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Orthogonal Stability of Additive Functional Equation in Fuzzy β-Normed Spaces

M. Nila[1], **A. Vijayalakshmi**[2], **M. Balamurugan**[3,*], **S. Gayathri**[4], **K. Thangapandi**[5], **K. Punniyamoorthy**[6], **N. Prabaharan**[7]

[1] *Department of Mathematics, PSGR Krishnammal College for Women, Coimbatore - 641004, Tamil Nadu, India*

[2] *Department of Mathematics, GRT Institute of Engineering and Technology, Tiruttani, Tiruvallur - 631209, Tamil Nadu, India*

[3] *Department of Mathematics, Vel Tech Rangarajan Dr. Sagunthala R&D Institute of Science and Technology, Chennai - 600 062, Tamil Nadu, India*

[4] *Department of Mathematics, Aarupadai veedu institute of technology, Vinayaka Mission's Research Foundation, Paiyanoor - 603104, Tamil Nadu, India*

[5] *Department of Mathematics, School of advanced Sciences, Kalasalingam Academy of Research and Education, Krishnankoil - 626126, Tamil Nadu, India*

[6] *Department of Mathematics, School of Engineering and Technology, Dhanalakshmi Srinivasan University, Perambalur 621 212, Tamilnadu, India*

[7] *Department of Mathematics, R.M.D. Engineering College, Kavaraipettai, Tiruvallur - 601206, Tamil Nadu, India*

**Corresponding author:* balamurugansvm@gmail.com

Abstract. The primary objective of this study is to present a novel form of generalised additive functional equation $\phi\Big(\sum_{j=1}^{l} jv_j\Big) = \sum_{j=1}^{l} j\phi(v_j)$, where $l \geq 2$ with each $v_i \perp v_j$; $i \neq j = 1, 2, \cdots, l$, and derive its solution. Mainly, we examine the Hyers-Ulam-Rassias orthogonal stability of this equation by utilizing two different approaches.

1. Introduction

The concept of orthogonal additivity emerged in the early 20th century. Readers interested in a comprehensive review might consult the work of Paganoni and Ratz (see [22]). It is worth noting that Birkhoff and James introduced a concept known as the Birkhoff-James orthogonality,

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which establishes a connection between orthogonality in normed linear spaces and inner product spaces (see [3]). In the course of the research, various types of orthogonality on linear spaces were introduced (see [13]). In 1975, researchers Gudder and Strawther [10] proposed an axiomatic framework to describe the concept of orthogonality. This idea was further developed by Rätz and Szabò [25,26]. The researchers proposed the following maxims.

Definition 1.1. *[10] Take into consideration that A is a real vector space with dimensions equal to or greater than two, and that $\perp$ is a binary relation on A that possesses the following properties:*

(G1) *Totality of $\perp$ for zero:*
$v \perp 0, 0 \perp v$ *for every* $v \in A$;

(G2) *Independance:*
If $v, w \in A - \{0\}$, *then* v, w *are linearly independent;*

(G3) *Homogeneity:*
If $v, w \in A$, $v \perp w$, *then* $\alpha v \perp \beta w \ \forall \ \alpha, \beta \in \mathbb{R}$;

(G4) *The Thalesian property:*
Assuming that T is a subspace of A that is two-dimensional, $v \in T$, α in $\mathbb{R}^+$, then there is $w_0 \in T$ fulfills $v \perp w_0$ and $v + w_0 \perp \alpha v - w_0$.

Such a pair $(A, \perp)$ is referred to as an orthogonality space.

Felbin is the one who first introduced the idea of a fuzzy normed linear space. In a linear space with a finite number of dimensions, it was proven to him that fuzzy norms are identical to fuzzy proportionality up to a certain point. A fuzzy normed linear space's finite-dimensional fuzzy subspaces turned out to be essentially complete fuzzy normed linear spaces because of their basic completeness. The notions of fuzy theory was investigated in breifly by many authors [2,4,8].

The following is an example of an important equation that is included in the functional equations theory: Under what conditions does a function that approximately satisfies a functional equation must approximate a precise solution of the problem?

We consider an equation to be stable as soon as it is possible to find a solution that is unique to the situation. In the beginning, Ulam [28] brought up the major stability issue that pertains to group homomorphisms, and Hyers [11] subsequently confirmed that this issue was indeed present. Researchers have comprehensively examined the stability concerns associated with numerous functional equations by employing fixed point methods. These investigations have been conducted by multiple authors, as documented in references [5, 6, 12, 24]. In recent studies, researchers Choonkil Park and Reza Saadati have shown the Hyers-Ulam stability of the orthogonally additive-quadratic functional equation in orthogonality spaces using the fixed point method [18,23].

Initially, Gudder and Strawther [10] was examined the orthogonal Cauchy functional equation. Then, Vajzovic [29] was investigated orthogonally quadratic equation. Afterward, Moslehian and Rassias [20,21], Szabo [27], Moslehian [19], Fochi [9] and Drljevic [7] have examined the orthogonal stability of functional equations. In their study, Ashish [1] examined the Hyers-Ulam-Rassias (H-U-R) stability in the context of Ratz orthogonality for the orthogonally cubic and quartic functional equation. In recent years, there has been extensive research conducted by multiple researchers on the stability issues of various functional equations [14,15,17].

