Int. J. Anal. Appl. (2024), 22:192

International Journal of Analysis and Applications

Exploring Double Composed Partial Metric Spaces: A Novel Approach to Fixed Point Theorems

Fatima M. Azmi*, **Irshad Ayoob**, **Nabil Mlaiki**

Department of Mathematics and Sciences, Prince Sultan University, P.O. Box 66833, Riyadh 11586, Saudi Arabia

**Corresponding author:* fazmi@psu.edu.sa

Abstract. This paper innovatively extends partial metric spaces to introduce double composed partial metric space (DCPMS). Unlike traditional metrics, DCPMS replaces the triangle inequality with a nuanced form, integrating control functions into the metric. Building upon Ayoob et al.'s work, this novel generalization focuses on establishing fixed point theorems for DCPMS, contributing to the evolving landscape of mathematical analysis in this unique domain.

1. Introduction

In recent decades, the landscape of fixed point theory has experienced a profound evolution, marked by a surge in generalizations extending beyond the traditional metric space framework. This transformative trajectory is characterized by two primary avenues of exploration: the adaptation of underlying spaces and the refinement of contraction conditions. The investigation into diverse spaces has given rise to the emergence of generalized metric spaces, where a notable departure from convention involves the introduction of a constant in the right-hand side of the triangle inequality. Pioneering this novel concept, Czerwik [6] and Bakhtin [7] introduced the captivating notion of b-metric spaces, ushering in a new realm of mathematical analysis with a topology that significantly diverges from familiar metric spaces.

In 2017, Kamran et al. [8] extended the concept of b-metric spaces, giving rise to what is now recognized as extended b-metric spaces, accompanied by corresponding fixed point theorems. Building upon this extension, Mlaiki et al. [9] took a further step in 2018 by advancing extended b-metric spaces into the domain of controlled metric spaces. This extension involved the introduction of a binary control function on the right side of the triangle inequality, paving the way for the

Received: Sep. 7, 2024.

2020 *Mathematics Subject Classification.* 47H10, 54H25, 54E50.

Key words and phrases. partial metric spaces, partial b-metric spaces; controlled metric spaces; double composed metric spaces; fixed point; double composed partial metric spaces.

https://doi.org/10.28924/2291-8639-22-2024-192
ISSN: 2291-8639

establishment of new fixed point results. Subsequently, in 2019, Lattef [10] contributed to the field by establishing a Kannan-type fixed point result specifically tailored for controlled metric spaces. In 2020, Ahmad et al. [11] further enriched this evolving landscape by establishing a fixed point result for Reich-type contractions in controlled metric spaces.

Venturing beyond controlled metric spaces, Abdeljawad et al. [12] introduced the concept of double controlled metric type spaces in 2021. This novel extension involves the incorporation of two binary control functions on the right side of the triangle inequality. The authors not only introduced this concept but also established corresponding Banach-type and Kannan-type fixed point results within this framework. Numerous researchers have since delved into the realm of double controlled metric type spaces, employing various contraction mappings to establish fixed point results. For instance, Azmi [15] explored the space using two contraction mappings, namely 'Ciri'c-Reich-Rus-type and Θ-contraction, unveiling compelling results on the existence and uniqueness of fixed points. More recently, Ayoob et al. [4] introduced a groundbreaking generalization of metric spaces named the double composed metric space (DCMS). The topology of this newly introduced space diverges from that of double controlled metric type spaces, offering fresh perspectives in mathematical analysis.

Matthews [16] introduced the concept of partial metric spaces, presenting a departure from conventional metric spaces. In this framework, the typical metric is replaced by a partial metric, characterized by the intriguing property that the self-distance of any point within the space may not necessarily be zero. Building upon this novel foundation, Matthews demonstrated the applicability of the Banach contraction principle in partial metric spaces. Controlled partial metric type spaces were introduced by Souayah et al. in 2019, as documented in their work [5]. This groundbreaking insight not only broadens the mathematical landscape but also finds practical utility, as evidenced by its application in program verification.

This paper contributes to this evolving landscape by introducing an innovative extension of partial metric spaces, termed the double composed partial metric space (DCPMS). Differing from traditional metric spaces, the triangle inequality in DCPMS is replaced by $\mathcal{D}(\eta,\tau) \leq \alpha\left(\mathcal{D}(\eta,\theta)\right) + \beta\left(\mathcal{D}(\theta,\tau)\right)$ for all $\eta,\theta,\tau \in X$, where the control functions $\alpha,\beta : [0,\infty) \to [0,\infty)$ are intricately woven into the metric $\mathcal{D}$. This novel generalization builds upon the work presented in [4], extending the scope to partial metric spaces. The primary focus of this paper is to establish a fixed point theorem specifically tailored for double composed partial metric spaces, thereby contributing to the evolving landscape of mathematical analysis in this distinct domain.

2. Preliminaries

We start by defining the extended $b-$metric spaces as initially introduced by Kamran et al. in their work [8].

Definition 2.1. *Let $\mho$ be a non empty set and $\xi : \mho \times \mho \to [1,\infty)$. A function $\lrcorner : \mho \times \mho \to [0,\infty)$ is called an extended $b-$metric type if it satisfies:*

(1) $\daleth(\dot{\varrho},\dot{y}) = 0$ *if and only if* $\dot{\varrho} = \dot{y}$ *for all* $\dot{\varrho},\dot{y} \in \mho$;
(2) $\daleth(\dot{\varrho},\dot{y}) = \daleth(\dot{y},\dot{\varrho})$ *for all* $\dot{\varrho},\dot{y} \in \mho$;
(3) $\daleth(\dot{\varrho},\hat{l}) \leq \xi(\dot{\varrho},\dot{y})[\daleth(\dot{\varrho},\dot{y}) + \daleth(\dot{y},\hat{l})]$ *for all* $\dot{\varrho},\dot{y},\hat{l} \in \mho$.

The pair $(\mho,\daleth)$ *is called extended* $b-$*metric space.*

Mlaiki et al. introduced a novel generalization of extended $b-$metric spaces, termed controlled metric type spaces, as outlined in their work [9].

Definition 2.2. *Let* $\mho$ *be a non empty set and* $\xi : \mho\times\mho \to [1,\infty)$. *A function* $\daleth : \mho\times\mho \to [0,\infty)$ *is called a controlled metric type if it satisfies:*

(1) $\daleth(\dot{\varrho},\dot{y}) = 0$ *if and only if* $\dot{\varrho} = \dot{y}$ *for all* $\dot{\varrho},\dot{y} \in \mho$;
(2) $\daleth(\dot{\varrho},\dot{y}) = \daleth(\dot{y},\dot{\varrho})$ *for all* $\dot{\varrho},\dot{y} \in \mho$;
(3) $\daleth(\dot{\varrho},\hat{l}) \leq \xi(\dot{\varrho},\dot{y})\daleth(\dot{\varrho},\dot{y}) + \xi(\dot{y},\hat{l})\daleth(\dot{y},\hat{l})$ *for all* $\dot{\varrho},\dot{y},\hat{l} \in \mho$.

