

Modeling the Process of Critical Exhaustion of Oxygen in a River Caused by Exponential Source of Pollution

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ABSTRACT. The analysis and improvement of the river's change dynamics are critical to the global investigation of water quality issues. This study investigates the mathematical model representation of one dimension of a pair of advection-dispersion equations. These equations pertain to the concentrations of both pollutants and dissolved oxygen. Certain terms, which demonstrate an increase in the decay rate of pollution through exponential sources and assume a decreasing river's cross-sectional measurement, lead to the diminished oxygen levels observed in this study. The crucial points are demonstrated using both analytical and numerical analysis of the steady-state model, which allows for examining multiple situations about the elimination of oxygen and aquatic survival in a river with severe pollution.

1. Introduction

Mathematical modelling is a crucial role in many real problems such as simulating, predicting, and others. The predicting water quality for effective water management globally is also popular. Thailand's primary pollution sources include home sewage, industrial discharges, and agricultural effluents. An analysis of the main rivers has revealed concerns regarding water quality due to a decrease in dissolved oxygen (DO), the death of fish, increased levels of ammonium and nitrogen into the air, also known as increased coliform bacteria contamination, and instances of nutrient enrichment events.[1]. A significant water quality issue in Thailand is the presence of low dissolved oxygen levels, specifically at an amount of (12%). The Tha Chin

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River is the specific river under investigation in this study. It was one of Thailand's most extensively contaminated rivers. [2].

A state of the fluctuation in dissolved oxygen levels downstream of wastewater discharge points was conducted by Chapra [3], who outlined the fundamental equations of water contamination. This involved the utilization of advection-diffusion equations to determine the levels of contaminants and dissolved oxygen. This research presents a mathematical model focusing on aeration in rivers to address pollution issues. The model utilizes coupled advection-dispersion equations to represent the levels of pollutants and dissolved oxygen concentrations. This model builds upon previous research conducted in ([4, 5]). The model comprises two equations: one represents the rate at which pollutant concentration removed through aeration, and the other provides a oxygen mass flow analysis, which is consumed during the process of pollutant removal. The rate of added pollution is assumed as exponentially increasing forms. To simplify the problem, the river is conceptualized as one-dimensional, and a steady-state solution is assumed. This investigation aims to anticipate a river critical point allowing for analytical solutions and simulations to explore scenarios like oxygen depletion and fish survival in polluted rivers. Both analytical and numerical analysis have been developed to solve the model.

2. The governing equations

The dynamics of unsteady flow within the river are represented by the one dimension of the coupled model of advection-dispersion equations, which the spatial x (km) and temporal variables t (day) are incorporated through Equations 2.1 and 2.2 in ([4], [5]).

$$\frac{\partial(AP)}{\partial t} = D_p \frac{\partial^2(AP)}{\partial x^2} - \frac{\partial(vAP)}{\partial x} - K_1 \frac{X}{X+k} AP + q(1 - \exp(-\lambda x)), (0 \leq x < L < \infty, t \geq 0) \quad (2.1)$$

$$\frac{\partial(AX)}{\partial t} = D_x \frac{\partial^2(AX)}{\partial x^2} - \frac{\partial(vAX)}{\partial x} - K_2 \frac{X}{X+k} AP + \alpha(S - X), (0 \leq x < L < \infty, t \geq 0) \quad (2.2)$$

where v represents the horizontal water velocity, D_p represents the pollutant dispersal constant coefficient, D_x represents the dissolved oxygen dispersal constant coefficient, S represents the concentration of oxygen in saturation, K_1 represents the constant coefficient that determines the rate at which the pollutant degrades, K_2 represents the constant coefficient that determines the pace at which dissolved oxygen degrades, α represents the process of oxygen transfer from air into water, q represents the pollutant influx rate along the river, k represents the concentration at which oxygen demand is half-saturated for pollutant decay, A represents the river's cross-sectional measurement and λ represents a constant value of exponentially term of source [4]. The analysis focuses on the case where k is no significant ($k \approx 0$). In this scenario, Equations 2.1 and 2.2 can be expressed as follows.

