s) \geq \alpha, \mathfrak{B}^N(\hbar \vee s) \leq \beta$ and $\mathfrak{B}^P(\hbar \wedge s) \geq \alpha, \mathfrak{B}^N(\hbar \wedge s) \leq \beta$. Thus, $\mathfrak{B}^P(\hbar \vee s) \geq \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s)\}$, $\mathfrak{B}^P(\hbar \wedge s) \geq \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s)\}$, $\mathfrak{B}^N(\hbar \vee s) \leq \max\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s)\}$, $\mathfrak{B}^N(\hbar \wedge s) \leq \max\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s)\}$. □

Theorem 3.2. *Consider* $\mathbb{L}$ *and* $\mathfrak{B} =< \mathfrak{B}^P, \mathfrak{B}^N >$ *where* $\mathfrak{B}^P : \mathbb{L} \to [0,1]$ *and* $\mathfrak{B}^N : \mathbb{L} \to [-1,0]$ *is a BFS of* $\mathbb{L}$*. If* $\mathfrak{B}$ *forms a BFL of* $\mathbb{L}$*, then* $\mathrm{Supp}(\mathfrak{B})$ *forms a crisp sublattice of* $\mathbb{L}$.

Proof. Suppose $\mathfrak{B} = \{< \hbar, \mathfrak{B}^P(\hbar), \mathfrak{B}^N(\hbar) >| \hbar \in \mathbb{L}\} \in BFS(\mathbb{L})$ and $\hbar, s \in \mathrm{Supp}(B)$. Suppose that $\mathfrak{B}^P(\hbar) \neq 0$ or $\mathfrak{B}^N(\hbar) \neq 0$. Given $\mathfrak{B}$ is a BFSL of $\mathbb{L}$. Then $\mathfrak{B}^P(\hbar \vee s) \geq \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s)\}$. Thus, $\mathfrak{B}^P(\hbar \vee s) \neq 0$. Likewise, we can obtain $\mathfrak{B}^N(\hbar \vee s) \neq 0, \mathfrak{B}^P(\hbar \wedge s) \neq 0, \mathfrak{B}^N(\hbar \wedge s) \neq 0$. This gives us $\hbar \vee s \in \mathrm{Supp}(\mathfrak{B})$ and $\hbar \wedge s \in \mathrm{Supp}(\mathfrak{B})$. Hence, $\mathrm{Supp}(\mathfrak{B})$ is a crisp sublattice of $\mathbb{L}$. □

Remark 3.1. *The converse part of Theorem 3.2 does not hold in general. Let* $\mathfrak{B} = \{< 1, 0.5, -0.1 >, < 2, 0.7, -0.2 >, < 5, 0.8, -0.05 >, < 10, 0.4, -0.01 >\}$ *be a BFS in* $\mathbb{L} = \{1, 2, 5, 10\}$ *with divisors of 10. Then* $\mathrm{Supp}(\mathfrak{B}) = \{1, 2, 5, 10\}$ *is a crisp sublattice of* $\mathbb{L}$*. But* $\mathfrak{B}^P(2 \vee 5) = \mathfrak{B}^P(10) = 0.4 < \min\{\mathfrak{B}^P(2), \mathfrak{B}^P(5)\} = 0.7$ *and* $\mathfrak{B}^P(2 \wedge 5) = \mathfrak{B}^P(1) = 0.5 < \min\{\mathfrak{B}^P(2), \mathfrak{B}^P(5)\} = 0.7$*, which is a contradiction to the property of* $\mathfrak{B}$ *is a BFSL of* $\mathbb{L}$*.*