The pair $(\mho,\daleth)$ *is called controlled metric type space.*

In 2019, Souayah et al. introduced controlled partial metric type spaces, defining them as outlined below in their work [5].

Definition 2.3. *Let* $\mho$ *be a non empty set and* $\xi : \mho\times\mho \to [1,\infty)$. *The function* $\daleth : \mho\times\mho \to [0,\infty)$ *is called a controlled partial metric type if it satisfies:*

(1) $\dot{\varrho} = \dot{y}$ *if and only if* $\daleth(\dot{\varrho},\dot{y}) = \daleth(\dot{\varrho},\dot{\varrho}) = \daleth(\dot{y},\dot{y})$, *for all* $\dot{\varrho},\dot{y} \in \mho$;
(2) $\daleth(\dot{\varrho},\dot{y}) = \daleth(\dot{y},\dot{\varrho})$ *for all* $\dot{\varrho},\dot{y} \in \mho$;
(3) $\daleth(\dot{\varrho},\dot{\varrho}) \leq \daleth(\dot{\varrho},\dot{y})$
(4) $\daleth(\dot{\varrho},\hat{l}) \leq \xi(\dot{\varrho},\dot{y})\daleth(\dot{\varrho},\dot{y}) + \xi(\dot{y},\hat{l})\daleth(\dot{y},\hat{l})$ *for all* $\dot{\varrho},\dot{y},\hat{l} \in \mho$.

The pair $(\mho,\daleth)$ *is called controlled partial metric type space.*

A recent development by Ayoob et al. [4] introduces a novel generalization of a metric space, termed the double composed metric space (DCMS).

Definition 2.4. *Let* $\mho$ *be a non empty set and* $\pi,\varkappa : [0,\infty) \to [0,\infty)$ *be two non-constant functions. A function* $\daleth : \mho\times\mho \to [0,\infty)$ *is called a double composed metric if it satisfies:*

(1) $\daleth(\dot{\varrho},\dot{y}) = 0$ *if and only if* $\dot{\varrho} = \dot{y}$ *for all* $\dot{\varrho},\dot{y} \in \mho$
(2) $\daleth(\dot{\varrho},\dot{y}) = \daleth(\dot{y},\dot{\varrho})$ *for all* $\dot{\varrho},\dot{y} \in \mho$
(3) $\daleth(\dot{\varrho},\hat{l}) \leq \pi\Big(\daleth(\dot{\varrho},\dot{y})\Big) + \varkappa\Big(\daleth(\dot{y},\hat{l})\Big)$ *for all* $\dot{\varrho},\dot{y},\hat{l} \in \mho$.

The pair $(\mho,\daleth)$ *is a called double composed metric space.*

Next, we present a new concept: the double composed partial metric space.

Definition 2.5. *Let* $\mho$ *be a non empty set and* $\pi,\varkappa : [0,\infty) \to [0,\infty)$ *be two non-constant functions. A function* $\daleth : \mho\times\mho \to [0,\infty)$ *is called a double composed partial metric space if it satisfies the following:*

(1) $\dot{\varrho} = \dot{y}$ *if and only if* $\daleth(\dot{\varrho},\dot{y}) = \daleth(\dot{\varrho},\dot{\varrho}) = \daleth(\dot{y},\dot{y})$, *for all* $\dot{\varrho},\dot{y} \in \mho$;
(2) $\daleth(\dot{\varrho},\dot{y}) = \daleth(\dot{y},\dot{\varrho})$ *for all* $\dot{\varrho},\dot{y} \in \mho$

(3) $\daleth(\dot{\varrho},\dot{\varrho}) \leq \daleth(\dot{\varrho},\dot{y})$

(4) $\daleth(\dot{\varrho},\hat{l}) \leq \pi\Big(\daleth(\dot{\varrho},\dot{y})\Big) + \varkappa\Big(\daleth(\dot{y},\hat{l})\Big)$ *for all* $\dot{\varrho},\dot{y},\hat{l} \in \mho$.

The pair $(\mho,\daleth)$ *is a called double composed partial metric space (DCPMS).*

Remark 2.1. *It is important to note that the double composed partial metric space encompasses a broader scope compared to the double composed metric space. This is because a double composed partial metric space may not necessarily qualify as a double composed metric space. The following example serves to illustrate this distinction.*

Remark 2.2. *Every partial metric space is a DCPMS, by taking* $\pi(\dot{\varrho}) = \varkappa(\dot{\varrho}) = \dot{\varrho}$ *but the converse need not be true as shown by the following example.*

Example 2.1. *Let* $\mho = \mathbb{N} = \{1,2,,...\}$. *Define* $\daleth : \mho\times\mho \to [0,\infty)$ *by*

$$\daleth(\dot{\varrho},\dot{\varrho}) = \begin{cases} \frac{1}{\dot{\varrho}^2} & \text{if } \dot{\varrho} \text{ is even,} \\ 0 & \text{if } \dot{\varrho} \text{ is odd.} \end{cases}$$

and if $\dot{\varrho} \neq \dot{y}$, *then*

$$\daleth(\dot{\varrho},\dot{y}) = \begin{cases} \frac{1}{\dot{\varrho}}, & \text{if } \dot{\varrho} \text{ is even and } \dot{y} \text{ is odd} \\ \frac{1}{\dot{y}}, & \text{if } \dot{\varrho} \text{ is odd and } \dot{y} \text{ is even} \\ 1, & \text{otherwise .} \end{cases}$$

Define functions $\pi,\varkappa : [0,\infty) \to [0,\infty)$ *by*

$$\pi(a) = \varkappa(a) = \begin{cases} 0, & \text{if } a = 0 \\ \frac{1}{a}, & \text{otherwise.} \end{cases}$$

Then $(\mho,\daleth)$ *is a double composed partial metric space, which is not a partial metric space, and also not a double composed metric space.*

Proof. Conditions 1 and 2 of Definition 2.5 are easily verified, we will prove conditions 3 and 4. Note that $\daleth(\dot{\varrho},\dot{\varrho}) \leq \daleth(\dot{\varrho},\dot{y})$ for all $\dot{\varrho} \neq \dot{y}$. Since

case 1: if $\dot{\varrho}$ is even, then $\daleth(\dot{\varrho},\dot{\varrho}) = \frac{1}{\dot{\varrho}^2}$, while $\daleth(\dot{\varrho},\dot{y}) = \frac{1}{\dot{\varrho}}$ or 1, in both cases $\daleth(\dot{\varrho},\dot{\varrho}) \leq \daleth(\dot{\varrho},\dot{y})$.
case 2: if $\dot{\varrho}$ is odd, then clearly $0 = \daleth(\dot{\varrho},\dot{\varrho}) \leq \daleth(\dot{\varrho},\dot{y})$.