$$\frac{\partial(AP)}{\partial t} = D_p \frac{\partial^2(AP)}{\partial x^2} - \frac{\partial(vAP)}{\partial x} - K_1 AP + q(1 - \exp(-\lambda x)), \quad (2.3)$$

$$\frac{\partial(AX)}{\partial t} = D_x \frac{\partial^2(AX)}{\partial x^2} - \frac{\partial(vAX)}{\partial x} - K_2 AX + \alpha(S - X). \quad (2.4)$$

The temporal variation is zero and the steady-state conditions were considered. Consequently, Equations 2.3 and 2.4 can be simplified to the second-order ordinary differential equations Equations 2.5 and 2.6, where pollution $P(x, t)$ denoted by $P_s(x)$ (the pollutant concentration at steady state ($kg\ km^{-3}$)) and for dissolved oxygen $X(x, t)$ denoted by $X_s(x)$ (dissolved oxygen concentration at steady state ($kg\ km^{-3}$)) [6].

$$D_p \frac{d^2(AP_s)}{dx^2} - \frac{d(vAP_s)}{dx} - K_1 AP_s + q(1 - \exp(-\lambda x)) = 0 \quad (2.5)$$

$$D_x \frac{d^2(AX_s)}{dx^2} - \frac{d(vAX_s)}{dx} - K_2 AX_s + \alpha(S - X) = 0 \quad (2.6)$$

Considering the actual conditions of the river, when the concentration of pollutants in the river exceeds the permissible limits, the concentration of dissolved oxygen levels may reach zero downstream, it is imperative to identify the river critical position (x_c) where DO exhausts to zero $X_s(x_c) = 0$. Subsequently, the river is segmented into two parts to derive further insights: the first part is $0 \leq x < x_c$ and the second part is $x_c \leq x < \infty$. In order to ascertain the solution in the second section of the river, we need to measure the dissolved oxygen deficit at the transitional point where we have consumed all the substrate.

When the period exceeds a certain point, there are no more substances available, resulting the parameter K_1 in the pollution Equation 2.5 being assigned to zero under the assumption that the pollution level in this section exceeds the regulatory limits. Hence the Equation 2.5 is converted to

$$D_p \frac{d^2(AP_s)}{dx^2} - \frac{d(vAP_s)}{dx} + q(1 - \exp(-\lambda x)) = 0. \quad (2.7)$$

3. Results and interpretations

3.1 Analytical Results

A couple of linear second-order ordinary differential equations Equations 2.5, 2.6, and 2.7 are solved by using the method of undetermined coefficients. Solve the corresponding homogeneous equation by setting the right-hand side to zero. This involves finding the roots of the characteristic equation. The form of particular solutions is based on the exponential term of pollution source. Start by finding the solution for pollution, then substitute the particular solution of pollution to find the solution for dissolved oxygen in the same way. Through the utilization of

analytical methodology, we can derive the steady state solutions that lead to the division of the river into two parts as followed.

$$P_s(x) = \begin{cases} A_1 e^{(\gamma-\beta)x} + \frac{q}{AK_1} + \frac{qe^{-\lambda x}}{AK_4}, & 0 \leq x < x_c \\ A_2 + \frac{q}{vA}x + \frac{qe^{-\lambda x}}{A(D_p\lambda^2 + v\lambda)}, & x_c \leq x < \infty \end{cases} \quad (3.1)$$

$$X_s(x) = \begin{cases} S + B_1 e^{(\delta-\eta)x} - \frac{K_2 q}{K_1 \alpha} + \left(\frac{K_2 A_1 A}{K_5} \right) e^{(\gamma-\beta)x} + \left(\frac{K_2 q}{K_4 K_6} \right) e^{-\lambda x}, & 0 \leq x < x_c \\ 0, & x_c \leq x < \infty \end{cases} \quad (3.2)$$

Where $\gamma = \frac{v}{2D_p}$, $\beta = \sqrt{\frac{v^2}{4D_p^2} + \frac{K_1}{D_p}}$, $\delta = \frac{v}{2D_x}$, $\eta = \sqrt{\frac{v^2}{4D_x^2} + \frac{\alpha}{AD_x}}$, $K_4 = D_p\lambda^2 + v\lambda - K_1$,

$K_5 = AD_x(\gamma - \beta)^2 - vA(\gamma - \beta) - \alpha$, $K_6 = AD\lambda^2 + vA\lambda - \alpha$. A_1 , A_2 , B_1 and X_c are constants.