Definition 3.1. *Suppose* $\mathfrak{B} =< \mathfrak{B}^P, \mathfrak{B}^N >$ *where* $\mathfrak{B}^P : \mathbb{L} \to [0, 1]$ *and* $\mathfrak{B}^N : \mathbb{L} \to [-1, 0]$ *is a BFS of* $\mathbb{L}$ *is a BFS in* $\mathbb{L}$ *and* $(\alpha, \beta) \in [0, 1]$*. We denote* $\triangledown = -1 - \inf\{\mathbb{B}^N(\mathcal{T}) \mid \mathcal{T} \in \mathbb{D}\}$ *and* $\triangle = 1 - \sup\{\mathbb{B}^P(\mathcal{T}) \mid \mathcal{T} \in \mathbb{D}\}$*. Let* $(\theta, \vartheta) \in [\triangledown, 0] \times [0, \triangle]$*. By a BFMT of* $\mathfrak{B} =< \mathfrak{B}^P, \mathfrak{B}^N >$*, we mean a BFS* $M = \{< r, B^P_{(\beta,\vartheta)}(r), B^N_{(\alpha,\theta)}(r) >| r \in \mathbb{L}\}$ *or simply as* $M = \{< r, B^P_M(r), B^N_M(r) >| r \in \mathbb{L}\}$*, where* $\mathfrak{B}^P_M(r) = B^P_{(\beta,\vartheta)} : \mathbb{L} \to [0, 1]$ *and* $\mathfrak{B}^N_M = B^N_{(\alpha,\theta)} : \mathbb{L} \to [-1, 0]$ *and defined by* $\mathfrak{B}^P_M(r) = B^P_{(\beta,\vartheta)}(r) = \beta\mathfrak{B}^P(r) + \vartheta$ *for all* $r \in \mathbb{L}$ *and* $\mathfrak{B}^N_M(r) = B^N_{(\alpha,\theta)}(r) = \alpha\mathfrak{B}^N(r) + \theta$*.*

Example 3.1. *Let* $\mathbb{L} = \{1, 2, 3, 6\}$ *of divisors of 6 and let* $\mathfrak{B} = \{< 1, 0.3, -0.2 >, < 2, 0.4, -0.3 >, < 3, 0.5, -0.1 >, < 6, 0.3, -0.1 >\}$*. Let* $\theta \in [-0.9, 0]$ *and* $\vartheta \in [0, 0.5]$*. Let* $\alpha = 0.1, \beta = 0.2, \theta = -0.8, \vartheta = 0.2$*. Hence, the BFMT* $M = \{< 1, 0.26, -0.8 >, < 2, 0.36, -0.83 >, < 3, 0.3, -0.81 >< 6, 0.26, -0.81 >\}$*.*

Theorem 3.3. *Let* $\mathfrak{B} =< \mathfrak{B}^P, \mathfrak{B}^N >$ *where* $\mathfrak{B}^P : \mathbb{L} \to [0, 1]$ *and* $\mathfrak{B}^N : \mathbb{L} \to [-1, 0]$ *be a BFS of* $\mathbb{L}$*. Then* $\mathfrak{B}$ *forms a BFL of* $\mathbb{L}$ *if and only if the BFMT* M *of* $\mathfrak{B}$ *is a BFL of* $\mathbb{L}$*.*

Proof. Assume that $\mathfrak{B}$ is BFL of $\mathbb{L}$ and M is a BFMT of $\mathfrak{B}$. Suppose $\hbar, s \in \mathbb{L}$. Now,

$$\begin{aligned}
\mathfrak{B}^P_{(\beta,\vartheta)}(\hbar \vee s) &= \beta\mathfrak{B}^P(\hbar \vee s) + \vartheta \\
&\geq \beta \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s)\} + \vartheta \\
&= \min\{\beta\mathfrak{B}^P(\hbar) + \vartheta, \beta\mathfrak{B}^P(s) + \vartheta\} \\
&= \min\{B^P_{(\beta,\vartheta)}(\hbar), B^P_{(\beta,\vartheta)}(s)\}, \\
\mathfrak{B}^P_{(\beta,\vartheta)}(\hbar \wedge s) &= \beta\mathfrak{B}^P(\hbar \wedge s) + \vartheta \\
&\geq \beta \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s)\} + \vartheta \\
&= \min\{\beta\mathfrak{B}^P(\hbar) + \vartheta, \beta\mathfrak{B}^P(s) + \vartheta\} \\
&= \min\{B^P_{(\beta,\vartheta)}(\hbar), B^P_{(\beta,\vartheta)}(s)\}, \\
\mathfrak{B}^N_{(\alpha,\theta)}(\hbar \vee s) &= \alpha\mathfrak{B}^N(\hbar \vee s) + \theta \\
&\leq \alpha \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(s)\} + \theta \\
&= \max\{\alpha\mathfrak{B}^N(\hbar) + \theta, \alpha\mathfrak{B}^N(s) + \theta\} \\
&= \max\{B^N_{(\alpha,\theta)}(\hbar), B^N_{(\alpha,\theta)}(s)\},
\end{aligned}$$