Definition 1.2. *[30] Consider A as a vector space over the field of real numbers. The function N_β is defined as:*
The function maps from the Cartesian product of A and the set of real numbers to the closed interval from 0 to 1. The symbol N_β is referred to as a fuzzy β-norm on A, where $0 < \beta \leq 1$, if for any $k, w \in A$ and $p, r \in \mathbb{R}$,

(F1) $N_\beta(k, r) = 0 \ \forall \ r \leq 0$;
(F2) $k = 0 \Leftrightarrow N_\beta(k, r) = 1 \ \forall \ r > 0$;
(F3) $N_\beta(ck, r) = N_\beta(k, \frac{r}{|c|^\beta})$ *if* $c \neq 0$;
(F4) $N_\beta(k + w, p + r) \geq \min\{N_\beta(k, p), N_\beta(w, r)\}$;
(F5) $N_\beta(k, \cdot)$ *is a monotonically increasing function on the real numbers* $\mathbb{R}$ *and*

$$\lim_{r\to\infty} N_\beta(k, r) = 1;$$

(F6) *for any non-zero value of k, the function $N_\beta(k, \cdot)$ is continuous on the set of real numbers, denoted by* $\mathbb{R}$.

The pair (A, N_β) is referred to as a fuzzy β-normed vector space.

It is worth noting that when $\beta = 1$, the pair (A, N_β) represents a fuzzy normed space denoted as (A, N).

Example 1.1. *[30] Let a pair $(A, \|\cdot\|_\beta)$ be a β-normed linear space with $0 < \beta \leq 1$ and $c, \gamma > 0$. Then*

$$N_\beta(v, p) = \begin{cases} \frac{cp}{cp+\gamma\|v\|_\beta}, & p > 0, \ v \in A; \\ 0, & p \leq 0, v \in A, \end{cases}$$

is a fuzzy β-norm on A.

Definition 1.3. *An odd function $\phi : A \to B$ is called an orthogonally additive function if*

$$\phi\Big(\sum_{1\leq j\leq l} jv_j\Big) = \sum_{j=1}^{l} j\phi(v_j)$$

for every $v_1, v_2, \cdots, v_l \in A$ with $v_i \perp v_j$ for each $i \neq j = 1, 2, \cdots, l$ in the sense of Rätz.

Theorem 1.1. *[16] Consider a generalised complete metric space (A, d) and a strongly contractive mapping $C : A \to A$ with a Lipschitz constant $L < 1$. For each element s in set A, one of the following conditions holds:*

$$d\Big(C^n s, C^{n+1} s\Big) = \infty, \ \forall \ n \geq n_0;$$

or there is $n_0 > 0$ fulfills

(i) $d\Big(C^n s, C^{n+1} s\Big) < \infty, \ \forall \ n \geq n_0$;
(ii) *there is a fixed point s^* of C where the sequence $\{C^n s\}$ converges;*
(iii) *s^* is the only one fixed point of C in $A^* = \{v \in A | d(C^{n_0} s, v) < \infty\}$;*

(iv) $d(v, s^*) \leq \frac{1}{1-L} d(Cv, v),\ \forall\ v \in A^*$.

In this study, the additive functional equation is introduced and generalised for any positive integer l in $\mathbb{R}^+$. The equation is of the form

$$\phi\Big(\sum_{1\leq j\leq l} jv_j\Big) = \sum_{1\leq j\leq l} j\phi(v_j), \tag{1.1}$$

where $l \geq 2$ with each $v_i \perp v_j;\ i \neq j = 1, 2, \cdots, l$, and derive its general solution. Mainly, we examine the Hyers-Ulam-Rassias orthogonal stability of the above equation by direct and fixed-point approaches.

In this case, the orthogonality in the Ratz sense is $\perp$.

2. General Solution

Let A and B are any two real vector spaces.

Theorem 2.1. *If a mapping $\phi : A \to B$ satisfies the functional equation (1.1) for all $v_1, v_2, \cdots, v_l \in A$, then ϕ is additive. i.e., the function ϕ satisfies*

$$\phi(s + k) = \phi(s) + \phi(k)$$

for all $s, k \in A$.

Proof. Consider the mapping $\phi : A \to B$ fulfils the equation (1.1). Now, replacing $(v_1, v_2, \cdots, v_l)$ by $(0, 0, \cdots, 0)$ in (1.1), we have $\phi(0) = 0$. Next, we need to prove that the function ϕ is odd function and which is additive. Setting $(v_1, v_2, \cdots, v_l)$ by $(v, -v, 0, \cdots, 0)$ in equation (1.1), we get $\phi(-v) = -\phi(v)$ for all $v \in A$, which shows that the function ϕ is odd. Switching $v_1 = 0,\ v_2 = v,\ v_3 = \cdots = v_l = 0$ in (1.1), we obtain

$$2\phi(v) = \phi(2v), \tag{2.1}$$

for all $v \in A$. Replacing v by $2v$ in (2.1), we obtain

$$2^2\phi(v) = \phi(2^2 v), \tag{2.2}$$

for all $v \in A$. Switching v by $2v$ in equation (2.2), we arrive

$$2^3\phi(v) = \phi(2^3 v), \tag{2.3}$$

for all $v \in A$. In this way, we can conclude that for any integer $n \in \mathbb{R}^+$, that

$$\phi(2^n v) = 2^n \phi(v), \tag{2.4}$$

for all $v \in A$. Similarly, we have that

$$\phi\Big(\frac{v}{2^n}\Big) = \frac{1}{2^n}\phi(v), \tag{2.5}$$

for all $v \in A$. Replacing $(v_1, v_2, \cdots, v_l)$ by $(s, \frac{k}{2}, 0, \cdots, 0)$ in (1.1), we reach our needed outcome of the function ϕ is additive. □

Define a difference operator $\Delta\phi : A^l \to B$ by

$$\Delta\phi(v_1, v_2, \cdots, v_l) = \phi\Big(\sum_{1\le j\le l} jv_j\Big) - \sum_{1\le j\le l} j\phi(v_j),\ \forall\ v_1, v_2, \cdots, v_l \in A,$$

with each $v_i \perp v_j\ ; i \neq j = 1, 2, \cdots, l$.

