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Approximate Solutions of the Coupled M-Truncated Fractional mKdV System Using the Adomian Decomposition Technique

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ABSTRACT. This manuscript employs the Adomian decomposition technique (ADT) to develop solutions for the fractional space-time nonlinear mKdV system, incorporating an M-truncated fractional order and supposed initial conditions. The technique yields a power series expansion solution without the need for linearization, weak nonlinear assumptions, or perturbation theory. Software such as Maple or Mathematica was utilized to compute the Adomian formulas for the solution expansion. This technique can also be utilized for a range of nonlinear fractional-order models in mathematical physics. A graphical analysis is provided to demonstrate the behavior of Adomian solutions and how variations in non-integer order values influence the results. The technique is straightforward, clear, and widely applicable to other nonlinear fractional problems in both physics and mathematics. It is believed that these studies significantly advance our understanding of the nonlinear coupled fractional mKdV system and its potential applications in physics and engineering.

1. Introduction

It is widely recognized that nonlinear complex physical phenomena are frequently described by nonlinear evolution equations (NLEEs), which are relevant across various fields such as mathematical physics and engineering. Investigating exact solutions to these PDEs

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enhances our understanding of these phenomena [1]. For instance, they are used to model and understand complex behaviors in oceanography, optics, biology, chemistry, and physics [1–6]. Among the various explicit solutions, soliton and solitary wave solutions stand out due to their exponential localization in specific spatial and temporal directions, making them crucial for accurately representing wave behaviors in diverse applications [7–12]. For instance, solitons play a crucial role in understanding pulse propagation in optical fibers, while solitary waves are used to model waves in shallow water and plasma physics. Both types of solutions also enhance our understanding of reaction-diffusion processes in biological and chemical systems. Several techniques have been established to obtain explicit solutions for NLEEs. These methods include the Darboux transformation, inverse scattering transform [1], Bäcklund transformation, Hirota bilinear technique [2], Painlevé analysis, tanh function technique [10], sine-cosine procedure, and homogeneous balance method, among others [7-12].

The ADT is a robust method for obtaining both numerical and analytical solutions to integer and fractional differential models commonly encountered in natural phenomena modeling. Developed by George Adomian between the 1970s and 1990s, this method utilizes the Adomian polynomial [13]. This polynomial ensures the convergence of series solutions without requiring linearization or discretization of the nonlinear terms. It derives a Maclaurin series expansion around an arbitrary parameter, offering greater flexibility compared to the Taylor series method. Unlike traditional numerical methods, the ADT produces closed-form solutions based solely on initial conditions. The benefits of the ADM compared to the Taylor series technique, homotopy perturbation procedure, and Picard's technique are examined in [14–16].

Adesanya et al. [17] investigated the behavior of Bratu's model by means of the ADT. They discovered that the model has two solutions, both of which are convergent and exhibit desirable behavior. In [18], Adesanya applied the ADT to investigate the linear stability of hydromagnetic Plane-Poiseuille flow, with results that were in agreement with established Multideck asymptotic technique. Additionally, Ahmed et al. in [19] used the ADT to derive the explicit solution for the Biswas-Milowic model, a general form of the nonlinear Schrödinger model used in the fiber optics field. Aswhad et al. in [20] applied the ADT to solve Fisher's equations and compared their results. In 2015, Wazwaz et al. [21] discussed the use of the ADT to demonstrate that a class of nonlinear boundary value problems can have two distinct solutions.

Bougoffa et al. in [22] discussed the function of Green to alongside the ADT for solving a linear differential equation of fourth-order with varied boundaries. In 2017, Alshaery et al. [23] applied the ADT to solve the hyperbolic Kepler's equation featuring sine hyperbolic nonlinearity. The year after, Jaradat et al. [24] addressed issues related to the simple harmonic quantum oscillator. Between 2020 and 2021, researchers [25–27] explored advanced ADT techniques for singular boundary and initial value problems with both unequal and equal partition step sizes, showcasing the convergence of these methods.

A diverse array of fractional order models—be they nonlinear or linear, and whether involving partial or ordinary differential equations—can be effectively and accurately addressed using the ADM [28, 29]. This method yields approximations that quickly converge to precise solutions. It is especially adept at managing nonlinear physical models, as it sidesteps the potential inaccuracies introduced by unnecessary linearization. This study specifically investigates solutions derived from the ADM for fractional nonlinear models, including the coupled mKdV model with M-truncated fractional order

The M-truncated fractional differential equations are used in modeling complex systems where memory effects and non-local interactions are significant. They find applications in various fields examples include signal processing, control theory, and physics, where they help describe phenomena like diffusion processes and wave propagation with non-standard behaviors. Their ultimate advantage lies in their ability to capture intricate dynamics that traditional integer-order models might miss. Various methods have been utilized to define and develop fractional differentials, including the Caputo, Kolwankar-Gangal, Chen's fractal differentials, Riemann-Liouville (RL), conformable fractional differentials, modified RL, and Cresson's approaches [30-43]. Building on earlier mathematical discoveries, fractional calculus is emerging as the calculus of the twenty-first century. Recent advancements and applications in fractional calculus have made it a compelling and increasingly popular field of study. Fractional differential models offer accurate and precise representations of many real-world phenomena [30-43].

The research leave is structured as follows: Section 2 presents the M-truncated fractional basic equations. Section 3 provides an overview of the ADM. Section 4 discusses the application of this technique. Finally, we conclude with a summary and suggestions for future work.