$$\begin{aligned}\mathfrak{B}^N_{(\alpha,\theta)}(\hbar \wedge s) &= \alpha\mathfrak{B}^N(\hbar \wedge s) + \theta \\ &\leq \alpha \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(s)\} + \theta \\ &= \max\{\alpha\mathfrak{B}^N(\hbar) + \theta, \alpha\mathfrak{B}^N(s) + \theta\} \\ &= \max\{B^N_{(\alpha,\theta)}(\hbar), B^N_{(\alpha,\theta)}(s)\}.\end{aligned}$$

Hence, the BFMT of a BFL is again a BFL of $\mathbb{L}$.

Conversely, assume that the BFMT M of $\mathfrak{B}$ is a BFL of $\mathbb{L}$. Then

$$\begin{aligned}\mathfrak{B}^P(\hbar \vee s) &= \frac{1}{\beta}(B^P_{(\beta,\vartheta)}(\hbar \vee s) - \vartheta) \\ &\geq \frac{1}{\beta}(\min\{B^P_{(\beta,\vartheta)}(\hbar), B^P_{(\beta,\vartheta)}(s)\} - \vartheta) \\ &= \min\{\frac{1}{\beta}(B^P_{(\beta,\vartheta)}(\hbar) - \vartheta), \frac{1}{\beta}(B^P_{(\beta,\vartheta)}(s) - \vartheta)\} \\ &= \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s)\},\end{aligned}$$

$$\begin{aligned}\mathfrak{B}^P(\hbar \wedge s) &= \frac{1}{\beta}(B^P_{(\beta,\vartheta)}(\hbar \wedge s) - \vartheta) \\ &\geq \frac{1}{\beta}(\min\{B^P_{(\beta,\vartheta)}(\hbar), B^P_{(\beta,\vartheta)}(s)\} - \vartheta) \\ &= \min\{\frac{1}{\beta}(B^P_{(\beta,\vartheta)}(\hbar) - \vartheta), \frac{1}{\beta}(B^P_{(\beta,\vartheta)}(s) - \vartheta)\} \\ &= \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s)\}.\end{aligned}$$

In a similar way, we can prove that $\mathfrak{B}^N(\hbar \vee s) \leq \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(s)\}$ and $\mathfrak{B}^N(\hbar \vee s) \leq \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(s)\}$. Hence, $\mathfrak{B}$ is BFL of $\mathbb{L}$. □

Now, we explore the theory of BFLs under lattice homomorphism.

Definition 3.2. *Let $\varphi : \mathbb{L} \to \mathbb{L}^1$ be a mapping from lattices $\mathbb{L}$ to $\mathbb{L}^1$ and $\mathfrak{B} = (\mathfrak{B}^P, \mathfrak{B}^N)$ where $\mathfrak{B}^P : \mathbb{L} \to [0,1]$ and $\mathfrak{B}^N : \mathbb{L} \to [-1,0]$ be a BFS in $\mathbb{L}$. Then the image $\varphi(\mathfrak{B})$ is defined as $\varphi(\mathfrak{B}) = \{< s, \varphi(\mathfrak{B}^P)(s), \varphi(\mathfrak{B}^N)(s) >\mid s \in \mathbb{L}^1\}$,*

$$\varphi(\mathfrak{B}^P)(s) = \begin{cases}\sup\{\mathfrak{B}^P(\hbar) \mid \hbar \in \varphi^{-1}(s)\} & \text{if } \varphi^{-1}(s) \neq \emptyset, \\ 0 & \text{otherwise}\end{cases}$$

and

$$\varphi(\mathfrak{B}^N)(s) = \begin{cases}\inf\{\mathfrak{B}^N(\hbar) \mid \hbar \in \varphi^{-1}(s)\} & \text{if } \varphi^{-1}(s) \neq \emptyset, \\ 0 & \text{otherwise.}\end{cases}$$

Similarly, if $C = (C^P, C^N)$ is a BFS in $\mathbb{L}^1$, then $\varphi^{-1}(C) = \{< \hbar, \varphi^{-1}(C^P(\hbar)), \varphi^{-1}(C^N(\hbar)) >\mid \hbar \in \mathbb{L}\}$, where $\varphi^{-1}(C^P(\hbar)) = C^P(\varphi(\hbar))$ and $\varphi^{-1}(C^N(\hbar)) = C^N(\varphi(\hbar))$.

