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New Properties of Frames

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Abstract. Let H be a finite-dimentional complex Hilbert space and $l^2(H)$ is the space of square summable sequences in H. We will give a new characterization of a frame for H, we give our definition of a frame for the Hilbert space $l^2(H)$, we also define and give the properties of the frame operator. We equally show that our definition is equivalent to the definition of a frame for the Hilbert space H. Finally, we give a way to construct frames for $l^2(H^n)$ from frames for $l^2(H^p)$ such that $p < n$ via fusion frame theory.

1. Introduction

One of the important concepts in the study of vector spaces is the concept of a basis for the vector space, which allows every vector to be uniquely represented as a linear combination of the basis elements. However, the linear independence property for a basis is restrictive; sometimes it is impossible to find vectors which both fulfill the basis requirements and also satisfy external conditions demanded by applied problems. For such purposes, we need to look for more flexible types of spanning sets. Frames provide these alternatives. They not only have great variety for use in applications, but also have a rich theory from a pure analysis point of view. A frame is a set of vectors in a Hilbert space that can be used to reconstruct each vector in the space from its inner products with the frame vectors. These inner products are generally called the frame coeficients of the vector. But unlike an orthonormal basis each vector may have infinitely many diferent representations in terms of its frame coeficients.

Frames is a notion that was introduced in 1952 by Duffin and Shaeffer [7] to study some deep problems in nonharmonic Fourier series.This idea seemed to have been unnoticed outside this area until Daubechies, Grossmann and Meyer [6] brought it into light in 1986. The latters' showed that Duffin and Schaeffer's definition was an abstraction of the concept introduced by Gabor [8] in 1946

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for doing signal analysis. Recently, the frames that have been introduced by Gabor are referred to as Gabor frames or Weyl-Heisenberg frames, and they play a vital role in signal analysis

Nowadays, frames have received great attention due to their wide range of applications in both pure and applied mathematics, specially that it has been extensively used in many fields such as filter bank theory, signal and image processing, coding and communication [10] and other areas. We refer to [1,4,5,9] for an introduction to frame theory and its applications.

Frames in finite dimensional spaces, (finite frames), are a very important class of frames due to their significant relevance in applications. It makes the basic idea more transparent. It also gives the right feeling about the infinite-dimensional setting. Moreover, every *real* application of frames has to be performed in a finite-dimensional vector space. The book [3] is the first comprehensive introduction to both the theory and applications of finite frames. For the above reasons, we are motivated to contribute to this area.

This paper falls into 4 sections: Section 2 will be devoted to giving sufficient conditions for a family (finite or infinite) of elements in H to be a frame for H. This will be illustrated by providing many examples and counterexamples. In section 3, we define frames for $l^2(H)$ and we define and give the properties of the frame operator. We equally show that they are also frames for the Hilbert space H . Section 4 will tackle the construction of frames for $l^2(H)$ from smaller spaces. This of course is owing to fusion frame theory introduced in [2] and [3]

Formally, a frame in a separable Hilbert space H is a sequence $\{x_i\}_{i\in I}$ for which there exist positive constants $A, B > 0$ such that:
$A\|x\|^2 \leq \sum_{i\in I} |\langle x, x_i\rangle|^2 \leq B\|x\|^2$, for all $x \in H$. The constants A, B are called lower and upper bounds, respectively. If $A = B$, it is called a tight frame and it is said to be a normalized tight or Parseval frame if $A = B = 1$. The collection $\{x_i\}_{i\in I} \subset H$ is called Bessel if the above second inequality holds. In this case, B is called the Bessel bound.

The largest number $A > 0$ and smallest number $B > 0$ satisfying the frame inequalities for all $x \in H$ are called the optimal frame bounds and they are noted A_{op} and B_{op}.

Throughout this paper, we will also adopt the following notations: H is a complex p-dimensional Hilbert space with the inner product on H: $\langle x, y\rangle = \sum_{i=1}^{p} x^i \overline{y^i}$ (every $x \in H$ is denoted: $x = (x^1, x^2, ..., x^p)$.