2. M-truncated Fractional derivative

In twenty-first-century mathematics, fractional and non-integer calculus have emerged as significant extensions of traditional concepts, paving the way for new research and practical applications. Fractional calculus, in particular, has established to be a versatile tool with a broad range of applications across various fields. This area of study is especially valuable for modeling complex real-world phenomena [30- 43]. For example, fractional differential models are used to understand anomalous diffusion in porous materials, characterize viscoelastic behavior in material science, and describe complex biological processes such as cell growth and tumor dynamics. In finance, fractional calculus helps analyze long-term dependencies and volatility patterns in time series data.

The truncated Mittag-Leffler function (MLF) can be defined as [44]:

$$ {}_{l}\mathrm{E}_{\beta}(\varepsilon s^{\alpha}) = \sum_{j=0}^{l} \frac{(\varepsilon s^{\alpha})^{j}}{\Gamma(\beta j+1)}, \quad \beta > 0, \quad s \in \mathbb{C}. \tag{1} $$

Definition 1: Let $\psi:[0,\infty)\to\Re$ be a function, the local truncated *M*-fractional differential (MFD) of ψ with respect to *y* is given [44]:

$$ {}_{l}D_{M,\mathrm{t}}^{\alpha,\beta}\psi(s) = \lim_{\delta\to 0}\frac{\psi(s\,{}_{l}\mathrm{E}_{\beta}(\delta s^{-\alpha}))-\psi(s)}{\delta}, \quad \forall\,\beta, s>0, \quad \alpha\in(0,1). \tag{2} $$

The MFD adheres to the following axioms:

$$ {}_{l}D_{M}^{\alpha,\beta}t^{m} = \frac{m}{\Gamma(\beta+1)}t^{m-\alpha}, \ \ m\in R, \qquad {}_{l}D_{M}^{\alpha,\beta}c = 0, \quad \forall\,\psi(t)=c, \tag{3} $$

$$ {}_{l}D_{M}^{\alpha,\beta}(c_{1}\psi+c_{2}\varphi) = c_{1}\,{}_{l}D_{M}^{\alpha,\beta}\psi + c_{2}\,{}_{l}D_{M}^{\alpha,\beta}\varphi, \qquad \forall\, c_{1},c_{2}\in\Re, \tag{4} $$

$$ {}_{l}D_{M}^{\alpha,\beta}(\varphi\psi) = \varphi\,{}_{l}D_{M}^{\alpha,\beta}\psi + \psi\,{}_{l}D_{M}^{\alpha,\beta}\psi, \tag{5} $$

$$ {}_{l}D_{M}^{\alpha,\beta}\left(\frac{\varphi}{\psi}\right) = \frac{\psi\,{}_{l}D_{M}^{\alpha,\beta}\varphi - \varphi\,{}_{l}D_{M}^{\alpha,\beta}\psi}{\psi^{2}}, \tag{6} $$

$$ {}_{l}D_{M}^{\alpha,\beta}\,\varphi(\psi) = \frac{d\varphi}{d\psi}\,{}_{l}D_{M}^{\alpha,\beta}\psi, \qquad {}_{l}D_{M}^{\alpha,\beta}\,\varphi(y) = \frac{y^{1-\alpha}}{\Gamma(\beta+1)}\frac{d\varphi}{dy}, \tag{7} $$

With φ, ψ represents two $\alpha-$differentiable functions of a dependent variable, the above relations are proved in reference [44].

Choosing $\beta=1$ and $l=1$ on the two sides of Eq.(1), we have

$$ {}_1D_{M,t}^{\alpha,1}\psi(s)=\lim_{\delta\to 0}\frac{\psi(s\,{}_1\mathrm{E}_1(\delta s^{-\alpha}))-\psi(s)}{\delta}, \quad \forall s>0, \quad \alpha\in(0,1) $$

But, it is know that

$$ {}_1\mathrm{E}_1(\delta s^{-\alpha})=\sum_{r=0}^{1}\frac{(\delta s^{-\alpha})^r}{\Gamma(2)}=1+\delta s^{-\alpha}. $$

Thus, we conclude that

$$ {}_1D_{M,s}^{\alpha,1}\psi(s)=\lim_{\delta\to 0}\frac{\psi(s+\delta s^{1-\alpha})-\psi(s)}{\delta}=D_t^{\alpha}\psi(s), \quad \forall s>0, \quad \alpha\in(0,1] $$

which is exactly the conformable fractional derivative. Simply we write ${}_1D_M^{\alpha,\beta}$ as $D_M^{\alpha,\beta}$. The MFD of some functions [44]

$$ D_{M,s}^{\alpha,\beta}e^{cs}=\frac{c\,s^{1-\alpha}}{\Gamma(\beta+1)}e^{cs}, \qquad D_{M,s}^{\alpha,\beta}\sin(cx)=\frac{c\,s^{1-\alpha}}{\Gamma(\beta+1)}\cos(cs), $$

$$ D_{M,s}^{\alpha,\beta}\cos(cs)=-\frac{c\,s^{1-\alpha}}{\Gamma(\beta+1)}\sin(cs), \qquad D_{M,s}^{\alpha,\beta}e^{c\,x^{\alpha}}=\frac{c\,\alpha}{\Gamma(\beta+1)}e^{c\,s^{\alpha}}, $$

$$ D_{M,s}^{\alpha,\beta}\sin(cs^{\alpha})=\frac{c\,\alpha}{\Gamma(\beta+1)}\cos(cs^{\alpha}), \qquad D_{M,s}^{\alpha,\beta}\cos(cs^{\alpha})=-\frac{c\alpha}{\Gamma(\beta+1)}\sin(cs^{\alpha}). $$

The MFD can be used for non-differentiable functions, making it suitable for applications involving discontinuous media.