Theorem 3.4. *Let $\varphi : \mathbb{L} \to \mathbb{L}^1$ be a lattice epimorphism. If $\mathfrak{B} =< \mathfrak{B}^P, \mathfrak{B}^N >$ where $\mathfrak{B}^P : \mathbb{L} \to [0,1]$ and $\mathfrak{B}^N : \mathbb{L} \to [-1,0]$ is a BFL of $\mathbb{L}$, then $\varphi(B)$ is a BFL of $\mathbb{L}^1$.*

Proof. Let $\mathfrak{B} = (\mathfrak{B}^P, \mathfrak{B}^N)$ be a BFL of $\mathbb{L}$. Let $s, w \in \mathbb{L}^1$. Then

$$\begin{aligned}
\varphi(\mathfrak{B}^P)(s \vee w) &= \sup\{\mathfrak{B}^P(\hbar) \mid \hbar \in \varphi^{-1}(s \vee w)\} \\
&\geq \sup\{\mathfrak{B}^P(u \vee \xi) \mid u \in \varphi^{-1}(s), \xi \in \varphi^{-1}(w)\} \\
&\geq \sup\{\min\{\mathfrak{B}^P(u), \mathfrak{B}^P(\xi)\} \mid u \in \varphi^{-1}(s), \xi \in \varphi^{-1}(w)\} \\
&= \min\{\sup\{\mathfrak{B}^P(u) \mid u \in \varphi^{-1}(s)\}, \sup\{\mathfrak{B}^P(\xi) \mid \xi \in \varphi^{-1}(w)\}\} \\
&= \min\{\varphi(\mathfrak{B}^P)(s), \varphi(\mathfrak{B}^P)(w)\},
\end{aligned}$$

$$\begin{aligned}
\varphi(\mathfrak{B}^P)(s \wedge w) &= \sup\{\mathfrak{B}^P(\hbar) \mid \hbar \in \varphi^{-1}(s \wedge w)\} \\
&\geq \sup\{\mathfrak{B}^P(u \wedge \xi) \mid u \in \varphi^{-1}(s), \xi \in \varphi^{-1}(w)\} \\
&\geq \sup\{\min\{\mathfrak{B}^P(u), \mathfrak{B}^P(\xi)\} \mid u \in \varphi^{-1}(s), \xi \in \varphi^{-1}(w)\} \\
&= \min\{\sup\{\mathfrak{B}^P(u) \mid u \in \varphi^{-1}(s)\}, \sup\{\mathfrak{B}^P(\xi) \mid \xi \in \varphi^{-1}(w)\}\} \\
&= \min\{\varphi(\mathfrak{B}^P)(s), \varphi(\mathfrak{B}^P)(w)\},
\end{aligned}$$

$$\begin{aligned}
\varphi(\mathfrak{B}^N)(s \vee w) &= \inf\{\mathfrak{B}^N(\hbar) \mid \hbar \in \varphi^{-1}(s \vee w)\} \\
&\leq \inf\{\mathfrak{B}^N(u \vee \xi) \mid u \in \varphi^{-1}(s), \xi \in \varphi^{-1}(w)\} \\
&\leq \inf\{\max\{\mathfrak{B}^N(u), \mathfrak{B}^N(\xi)\} \mid u \in \varphi^{-1}(s), \xi \in \varphi^{-1}(w)\} \\
&= \max\{\inf\{\mathfrak{B}^N(u) \mid u \in \varphi^{-1}(s)\}, \inf\{\mathfrak{B}^N(\xi) \mid \xi \in \varphi^{-1}(w)\}\} \\
&= \max\{\varphi(\mathfrak{B}^N)(s), \varphi(\mathfrak{B}^N)(w)\},
\end{aligned}$$