To prove property 4, we consider several cases
case 1: if $\dot{y} = \dot{\varrho}$ or $\dot{y} = \hat{l}$, then clearly we have $\daleth(\dot{\varrho},\hat{l}) \leq \pi\Big(\daleth(\dot{\varrho},\dot{y})\Big) + \varkappa\Big(\daleth(\dot{y},\hat{l})\Big)$.
case 2: if $\dot{y} \neq \dot{\varrho}$ and $\dot{y} \neq \hat{l}$, and suppose $\dot{\varrho} = \hat{l}$, then if $\dot{\varrho}$ is even, we have
$\frac{1}{\dot{\varrho}^2} = \daleth(\dot{\varrho},\dot{\varrho}) \leq \pi\Big(\daleth(\dot{\varrho},\dot{y})\Big) + \varkappa\Big(\daleth(\dot{y},\dot{\varrho})\Big) = 2\dot{\varrho}$.

case 3: : if $\dot{y} \neq \dot{\varrho} \neq \hat{l}$, then we verify condition (4), by considering several sub cases:

1) if all $\dot{y}$, $\dot{\varrho}$, and $\hat{l}$ are odd. Then clearly condition holds.

2) if $\dot{\varrho}, \hat{l}$ are odd but $\dot{y}$ is even, then
$1 = \Finv(\dot{\varrho}, \hat{l}) \leq \pi\Big(\Finv(\dot{\varrho}, \dot{y})\Big) + \varkappa\Big(\Finv(\dot{y}, \hat{l})\Big) = 2\dot{y}.$

3) if all $\dot{y}$, $\dot{\varrho}$, and $\hat{l}$ are even, then
$1 = \Finv(\dot{\varrho}, \hat{l}) \leq \pi\Big(\Finv(\dot{\varrho}, \dot{y})\Big) + \varkappa\Big(\Finv(\dot{y}, \hat{l})\Big) = 2.$

4) if $\dot{\varrho}, \dot{y}$ are odd but $\hat{l}$ is even, then
$\frac{1}{\hat{l}} = \Finv(\dot{\varrho}, \hat{l}) \leq \pi\Big(\Finv(\dot{\varrho}, \dot{y})\Big) + \varkappa\Big(\Finv(\dot{y}, \hat{l})\Big) = 2\hat{l}.$ (similar the case of $\hat{l}, \dot{y}$ are odd but $\dot{\varrho}$ is even)

5) if $\dot{\varrho}, \dot{y}$ are even but $\hat{l}$ is odd, then
$\frac{1}{\hat{l}} = \Finv(\dot{\varrho}, \hat{l}) \leq \pi\Big(\Finv(\dot{\varrho}, \dot{y})\Big) + \varkappa\Big(\Finv(\dot{y}, \hat{l})\Big) = 1 + \dot{y}.$ (similar is the case of $\hat{l}, \dot{y}$ are even but $\dot{\varrho}$ is odd, and the case $\dot{\varrho}$ is even but $\dot{y}, \hat{l}$ are odd).

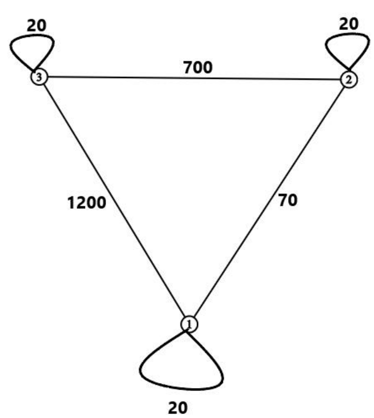
Thus we have shown that $(\mho, \Finv)$ is a double composed partial metric space.
Next, to show $(\mho, \Finv)$ is not a partial metric space, take $\dot{\varrho} = 6$, $\hat{l} = 4$ and let $\dot{y} = 3$, then

$$1 = \Finv(6, 4) > \Finv(6, 3) + \Finv(3, 4) = \frac{1}{6} + \frac{1}{4}.$$

Clearly $(\mho, \Finv)$ is not a double composed metric space, since $\Finv(2, 2) = \frac{1}{4} \neq 0$.

□

Example 2.2. *Let* $\mho = \{1, 2, 3\}$. *Two non-constant functions* $\pi, \varkappa : [0, \infty) \to [0, \infty)$ *are defined by* $\pi(\dot{\varrho}) = \dot{\varrho} + 700$ *and* $\varkappa(\dot{\varrho}) = \dot{\varrho}$. *Define a function* $\Finv : \mho \times \mho \to [0, \infty)$ *by*



$$\Finv(1, 1) = \Finv(2, 2) = \Finv(3, 3) = 20$$

$$\Finv(1, 2) = \Finv(2, 1) = 70$$

$$\Finv(1, 3) = \Finv(3, 1) = 1200$$

$$\Finv(2, 3) = \Finv(3, 2) = 700$$

Then $(\mho, \Finv)$ *is a double composed partial metric space.*

Proof. To prove that $(\mho, \daleth)$ double composed partial metric space, observe that the first three conditions of Definition 2.5 are easily satisfied by $\daleth$. We establish the triangle inequality.

$$\pi(\daleth(1,3)) + \varkappa(\daleth(3,2)) = \pi(1200) + \varkappa(700) = (1200 + 700) + 700 \geq 70 = \daleth(1,2).$$

$$\pi(\daleth(1,2)) + \varkappa(\daleth(2,3)) = \pi(70) + \varkappa(700) = (70 + 700) + 700 \geq 1200 = \daleth(1,3).$$

$$\pi(\daleth(2,1)) + \varkappa(\daleth(1,3)) = \pi(70) + \varkappa(1200) = (70 + 700) + 1200 \geq 700 = \daleth(2,3).$$

Also,

$$\pi(\daleth(1,2)) + \varkappa(\daleth(2,1)) = \pi(1200) + \varkappa(700) = (70 + 700) + 70 \geq 20 = \daleth(1,1)$$

Similarly the rest of the cases. Hence, for all $\dot{\varrho}, \dot{y}, \hat{l} \in \mho$, we have

$$\daleth(\dot{\varrho}, \hat{l}) \leq \pi\Big(\daleth(\dot{\varrho}, \dot{y})\Big) + \varkappa\Big(\daleth(\dot{y}, \hat{l})\Big).$$

□

A double composed partial metric space $(\mho, \daleth)$ generates a T_0-Topology on $\mho$, with a base of the family of open balls $\{B(x,r) : x \in \mho,\ r > 0\}$, where

$$B(x,r) = \{y \in \mho : \daleth(x,y) < \daleth(x,x) + r\}.$$

Now, we define the convergence of sequences in double composed partial metric spaces.