$$l^2(\mathbb{C}) = \{\lambda = (\lambda_n)_{n\in\mathbb{N}}; \sum_{n\in\mathbb{N}} |\lambda_n|^2 < \infty\}$$

with inner product: $\langle \lambda, \mu\rangle_{l^2(\mathbb{C})} = \sum_{n\in\mathbb{N}} \lambda_n \overline{\mu_n}$ and the norm $\|\lambda\| = \sqrt{\langle \lambda, \lambda\rangle}$;

$$l^2_{\mathbb{N}^2}(\mathbb{C}) = \{\{\lambda_{n,m}\}_{n,m)\in\mathbb{N}^2}; \sum_{(n,m)\in\mathbb{N}2} |\lambda_{n,m}|^2 < \infty\}$$

$$l^2(H) = \{u = (u_n)_{n\in\mathbb{N}};\ u_n \in H; \sum_{n\in\mathbb{N}} \|u_n\|^2 < \infty\}$$

with inner product:
$\langle u, v\rangle_{l^2(H)} = \sum_{n\in\mathbb{N}} \langle u_n, v_n\rangle_H$ and the corresponding norm: $\|u\|_{l^2(H)} = \left(\sum_{n\in\mathbb{N}} \|u_n\|_H^2\right)^{\frac{1}{2}}$.

Remark 1.1. *It is easy to verify that $l^2(H)$ endowed with the inner product defined above is a Hilbert space.*

2. A characterization of frame for H^p

We begin with the following remarks:

The most important question in frames theory is: when is the lower frame bound condition achived?.

As we are working with elements from the space $l^2(H)$ in which the upper frame bound is always satisfied We will try to discus the lower frame condition.

$l^2(H)$ is the space of Bessel sequences in H.

We can see the space H as a subspace of $l^2(H)$ of sequences that has only the first term nonzero.

If $u = \{u_n\}_{n\in\mathbb{N}}$ has only finite nonzero terms, then u is a finite sequence ($u = \{u_n\}_{n=1}^{N}$). In this case, if $N < p$, $\{u_u\}_{i=1}^{N}$ can't be a frame, because N vectors can at most span N-dimensional space. If $N \geq p$, then $\{u_u\}_{i=1}^{N}$ is a frame for H^p iff its associated analysis operator is injective iff its synthesis operator is surjective iff $\{u_u\}_{i=1}^{N}$ spans H. for more details about finite frame theory, we refer to [2] and [5](chapter 1).

Definition 2.1. *Let H be a complex finite-dimentional Hilbert space (dimH = p). A sequence $u \in l^2(H)$ is called a (finite or infinite) frame for H if there exists $A; B > 0$ such that:*

$$A\|x\| \leq \sum_{n\in\mathbb{N}} |\langle u_n, x\rangle|^2 \leq B\|x\|^2,\ \forall x \in H.$$

In the following Theorem, we will give sufficient condition for a family in $l^2(H)$ to be a frame for H.

Theorem 2.1. *Let $u \in l^2(H)$, with*

$$u = (u^1, u^2, ..., u^p) = \begin{pmatrix} u_1^1 & u_1^2 & ... & u_1^p \\ u_2^1 & u_2^2 & ... & u_2^p \\ . & . & ... & . \\ . & . & ... & . \\ . & . & ... & . \end{pmatrix}.$$

If

$$min(\|u^1\|^2, ..., \|u^p\|^2) > \sum_{1=i<j}^{j=p} |\langle u^i, u^j\rangle|;$$

then u is a frame for H.

Proof. It is easy to see that the right hand side of the inequality holds, by Cauchy-Schwartz inequality and the fact that u is in $l^2(H)$.

For a fixed $n \in \mathbb{N}$ and $h \in H$: $\langle u_n, h\rangle = \sum_{i=1}^{p} u_n^i \overline{h^i}$.

Then;

$$|\langle u_n, h\rangle|^2 = \sum_{i,j=1}^{p} u_n^i \overline{h^i} \overline{u_n^j \overline{h^j}} = \sum_{i,j=1}^{p} u_n^i \overline{u_n^j} h^j \overline{h^i}.$$

So:

$$\begin{aligned}
\sum_{n\in\mathbb{N}} |\langle u_n, h\rangle|^2 &= \sum_{n\in\mathbb{N}} \sum_{i,j=1}^{p} u_n^i \overline{u_n^j} h^j \overline{h^i} \\
&= \sum_{i,j=1}^{p} \left(\sum_{n\in\mathbb{N}} u_n^i \overline{u_n^j} \right) h^j \overline{h^i} \\
&= \sum_{i,j=1}^{p} \langle u^i, u^j \rangle_{l^2(\mathbb{C})} h^j \overline{h^i} \\
&= \sum_{i=1}^{p} \|u^i\|^2 |h^i|^2 + 2Re \sum_{1=i<j}^{j=k} \langle u^i, u^j \rangle_{l^2(\mathbb{C})} h^j \overline{h^i} \\
&= \sum_{i=1}^{p} \|u^i\|^2 |h^i|^2 - 2Re \sum_{1=i<j}^{j=k} (-\langle u^i, u^j \rangle_{l^2(\mathbb{C})}) h^j \overline{h^i} \\
&\geq min(\|u^1\|^2, ..., \|u^p\|^2) \left(\sum_{i=1}^{p} |h^i|^2 \right) - 2Re \sum_{1=i<j}^{j=k} (-\langle u^i, u^j \rangle_{l^2(\mathbb{C})}) h^j \overline{h^i} \\
&= min(\|u^1\|^2, ..., \|u^p\|^2) \|h\|^2 - 2Re \sum_{1=i<j}^{j=k} (-\langle u^i, u^j \rangle_{l^2(\mathbb{C})}) h^j \overline{h^i}
\end{aligned}$$