3.2 Analytical and Numerical Simulation

In this model, P_0 depicts the source concentration of pollution, while X_0 represents the initial level of dissolved oxygen [4]. There are no point sources of pollutants, making $P_s(x)$ as continuous. The parameters in this study are taken as constants which their values are the same as [5], given by Table 3. The four constants values A_1 , A_2 , B_1 , and x_c can be determined by the continuity's conditions of $P_s(x)$ and $X_s(x)$ at $x = x_c$ with applying boundary conditions $P_s(0) = P_0$, $X_s(0) = S$, and $P_s(\infty) < \infty$. Afterwards, three equations to compute for the three constant A_1 , A_2 , and B_1 was used in Equations 3.6 and 3.7 are as follows:

$$A_1 = P_0 - \left(\frac{qK_4 + qK_1}{AK_1K_4} \right) \quad (3.3)$$

$$A_2 = A_1 e^{(\gamma-\beta)x_c} + \frac{qK_4 + qK_1 e^{-\lambda x_c}}{AK_1K_4} - \frac{q}{vA}x_c - \frac{qe^{-\lambda x_c}}{A(D_p\lambda^2 + v\lambda)} \quad (3.4)$$

$$B_1 = \frac{K_2 q}{K_1 \alpha} - \left(\frac{K_2 A_1 A}{K_5} \right) - \left(\frac{K_2 q}{K_4 K_6} \right) \quad (3.5)$$

When obtaining the analytical solutions, we use the bisection method to calculate the unknown parameter x_c in Equation 3.6. The bisection method is a systematic strategy used to locate the root of a continuous function. The method relies on identifying an interval where a zero is

already known to exist, splitting the interval into two equal subintervals, and detecting which subintervals contains the zero[7]. The critical point x_c indicates the threshold at which dissolved oxygen is no longer present. Nevertheless, our attention is also directed towards the point where the oxygen concentration diminishes to a level unsuitable for sustaining fish life.

$$B_1 e^{(\delta-\eta)x_c} + \left(\frac{K_2 A_1 A}{K_5}\right) e^{(\gamma-\beta)x_c} + \left(\frac{K_2 q}{K_4 K_6}\right) e^{-\lambda x_c} + S - \frac{K_2 q}{K_1 \alpha} = 0 \tag{3.6}$$

To ensure fish survival, it is crucial to maintain dissolved oxygen concentrations at a minimum of 30% of the saturated value in aquatic ecosystems, denoted as x_f . By altering the saturated oxygen concentration S by 30%, and evaluating the value of x_f undergoes the following modifications.

$$B_1 e^{(\delta-\eta)x_f} + \left(\frac{K_2 A_1 A}{K_5}\right) e^{(\gamma-\beta)x_f} + \left(\frac{K_2 q}{K_4 K_6}\right) e^{-\lambda x_f} + 0.7S - \frac{K_2 q}{K_1 \alpha} = 0 \tag{3.7}$$

To investigate the factors that influence the location of the crucial value points x_c and x_f . We have selected two primary constant variables: the rate at which pollutants are discharged into the river (q) and the cross-sectional area of the river (A). The solutions of performing numerical calculations for x_c and x_f across a range of q and A values, as depicted in Table 1 and in Table 2, respectively.