3. Clarification of the procedure

To address the fractional order coupled nonlinear mKdV model, we suppose that the space-time fractional partial differential system can be expressed in terms of an operator formula as follows:

$$ \begin{aligned} \Im_{\alpha_t}u+\Im_{\alpha_x}u+f(v,u)&=0,\\ \Im_{\alpha_t}v+\Im_{\alpha_x}v+g(v,u)&=0, \end{aligned} \tag{8} $$

with the operators of nonlinear terms are designated by the symbolizations $M(v,u)$, $N(v,u)$ and the linear *M*-truncated differential fractional operators by the representations

$\Im_{\alpha_t} = D_t^{\alpha,\beta}$, and $\Im_{\alpha_x} = D_x^{\alpha\alpha\alpha,\beta} = D_x^{\alpha,\beta} D_x^{\alpha,\beta} D_x^{\alpha,\beta}$. By means of equations (8) and take the inverse *M*-truncated differential fractional operator $\Im_{\alpha_t}^{-1} = \int_0^t (.)dt^{\alpha}$, , we have

$$\begin{aligned} u(x^{\alpha},t^{\alpha}) &= f_1(x^{\alpha}) - \Im_{\alpha_t}^{-1}\left[\Im_{\alpha_x} u + f(u,v)\right], \\ v(x^{\alpha},t^{\alpha}) &= g_1(x^{\alpha}) - \Im_{\alpha_t}^{-1}\left[\Im_{\alpha_x} v + g(u,v)\right], \end{aligned} \tag{9}$$

with $u(0,x^{\alpha}) = f_1(x^{\alpha})$, and $v(0,x^{\alpha}) = g_1(x^{\alpha})$, are functions determined by the given initial conditions. The ADTassumes that the unknown functions $u(t^{\alpha},x^{\alpha})$ and $v(t^{\alpha},x^{\alpha})$ can be expressed as an infinite series of the form

$$u(x^{\alpha},t^{\alpha}) = \sum_{s=0}^{\infty} u_s(x^{\alpha},t^{\alpha}), \qquad v(x^{\alpha},t^{\alpha}) = \sum_{s=0}^{\infty} v_s(x^{\alpha},t^{\alpha}), \tag{10}$$

Along with the nonlinear operators, the infinite sequence of Adomian polynomials used to express $f(v,u)$ and $g(v,u)$ is

$$f(u,v) = \sum_{s=0}^{\infty} M_s, \qquad g(u,v) = \sum_{s=0}^{\infty} N_s, \tag{11}$$

With the related Adomian polynomials, M_j and N_j, are derived using the technique described in [13]. For the reader's convenience, we usage the nonlinear term $f(v,u) = \sum_{j-0}^{\infty} M_j$ from the general formulation to derive the Adomian polynomials M_j. These are given by

$$f_j(u_0, ..., u_j, v_0, ..., v_j) = \frac{1}{j!} \frac{d^j}{d\lambda^j}\left[f\left(\sum_{s=0}^{j} \lambda u_s, \sum_{s=0}^{j} \lambda v_s\right)\right]_{\lambda=0}, \quad j \geq 1, \tag{12}$$

It is simple to use this formulation to tell the computer code to calculate as many polynomials as necessary for both the explicit and numerical solutions. We recommend the reader to [22,23] for a generic formula of Adomian polynomials and a full discussion of the Adomian decomposition technique.

The nonlinear formulae given by equation (8) is derived through a recursive relationship expression by means of the decomposition procedure

$$\begin{aligned} u_0(x^{\alpha},t^{\alpha}) &= f_1(x^{\alpha}), \qquad u_{j+1}(x^{\alpha},t^{\alpha}) = -\Im_{\alpha_t}^{-1}\left[\Im_{\alpha_x} u_j + f_j\right], \\ v_0(x^{\alpha},t^{\alpha}) &= g_1(x^{\alpha}), \qquad v_{j+1}(x^{\alpha},t^{\alpha}) = -\Im_{\alpha_t}^{-1}\left[\Im_{\alpha_x} v_j + g_j\right], \end{aligned} \tag{13}$$

when the initial conditions are in which functions $f_1(x^\alpha)$ and $g_1(x^\alpha)$ are derived. It is important to remember that the zero component, and being more prominent than the remainder components, $u_j(t^\alpha, x^\alpha)$ and $v_j(t^\alpha, x^\alpha), j \geq 1$ may be fully ascertained, meaning that every term can be computed by incorporating the previous terms. Consequently, the series solutions are completely established and the components $u_0, u_1, u_2, ...$ and $v_0, v_1, v_2, ...$ are well-known. Nevertheless, it is often possible to obtain the explicit solution in a closed form.

We developed the solutions $u(x^\alpha, t^\alpha)$ and $v(x^\alpha, t^\alpha)$ by means of numerical formulae in the form

$$u(x^\alpha, t^\alpha) = \lim_{j\to\infty}\sum_{k=0}^{j} u_k(x^\alpha, t^\alpha), \qquad v(x^\alpha, t^\alpha) = \lim_{j\to\infty}\sum_{k=0}^{j} v_k(x^\alpha, t^\alpha), \tag{14}$$

and equation (13) represents the recurrent relation. Additionally, in strictly physical contexts, the solutions of the decomposition series generally converge relatively quickly.

In the next section, we examine the space-time coupled fractional mKdV model to demonstrate the application of the ADT previously discussed.