$$\begin{aligned}
\varphi(\mathfrak{B}^N)(s \wedge w) &= \inf\{\mathfrak{B}^N(\hbar) \mid \hbar \in \varphi^{-1}(s \wedge w)\} \\
&\leq \inf\{\mathfrak{B}^N(u \wedge \xi) \mid u \in \varphi^{-1}(s), \xi \in \varphi^{-1}(w)\} \\
&\leq \inf\{\max\{\mathfrak{B}^N(u), \mathfrak{B}^N(\xi)\} \mid u \in \varphi^{-1}(s), \xi \in \varphi^{-1}(w)\} \\
&= \max\{\inf\{\mathfrak{B}^N(u) \mid u \in \varphi^{-1}(s)\}, \inf\{\mathfrak{B}^N(\xi) \mid \xi \in \varphi^{-1}(w)\}\} \\
&= \max\{\varphi(\mathfrak{B}^N)(s), \varphi(\mathfrak{B}^N)(w)\}.
\end{aligned}$$

Hence, $\varphi(\mathfrak{B})$ is a BFL of $\mathbb{L}^1$. □

Theorem 3.5. *Let $\varphi : \mathbb{L} \to \mathbb{L}^1$ be a homomorphism from $\mathbb{L}$ to $\mathbb{L}^1$. If $C =< C^P, C^N >$ where $C^P : \mathbb{L} \to [0,1]$ and $C^N : \mathbb{L} \to [-1,0]$ is a BFL of $\mathbb{L}^1$, then $\varphi^{-1}(C)$ is a BFL of $\mathbb{L}$.*

Proof. Let $C =< C^P, C^N >$ be a BFL of $\mathbb{L}^1$. Let $\hbar, s \in \mathbb{L}$. Then

$$\begin{aligned}
\varphi^{-1}(C^P)(\hbar \vee s) &= C^P(\varphi(\hbar \vee s)) \\
&= C^P\{(\varphi(\hbar) \vee \varphi(s)\} \\
&\geq \min\{C^P(\varphi(\hbar)), C^P(\varphi(s))\} \\
&= \min\{\varphi^{-1}(C^P)(\hbar), \varphi^{-1}(C^P)(s)\},
\end{aligned}$$

$$\begin{aligned}
\varphi^{-1}(C^P)(\hbar \wedge s) &= C^P(\varphi(\hbar \wedge s)) \\
&= C^P\{(\varphi(\hbar) \wedge \varphi(s)\} \\
&\geq \min\{C^P(\varphi(\hbar)), C^P(\varphi(s))\} \\
&= \min\{\varphi^{-1}(C^P)(\hbar), \varphi^{-1}(C^P)(s)\},
\end{aligned}$$

$$\begin{aligned}
\varphi^{-1}(C^N)(\hbar \vee s) &= C^N(\varphi(\hbar \vee s)) \\
&= C^N\{(\varphi(\hbar) \vee \varphi(s)\}\} \\
&\leq \max\{C^N(\varphi(\hbar)), C^N(\varphi(s)) \\
&= \max\{\varphi^{-1}(C^N)(\hbar), \varphi^{-1}(C^N)(s)\},
\end{aligned}$$

$$\begin{aligned}
\varphi^{-1}(C^N)(\hbar \wedge s) &= C^N(\varphi(\hbar \wedge s)) \\
&= C^N\{(\varphi(\hbar) \wedge \varphi(s)\} \\
&\leq \max\{C^N(\varphi(\hbar)), C^N(\varphi(s))\} \\
&= \max\{\varphi^{-1}(C^N)(\hbar), \varphi^{-1}(C^N)(s)\}.
\end{aligned}$$

Hence, $\varphi^{-1}(C)$ is a BFL of $\mathbb{L}$. □

Theorem 3.6. *Let $\mathbb{L}$ and $\mathbb{L}^1$ be any two lattices, and $\varkappa$ be a homomorphism from $\mathbb{L}$ to $\mathbb{L}^1$. Then the homomorphic image of a BFMT M of a BFL $\mathfrak{B}$ of $\mathbb{L}$ forms a BFL of $\mathbb{L}^1$.*