3. Orthogonal H-U Stability in Fuzzy β-Normed Spaces

Here, we consider (B, S_β), (Z, S'_β) and $(A, \perp)$ are complete fuzzy β-normed space, fuzzy β-normed space and real orthogonality vector space, respectively, with $0 < \beta \le 1$.

3.1. **Stability Results Using Direct Approach.**

Theorem 3.1. *If mapping $\phi : A \to B$ such that*

$$S_\beta(\Delta\phi(v_1, v_2, \cdots, v_l), \epsilon) \ge S'_\beta(\psi(v_1, v_2, \cdots, v_l), \epsilon), \tag{3.1}$$

with $v_i \perp v_j\ ; i \neq j = 1, 2, \cdots, l$, and a mapping $\psi : A^l \to Z$ such that

$$S'_\beta\Big(\psi\Big(\frac{v_1}{2^k}, \frac{v_2}{2^k}, \cdots, \frac{v_l}{2^k}\Big), \epsilon\Big) \ge S'_\beta\Big(\psi\Big(v_1, v_2, \cdots, v_l\Big), \frac{\epsilon}{\rho^{k\beta}}\Big),\ k \ge 0 \tag{3.2}$$

for all $v_1, v_2, \cdots, v_l \in A, \epsilon > 0$ with $v_i \perp v_j\ ; i \neq j = 1, 2, \cdots, l$ and for some constant $\rho \in \mathbb{R}$ with $0 < \rho < \frac{1}{2}$. Then there exists an unique orthogonal additive mapping $C_1 : A \to B$ satisfying

$$S_\beta\left(\phi(v) - C_1(v), \epsilon\right) \ge S'_\beta\left(\psi(0, v, 0, \cdots, 0), (\rho^{-\beta} - 2^\beta)\epsilon\right) \tag{3.3}$$

for all $v \in A, \epsilon > 0$.

Proof. Replacing $(v_1, v_2, \cdots, v_l)$ by $(0, 0, \cdots, 0)$ in (1.1), we have

$$S_\beta\left(\phi(2v) - 2\phi(v), \epsilon\right) \ge S'_\beta\left(\psi(0, v, 0, \cdots, 0), \epsilon\right), \tag{3.4}$$

for all $v \in A$ and $\epsilon > 0$. Replacing v by $\frac{v}{2}$ in (3.4), we obtain

$$S_\beta\left(\phi(v) - 2\phi\left(\frac{v}{2}\right), \epsilon\right) \ge S'_\beta\left(\psi(0, \frac{v}{2}, 0, \cdots, 0), \epsilon\right), \tag{3.5}$$

for all $v \in A$ and $\epsilon > 0$. Replacing v by $\frac{v}{2}$ in (3.5), we obtain

$$S_\beta\left(\phi\left(\frac{v}{2}\right) - 2\phi\left(\frac{v}{2^2}\right), \epsilon\right) \ge S'_\beta\left(\psi(0, \frac{v}{2^2}, 0, \cdots, 0), \epsilon\right), \tag{3.6}$$

for all $v \in A$ and $\epsilon > 0$. It follows from the inequality (3.6), that

$$S_\beta\left(2\phi\left(\frac{v}{2}\right) - 2^2\phi\left(\frac{v}{2^2}\right), 2^\beta\epsilon\right) \ge S'_\beta\left(\psi\left(0, \frac{v}{2^2}, 0, \cdots, 0\right), \epsilon\right), \tag{3.7}$$

for all $v \in A$ and $\epsilon > 0$. Replacing v by $\frac{v}{2}$ in (3.7), we get

$$S_\beta\left(2^2\phi\left(\frac{v}{2^2}\right) - 2^3\phi\left(\frac{v}{2^3}\right), 2^{2\beta}\epsilon\right) \ge S'_\beta\left(\psi\left(0, \frac{v}{2^3}, 0, \cdots, 0\right), \epsilon\right),\ \forall v \in A, \epsilon > 0. \tag{3.8}$$

In this way, we can generalize for any integer $k \in \mathbb{R}^+$, that

$$S_\beta\left(2^k\phi\left(\frac{v}{2^k}\right) - 2^{k+1}\phi\left(\frac{v}{2^{k+1}}\right), 2^{k\beta}\epsilon\right) \ge S'_\beta\left(\psi\left(0, \frac{v}{2^{k+1}}, 0, \cdots, 0\right), \epsilon\right), \tag{3.9}$$

for all $v \in A$ and $\epsilon > 0$. Using the inequality (3.2), we obtain that

$$S_\beta\left(2^k\phi\left(\frac{v}{2^k}\right)-2^{k+1}\phi\left(\frac{v}{2^{k+1}}\right),2^{k\beta}\epsilon\right)\geq S_\beta^{'}\left(\psi\left(0,v,0,\cdots,0\right),\frac{\epsilon}{\rho^{\beta(k+1)}}\right), \tag{3.10}$$

for all $v \in A$ and $\epsilon > 0$. Replacing ϵ by $\rho^{\beta(k+1)}\epsilon$ in (3.10), we have

$$S_\beta\left(2^k\phi\left(\frac{v}{2^k}\right)-2^{k+1}\phi\left(\frac{v}{2^{k+1}}\right),2^{k\beta}\rho^{\beta(k+1)}\epsilon\right)\geq S_\beta^{'}\left(\psi\left(0,v,0,\cdots,0\right),\epsilon\right), \tag{3.11}$$