Definition 2.6. *Let $(\mho, \daleth)$ be a double composed partial metric space. For each sequence $\{\hbar_n\} \in \mho$, we say*

1. *$\{\hbar_n\}$ is a Cauchy sequence if $\lim_{n,m\to\infty} \daleth(\hbar_n, \hbar_m)$ exists and is finite ;*
2. *$\{\hbar_n\}$ converges to $\hbar$ if and only if $\lim_{n\to\infty} \daleth(\hbar_n, \hbar) = \daleth(\hbar, \hbar)$;*
3. *$(\mho, \hbar)$ is complete if every Cauchy sequence in $\mho$ is convergent to some point in $\hbar \in \mho$, i.e. $\lim_{n,m\to\infty} \daleth(\hbar_n, \hbar_m) = \daleth(\hbar, \hbar)$.*

Proposition 2.1. *Let $(\mho, \daleth)$ be a double composed partial metric space with two non-constant and continuous control functions $\pi, \varkappa : [0,\infty) \to [0,\infty)$ satisfying $\pi(\daleth(s,s)) + \varkappa(\daleth(t,t)) = 0$, for all $s, t \in \mho$. Then every convergent sequence has a unique limit.*

Proof. Let $\{\hbar_n\} \in \mho$ be a sequence and suppose it converges to s and t in $\mho$. By the definition of convergence, we have $\lim_{n\to\infty} \daleth(\hbar_n, s) = \daleth(s,s)$ and $\lim_{n\to\infty} \daleth(\hbar_n, t) = \daleth(t,t)$. By the triangle inequality in DCPMS, we obtain

$$\daleth(s,t) \leq \pi(\daleth(\hbar_n, s)) + \varkappa(\daleth(\hbar_n, t))$$

Since π and $\varkappa$ are continuous, taking the limit in the above inequality, we obtain

$$\begin{aligned}\daleth(s,t) &\leq \pi(\lim_{n\to\infty} \daleth(\hbar_n, s)) + \varkappa(\lim_{n\to\infty} \daleth(\hbar_n, t)) \\ &= \pi(\daleth(s,s)) + \varkappa(\daleth(t,t)) \\ &= 0\end{aligned}$$

This implies that $\daleth(s,t) = 0$. As $\daleth(s,s) \leq \daleth(s,t) = 0$, this gives $\daleth(s,s) = 0$ and similarly $\daleth(t,t) = 0$. Thus, $\daleth(s,t) = \daleth(s,s) = \daleth(t,t) = 0$, which further implies that $s = t$. □

3. Main Result

Prior to presenting our initial main theorem, we will articulate the following lemma.

Lemma 3.1. *Let $(\mho, \Finv)$ be a complete double composed partial metric space with non-decreasing control functions $\pi, \varkappa : [0,\infty) \to [0,\infty)$ satisfying the conditions of Proposition 2.1 and morover $\varkappa$ is sub-additive. Let $F : \mho \to \mho$ be a continuous mapping satisfying the two conditions:*

1)

$$\Finv(F\dot{\varrho}, F\dot{y}) \leq k\Finv(\dot{\varrho}, \dot{y}) \tag{3.1}$$

for all $\dot{\varrho}, \dot{y} \in \mho$ and some $k \in (0,1)$.

2) For $\hbar_0 \in \mho$, we define a sequence $\{\hbar_n\}$ by $\hbar_n = F^n\hbar_0$, such that

$$\lim_{m,n\to\infty}\left[\sum_{i=m}^{n-2}\varkappa^{i-m}\Big(\pi\left(k^i\Finv(\hbar_0,\hbar_1)\right)\Big) + \varkappa^{n-m-1}\left(k^{n-1}\Finv(\hbar_0,\hbar_1)\right)\right] = 0, \tag{3.2}$$

where $\varkappa^{i-m}\Big(\pi\left(k^i\Finv(\hbar_0,\hbar_1)\right)\Big)$ and $\varkappa^{n-m-1}\left(k^{n-1}\Finv(\hbar_0,\hbar_1)\right)$ denote the composite functions. Then, the sequence $\{\hbar_n\}$ converges to some point in $\mho$.

Proof. Let $\hbar_0 \in \mho$. Define a sequence $\{\hbar_n\}$ in $\mho$ with $\hbar_n = F^n\hbar_0$ so that $\hbar_{n+1} = F\hbar_n$ for all $n \in \mathbb{N}$. We have

$$\begin{aligned}
\Finv(\hbar_n, \hbar_{n+1}) &= \Finv(F\hbar_{n-1}, F\hbar_n) \\
&\leq k\Finv(\hbar_{n-1}, \hbar_n) \\
&= k\Finv(F\hbar_{n-2}, F\hbar_{n-1}) \\
&\leq k^2\Finv(\hbar_{n-2}, \hbar_{n-1}) \\
&\;\;\vdots \\
&\leq k^n\Finv(\hbar_0, \hbar_1).
\end{aligned} \tag{3.3}$$

$$\Finv(\hbar_n, \hbar_{n+1}) \leq k^n\Finv(\hbar_0, \hbar_1) \tag{3.4}$$

For $n \geq m$, by using the triangle inequality repeatedly, and the sub-additivity of $\varkappa$, we obtain

$$\begin{aligned}
\Finv(\hbar_m, \hbar_n) &\leq \pi(\Finv(\hbar_m, x_{m+1})) + \varkappa(\Finv(x_{m+1}, \hbar_n)) \\
&\leq \pi(\Finv(\hbar_m, x_{m+1})) + \varkappa[\pi(\Finv(x_{m+1}, \hbar_{m+2})) + \varkappa(\Finv(\hbar_{m+2}, \hbar_n))] \\
&= \pi(\Finv(\hbar_m, x_{m+1})) + \varkappa\pi(\Finv(x_{m+1}, \hbar_{m+2})) + \varkappa^2[\Finv(\hbar_{m+2}, \hbar_n)] \\
&\leq \pi(\Finv(\hbar_m, x_{m+1})) + \varkappa\pi(\Finv(x_{m+1}, \hbar_{m+2})) + \varkappa^2[\pi(\Finv(\hbar_{m+2}, \hbar_{m+3})) + \varkappa(\Finv(\hbar_{m+3}, \hbar_n))] \\
&= \pi(\Finv(\hbar_m, x_{m+1})) + \varkappa\pi(\Finv(x_{m+1}, \hbar_{m+2})) + \varkappa^2\pi(\Finv(\hbar_{m+2}, \hbar_{m+3})) + \varkappa^3[\Finv(\hbar_{m+3}, \hbar_n)] \\
&\leq \pi(\Finv(\hbar_m, x_{m+1})) + \varkappa\pi(\Finv(x_{m+1}, \hbar_{m+2})) + \varkappa^2\pi(\Finv(\hbar_{m+2}, \hbar_{m+3})) \\
&\quad + \varkappa^3[\pi(\Finv(\hbar_{m+3}, \hbar_{m+4})) + \varkappa(\Finv(\hbar_{m+4}, \hbar_n))] \\
&= \pi(\Finv(\hbar_m, x_{m+1})) + \varkappa\pi(\Finv(x_{m+1}, \hbar_{m+2})) + \varkappa^2\pi(\Finv(\hbar_{m+2}, \hbar_{m+3})) \\
&\quad + \varkappa^3\pi(\Finv(\hbar_{m+3}, \hbar_{m+4})) + \varkappa^4[\Finv(\hbar_{m+4}, \hbar_n)]
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
&\vdots\\
&\leq \pi(\dashv(\hbar_m, x_{m+1})) + \varkappa\pi(\dashv(x_{m+1}, \hbar_{m+2})) + \varkappa^2\pi(\dashv(\hbar_{m+2}, \hbar_{m+3})) + \varkappa^3\pi(\dashv(\hbar_{m+3}, \hbar_{m+4}))\\
&+ \varkappa^4[\dashv(\hbar_{m+4}, \hbar_n)] + ... + \varkappa^{n-2}[\pi(\dashv(\hbar_{n-2}, \hbar_{n-1})) + \varkappa(\dashv(\hbar_{n-1}, \hbar_n))]\\
&= \pi(\dashv(\hbar_m, x_{m+1})) + \varkappa\pi(\dashv(x_{m+1}, \hbar_{m+2})) + \varkappa^2\pi(\dashv(\hbar_{m+2}, \hbar_{m+3})) + \varkappa^3\pi(\dashv(\hbar_{m+3}, \hbar_{m+4})) \qquad (3.6)\\
&+ \varkappa^4[\dashv(\hbar_{m+4}, \hbar_n)] + ... + \varkappa^{n-m-2}\pi(\dashv(\hbar_{n-2}, \hbar_{n-1})) + \varkappa^{n-m-1}(\dashv(\hbar_{n-1}, \hbar_n))\\
&= \sum_{i=m}^{n-2} \varkappa^{i-m}\pi\left(\dashv(\hbar_i, \hbar_{i+1})\right) + \varkappa^{n-m-1}\left(\dashv(\hbar_{n-1}, \hbar_n)\right)
\end{aligned}$$