Proof. Let $\mathbb{V} = \varkappa(M)$. Now, for $\varkappa(r)$ and $\varkappa(s)$ in $\mathbb{L}^1$, we have

$$\begin{aligned}
\mathbb{V}^P(\varkappa(r) \vee \varkappa(s)) &= \mathbb{V}^P(\varkappa(r \vee s)) \\
&\geq \mathfrak{B}^P_M(r \vee s) \\
&= \beta\mathfrak{B}^P(r \vee s) + \vartheta \\
&\geq \beta(\min\{\mathfrak{B}^P(r), \mathfrak{B}^P(s)\}) + \vartheta \\
&= \min\{\beta\mathfrak{B}^P(r) + \vartheta, \beta\mathfrak{B}^P(s) + \vartheta\} \\
&= \min\{\mathbb{V}^P(\varkappa(r), \mathbb{V}^P(\varkappa(s)\}.
\end{aligned}$$

Thus, $\mathbb{V}^P(\varkappa(r) \vee \varkappa(s)) \geq \min\{\mathbb{V}^P(\varkappa(r), \mathbb{V}^P(\varkappa(s)\}$. Similarly, we can prove the remaining three conditions. Thus, the homomorphic image of a BFMT M of a BFL $\mathfrak{B}$ of $\mathbb{L}$ is a BFL of $\mathbb{L}^1$. □

Theorem 3.7. *Let $\mathbb{L}$ and $\mathbb{L}^1$ be any two lattices. Then the homomorphic pre-image of a BFMT of a BFL $\mathfrak{B}$ of $\mathbb{L}^1$ forms a BFL of $\mathbb{L}$.*

Proof. The proof is similar to Theorem 3.5. □

Theorem 3.8. *Let $I =< I^P, I^N >$ be a BFS of $\mathbb{L}$. Then I is a BFI of $\mathbb{L}$ if and only if the nonempty level subset $I_{(\alpha,\beta)}$ is an ideal of $\mathbb{L}$ for each $\alpha \in [0,1]$ and $\beta \in [-1,0]$.*

Proof. Suppose that $I =< I^P, I^N >$ is a BFI of $\mathbb{L}$. Let $\hbar, s \in I_{(\alpha,\beta)}$ and $l \in \mathbb{L}$. Then $I_{(\alpha,\beta)}$ is a sublattice of $\mathbb{L}$ as $I_{(\alpha,\beta)}$ forms a BFI of $\mathbb{L}$. Now, for $l \in \mathbb{L}$, $I^P(\hbar \wedge l) \geq \max\{I^P(\hbar), I^P(l)\} \geq I^P(\hbar) \geq \alpha$ and $I^N(\hbar \wedge l) \leq \max\{I^N(\hbar), I^N(l)\} \leq I^N(\hbar) \leq \beta$. Thus, $\hbar \wedge l \in I_{(\alpha,\beta)}$. Hence, $I_{(\alpha,\beta)}$ forms an ideal of $\mathbb{L}$.

Conversely, assume that $I_{(\alpha,\beta)}$ is an ideal of $\mathbb{L}$. Then $I^P(\hbar), I^P(s) \in [0,1]$ and $I^N(\hbar), I^N(s) \in [-1,0]$. We must show that I is a BFI of $\mathbb{L}$. We know that every ideal is a sublattice of $\mathbb{L}$. Thus, $I_{(\alpha,\beta)}$ is a sublattice of $\mathbb{L}$. By Theorem 3.1, I is a BFL of $\mathbb{L}$. Choose $\alpha = \min\{I^P(\hbar), I^P(s)\}$ and $\beta = \max\{I^N(\hbar), I^N(s)\}$. Thus, $I^P(\hbar) \geq \alpha$, $I^P(s) \geq \alpha$, $I^N(\hbar) \leq \beta$, $I^N(s) \leq \beta$, so $\hbar, s \in I_{(\alpha,\beta)}$. As $I_{(\alpha,\beta)}$ is an ideal of $\mathbb{L}$, we have $\hbar \vee s \in I_{(\alpha,\beta)}$ and $\hbar \wedge s \in I_{(\alpha,\beta)}$. Hence, $I^P(\hbar \vee s) \geq \alpha$, $I^N(\hbar \vee s) \leq \beta$ and $I^P(\hbar \wedge s) \geq \alpha$, $I^N(\hbar \wedge s) \leq \beta$. Thus, we obtain $I^P(\hbar \vee s) \geq \min\{I^P(\hbar), I^P(s)\}$. Similarly, other conditions of BFI are also valid. Hence, I is a BFI of $\mathbb{L}$. □