for all $v \in A$ and $\epsilon > 0$. Then

$$\begin{aligned} S_\beta\left(2^l\phi\left(\frac{v}{2^l}\right)-\phi(v),\sum_{k=0}^{l-1}2^{k\beta}\rho^{(k+1)\beta}\epsilon\right) \geq\ & \min\cup_{k=0}^{l-1}\left\{S_\beta\left(2^{k+1}\phi\left(\frac{v}{2^{k+1}}\right)\right)-2^k\phi\left(\frac{v}{2^k}\right),2^{k\beta}\rho^{\beta(k+1)}\epsilon\right\} \\ \geq\ & S_\beta^{'}\left(\psi\left(0,v,0,\cdots,0\right),\epsilon\right), \end{aligned} \tag{3.12}$$

this implies

$$\begin{aligned} S_\beta\Big(2^{l+m}\phi\left(\frac{v}{2^{l+m}}\right) & -2^m\phi\left(\frac{v}{2^m}\right),\sum_{k=0}^{l-1}2^{\beta(k+m)}\rho^{(k+m+1)\beta}\epsilon\Big) \\ \geq\ & \min\cup_{k=0}^{l-1}\left\{S_\beta\left(2^{k+m+1}\phi\left(\frac{v}{2^{k+m+1}}\right)\right)-2^{k+m}\phi\left(\frac{v}{2^{k+m}}\right),2^{\beta(k+m)}\rho^{\beta(k+m+1)}\epsilon\right\} \\ \geq\ & S_\beta^{'}\left(\psi\left(0,v,0,\cdots,0\right),\epsilon\right),\ \forall v\in A,\epsilon>0, \end{aligned}$$

with $l > 0$, $m \geq 0$. Therefore,

$$S_\beta\left(2^{l+m}\phi\left(\frac{v}{2^{l+m}}\right)-2^m\phi\left(\frac{v}{2^m}\right),\epsilon\right)\geq S_\beta^{'}\left(\psi\left(0,v,0,\cdots,0\right),\frac{\epsilon}{\sum_{k=0}^{l-1}2^{\beta(m+k)}\rho^{\beta(m+k+1)}}\right) \tag{3.13}$$

for all $v \in A$, $\epsilon > 0$, and $l > 0$, $m \geq 0$. With $\sum_{k=0}^{l-1}2^{\beta k}\rho^{\beta k}$ as its convergent series. In inequality (3.13), we see that the sequence $\left\{2^l\phi\left(\frac{v}{2^l}\right)\right\}$ is a Cauchy sequence in (B, S_β) as $m \to \infty$. Thus, it converges in B. Next, we can define a mapping $C_1 : A \to B$ by

$$C_1(v) = S_\beta - \lim_{l\to\infty}2^l\phi\left(\frac{v}{2^l}\right),$$

for all $v \in A$. Clearly,

$$\lim_{l\to\infty}S_\beta\left(C_1(v)-2^l\phi\left(\frac{v}{2^k}\right),\epsilon\right)=1,$$

for all $v \in A$ and all $\epsilon > 0$. Replacing $(v_1, v_2, \cdots, v_l)$ by $(2^m v_1, 2^m v_2, \cdots, 2^m v_l)$ in inequality (3.1), we obtain

$$S_\beta\left(\Delta\phi\left(\frac{v_1}{2^m},\frac{v_2}{2^m},\cdots,\frac{v_l}{2^m}\right),\epsilon\right)\geq S_\beta^{'}\left(\psi\left(\frac{v_1}{2^m},\frac{v_2}{2^m},\cdots,\frac{v_l}{2^m}\right),\epsilon\right),$$

for all $v_1, v_2, \cdots, v_l \in A, \epsilon > 0$, with $v_i \perp v_j$; $i \neq j = 1, 2, \cdots, l$ and $m \in \mathbb{N}$. As $\frac{v_i}{2^m}\perp\frac{v_j}{2^m}$, we obtain

$$S_\beta\left(2^m\Delta\phi\left(\frac{v_1}{2^m},\frac{v_2}{2^m},\cdots,\frac{v_l}{2^m}\right),\epsilon\right)\geq S_\beta^{'}\left(\psi\left(\frac{v_1}{2^m},\frac{v_2}{2^m},\cdots,\frac{v_l}{2^m}\right),\frac{\epsilon}{2^{m\beta}}\right)$$

for all $v_1, v_2, \cdots, v_l \in A, \epsilon > 0$, with $v_i \perp v_j$; $i \neq j = 1, 2, \cdots, l$ and $m \in \mathbb{N}$. Since

$$\lim_{m\to\infty}S_\beta^{'}\left(\psi\left(v_1,v_2,\cdots,v_l\right),\frac{\epsilon}{(2\rho)^{m\beta}}\right)=1,\ \forall\, v_1,v_2,\cdots,v_l\in A,\epsilon>0,$$

with $v_i \perp v_j;\ i \neq j = 1, 2, \cdots, l$ and $m \in \mathbb{N}$, we get

$$S_\beta\left(C_1\Big(\sum_{1\le j\le l} jv_j\Big) - \sum_{1\le j\le l} jC_1(v_j), \epsilon\right) = 1,\ \forall\, v_1, v_2, \cdots, v_l \in A, \epsilon > 0,$$

with $v_i \perp v_j;\ i \neq j = 1, 2, \cdots, l$. We can conclude that, the mapping $C_1 : A \to B$ is additive. Since, the function $\phi(v)$ is an odd, so that $C_1(v)$ is an odd function.

Therefore, the additive function $C_1 : A \to B$ is orthogonal. Putting $m = 0$ and taking limit as l tends to ∞ in inequality (3.13), we get (3.3). Now, we want to prove that the uniqueness of C_1.