3. Application of the specified technique

Consider the general space time coupled fractional mKdV model:

$$D_t^{\alpha,\beta} u + 3u^2 D_x^{\alpha,\beta} u - 3D_x^{\alpha,\beta}(uv) = \varphi(t^\alpha, x^\alpha), \tag{15}$$

$$D_t^{\alpha,\beta} v - 3u^2 D_x^{\alpha,\beta} v - 3v\, D_x^{\alpha,\beta} v + 3D_x^{\alpha,\beta} u\, D_x^{\alpha,\beta} v = \psi(t^\alpha, x^\alpha), \tag{16}$$

under the initial condition

$$f(x^\alpha) = u(0, x^\alpha), \tag{17}$$

$$g(x^\alpha) = v(0, x^\alpha), \tag{18}$$

with $\psi(t^\alpha, x^\alpha), \varphi(t^\alpha, x^\alpha), f(x^\alpha),$ and $g(x^\alpha)$ are the given functions.

We rewrite Equations (15) and (16) in an operator formulae to solve them using the ADT

$$\Im_{\alpha_t} u = \varphi(t^\alpha, x^\alpha) + 3M(u, v) - 3F(u), \tag{20}$$

$$\Im_{\alpha_t} v = \psi(t^\alpha, x^\alpha) - 3[H(v, u) + N(v, u) + G(v)], \tag{21}$$

with the fractional order linear differential operator is represented by $\Im_{\alpha_t} = D_t^{\alpha,\beta}$, and the fractional inverse operator $\Im_{\alpha_t}^{-1}$ is provided by

$$\Im_{\alpha_t}^{-1} = \int_0^t (.) dt^{\alpha}, \tag{22}$$

Operating with $\Im_{\alpha_t}^{-1}$ t on the two sides of Eqs. (20) and (21), gives

$$u(x^{\alpha},t^{\alpha}) = u(0,x^{\alpha}) + \Im_{\alpha_t}^{-1}\left(\varphi(x^{\alpha},t^{\alpha}) + 3[M(v,u) - F(u)]\right), \tag{23}$$

$$v(x^{\alpha},t^{\alpha}) = v(0,x^{\alpha}) + \Im_{\alpha_t}^{-1}\left(\psi(x^{\alpha},t^{\alpha}) - 3[H(v,u) + N(v,u) + G(v)]\right), \tag{24}$$

The ADT assumes that an infinite series can represent the two unknown functions, $u(t^{\alpha},x^{\alpha})$ and $v(t^{\alpha},x^{\alpha})$ as

$$u(x^{\alpha},t^{\alpha}) = \sum_{s=0}^{\infty} u_s(x^{\alpha},t^{\alpha}), \tag{25}$$

$$v(x^{\alpha},t^{\alpha}) = \sum_{s=0}^{\infty} v_s(x^{\alpha},t^{\alpha}), \tag{26}$$

Equations (23) and (24) can be replaced with Eqs. (25) and (26) to obtain

$$u_{s+1}(x^{\alpha},t^{\alpha}) = u(0,x^{\alpha}) + \Im_{\alpha_t}^{-1}\left(\varphi(x^{\alpha},t^{\alpha}) + 3[M(v,u) - F(u)]\right), \tag{27}$$

$$v_{s+1}(x^{\alpha},t^{\alpha}) = v(0,x^{\alpha}) + \Im_{\alpha_t}^{-1}\left(\psi(x^{\alpha},t^{\alpha}) - 3[H(v,u) + N(v,u) + G(v)]\right), \tag{28}$$

where the functions $F(u,v) = u^2 D_x^{\alpha,\beta} u$, $G(u,v) = v\, D_x^{\alpha,\beta} v$, $M(u,v) = D_x^{\alpha,\beta}(uv)$, $H(u,v) = D_x^{\alpha,\beta} u\, D_x^{\alpha,\beta} v$, and $N(u,v) = u^2\, D_x^{\alpha,\beta} v$, are accompanying through the nonlinear term and are expressed in terms of the Adomian polynomials as follows: $F(u,v) = \sum_{s=0}^{\infty} F_s$, $G(u,v) = \sum_{s=0}^{\infty} G_s$, $M(u,v) = \sum_{s=0}^{\infty} M_s$, $H(u,v) = \sum_{s=0}^{\infty} H_s$, and $N(u,v) = \sum_{s=0}^{\infty} N_s$, where it is possible to compute the components F_s G_s, M_s, H_s, and N_s using the formulae

$$F(v,u) = \sum_{s=0}^{\infty} F_s = u^2 D_x^{\alpha,\beta} u = (u_0 + u_1\lambda + u_2\lambda^2 + ...)^2 D_x^{\alpha,\beta}(u_0 + u_1\lambda + u_2\lambda^2 + ...), \tag{29}$$

Since

$$F_s = \frac{1}{s!}\frac{d^s}{d\lambda^s}\left[(u_0 + u_1\lambda + u_2\lambda^2 + ...)^2 D_x^{\alpha,\beta}(u_0 + u_1\lambda + u_2\lambda^2 + ...)\right]_{\lambda=0}, \tag{30}$$

The initial four terms can be expressed as

$$
\begin{aligned}
F_0 &= u_0{}^2 D_x^{\alpha,\beta} u_0, \\
F_1 &= u_0{}^2 D_x^{\alpha,\beta} u_1 + 2u_0 u_1 D_x^{\alpha,\beta} u_0, \\
F_2 &= (u_1{}^2 + 2u_0 u_2) D_x^{\alpha,\beta} u_0 + u_0{}^2 D_x^{\alpha,\beta} u_2 + 2u_0 u_1 D_x^{\alpha,\beta} u_1, \\
F_3 &= (u_1{}^2 + 2u_0 u_2) D_x^{\alpha,\beta} u_1 + u_0{}^2 D_x^{\alpha,\beta} u_3 + 2u_0 u_1 D_x^{\alpha,\beta} u_2 + (2u_0 u_3 + 2u_1 u_2) D_x^{\alpha,\beta} u_1,
\end{aligned} \tag{31}
$$

$$
G(u,v) = \sum_{j=0}^{\infty} G_j = v D_x^{\alpha,\beta} v = (v_0 + v_1\lambda + v_2\lambda^2 + \ldots.) D_x^{\alpha,\beta} (v_0 + v_1\lambda + v_2\lambda^2 + \ldots.), \tag{32}
$$