Theorem 3.9. *Let $\mathbb{L}$ be a lattice and $\mathfrak{B}$ be a BFS of $\mathbb{L}$. If $\mathfrak{B}$ is a BFI of $\mathbb{L}$, then $\mathrm{Supp}(\mathfrak{B})$ forms a crisp ideal of $\mathbb{L}$.*

Proof. Suppose $\mathfrak{B}$ is a BFI of $\mathbb{L}$ and $\hbar, s \in \mathrm{Supp}(\mathfrak{B})$. Assume that $\mathfrak{B}^P(\hbar) \neq 0$ or $\mathfrak{B}^N(\hbar) \neq 0$. Given $\mathfrak{B}$ is a BFI of $\mathbb{L}$. Then $\mathfrak{B}^P(\hbar \vee s) \geq \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s))\}$, so $\mathfrak{B}^P(\hbar \vee s) \neq 0$. Similar to above, we can get $\mathfrak{B}^N(\hbar \vee s) \neq 0$, $\mathfrak{B}^P(\hbar \wedge s) \neq 0$, and $\mathfrak{B}^N(\hbar \wedge s) \neq 0$. This gives us $\hbar \vee s \in \mathrm{Supp}(\mathfrak{B})$ and $\hbar \wedge s \in \mathrm{Supp}(\mathfrak{B})$. Now, let $l \in \mathbb{L}$ and $\hbar \in \mathrm{Supp}(\mathfrak{B})$. Then $\mathfrak{B}^P(\hbar \wedge l) \geq \max\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(l)\} \geq \mathfrak{B}^P(\hbar) \neq 0$. Thus, $\hbar \wedge l \in \mathrm{Supp}(\mathfrak{B})$. Hence, $\mathrm{Supp}(\mathfrak{B})$ is a crisp ideal of $\mathbb{L}$. □

Remark 3.2. *The converse part of the above theorem does not necessarily hold in general. Consider $\mathfrak{B} = \{< 1, 0.5, -0.1 >, < 2, 0.7, -0.2 >, < 5, 0.8, -0.05 >, < 10, 0.4, -0.05 >\}$ is a BFS in $\mathbb{L} = \{1, 2, 5, 10\}$. Then $\mathrm{Supp}(\mathfrak{B}) = \{1, 2, 5, 10\}$ is a crisp sublattice of $\mathbb{L}$. But $\mathfrak{B}^P(2 \vee 5) = \mathfrak{B}^P(10) = 0.4 < \min\{\mathfrak{B}^P(2), \mathfrak{B}^P(5)\} = 0.7$, which is a contradiction to the property of BFI of $\mathbb{L}$.*

Theorem 3.10. *Suppose $\mathfrak{B}$ is a BFS of $\mathbb{L}$. Thus $\mathfrak{B}$ is a BFI of $\mathbb{L}$ if and only if the BFMT M of $\mathfrak{B}$ is a BFI of $\mathbb{L}$.*

Proof. Assume that $\mathfrak{B}$ is BFI of $\mathbb{L}$ and M is a BFMT of $\mathfrak{B}$. Let $\hbar, s \in \mathbb{L}$. Then