Suppose an another additive mapping $C_2 : A \to B$ satisfying (3.3). Hence

$$\begin{aligned}
S_\beta\Big(C_1(v) - C_2(v), \epsilon\Big) = \quad & S_\beta\left(2^m C_1\left(\frac{v}{2^m}\right) - 2^m C_2\left(\frac{v}{2^m}\right), \epsilon\right) \\
\geq \quad & \min\left\{ S_\beta\Big(2^m C_1\Big(\frac{v}{2^m}\Big) - 2^m \phi\Big(\frac{v}{2^m}\Big), \frac{\epsilon}{2}\Big),\right. \\
& \qquad\qquad \left. S_\beta\Big(2^m \phi\Big(\frac{v}{2^m}\Big) - 2^m C_2\Big(\frac{v}{2^m}\Big), \frac{\epsilon}{2}\Big)\right\} \\
\geq \quad & \min\left\{ S'_\beta\Big(\psi\Big(0, \frac{v}{2^m}, 0, \cdots, 0\Big), \frac{(\rho^\beta - 2^\beta \epsilon}{2(2^{m\beta})}\Big),\right. \\
& \qquad\qquad \left. S'_\beta\Big(\psi\Big(0, \frac{v}{2^m}, 0, \cdots, 0\Big), \frac{(\rho^\beta - 2^\beta \epsilon}{2(2^{m\beta})}\Big)\right\} \\
\geq \quad & S'_\beta\Big(\psi(0, v, 0, \cdots, 0), \frac{(\rho^{-\beta} - 2^\beta)\epsilon}{2} \frac{1}{(2\rho)^{m\beta}} \quad \to\ 1 \text{ as } m \to \infty,
\end{aligned}$$

for all $v \in A, \epsilon > 0$ and all $m \in \mathbb{N}$. Therefore, the mapping $C_1 : A \to B$ is unique. □

Theorem 3.2. *If an odd mapping $\phi : A \to B$ satisfying (3.1) with $v_i \perp v_j\,; i \neq j = 1, 2, \cdots, l$, and a mapping $\psi : A^l \to Z$ such that*

$$S'_\beta\Big(\psi\Big(v_1, v_2, \cdots, v_l\Big), \rho^{k\beta}\epsilon\Big) \geq S'_\beta\Big(\psi\Big(\frac{v_1}{2^k}, \frac{v_2}{2^k}, \cdots, \frac{v_l}{2^k}\Big), \epsilon\Big),\ k \geq 0 \tag{3.14}$$

for all $v_1, v_2, \cdots, v_l \in A, \epsilon > 0$ with $v_i \perp v_j\,; i \neq j = 1, 2, \cdots, l$ and for some constant $\rho \in \mathbb{R}$ with $0 < \rho < 2$. Then there exists an unique orthogonally additive mapping $C_1 : A \to B$ satisfying

$$S_\beta\left(\phi(v) - C_1(v), \epsilon\right) \geq S'_\beta\left(\psi(0, v, 0, \cdots, 0), (2^\beta - \rho^\beta)\epsilon\right) \tag{3.15}$$

for all $v \in A, \epsilon > 0$.

Proof. Replacing $(v_1, v_2, v_3, \cdots, v_l)$ by $(0, v, 0, \cdots, 0)$ in (1.1), we have

$$S_\beta\left(\phi(2v) - 2\phi(v), \epsilon\right) \geq S'_\beta\left(\psi(0, v, 0, \cdots, 0), \epsilon\right), \tag{3.16}$$

for all $v \in A$ and all $\epsilon > 0$. It follows from (3.16) that

$$S_\beta\left(\frac{\phi(2v)}{2} - \phi(v), \frac{\epsilon}{2^\beta}\right) \geq S'_\beta\left(\psi(0, v, 0, \cdots, 0), \epsilon\right), \tag{3.17}$$

for all $v \in A$ and all $\epsilon > 0$. Setting $v = 2v$ in (3.17), we arrive

$$S_\beta\left(\frac{\phi(2^2 v)}{2^2} - \frac{\phi(2v)}{2}, \frac{\epsilon}{2^{2\beta}}\right) \geq S_\beta^{'}\left(\psi(0, 2v, 0, \cdots, 0), \epsilon\right), \tag{3.18}$$

for all $v \in A$ and all $\epsilon > 0$. Switching v by $2v$ in (3.18), we get

$$S_\beta\left(\frac{\phi(2^3 v)}{2^3} - \frac{\phi\left(2^2 v\right)}{2^2}, \frac{\epsilon}{2^{3\beta}}\right) \geq S_\beta^{'}\left(\psi(0, 2^2 v, 0, \cdots, 0), \epsilon\right), \tag{3.19}$$

for all $v \in A$ and all $\epsilon > 0$. In this way, we can generalize for any integer $k \in \mathbb{R}^+$, that

$$S_\beta\left(\frac{\phi\left(2^{k+1} v\right)}{2^{k+1}} - \frac{\phi\left(2^k v\right)}{2^k}, \frac{\epsilon}{2^{(k+1)\beta}}\right) \geq S_\beta^{'}\left(\psi\left(0, 2^k v, 0, \cdots, 0\right), \epsilon\right), \tag{3.20}$$

for all $v \in A$ and all $\epsilon > 0$. Using the inequality (3.14), we obtain that

$$S_\beta\left(\frac{\phi\left(2^{k+1} v\right)}{2^{k+1}} - \frac{\phi\left(2^k v\right)}{2^k}, \frac{\epsilon}{2^{(k+1)\beta}}\right) \geq S_\beta^{'}\left(\psi\left(0, v, 0, \cdots, 0\right), \frac{\epsilon}{\rho^{k\beta}}\right), \tag{3.21}$$

for all $v \in A$ and all $\epsilon > 0$. Substituting ϵ by $\rho^{k\beta}\epsilon$ in inequality (3.21), we have