Table 1: Value of x_f and x_c by various value the pollutant influx rate along the river : q with $\lambda = 0.06289 \text{ km}^{-1}$

| $q \text{ (kg km}^{-1} \text{ day}^{-1})$ | $x_f \text{ (km)}$ | $x_c \text{ (km)}$ |
|---|--------------------|--------------------|
| 60 | 2483.40474 | 4007.62131 |
| 70 | 2056.67252 | 3229.04290 |
| 80 | 1757.04267 | 2709.87021 |
| 90 | 1534.78669 | 2337.45003 |
| 100 | 1363.23212 | 2056.67291 |

Table 2: Value of x_f and x_c by various of the river’s cross-sectional measurement : A

| $A \text{ (m}^2)$ | $x_f \text{ (km)}$ | $x_c \text{ (km)}$ |
|-------------------|--------------------|--------------------|
| 2100 | 2483.40474 | 4007.62131 |
| 1575 | 1867.841798 | 3011.010509 |
| 1050 | 1252.287457 | 2014.408311 |
| 525 | 636.7650940 | 1017.836521 |
| 100 | 1363.23212 | 2056.67291 |

Table 3: Parameter Values

| Parameters | |
|---|-----------|
| L is the distance from the source to the mouth of a river. (km) | 325* |
| D_p is the pollutant dispersal constant coefficient in the x direction. ($\text{km}^2 \text{day}^{-1}$) | 3.456* |
| D_x is the dissolved oxygen dispersal constant coefficient in the x direction. ($\text{km}^2 \text{day}^{-1}$) | 3.456* |
| v is the horizontal water velocity. (km day^{-1}) | 43.2* |
| A is the river's cross-sectional measurement. (m^2) | 2,100* |
| q is the pollutant influx rate along the river. ($\text{kg km}^{-1} \text{day}^{-1}$) | 60* |
| K_1 is the decay rate constant coefficient at 20° for pollutant. (day^{-1}) for dissolved oxygen. (day^{-1}) | 8.27*** |
| K_2 is the re-aeration rate constant coefficient at 20° | 44.10** |
| k is the oxygen demand concentration at half-saturation for pollutant degradation. (kg m^{-1}) | 0.007**** |
| α is the oxygen transfer across the air-water interface. ($\text{m}^2 \text{day}^{-1}$) | 16.50** |
| S is the saturation level of oxygen. (mg L^{-1}). | 0.01* |
| * [2], ** [8], **** estimated, | |
| *** based on the molecular weights in the chemical reaction $K_1 = \left(\frac{3}{16}\right) K_2$ | |

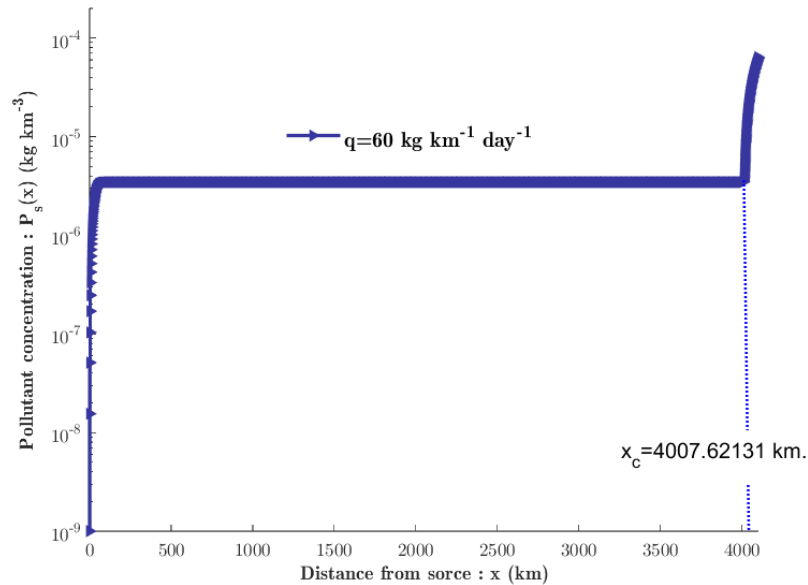


Figure 1: Behaviors of pollutant concentration P_s with $q = 60 \text{ kg km}^{-1} \text{day}^{-1}$ and $\lambda = 0.06289 \text{ km}^{-1}$