$$\begin{aligned}
\mathfrak{B}^P_{(\beta,\vartheta)}(\hbar \vee s) &= \beta\mathfrak{B}^P(\hbar \vee s) + \vartheta \\
&\geq \beta \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s))\} + \vartheta \\
&= \min\{\beta\mathfrak{B}^P(\hbar) + \vartheta, \beta\mathfrak{B}^P(s)) + \vartheta\} \\
&= \min\{\mathfrak{B}^P_{(\beta,\vartheta)}(\hbar), \mathfrak{B}^P_{(\beta,\vartheta)}(s))\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{B}^P_{(\beta,\vartheta)}(\hbar \wedge s) &= \beta\mathfrak{B}^P(\hbar \wedge s) + \vartheta \\
&\geq \beta \max\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s))\} + \vartheta \\
&= \max\{\beta\mathfrak{B}^P(\hbar) + \vartheta, \beta\mathfrak{B}^P(s)) + \vartheta\} \\
&= \max\{\mathfrak{B}^P_{(\beta,\vartheta)}(\hbar), \mathfrak{B}^P_{(\beta,\vartheta)}(s))\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{B}^N_{(\alpha,\theta)}(\hbar \vee s) &= \alpha\mathfrak{B}^N(\hbar \vee s) + \theta \\
&\leq \alpha \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(s))\} + \theta \\
&= \max\{\alpha\mathfrak{B}^N(\hbar) + \theta, \alpha\mathfrak{B}^N(s)) + \theta\} \\
&= \max\{\mathfrak{B}^N_{(\alpha,\theta)}(\hbar), \mathfrak{B}^N_{(\alpha,\theta)}(s))\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{B}^N_{(\alpha,\theta)}(\hbar \wedge s) &= \alpha\mathfrak{B}^N(\hbar \wedge s) + \theta \\
&\leq \alpha \min\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(s))\} + \theta \\
&= \min\{\alpha\mathfrak{B}^N(\hbar) + \theta, \alpha\mathfrak{B}^N(s)) + \theta\} \\
&= \min\{\mathfrak{B}^N_{(\alpha,\theta)}(\hbar), \mathfrak{B}^N_{(\alpha,\theta)}(s))\}.
\end{aligned}$$

Hence, the BFMT of a BFL is again a BFL of $\mathbb{L}$.

Conversely, assume that the BFMT M of $\mathfrak{B}$ is a BFI of $\mathbb{L}$. Then

$$\begin{aligned}
\mathfrak{B}^P(\hbar \vee s) &= \frac{1}{\beta}(\mathfrak{B}^P_{(\beta,\vartheta)}(\hbar \vee s) - \vartheta) \\
&\geq \frac{1}{\beta}(\min\{\mathfrak{B}^P_{(\beta,\vartheta)}(\hbar), \mathfrak{B}^P_{(\beta,\vartheta)}(s))\} - \vartheta) \\
&= \min\{\frac{1}{\beta}(\mathfrak{B}^P_{(\beta,\vartheta)}(\hbar) - \vartheta), \frac{1}{\beta}(\mathfrak{B}^P_{(\beta,\vartheta)}(s)) - \vartheta)\} \\
&= \min\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s))\},
\end{aligned}$$

$$\begin{aligned}
\mathfrak{B}^P(\hbar \wedge s) &= \frac{1}{\beta}(\mathfrak{B}^P_{(\beta,\vartheta)}(\hbar \wedge s) - \vartheta) \\
&\geq \frac{1}{\beta}(\max\{\mathfrak{B}^P_{(\beta,\vartheta)}(\hbar), \mathfrak{B}^P_{(\beta,\vartheta)}(s))\} - \vartheta) \\
&= \max\{\frac{1}{\beta}(\mathfrak{B}^P_{(\beta,\vartheta)}(\hbar) - \vartheta), \frac{1}{\beta}(\mathfrak{B}^P_{(\beta,\vartheta)}(s)) - \vartheta)\} \\
&= \max\{\mathfrak{B}^P(\hbar), \mathfrak{B}^P(s))\}.
\end{aligned}$$

Similarly, we can prove $\mathfrak{B}^N(\hbar \vee s) \leq \max\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(s))\}$ and $\mathfrak{B}^N(\hbar \vee s) \leq \min\{\mathfrak{B}^N(\hbar), \mathfrak{B}^N(s))\}$. Hence, $\mathfrak{B}$ is BFI of $\mathbb{L}$.

□

4. Conclusion

This study introduces BF level subsets in a lattice, establishing that such subsets, denoted as $\mathfrak{B}$, can form BFLs and BFIs. We prove that if $\mathfrak{B}$ forms a BFL within a lattice $\mathbb{L}$, then its support set Supp$(\mathfrak{B})$ is a crisp sublattice of $\mathbb{L}$, though the converse is not always true. We further explore BFMT and demonstrate that both the homomorphic image and pre-image of a BFMT preserve the structures of BFLs and BFIs, offering new insights into the interaction between lattice theory and fuzzy logic in bipolar systems.

Acknowledgment: This research was supported by University of Phayao and Thailand Science Research and Innovation Fund (Fundamental Fund 2025, Grant No. 5027/2567).

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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