$$S_\beta\left(\frac{\phi\left(2^{k+1} v\right)}{2^{k+1}} - \frac{\phi\left(2^k v\right)}{2^k}, \frac{\rho^{k\beta}\epsilon}{2^{\beta(k+1)}}\right) \geq S_\beta^{'}\left(\psi\left(0, v, 0, \cdots, 0\right), \epsilon\right), \tag{3.22}$$

for all $v \in A$ and all $\epsilon > 0$. Hence

$$S_\beta\left(\frac{\phi\left(2^l v\right)}{2^l} - \phi(v), \sum_{k=0}^{l-1} \frac{\rho^{k\beta}\epsilon}{2^{(k+1)\beta}}\right) \geq S_\beta^{'}\left(\psi\left(0, v, 0, \cdots, 0\right), \epsilon\right), \tag{3.23}$$

this implies

$$\begin{aligned} S_\beta\Big(\frac{\phi\left(2^{l+m} v\right)}{2^{l+m}} - \frac{\phi(2^m v)}{2^m}, \quad & \sum_{k=0}^{l-1} \frac{\rho^{(k+m)\beta}\epsilon}{2^{(k+m+1)\beta}}\Big) \\ \geq \quad & \min \cup_{k=0}^{l-1}\Big\{S_\beta \frac{\phi(2^{k+m+1} v)}{2^{k+m+1}} - \frac{\phi(2^{k+m} v)}{2^{k+m}}, \frac{\rho^{\beta(k+m)}\epsilon}{2^{\beta(k+m+1)}}\Big\} \\ \geq \quad & S_\beta^{'}\left(\psi\left(0, v, 0, \cdots, 0\right), \epsilon\right), \end{aligned}$$

for all $v \in A$ and all $\epsilon > 0$, with $l > 0$, $m \geq 0$. Following this, the proof is similar to the proof of the Theorem 3.1. □

Corollary 3.1. *Let $\lambda \geq 0$, $w \in \mathbb{R}^+$ with $w > 1$, $(A, \perp)$ be a real orthogonality vector space with norm $\|\cdot\|$ with $0 < \beta \leq 1$ and $(\mathbb{R}, S^{'})$ be a complete fuzzy β-normed space. If a mapping $\phi : A \to B$ such that*

$$S_\beta\left(\Delta\phi(v_1, v_2, \cdots, v_l), \epsilon\right) \geq S_\beta^{'}\left(\lambda\left(\sum_{1 \leq j \leq l} \|v_j\|^w\right), \epsilon\right), \tag{3.24}$$

with each $v_i \perp v_j$; $i \neq j = 1, 2, \cdots, l$, there exists an unique orthogonally additive mapping $C_1 : A \to B$ satisfying

$$S_\beta\left(\phi(v) - C_1(v), \epsilon\right) \geq S_\beta^{'}\left(\lambda\|v\|^w, (2^{w\beta} - 2^\beta)\epsilon\right),$$

for all $v \in A$ and all $\epsilon > 0$.

Proof. From Theorem 3.1 by considering $\psi(v_1, v_2, \cdots, v_l) = \lambda \sum_{j=0}^{l} \|v_j\|^w$ and choosing $\rho = 2^{-w}$, we get our needed outcome. □

Corollary 3.2. *Assume the same hypotheses as in Corollary 3.1 except that there exists an unique orthogonally additive mapping $C_1 : A \to B$ satisfying*

$$S_\beta\left(\phi(v) - C_1(v), \epsilon\right) \geq S'_\beta\left(\lambda\|v\|^w, (2^\beta - 2^{w\beta})\epsilon\right), \ \forall\, v \in A, \epsilon > 0.$$

Proof. From Theorem 3.2 by considering $\psi(v_1, v_2, \cdots, v_l) = \lambda \sum_{j=0}^{l} \|v_j\|^w$ and choosing $\rho = 2^{w}$, we get our needed outcome. □

3.2. **Stability Results Using Fixed Point Approach.**

Theorem 3.3. *If a mapping $\psi : A^3 \to [0, \infty)$, there exists a Lipschitz constant $L(0 < L < 1)$ such that*

$$\psi\left(2^k v_1, 2^k v_2, \cdots, 2^k v_l\right) \leq 2^{k\beta} L^k \psi\left(v_1, v_2, \cdots, v_l\right), \quad \forall v_1, v_2, \cdots, v_l \in A, \tag{3.25}$$

with each $v_i \perp v_j$; $i \neq j = 1, 2, \cdots, l$. Suppose an odd mapping $\phi : A \to B$ such that

$$S_\beta\left(\Delta\phi\left(v_1, v_2, \cdots, v_l\right), \epsilon\right) \geq \frac{\epsilon}{\epsilon + \psi\left(v_1, v_2, \cdots, v_l\right)}, \quad \forall v_1, v_2, \cdots, v_l \in A, \tag{3.26}$$

with every $v_i \perp v_j$; $i \neq j = 1, 2, \cdots, l$, then there exists an unique additive mapping $C_1 : A \to B$ satisfying

$$S_\beta\left(\phi(v) - C_1(v), \epsilon\right) \geq \frac{2^\beta \epsilon(1-L)}{2^\beta \epsilon(1-L) + \psi(0, v, 0, \cdots, 0)} \tag{3.27}$$

for all $v \in A, \epsilon > 0$.