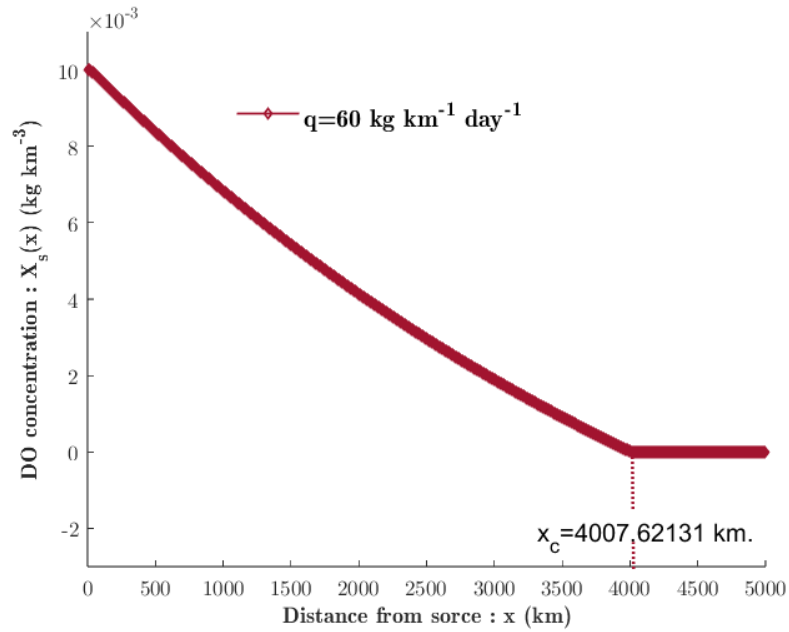


Figure 2: Behaviors of DO concentration X_s with $q = 60 \text{ kg km}^{-1} \text{ day}^{-1}$ and $\lambda = 0.06289 \text{ km}^{-1}$

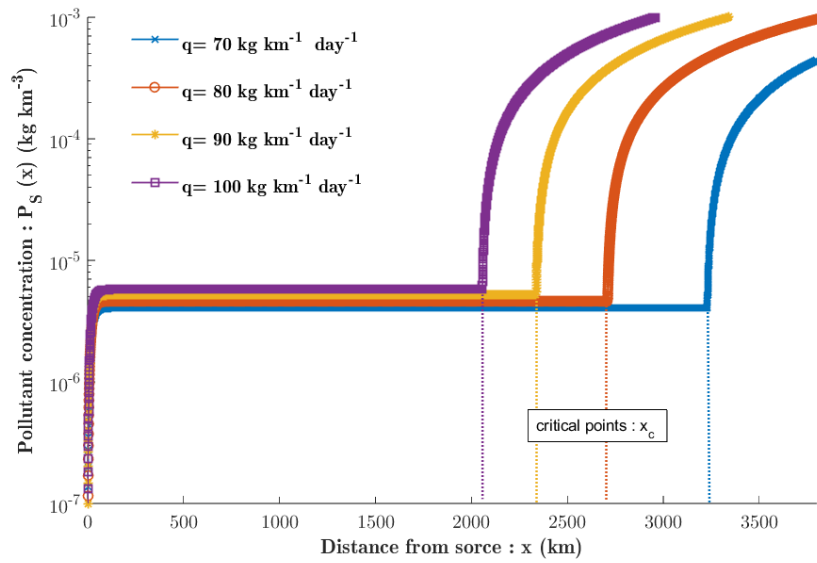


Figure 3: Log plot of pollutant concentrations against x by various of the pollutant influx rate (q) at $70 \leq q \leq 100 \text{ kg km}^{-1} \text{ day}^{-1}$ and $\lambda = 0.06289 \text{ km}^{-1}$