Since

$$
G_j = \frac{1}{j!} \frac{d^j}{d\lambda^j} \left[(v_0 + v_1\lambda + v_2\lambda^2 + \ldots.) D_x^{\alpha,\beta} (v_0 + v_1\lambda + v_2\lambda^2 + \ldots.) \right]_{\lambda=0}, \tag{33}
$$

The initial four terms can be expressed as

$$
\begin{aligned}
G_0 &= 2v_0 D_x^{\alpha,\beta} v_0, \\
G_1 &= 2v_0 D_x^{\alpha,\beta} v_1 + 2v_1 D_x^{\alpha,\beta} v_0, \\
G_2 &= 2v_0 D_x^{\alpha,\beta} v_2 + v_1 D_x^{\alpha,\beta} v_1 + 2v_2 D_x^{\alpha,\beta} v_0, \\
G_3 &= 2v_0 D_x^{\alpha,\beta} v_3 + 2v_1 D_x^{\alpha,\beta} v_2 + 2v_2 D_x^{\alpha,\beta} v_1 + 2v_3 D_x^{\alpha,\beta} v_0,
\end{aligned} \tag{34}
$$

$$
M(u,v) = \sum_{s=0}^{\infty} M_s = D_x^{\alpha,\beta}(v\,u) = D_x^{\alpha,\beta} \left[(v_0 + v_1\lambda + v_2\lambda^2 + \ldots.)(u_0 + u_1\lambda + u_2\lambda^2 + \ldots.) \right], \tag{35}
$$

Since

$$
M_s = \frac{1}{s!} \frac{d^s}{d\lambda^s} \left[D_x^{\alpha,\beta} \left[(u_0 + u_1\lambda + u_2\lambda^2 + \ldots.)(v_0 + v_1\lambda + v_2\lambda^2 + \ldots.) \right] \right]_{\lambda=0}, \tag{36}
$$

The initial four terms can be expressed as

$$
\begin{aligned}
M_0 &= u_0 D_x^{\alpha,\beta} v_0 + v_0 D_x^{\alpha,\beta} u_0, \\
M_1 &= u_0 D_x^{\alpha,\beta} v_1 + v_1 D_x^{\alpha,\beta} u_0 + v_0 D_x^{\alpha,\beta} u_1 + u_1 D_x^{\alpha,\beta} v_0, \\
M_2 &= u_0 D_x^{\alpha,\beta} v_2 + v_2 D_x^{\alpha,\beta} u_0 + u_1 D_x^{\alpha,\beta} v_1 + v_1 D_x^{\alpha,\beta} u_1 + u_2 D_x^{\alpha,\beta} v_0 + v_0 D_x^{\alpha,\beta} u_2, \\
M_3 &= u_0 D_x^{\alpha,\beta} v_3 + v_3 D_x^{\alpha,\beta} u_0 + u_1 D_x^{\alpha,\beta} v_2 + v_2 D_x^{\alpha,\beta} u_1 + u_2 D_x^{\alpha,\beta} v_1 + v_1 D_x^{\alpha,\beta} u_2 \\
&\quad + u_3 D_x^{\alpha,\beta} v_0 + v_0 D_x^{\alpha,\beta} u_3,
\end{aligned} \tag{37}
$$

$$
\begin{aligned}
H(v,u) &= \sum_{s=0}^{\infty} H_s = D_x^{\alpha,\beta} v D_x^{\alpha,\beta} u \\
&= D_x^{\alpha,\beta} (v_0 + v_1\lambda + v_2\lambda^2 + \ldots.) D_x^{\alpha,\beta} (u_0 + u_1\lambda + u_2\lambda^2 + \ldots.),
\end{aligned} \tag{38}
$$

Since

$$
H_s = \frac{1}{s!} \frac{d^s}{d\lambda^s} \left[D_x^{\alpha,\beta} (u_0 + u_1\lambda + u_2\lambda^2 + \ldots.) D_x^{\alpha,\beta} (v_0 + v_1\lambda + v_2\lambda^2 + \ldots.) \right]_{\lambda=0}, \tag{39}
$$

The initial four terms can be expressed as

$$\begin{aligned} H_0 &= D_x^{\alpha,\beta}u_0 D_x^{\alpha,\beta}v_0, \\ H_1 &= D_x^{\alpha,\beta}u_0 D_x^{\alpha,\beta}v_1 + D_x^{\alpha,\beta}u_1 D_x^{\alpha,\beta}v_0, \\ H_2 &= D_x^{\alpha,\beta}u_0 D_x^{\alpha,\beta}v_2 + D_x^{\alpha,\beta}u_1 D_x^{\alpha,\beta}v_1 + D_x^{\alpha,\beta}u_2 D_x^{\alpha,\beta}v_0, \\ H_3 &= D_x^{\alpha,\beta}u_0 D_x^{\alpha,\beta}v_3 + D_x^{\alpha,\beta}u_1 D_x^{\alpha,\beta}v_2 + D_x^{\alpha,\beta}u_2 D_x^{\alpha,\beta}v_1 + D_x^{\alpha,\beta}u_3 D_x^{\alpha,\beta}v_0, \end{aligned} \tag{40}$$

$$N(v,u) = \sum_{s=0}^{\infty} N_s = u^2 D_x^{\alpha,\beta}v = (u_0 + u_1\lambda + u_2\lambda^2 + ...)^2 D_x^{\alpha,\beta}(v_0 + v_1\lambda + v_2\lambda^2 + ...), \tag{41}$$