Proof. Setting $v_1 = 0$, $v_2 = v$, $v_3 = \cdots = v_l = 0$ in (3.26), we obtain

$$S_\beta\left(\phi(2v) - 2\phi(v), \epsilon\right) \geq \frac{\epsilon}{\epsilon + \psi(0, v, 0, \cdots, 0)}, \tag{3.28}$$

for all $v \in A$ and all $\epsilon > 0$. Using $(F3)$, we get

$$S_\beta\left(\frac{\phi(2v)}{2} - \phi(v), \frac{\epsilon}{2^\beta}\right) \geq \frac{\epsilon}{\epsilon + \psi(0, v, 0, \cdots, 0)}, \ \forall v \in A, \epsilon > 0. \tag{3.29}$$

Suppose, we can define,

$$G = \{f : A \to B | f(0) = 0\}$$

and denotes a generalized metric on G as

$$d(f, g) = \inf\left\{\tau \in [0, \infty] | S_\beta\left(f(v) - g(v), \tau\epsilon\right) \geq \frac{\epsilon}{\epsilon + \psi(0, v, 0, \cdots, 0)}, \ v \in A, \epsilon > 0\right\}.$$

Then (G, d) is complete. Suppose a mapping $T : G \to G$ by

$$(Tg)(v) = \frac{g(2v)}{2}, \ \forall v \in A.$$

Now, we want to show that T is strictly contractive on G. For all $f,g \in G$, consider $d(f,g)=\rho$. Therefore,

$$S_\beta\left(f(v)-g(v),\rho\epsilon\right) \geq \frac{\epsilon}{\epsilon+\psi(0,v,0,\cdots,0)},\ \forall v \in A, \epsilon>0.$$

Hence,

$$\begin{aligned} S_\beta\left(Tf)(v)-(Tg)(v),\rho L\epsilon\right) = & \quad S_\beta\left(\frac{f(2v)}{2}-\frac{g(v)}{2},\rho L\epsilon\right) \\ = & \quad S_\beta\left(f(2v)-g(2v),2^\beta\rho L\epsilon\right) \\ \geq & \quad \frac{2^\beta L\epsilon}{2^\beta L\epsilon+\psi(0,2v,0,\cdots,0)} \\ \geq & \quad \frac{\epsilon}{\epsilon+\psi(0,v,0,\cdots,0)},\ \forall v \in A, \epsilon>0. \end{aligned}$$

We conclude that $d(Tf,Tg) \leq \rho L$. We obtain

$$d\left(Tf,Tg\right) \leq Ld(f,g),$$

for all $f,g \in V$. Next, we show that $d(T\phi,\phi)<\infty$. From (2.4), we have

$$\begin{aligned} S_\beta\left((T\phi)(v)-\phi(v),\frac{\epsilon}{2^\beta}\right) & \quad \geq \frac{\epsilon}{\epsilon+\psi(0,v,0,\cdots,0)}, \\ \Rightarrow d(T\phi,\phi) & \quad \leq \frac{1}{2^\beta}<\infty, \end{aligned}$$

for all $v \in A$ and all $\epsilon>0$. By Theorem 1.1, there exists a fixed point C_1 of T satisfies $T^k\phi \to C_1$, namely,

$$C_1(v)=\lim_{k\to\infty}\frac{\phi(2^kv)}{2^k},\ \forall v \in A.$$

By using Theorem 1.1, we get

$$\begin{aligned} d(\phi,C_1) & \leq \frac{1}{1-L}d(T\phi,\phi) \\ & \leq \frac{1}{2^\beta}\frac{1}{1-L}, \end{aligned}$$

for all $v \in A$. So that,

$$S_\beta\left(\phi(v)-C_1(v),\frac{1}{2^\beta}\frac{1}{1-L}\epsilon\right) \geq \frac{\epsilon}{\epsilon+\psi(0,v,0,\cdots,0)},\ \forall v \in A, \epsilon>0.$$

From the above, inequality (3.27) is holds for all $v \in A$. From the inequality (3.25) and (3.26), we obtain

$$S_\beta\left(\frac{1}{2^k}\Delta\phi(2^kv_1,2^kv_2,\cdots,2^kv_l),\frac{\epsilon}{2^{k\beta}}\right) \geq \frac{\epsilon}{\epsilon+\psi(2^kv_1,2^kv_2,\cdots,2^kv_l)}$$

for all $v_1, v_2, \cdots, v_l \in A$, $\epsilon > 0$ and $k \in \mathbb{N}$, with each $2^k v_i \perp 2^k v_j$; $i \neq j = 1, 2, \cdots, l$. Using inequality (3.25), we obtain

$$\begin{aligned} S_\beta\Big(\frac{1}{2^k}\Delta\phi(2^k v_1, 2^k v_2, \cdots, 2^k v_l), \epsilon\Big) &\geq \frac{2^{k\beta}\epsilon}{2^{k\beta}\epsilon + 2^{k\beta}L^k\psi(v_1, v_2, \cdots, v_l)} \\ &\geq \frac{\epsilon}{\epsilon + L^k\psi(v_1, v_2, \cdots, v_l)},\ \forall v_1, v_2, \cdots, v_l \in A, \epsilon > 0 \end{aligned}$$

with each $v_i \perp v_j$; $i \neq j = 1, 2, \cdots, l$, and $k \in \mathbb{N}$. Since,

$$\lim_{k\to\infty} \frac{\epsilon}{\epsilon + L^k\psi(v_1, v_2, \cdots, v_l)} = 1,\ \forall v_1, v_2, \cdots, v_l \in A, \epsilon > 0 \tag{3.30}$$

with each $v_i \perp v_j$; $i \neq j = 1, 2, \cdots, l$, we get

$$C_1\Big(\sum_{j=1}^{l} j v_j\Big) = \sum_{j=1}^{l} j C_1(v_j),\ \forall v_1, v_2, \cdots, v_l \in A$$

with for each $v_i \perp v_j$; $i \neq j = 1, 2, \cdots, l$. Suppose that the inequality (3.27) is also satisfies the another additive mapping $C_2 : A \to B$ besides C_1. C_2 fulfils $C_2(v) = \frac{1}{2}C_2(2v) = (TC_2)(v)$, C_2 is a fixed point of T.