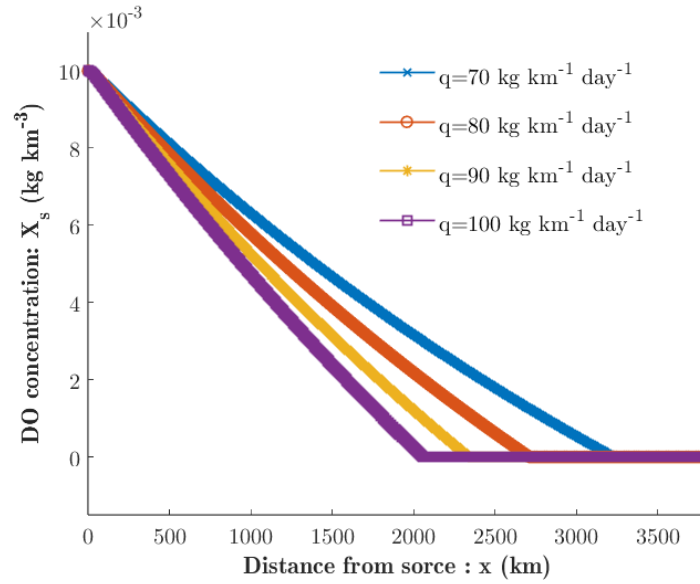


Figure 4: DO concentrations against x by various of the pollutant influx rate (q) at $70 \leq q \leq 100 \text{ kg km}^{-1} \text{ day}^{-1}$ and $\lambda = 0.06289 \text{ km}^{-1}$

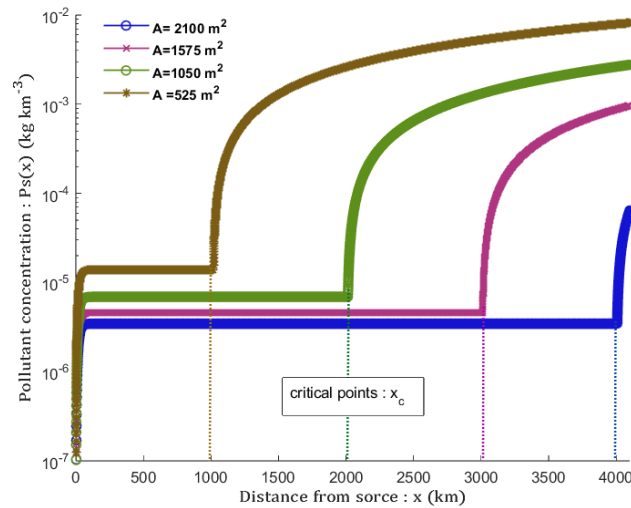


Figure 5: Log plot of pollutant concentrations against x by various of the river's cross-sectional measurement (A) at $525 \leq A \leq 2100 \text{ m}^2$

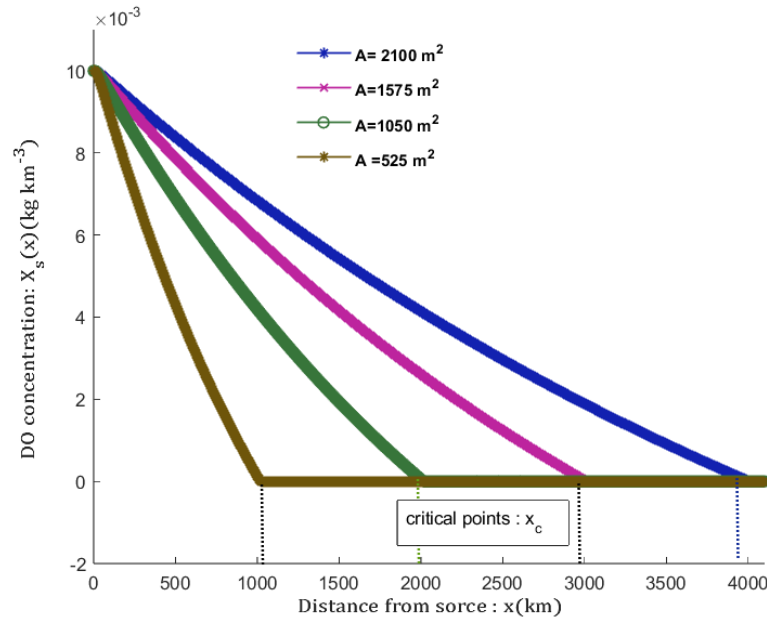


Figure 6: DO concentrations against x by various the river’s cross-sectional measurement (A) at $525 \leq A \leq 2100 \text{ m}^2$