Since

$$N_s = \frac{1}{s!}\frac{d^s}{d\lambda^s}\left[(u_0 + u_1\lambda + u_2\lambda^2 + ...)^2 D_x^{\alpha,\beta}(v_0 + v_1\lambda + v_2\lambda^2 + ...)\right]_{\lambda=0}, \tag{42}$$

The initial four terms can be expressed as

$$\begin{aligned} N_0 &= u_0{}^2 D_x^{\alpha,\beta}v_0, \\ N_1 &= u_0{}^2 D_x^{\alpha,\beta}v_1 + 2u_0u_1 D_x^{\alpha,\beta}v_0, \\ N_2 &= (u_1{}^2 + 2u_0u_2) D_x^{\alpha,\beta}v_0 + u_0{}^2 D_x^{\alpha,\beta}v_2 + 2u_0u_1 D_x^{\alpha,\beta}v_1, \\ N_3 &= (u_1{}^2 + 2u_0u_2) D_x^{\alpha,\beta}v_1 + u_0{}^2 D_x^{\alpha,\beta}v_3 + 2u_0u_1 D_x^{\alpha,\beta}v_2 + (2u_0u_3 + 2u_1u_2) D_x^{\alpha,\beta}v_1, \end{aligned} \tag{43}$$

For a given

$$\varphi(t^\alpha, x^\alpha) = \frac{1}{2}D_x^{\alpha\alpha\alpha,\beta}u + \frac{3}{2}D_x^{\alpha\alpha,\beta}v - 3aD_x^{\alpha,\beta}u, \tag{44}$$

$$\psi(t^\alpha, x^\alpha) = -D_x^{\alpha\alpha\alpha,\beta}v + 3aD_x^{\alpha,\beta}v, \tag{45}$$

$$u(0, x^\alpha) = f(x^\alpha) = \frac{b_1}{2k} + k\tanh\left(\frac{\Gamma(\beta+1)kx^\alpha}{\alpha}\right), \tag{46}$$

$$\mathrm{v}(0, x^\alpha) = g(x^\alpha) = \frac{a}{2}\left(1 + \frac{k}{b_1}\right) + b_1\tanh\left(\frac{\Gamma(\beta+1)kx^\alpha}{\alpha}\right). \tag{47}$$

The residue components $u_s(x^\alpha, t^\alpha)$ and $v_s(x^\alpha, t^\alpha)$, $s > 0$ can be calculated using the recursive relations with constant values of $a = k = b_1 = 1$ in the subsequent way, considering the theoretical aspects of Eqs. (28) and (29)

$$u_0 = \frac{1}{2} + \tanh\left(\frac{\Gamma(\beta+1)x^\alpha}{\alpha}\right), \tag{48}$$

$$v_0 = 1 + \tanh\left(\frac{\Gamma(\beta+1)x^\alpha}{\alpha}\right), \tag{49}$$

$$u_1=\left[-1+\tanh^2\left(\frac{\Gamma(\beta+1)x^{\alpha}}{\alpha}\right)\right]\frac{\Gamma(\beta+1)t^{\alpha}}{4\alpha}, \tag{50}$$

$$v_1=\left[-1+\tanh^2\left(\frac{\Gamma(\beta+1)x^{\alpha}}{\alpha}\right)\right]\frac{\Gamma(\beta+1)t^{\alpha}}{4\alpha}, \tag{51}$$

$$u_2=\left[-\tanh\left(\frac{\Gamma(\beta+1)x^{\alpha}}{\alpha}\right)+\tanh^3\left(\frac{\Gamma(\beta+1)x^{\alpha}}{\alpha}\right)\right]\frac{\Gamma(\beta+1)^2t^{2\alpha}}{16\alpha^2}, \tag{52}$$

$$v_2=\left[-\tanh\left(\frac{\Gamma(\beta+1)x^{\alpha}}{\alpha}\right)+\tanh^3\left(\frac{\Gamma(\beta+1)x^{\alpha}}{\alpha}\right)\right]\frac{\Gamma(\beta+1)^2t^{2\alpha}}{16\alpha^2}, \tag{53}$$

and so on. This allows for the complete determination of the remainder components $u_s(x^{\alpha},t^{\alpha})$ and $v_s(x^{\alpha},t^{\alpha})$, $s>0$, with each term being determined based on the preceding one.

The ADT solutions of $u(t^{\alpha},x^{\alpha})$ and $v(t^{\alpha},x^{\alpha})$ are obtained in power series form by substituting the expressions $v_0,v_1,v_2,\ldots.$ and $u_0,u_1,u_2,\ldots.$ into the summation $\sum_{s=0}^{\infty}u_s(x^{\alpha},t^{\alpha})$ and $\sum_{s=0}^{\infty}v_s(x^{\alpha},t^{\alpha})$, we have

$$\begin{aligned}u(x^{\alpha},t^{\alpha})=\frac{1}{2}+\tanh\left(\frac{x^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\frac{t^{\alpha}\Gamma(\beta+1)}{4\alpha}\left[-1+\tanh^2\left(\frac{x^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right]\\+\left(\frac{t^{\alpha}\Gamma(\beta+1)}{4\alpha}\right)^2\left[-\tanh\left(\frac{x^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\tanh^3\left(\frac{x^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right]+\ldots,\end{aligned} \tag{54}$$

$$\begin{aligned}v(x^{\alpha},t^{\alpha})=1+\tanh\left(\frac{x^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\frac{t^{\alpha}\Gamma(\beta+1)}{4\alpha}\left[-1+\tanh^2\left(\frac{x^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right]\\+\left(\frac{t^{\alpha}\Gamma(\beta+1)}{4\alpha}\right)^2\left[-\tanh\left(\frac{x^{\alpha}\Gamma(\beta+1)}{\alpha}\right)+\tanh^3\left(\frac{x^{\alpha}\Gamma(\beta+1)}{\alpha}\right)\right]+\ldots,\end{aligned} \tag{55}$$