Using condition (3.27) and the definition of d, we have

$$d\,(\phi, C_2) \leq \frac{1}{2^\beta}\left(\frac{1}{1-L}\right) < \infty.$$

Now,

$$\begin{aligned} d\,(T\phi, C_2) &\leq d\,(T\phi, \phi) + d\,(\phi, C_2) \\ &\leq \frac{1}{2^\beta} + \frac{1}{2^\beta}\left(\frac{1}{1-L}\right) < \infty. \end{aligned}$$

Thus, $C_2 \in G = \{s \in G \mid d(T\phi, s) < \infty\}$. Again, by the sense of Theorem 1.1, implies that $C_1 = C_2$. This shows the uniqueness of C_1. □

Theorem 3.4. *If an odd mapping $\psi : A^3 \to [0, \infty)$ and there exists $L(0 < L < 1)$ such that*

$$\psi\left(\frac{v_1}{2^k}, \frac{v_2}{2^k}, \cdots, \frac{v_l}{2^k}\right) \leq \frac{L^k}{2^{k\beta}}\psi\,(v_1, v_2, \cdots, v_l) \tag{3.31}$$

for all $v_1, v_2, \cdots, v_l \in A$, with each $v_i \perp v_j$; $i \neq j = 1, 2, \cdots, l$. Suppose an odd mapping $\phi : A \to B$ satisfies the inequality (3.26) with each $v_i \perp v_j$; $i \neq j = 1, 2, \cdots, l$, then there exists an unique orthogonally additive mapping $C_1 : A \to B$ satisfying

$$S_\beta\,(\phi(v) - C_1(v), \epsilon) \geq \frac{2^\beta((L)^{-1} - 1)\epsilon}{2^\beta((L)^{-1} - 1)\epsilon + \psi(0, v, 0, \cdots, 0)} \tag{3.32}$$

for all $v \in A, \epsilon > 0$.

Proof. Setting $v_1 = 0,\ v_2 = v,\ v_3 = \cdots = v_l = 0$ in (3.26), we obtain

$$S_\beta\left(\phi(2v) - 2\phi(v), \epsilon\right) \geq \frac{\epsilon}{\epsilon + \psi(0, v, 0, \cdots, 0)}, \tag{3.33}$$

for all $v \in A$ and all $\epsilon > 0$. Switching v by $\frac{v}{2}$ in (3.33), we arrive

$$\begin{aligned} S_\beta\left(\phi(v) - 2\phi\left(\frac{v}{2}\right), \epsilon\right) &\geq \frac{\epsilon}{\epsilon + \psi\left(0, \frac{v}{2}, 0, \cdots, 0\right)} \\ \Rightarrow S_\beta\left(\phi(v) - 2\phi\left(\frac{v}{2}\right), \epsilon\right) &\geq \frac{2^\beta \epsilon}{2^\beta \epsilon + L\psi\left(0, v, 0, \cdots, 0\right)}, \end{aligned}$$

for all $v \in A$ and all $\epsilon > 0$. As is the case with the proof of Theorem 3.3, the other parts of the proof are exactly the same. □

Corollary 3.3. *Let $\lambda \geq 0$, $w \in \mathbb{R}^+$ with $w < 1$, and $(A, \perp)$ be a real orthogonality vector space with β-norm $\|\cdot\|_\beta$ with $0 < \beta \leq 1$. Suppose that $\phi : A \to B$ is an odd mapping satisfies*

$$S_\beta\left(\Delta\phi(v_1, v_2, \cdots, v_l), \epsilon\right) \geq \frac{\epsilon}{\epsilon + \lambda\left(\sum_{j=1}^{l} \|v_j\|_\beta^w\right)}, \tag{3.34}$$

for all $v_1, v_2, \cdots, v_l \in A, \epsilon > 0$, with for each $v_i \perp v_j$; $i \neq j = 1, 2, \cdots, l$. Then there exists an unique orthogonally additive mapping $C_1 : A \to B$ satisfying

$$S_\beta\left(\phi(v) - C_1(v), \epsilon\right) \geq \frac{(2^\beta - 2^{w\beta})\epsilon}{(2^\beta - 2^{w\beta})\epsilon + \lambda\|v\|_\beta^w}$$

for all $v \in A, \epsilon > 0$.

Proof. From Theorem 3.3 by considering $\psi(v_1, v_2, \cdots, v_l) = \lambda \sum_{j=0}^{l} \|v_j\|_\beta^w$ and choosing $L = \left(\frac{2^w}{2}\right)^\beta$, we get our needed outcome. □

Corollary 3.4. *Assume the same hypotheses as in Corollary 3.3 except that $w > 1$, and there exists an unique orthogonally additive mapping $C_1 : A \to B$ satisfying*

$$S_\beta\left(\phi(v) - C_1(v), \epsilon\right) \geq \frac{(2^{w\beta} - 2^\beta)\epsilon}{(2^{w\beta} - 2^\beta)\epsilon + \lambda\|v\|_\beta^w}$$

for all $v \in A, \epsilon > 0$.

Proof. From Theorem 3.4 by considering $\psi(v_1, v_2, \cdots, v_l) = \lambda \sum_{j=0}^{l} \|v_j\|_\beta^w$ and choosing $L = \left(\frac{2}{2^w}\right)^\beta$, we get our needed outcome. □

4. Conclusion

We found the general solution of a new finite-dimensional additive functional equation (1.1) in this work. In order to investigate whether this quadratic functional equation is stable in both fuzzy β-normed spaces, our study mostly used direct and fixed point approaches.

In addition, we demonstrated real-world examples where the stability of the additive functional equation may be controlled by adding and multiplying norm powers.

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