Since π and $\varkappa$ are non-decreasing functions so that the compositions $\varkappa^{i-m}\pi\left(\dashv(\hbar_i, \hbar_{i+1})\right)$ and $\varkappa^{n-m-1}\left(\dashv(\hbar_{n-1}, \hbar_n)\right)$ are also non-decreasing. Using inequality (3.4) in (3.6), we obtain

$$\dashv(\hbar_m, \hbar_n) \leq \sum_{i=m}^{n-2} \varkappa^{i-m}\pi\left(k^i\dashv(\hbar_0, \hbar_1))\right) + \varkappa^{n-m-1}\left(k^{n-1}\dashv(\hbar_0, \hbar_1)\right) \qquad (3.7)$$

Taking the limit as m, n tends to infinity in (3.7), and utilizing condition 3.2, we obtain

$$\lim_{n,m\to\infty} \dashv(\hbar_m, \hbar_n) = 0. \qquad (3.8)$$

Therefore, the sequence $\{\hbar_n\}$ is Cauchy in $\mho$. Since $\mho$ is a complete double composed partial metric space, the sequence $\{\hbar_n\}$ converges to some point $s \in \mho$, that is,

$$\lim_{n\to\infty} \dashv(\hbar_n, s) = \dashv(s, s). \qquad (3.9)$$

□

Theorem 3.1. *Let $(\mho, \dashv)$ be a complete double composed partial metric space with non-decreasing control functions $\pi, \varkappa : [0, \infty) \to [0, \infty)$ and the continuous mapping $F : \mho \to \mho$ satisfying all the conditions of Lemma 3.1. Furthermore, let $\{\hbar_n\}$ be the convergent sequence in $\mho$ as in Lemma 3.1 which converges to $s \in \mho$, assume the following holds:*

$$\pi(\dashv(s, s)) + \varkappa(k\dashv(s, s)) \leq \dashv(s, s), \qquad (3.10)$$

and

$$\pi(\dashv(Fs, Fs)) + \varkappa(k\dashv(Fs, Fs)) \leq \dashv(Fs, Fs). \qquad (3.11)$$

Then, F has a unique fixed point.

Proof. In Lemma 3.1 we proved the sequence $\{\hbar_n\}$ converges to s, i.e. $\lim_{n\to\infty} \dashv(\hbar_n, s) = \dashv(s, s)$. Next, we prove that s is a fixed point of F, that is, $Fs = s$. Thus we need to show that $\dashv(s, Fs) = \dashv(s, s) = \dashv(Fs, Fs)$. To establish this, it is enough to show $\dashv(s, Fs) \leq \dashv(s, s)$ and $\dashv(s, Fs) \leq \dashv(Fs, Fs)$.

By the triangle inequality, and using condition 3.1, we have

$$\begin{aligned}\daleth(s, Fs) &\leq \pi(\daleth(s, \hbar_n)) + \varkappa(\daleth(\hbar_n, Fs)) \\ &= \pi(\daleth(s, \hbar_n)) + \varkappa(\daleth(F\hbar_{n-1}, Fs)) \\ &\leq \pi(\daleth(s, \hbar_n)) + \varkappa(k\daleth(\hbar_{n-1}, s))\end{aligned} \tag{3.12}$$

Since π and $\varkappa$ are continuous, taking the limit as n tends to infinity and utilizing condition 3.10, we obtain

$$\begin{aligned}\daleth(s, Fs) &\leq \lim_{n\to\infty}\Big[\pi(\daleth(s, \hbar_n)) + \varkappa(k\daleth(\hbar_{n-1}, s))\Big] \\ &= \pi(\lim_{n\to\infty}\daleth(s, \hbar_n)) + \varkappa(\lim_{n\to\infty}k\daleth(\hbar_{n-1}, s)) \\ &= \pi(\daleth(s, s)) + \varkappa(k\daleth(s, s)) \leq \daleth(s, s)\end{aligned} \tag{3.13}$$

Next, we establish $\daleth(s, Fs) \leq \daleth(Fs, Fs)$, note that

$$\begin{aligned}\daleth(\hbar_n, F\hbar_n) &\leq \pi(\daleth(\hbar_n, \hbar_n)) + \varkappa(\daleth(\hbar_n, F\hbar_n)) \\ &= \pi(\daleth(F\hbar_{n-1}, F\hbar_{n-1})) + \varkappa(\daleth(F\hbar_{n-1}, F\hbar_n)) \\ &\leq \pi(\daleth(F\hbar_{n-1}, F\hbar_{n-1})) + \varkappa(k\daleth(F\hbar_{n-2}, F\hbar_{n-1}))\end{aligned} \tag{3.14}$$

Continuity of F implies $\lim_{n\to\infty}\daleth(\hbar_n, F\hbar_n) = \daleth(s, Fs)$, thus taking the limit as n tends to infinity in 3.14 and utilizing condition 3.11 and the fact π, and $\varkappa$ are continuous, we obtain

$$\daleth(s, Fs) \leq \pi(\daleth(Fs, Fs)) + \varkappa(k\daleth(Fs, Fs)) \leq \daleth(Fs, Fs) \tag{3.15}$$

Thus we have established $\daleth(s, Fs) = \daleth(s, s) = \daleth(Fs, Fs)$ and this implies that $Fs = s$.

Finally, we prove that F has a unique fixed point. Suppose there are two fixed points $s, t \in \mho$, such that $s \neq t$. We have $\daleth(s, t) = \daleth(Fs, Ft) \leq k\daleth(s, t)$ which implies $\daleth(s, t) = 0$ since $k \in (0, 1)$. As $\daleth(s, s) \leq \daleth(s, t) = 0$, this gives $\daleth(s, s) = 0$. Similarly one can show $\daleth(t, t) = 0$. Thus, $\daleth(s, t) = \daleth(s, s) = \daleth(t, t) = 0$, which further implies that $s = t$.

□

The subsequent Theorem bears resemblance to the Kannan type fixed point theorem. However, before delving into its statement, we present a lemma crucial for the subsequent proof of the Theorem.