This is compactly expressed as

$$u(t^{\alpha},x^{\alpha})=\frac{1}{2}+\tanh\left(\delta+\frac{\Gamma(\beta+1)}{\alpha}\left(t^{\alpha}+x^{\alpha}\right)\right), \tag{56}$$

$$v(t^{\alpha},x^{\alpha})=1+\tanh\left(\delta+\frac{\Gamma(\beta+1)}{\alpha}\left(t^{\alpha}+x^{\alpha}\right)\right), \tag{57}$$

where δ is an arbitrary constant known as the phase shift. That is represented solitary wave solution. A solitary wave is a type of wave that maintains its shape while traveling at a constant speed. Unlike typical wave solutions which may disperse or change shape over time, solitary waves are localized, meaning they are confined to a specific region in space and do not spread out. A solitary wave retains its shape as it moves, which is a defining feature. It does not change in width or amplitude as it travels through a medium. Solitary waves often occur in nonlinear systems where the medium's response to disturbances is nonlinear. This nonlinearity allows the wave to sustain itself without dispersing. Solitary waves can be observed in shallow water where they manifest as waves that travel without changing shape. In plasmas, solitary waves are often referred to as "solitons" and can describe certain types of localized disturbances in the plasma. A special class of solitary waves, known as solitons, are solutions to integrable nonlinear equations. Solitons not only maintain their shape and speed but can also interact with other solitons and emerge from collisions unchanged. solitary waves are crucial for understanding many physical phenomena where wave-like disturbances remain stable and localized over time, providing valuable insights into nonlinear wave dynamics.

A Graphical analysis for the ADM evolutionary behavior of the solution of u described by equation (56), is conducted by varying fractional order values α and the parameter β with suitable choose of the phase shift δ. Figure (1) illustrates the kink-type wave solution of the function u for $\alpha = 1, 0.98, 0.96, 0.92, 0.9$, $\beta = 1$ and $\delta = -20$. Figure (1-b) is the 2-dimensional space representation of u when t=15. Figure (2) illustrates the kink-type wave solution of the function u for $\alpha = 1$, $\beta = 2, 1.5, 1, 0.9, 0.7, 0.5$ and $\delta = -20$. Figure (2-b) is the 2-dimensional space representation of u when t=5. Figure (3) illustrates the kink-type wave solution of the function u for $\alpha = \beta = 1$, $0.9, 0.8, 0.7, 0.6, 0.5$ and $\delta = -20$. Figure (3-b) is the 2-dimensional space representation of u when t=5.

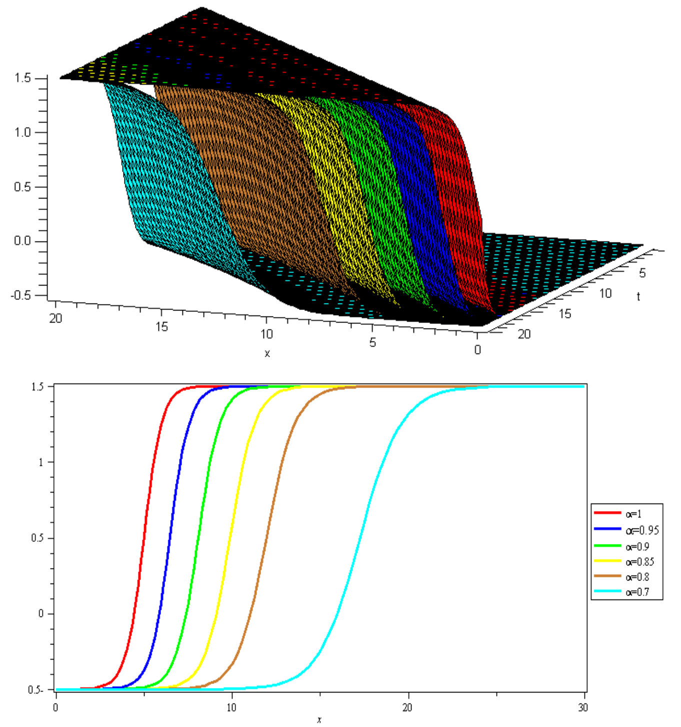

Fig (1) the evolution behavior of the function u when $\alpha = 1, 0.95, 0.9, 0.85, 0.8, 0.7$ and $\beta = 1$ with $\delta = -20$. b) is the cross section when t=15.