Figure 1 and Figure 2 show pollutant concentration and the dissolved oxygen from the analytical and numerical computations with $q = 60 \text{ kg km}^{-1} \text{ day}^{-1}$ and $\lambda = 0.06289 \text{ km}^{-1}$, respectively. The illustration in Figure 3 presents the pollution concentration resulting from different rates of pollutant discharge (q) at a value of $0.07 \leq q \leq 0.10$ and $\lambda = 0.06289 \text{ km}^{-1}$. It is evident that the concentrations exhibit variation with q , particularly in two distinct sections of the river. The depiction illustrates the location where the river exhibits an excessive number of contaminants at the crucial position x_c . The concentrations increasingly vary obviously with q , an increase in the value of q will lead to the critical point of the river becoming shorter. Figure 4 illustrates the DO concentration depicting the point where the second section approaches zero downstream beyond the critical point x_c . As the parameter q is increased, it is observable that the distances of both x_f and x_c decrease. The river has a cross-sectional area of 2100 square meters. To simplify the analysis, we are reducing the area by a quarter. Figure 5 displays the pollutant concentration that arises from varying the cross-sectional area of the river (A) by $525 \leq A \leq 2100 \text{ m}^2$ and $\lambda = 0.06289 \text{ km}^{-1}$. The relationship between the cross-sectional area of the river and the critical distance is clear: as the river’s cross-sectional area declines, the critical distance also reduces due to the higher concentration of pollutants. Figure 6 exhibits the dissolved oxygen (DO) concentration, displaying the location where the second section reaches zero downstream after the critical point x_c . As the value of A increases, it is evident that both the distances of x_f and x_c diminish. The length of the river extends to 325 km , with $q = 60 \text{ kg km}^{-1} \text{ day}^{-1}$, $A = 2100 \text{ m}^2$ and $\lambda = 0.06289 \text{ km}^{-1}$. This demonstrates that for a river with approximately this length and

cross-sectional, the oxygen level is maintained above the critical threshold necessary for fish survival. The investigation highlights the importance of utilizing this specific model, which indicates that for a river of this value, the oxygen levels remain above both of x_f and x_c .

4. Conclusions

In conclusion, the main aim of this study is focused on the development of steady-state transport modeling by studied the one-dimensional advection-dispersion equations with sink and sources term. The model includes two equations, one describing the rate at which pollutant concentration is reduced using aeration and another representing the physical balance for oxygen consumption to eliminate the pollutant. This model is built by taking into account the exponential growth of pollutant discharge along the river. In this steady-state condition, we simulate from the idea, which the degradation rate coefficient for pollutant value as zero and vary q between 70 to 100 $kg\ km^{-1}\ day^{-1}$ and cross-section area of the river between between 525 to 2100 m^2 , to observe the dissolved oxygen and pollutant behavior and where the critical points are reached. By the observation, the distance of x_f and x_c are proportional to the rate of increase of pollutants along the river and the cross-section area of the river. In future investigations, we may come across scenarios where the model has multiple parameters, either constant coefficients or variable coefficients. We can model pollutant insertion as a function of distance, which is similar to what happens in reality. These findings point to a decline in dissolved oxygen levels due to rising pollution levels without any remediation through water treatment or aeration. In future investigations, we may come across scenarios where the model has multiple parameters, either constant coefficients or variable coefficients. We can model pollutant insertion as a function of distance, which is similar to what happens in reality. Additionally, in order to gain a comprehensive understanding of the actual environmental condition, it would be beneficial to examine this model under non-steady state conditions. Finally, converting this deterministic model into its stochastic equivalent could provide additional insights into how to understand water pollution in practical situations.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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