Lemma 3.2. *Let $(\mho, \daleth)$ be a complete double composed partial metric space with non-decreasing control functions $\pi, \varkappa : [0, \infty) \to [0, \infty)$. Let $F : \mho \to \mho$ be a continuous mapping satisfying:*

$$\daleth(F\dot{\varrho}, F\dot{y}) \leq k[\daleth(\dot{\varrho}, F\dot{\varrho}) + \daleth(\dot{y}, F\dot{y})], \tag{3.16}$$

for all $\dot{\varrho}, \dot{y} \in \mho$ and $k \in (0, \frac{1}{2})$. For $\hbar_0 \in \mho$, define a sequence $\{\hbar_n\}$ by $\hbar_n = F^n\hbar_0$. Suppose that the following conditions are satisfied

$$\lim_{m,n\to\infty}\left[\sum_{i=m}^{n-2}\varkappa^{i-m}\Big(\pi\left(k^i\daleth(\hbar_0, \hbar_1)\right)\Big) + \varkappa^{n-m-1}\left(k^{n-1}\daleth(\hbar_0, \hbar_1)\right)\right] = 0, \tag{3.17}$$

where $r = \frac{k}{1-k}$, and $\varkappa^{i-m}\Big(\pi\left(k^i \exists(\hbar_0, \hbar_1)\right)\Big)$ and $\varkappa^{n-m-1}\left(k^{n-1}\exists(\hbar_0, \hbar_1)\right)$ denote the composite functions. Then the sequence $\{\hbar_n\}$ is convergent.

Proof. Let $\hbar_0 \in \mho$ and define a sequence $\{\hbar_n\}$ in $\mho$ inductively by taking $\hbar_n = F\hbar_{n-1}, n \geq 1$. Set $\exists_n = \exists(\hbar_n, \hbar_{n+1})$,then we have,

$$\begin{aligned} \exists_n = \exists\left(\hbar_n, \hbar_{n+1}\right) &= \exists\left(F\hbar_{n-1}, F\hbar_n\right). \\ &\leq k\Big[\exists(\hbar_{n-1}, F\hbar_{n-1}) + \exists(\hbar_n, F\hbar_n)\Big]. \\ &= k\Big[\exists(\hbar_{n-1}, \hbar_n) + \exists(\hbar_n, \hbar_{n+1})\Big] \\ &\leq k\Big[\exists_{n-1} + \exists_n)\Big]. \end{aligned} \tag{3.18}$$

Which implies,

$$\exists_n \leq r\exists_{n-1}, \tag{3.19}$$

where $r = \frac{k}{1-k} < 1$ as $k \in (0, \frac{1}{2})$.
Thus we have,

$$\exists\left(\hbar_n, \hbar_{n+1}\right) = \exists_n \leq r\exists_{n-1} \leq r^2\exists_{n-2} \leq ... \leq r^n\exists\left(\hbar_0, \hbar_1\right). \tag{3.20}$$

Similar to inequality (3.6) in Theorem 3.1, for all $n \geq m$, we have

$$\exists(\hbar_m, \hbar_n) \leq \sum_{i=m}^{n-2} \varkappa^{i-m}\Big(\pi\left(\exists(\hbar_i, \hbar_{i+1})\right)\Big) + \varkappa^{n-m-1}\left(\exists(\hbar_{n-1}, \hbar_n)\right) \tag{3.21}$$

Using inequality (3.20) in (3.21), we obtain

$$\exists(\hbar_m, \hbar_n) \leq \sum_{i=m}^{n-2} \varkappa^{i-m}\Big(\pi\left(k^i\exists(\hbar_0, \hbar_1)\right)\Big) + \varkappa^{n-m-1}\left(k^{n-1}\exists(\hbar_0, \hbar_1)\right). \tag{3.22}$$

Letting m, n tends to infinity in (3.22), and by equation 3.17, we obtain

$$\lim_{n,m\to\infty} \exists(\hbar_m, \hbar_n) = 0. \tag{3.23}$$

Therefore, the sequence $\{\hbar_n\}$ is Cauchy in $\mho$. Since $\mho$ is a complete double composed partial metric space, the sequence $\{\hbar_n\}$ converges to a point $s \in \mho$, that is,

$$\lim_{n\to\infty} \exists(\hbar_n, s) = 0. \tag{3.24}$$

□

Theorem 3.2. *Let $(\mho, \exists)$ be a complete double composed partial metric space with continuous and non-decreasing control functions $\pi, \varkappa : [0, \infty) \to [0, \infty)$. Let $F : \mho \to \mho$ be a continuous mapping satisfying all the conditions of Lemma 3.2 and let $\{\hbar_n\}$ be the convergent sequence which converges to $s \in \mho$. Assume these conditions hold:*

$$\pi(\exists(s, s)) + \varkappa(\exists(s, s)) \leq \exists(s, s), \tag{3.25}$$

and

$$\pi(\exists(Fs, Fs)) + \varkappa(\exists(Fs, Fs)) \leq \exists(Fs, Fs). \tag{3.26}$$

Then, F has a unique fixed point.

Proof. In Lemma 3.2 we proved the sequence $\{\hbar_n\}$ converges to s, i.e.

$$\lim_{n\to\infty} \daleth(\hbar_n, s) = \daleth(s,s).$$

Next, we prove that s is a fixed point of F, that is, $Fs = s$. Thus we need to show that $\daleth(s, Fs) = \daleth(s,s) = \daleth(Fs, Fs)$. To establish this, it is enough to show $\daleth(s, Fs) \leq \daleth(s,s)$ and $\daleth(s, Fs) \leq \daleth(Fs, Fs)$. First we establish $\daleth(s, Fs) \leq \daleth(s,s)$. By the triangle inequality, we have

$$\begin{aligned} \daleth(\hbar_n, F\hbar_n) &\leq \pi(\daleth(\hbar_n, \hbar_{n+2})) + \varkappa(\daleth(\hbar_{n+2}, F\hbar_n)). \\ &\leq \pi(\daleth(\hbar_n, \hbar_{n+2})) + \varkappa(\daleth(\hbar_{n+2}, \hbar_{n+1})). \end{aligned} \tag{3.27}$$

Continuity of F implies $\lim_{n\to\infty} \daleth(\hbar_n, F\hbar_n) = \daleth(s, Fs)$. Thus taking the limit as n tends to infinity in 3.27 and utilizing condition 3.25 and the fact π, $\varkappa$ are continuous, we obtain

$$\daleth(s, Fs) \leq \pi(\daleth(s,s)) + \varkappa(\daleth(s,s)) \leq \daleth(s,s). \tag{3.28}$$

Next, we establish $\daleth(s, Fs) \leq \daleth(Fs, Fs)$. Observe

$$\begin{aligned} \daleth(\hbar_n, F\hbar_n) &\leq \pi(\daleth(\hbar_n, \hbar_n)) + \varkappa(\daleth(\hbar_n, F\hbar_n)). \\ &= \pi(\daleth(F\hbar_{n-1}, F\hbar_{n-1})) + \varkappa(\daleth(F\hbar_{n-1}, F\hbar_n)). \end{aligned} \tag{3.29}$$