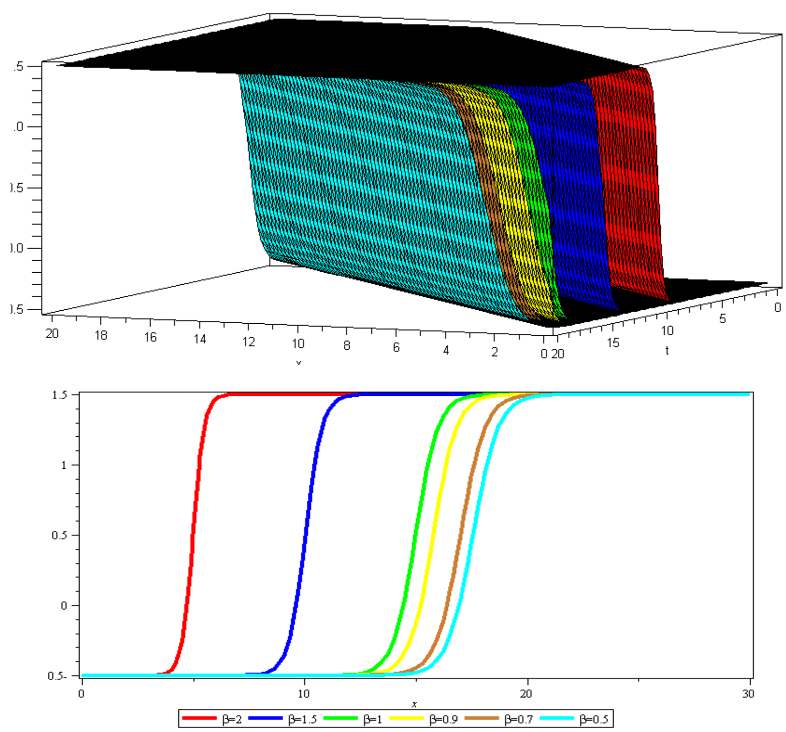

Fig (2) the evolution behavior of the function u when $\beta = 2, 1.5, 1, 0.9, 0.7, 0.5$ and $\alpha = 1$ with $\delta = -20$. b) is the cross section when t=5.

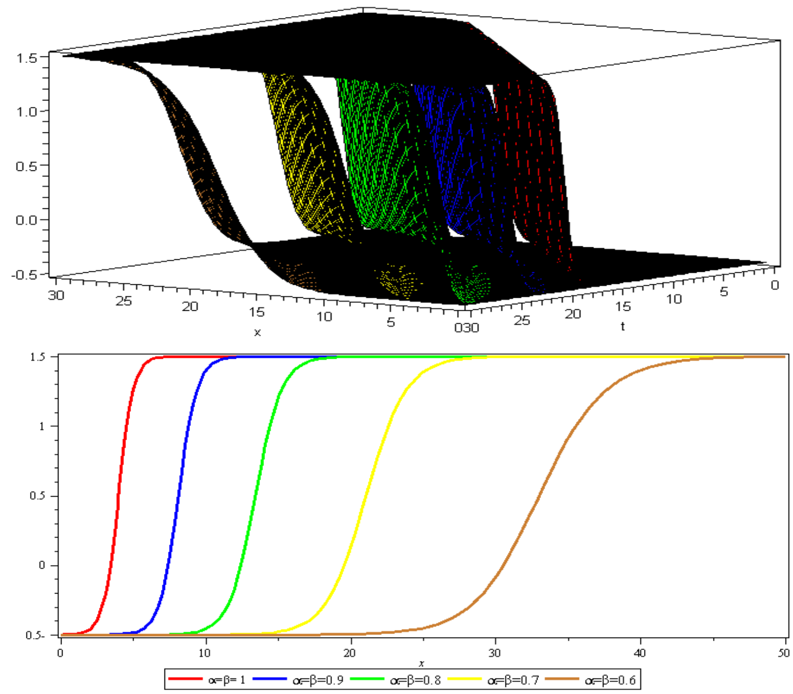

Fig (3) the evolution behavior of the function u when $\alpha = \beta = 1, 0.9,\ 0.8, 0.7, 0.6$ with $\delta = -20$. b) is the cross section when t=15.

From the figures we can say that, the kink type wave solution arises in nonlinear systems where the wave equation supports such localized structures. It is a localized, smooth transition between two distinct states or values, often observed in various physical and mathematical contexts. The nonlinearity of the system allows for the formation of these distinctive wave profiles. It is a type of solitary wave solution often found in nonlinear integer or fractional partial differential equations, particularly in models describing phenomena like fluid dynamics, plasma physics, or quantum mechanics. The kink wave has a distinctive profile where it smoothly transitions between two different levels. This transition typically looks like a sharp bend or "kink" in the wave, hence the name. The wave profile is often similar to a step function but with a smooth, continuous transition. It smoothly transitions from one constant value to another, often resembling a steep, localized change in the wave amplitude. Kink waves occur in nonlinear systems where the wave equation allows for such localized and stable structures. The nonlinearity enables the wave to maintain its shape and transition characteristics over time. So the kink-type wave solutions are important in various physical and mathematical contexts, providing insight into the behavior of nonlinear systems and the formation of localized wave structures. In particle physics and field theory, kink solutions represent domain walls or topological defects, which can describe transitions between different vacuum states. Kink waves can model domain walls or magnetic defects in ferromagnetic materials. Kink-type waves can also describe certain types of shock waves or other localized disturbances in fluids. Kink waves often model transitions or boundaries between different phases or states in a system. For example, in a ferromagnetic material, a kink wave might represent the boundary between regions of different magnetization. Kink waves are significant in both theoretical and applied contexts for their ability to represent sharp, localized transitions in a variety of nonlinear systems.

Remark that all the results obtained in [29] are recovered when $\beta = 1$. In addition, all the results given in [45] are recovered when $\alpha = \beta = 1$.

4. Conclusions and discussion

It has been found that non-classical calculus techniques, like fractional calculus and non-integer order calculus, are useful in explaining key physical phenomena, partly due to the rapid progress in advanced applied sciences. This work explores the Adomian decomposition approach as a potential analytical tool for examining these systems. The symbolic computation of this method in non-integer calculus, including Maple packages and additional numerical methods derived from Adomian decomposition, will be considered. The authors suggest that the Adomian decomposition approach might eventually be as significant as classical calculus. Future work will involve developing a Matlab or Maple software application to solve fractional differential models using this technique. It is believed that these studies significantly advance our understanding of the nonlinear coupled fractional mKdV system and its potential applications in physics and engineering.

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