Continuity of F implies $\lim_{n\to\infty} \daleth(F\hbar_n, F\hbar_n) = \daleth(Fs, Fs)$, thus taking the limit as n tends to infinity in 3.29 and utilizing condition 3.26 and the fact π, and $\varkappa$ are continuous, we obtain

$$\daleth(s, Fs) \leq \pi(\daleth(Fs, Fs)) + \varkappa(\daleth(Fs, Fs)) \leq \daleth(Fs, Fs). \tag{3.30}$$

Thus, we have established $\daleth(s, Fs) = \daleth(s,s) = \daleth(Fs, Fs)$ and this implies that $Fs = s$, i.e. s is the fixed point of F. □

4. Application to Integral Equations in Double Composed Partial Metric Spaces

Example 4.1. *Consider the space of all continuous real valued functions $\mho = C[0,1]$, with $\daleth : \mho \times \mho \longrightarrow [0, +\infty)$ be defined as*

$$\daleth(r, h) = \sup_{\tau \in [0,1]} |r(\tau) - h(\tau)|^2. \tag{4.1}$$

Define the functions $\pi, \varkappa : [0, +\infty) \longrightarrow [0, +\infty)$ by

$$\pi(x) = \varkappa(x) = 2x. \tag{4.2}$$

Then $(\mho, \daleth)$ is a complete double composed partial metric space.

Proof. Note that the equation (4.1) does not define a metric in the usual sense on $C[0,1]$. To prove that $(\mho, \daleth)$ is a double composed partial metric space, we need to establish the triangle inequality

of Definition (2.5). The other conditions are trivial.
Note that the following inequality hold true for all $\alpha, \beta \in \mathbb{R}$,

$$| \alpha + \beta |^2 \leq 2 | \alpha |^2 + 2 | \beta |^2 \tag{4.3}$$

For all $r, h, j \in C[0,1]$, using inequality 4.3 we obtain

$$\begin{aligned} | r(\tau) - h(\tau) |^2 &= | r(\tau) - j(\tau) + j(\tau) - h(\tau) |^2 \\ &\leq 2 | r(\tau) - j(\tau) |^2 + 2 | j(\tau) - h(\tau) |^2 \end{aligned} \tag{4.4}$$

Taking supremum on both sides, we get

$$\sup_{\tau \in [0,1]} | r(\tau) - h(\tau) |^2 \leq 2 \sup_{\tau \in [0,1]} | r(\tau) - j(\tau) |^2 + 2 \sup_{\tau \in [0,1]} | j(\tau) - h(\tau) |^2, \tag{4.5}$$

that is,

$$\dashv(r,h) \leq \pi\left(\dashv(r,j)\right) + \varkappa\left(\dashv(j,h)\right).$$

This proves the triangle inequality. It is not difficult to show that $(\mho, \dashv)$ is a complete space. □

Theorem 4.1. *Let $(\mho = C[0,1], \dashv)$ be a complete double composed partial metric space as defined in Example 4.1. Consider the following Fredholm integral equation, where $L : \mho \longrightarrow \mho$ is given by*

$$L(r(\tau)) = z(\tau) + \int_0^1 l(\tau, \psi, r(\tau)) d\psi, \tag{4.6}$$

where $l(\tau, \psi, r(\tau)) : [0,1] \times [0,1] \longrightarrow \mathbb{R}$ is a given continuous function satisfying the following condition for all $r, h, z \in \mho$, and $\tau, \psi \in [0,1]$:

$$|l(\tau, \psi, r(\tau)) - l(\tau, \psi, h(\tau))| \leq \sqrt{Z_1(\tau)}$$

where

$$Z_1(\tau) \leq k[\dashv(r(\tau), Lr(\tau)) + \dashv(h(\tau), Lh(\tau))], \quad with \ \ 0 < k < \frac{1}{2}.$$

Then the integral equation (4.6) has a unique solution.

Proof. Let $L : C[0,1] \longrightarrow C[0,1]$ be defined by $L(r(\tau)) = z(\tau) + \int_0^1 l(\tau, \psi, r(\tau)) d\psi$ then

$$\begin{aligned} \dashv(Lr(\tau), Lh(\tau)) &= \sup_{\tau \in [0,1]} |Lr(\tau) - Lh(\tau)|^2. \\ &= \sup_{\tau \in [0,1]} |z(\tau) + \int_0^1 l(\tau, \psi, r(\tau)) d\psi - z(\tau) - \int_0^1 l(\tau, \psi, h(\tau)) d\psi|^2. \\ &\leq \sup_{\tau \in [0,1]} \int_0^1 |l(\tau, \psi, r(\tau)) - l(\tau, \psi, h(\tau))|^2 d\psi. \\ &\leq \sup_{\tau \in [0,1]} \int_0^1 | \sqrt{Z_1(\tau)}|^2 d\psi. \\ &\leq \sup_{\tau \in [0,1]} |Z_1(\tau)| \int_0^1 d\psi. \end{aligned} \tag{4.7}$$

$$\begin{aligned}&\leq \sup_{\tau\in[0,1]} Z_1(\tau).\\&\leq k[\daleth(r(\tau),Lr(\tau))+\daleth(h(\tau),Lh(\tau))].\end{aligned}\tag{4.8}$$

Our goal is to apply Theorem (3.2) to the operator L, for which we need to establish the condition (3.17) of the Lemma (3.2). Let $\hbar_0=\hbar_0(\tau)\in C[0,1]$ and define a sequence of continuous functions $\{\hbar_n=\hbar_n(\tau)\}$ in $\mho$ inductively by taking $\hbar_n=L\hbar_{n-1}, n\geq 1$.
Since $\pi(x)=\varkappa(x)=2x$, we observe that the composition $\varkappa^i(x)=2^ix$. Therefore, we have

$$\varkappa^{i-m}\Big(\pi\left(k^i\daleth(\hbar_0,\hbar_1)\right)\Big)=2^{i-m+1}k^i\daleth(\hbar_0,\hbar_1),\tag{4.9}$$

and

$$\varkappa^{n-m-1}\left(k^{n-1}\daleth(\hbar_0,\hbar_1)\right)=2^{n-m-1}k^{n-1}\daleth(\hbar_0,\hbar_1).\tag{4.10}$$

Using the equations (4.9) and (4.10), we have

$$\begin{aligned}&\lim_{m,n\to\infty}\left[\sum_{i=m}^{n-2}\varkappa^{i-m}\Big(\pi\left(k^i\daleth(\hbar_0,\hbar_1)\right)\Big)+\varkappa^{n-m-1}\left(k^{n-1}\daleth(\hbar_0,\hbar_1)\right)\right]\\&=\lim_{m,n\to\infty}\left[\sum_{i=m}^{n-2}2^{i-m+1}k^i\daleth(\hbar_0,\hbar_1)+2^{n-m-1}k^{n-1}\daleth(\hbar_0,\hbar_1)\right].\\&=\daleth(\hbar_0,\hbar_1)\lim_{m,n\to\infty}\left[\sum_{i=m}^{n-2}2^{i-m+1}k^i+2^{n-m-1}k^{n-1}\right].\end{aligned}\tag{4.11}$$

Note the following summation,

$$\sum_{i=m}^{n-2}2^{i-m+1}k^i=\frac{2^{n-m}k^{n-1}-2k^m}{2k-1}.\tag{4.12}$$

Using equation (4.12) in (4.11), we obtain

$$\begin{aligned}&\lim_{m,n\to\infty}\left[\sum_{i=m}^{n-2}\varkappa^{i-m}\Big(\pi\left(k^i\daleth(\hbar_0,\hbar_1)\right)\Big)+\varkappa^{n-m-1}\left(k^{n-1}\daleth(\hbar_0,\hbar_1)\right)\right]\\&=\daleth(\hbar_0,\hbar_1)\lim_{m,n\to\infty}\left[\frac{2^{n-m}k^{n-1}-2k^m}{2k-1}+2^{n-m-1}k^{n-1}\right]=0.\end{aligned}\tag{4.13}$$

Therefore the condition (3.17) of the Lemma (3.2) is satisfied, and therefore the sequence $\hbar_0=\hbar_0(\tau)$ converges to continuous function, say, $s\in C[0,1]$. Note that such a function s is unique, suppose there are two functions $s,t\in\mho$, then the following condition of Proposition (2.1) holds,

$$\begin{aligned}\pi(\daleth(s,s))+\varkappa(\daleth(t,t))&=2\daleth(s,s)+2\daleth(t,t).\\&=2\sup_{\tau\in[0,1]}|s(\tau)-s(\tau)|^2+2\sup_{\tau\in[0,1]}|t(\tau)-t(\tau)|^2.\\&=0.\end{aligned}\tag{4.14}$$

Finally, as $\daleth(s,s) = \sup_{\tau\in[0,1]} |s(\tau) - s(\tau)|^2 = 0$ and $\daleth(Ls,Ls) = \sup_{\tau\in[0,1]} |Ls(\tau) - Ls(\tau)|^2 = 0$, we note that the conditions (3.25) and (3.26) of Theorem (3.2) are satisfied. Therefore L has a unique fixed point, which further implies that the integral equation (4.6) has a unique solution.

□

5. Conclusion

In conclusion, this paper introduces the concept of a double composed partial metric space, expanding the scope of mathematical spaces with potential practical applications. By providing illustrative examples and establishing key fixed point theorems, including Banach and Kannan types, the study lays a solid theoretical foundation for future research. This work opens new avenues for exploration and a deeper understanding of mathematical analysis.

Acknowledgements: The authors express their sincere gratitude to Prince Sultan University for its support and for covering the publication costs through the TAS Research Lab.

Author Contribution: F.M. A., N. M., and I. A. wrote the main manuscript text. All authors reviewed the manuscript.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] A. Al-Rawashdeh, H. Aydi, A. Felhi, S. Sahmim, W. Shatanawi, On Common Fixed Points for α-F-Contractions and Applications, J. Nonlinear Sci. Appl. 09 (2016), 3445–3458. https://doi.org/10.22436/jnsa.009.05.128.

[2] W. Shatanawi, Z. Mustafa, N. Tahat, Some Coincidence Point Theorems for Nonlinear Contraction in Ordered Metric Spaces, Fixed Point Theory Appl. 2011 (2011), 68. https://doi.org/10.1186/1687-1812-2011-68.

[3] W. Shatanawi, Some Fixed Point Results for a Generalized Ψ-Weak Contraction Mappings in Orbitally Metric Spaces, Chaos Solitons Fractals 45 (2012), 520–526. https://doi.org/10.1016/j.chaos.2012.01.015.

[4] I. Ayoob, N.Z. Chuan, N. Mlaiki, Double-Composed Metric Spaces, Mathematics 11 (2023), 1866. https://doi.org/10.3390/math11081866.

[5] N. Souayah, M. Mrad, On Fixed-Point Results in Controlled Partial Metric Type Spaces with a Graph, Mathematics 8 (2019), 33. https://doi.org/10.3390/math8010033.

[6] S. Czerwik, Contraction Mappings in b-Metric Spaces, Acta Math. Inf. Univ. Ostrav. 1 (1993), 5–11. http://dml.cz/dmlcz/120469.

[7] I.A. Bakhtin, The Contraction Mapping Principle in Almost Metric Spaces, Funct. Anal. 30 (1989), 26–37.

[8] T. Kamran, M. Samreen, Q. Ul Ain, A Generalization of b-Metric Space and Some Fixed Point Theorems, Mathematics 5 (2017), 19. https://doi.org/10.3390/math5020019.

[9] N. Mlaiki, H. Aydi, N. Souayah, T. Abdeljawad, Controlled Metric Type Spaces and the Related Contraction Principle, Mathematics 6 (2018), 194. https://doi.org/10.3390/math6100194.

[10] D. Lateef, Kannan Fixed Point Theorem in C-Metric Spaces, J. Math. Anal. 10 (2019), 34–40.

[11] J. Ahmad, A.E. Al-Mazrooei, H. Aydi, M. De La Sen, On Fixed Point Results in Controlled Metric Spaces, J. Function Spaces 2020 (2020), 2108167. https://doi.org/10.1155/2020/2108167.

[12] T. Abdeljawad, N. Mlaiki, H. Aydi, N. Souayah, Double Controlled Metric Type Spaces and Some Fixed Point Results, Mathematics 6 (2018), 320. https://doi.org/10.3390/math6120320.

[13] N. Mlaiki, Double Controlled Metric-like Spaces, J. Ineq. Appl. 2020 (2020), 189. https://doi.org/10.1186/s13660-020-02456-z.

[14] G.E. Hardy, T.D. Rogers, A Generalization of a Fixed Point Theorem of Reich, Canadian Math. Bull. 16 (1973), 201–206. https://doi.org/10.4153/CMB-1973-036-0.

[15] F.M. Azmi, Generalized Contraction Mappings in Double Controlled Metric Type Space and Related Fixed Point Theorems, J. Ineq. Appl. 2023 (2023), 87. https://doi.org/10.1186/s13660-023-02999-x.

[16] S.G. Matthews, Partial Metric Topology, Ann. N. Y. Acad. Sci. 728 (1994), 183–197. https://doi.org/10.1111/j.1749-6632.1994.tb44144.x.

[17] F.M. Azmi, New Fixed Point Results in Double Controlled Metric Type Spaces with Applications, AIMS Math. 8 (2023), 1592–1609. https://doi.org/10.3934/math.2023080.

[18] M.S. Aslam, M.S.R. Chowdhury, L. Guran, I. Manzoor, T. Abdeljawad, D. Santina, N. Mlaiki, Complex-Valued Double Controlled Metric like Spaces with Applications to Fixed Point Theorems and Fredholm Type Integral Equations, AIMS Math. 8 (2023), 4944–4963. https://doi.org/10.3934/math.2023247.

[19] M.B. Zada, M. Sarwar, T. Abdeljawad, A. Mukheimer, Coupled Fixed Point Results in Banach Spaces with Applications, Mathematics 9 (2021), 2283. https://doi.org/10.3390/math9182283.

[20] N. Alamgir, Q. Kiran, H. Aydi, A. Mukheimer, A Mizoguchi–Takahashi Type Fixed Point Theorem in Complete Extended b-Metric Spaces, Mathematics 7 (2019), 478. https://doi.org/10.3390/math7050478.