And for each i, j ($i < j$), we have:

$$2|Re(-\langle u^i, u^j \rangle_{l^2(\mathbb{C})}) h^j \overline{h^i}| \leq 2|(-\langle u^i, u^j \rangle_{l^2(\mathbb{C})}) h^j \overline{h^i}| \leq 2|\langle u^i, u^j \rangle_{l^2(\mathbb{C})}|.|h^i||h^j| \tag{2.1}$$

Then:

$$2|Re(-\langle u^i, u^j \rangle_{l^2(\mathbb{C})}) h^j \overline{h^i}| \leq |\langle u^i, u^j \rangle|(|h^i|^2 + |h^j|^2) \leq |\langle u^i, u^j \rangle_{l^2(\mathbb{C})}| \left(\sum_{k=1}^{p} |h^k|^2 \right)$$

So:

$$2|Re(-\langle u^i, u^j \rangle_{l^2(\mathbb{C})}) h^j \overline{h^i}| \leq |\langle u^i, u^j \rangle_{l^2(\mathbb{C})}| \|h\|^2 \tag{2.2}$$

It follows that:

$$\sum_{n\in\mathbb{N}} |\langle u_n, h\rangle|^2 \geq \left(min(\|u^1\|^2, ..., \|u^p\|^2) - \sum_{1=i<j}^{j=p} |\langle u^i, u^j \rangle_{l^2(\mathbb{C})}| \right) \|h\|^2$$

So, the left hand side of the inequality holds because:

$$min(\|u^1\|^2, ..., \|u^p\|^2) - \sum_{1=i<j}^{j=p} |\langle u^i, u^j \rangle_{l^2(\mathbb{C})}| > 0$$

□

Example 2.1. *let* $f := \{f_n\}_{n\in\mathbb{N}} \in l^2(\mathbb{C})$.
We set:

$$u_1 = (f_1, 0, ..., 0)$$

$$u_2 = (0, f_2, 0, ..., 0)$$

$$\vdots$$

$$u_p = (0, ..., 0, f_p)$$

$$u_{p+1} = (f_{p+1}, 0, ..., 0)$$

$$\vdots$$

$$u_{2p} = (0, ..., 0, f_{2p})$$

$$\vdots$$

In other words:

$$u^i_{kp+i} = f_{kp+i} \text{ and } u^i_{kp+j} = 0 \; (\forall i \neq j)$$

Note that: $u \in l^2(\mathbb{C}^p)$ *and* $\langle u^i, u^j \rangle = 0$ *for all* $i \neq j$*, moreover:*

$$\|u^j\|^2 = \sum_{k=0}^{\infty} |f_{kp+j}|^2, \; \forall j \in \{1, 2, ..., p\}.$$

If $min_{1 \leq j \leq p} \left(\sum_{k=0}^{\infty} |f_{kp+j}|^2 \right) > 0$*, then* $u = (u^1, u^2, ..., u^p)$ *is a frame for* $\mathbb{C}^p$*.*

We will give an example in $l^2(\mathbb{C}^2)$ in which $\sum_{1=i<j}^{j=p} |\langle u^i, u^j \rangle_{l^2(\mathbb{C})}| \neq 0$

Example 2.2. *Let* $u = (u^1_n, u^2_n)_{n \geq 1} = \left(\frac{(-i)^2}{n}, \frac{1}{n} \right)_{n \geq 1} \in l^2(\mathbb{C}^2)$*.*

We have

$$\begin{aligned} \langle u^1, u^2 \rangle &= \sum_{n \in \mathbb{N}^*} \frac{(-i)^n}{n^2} \\ &= \sum_{k \in \mathbb{N}^*} \frac{(-i)^{2k}}{(2k)^2} + \sum_{k \in \mathbb{N}} \frac{(-i)^{2k+1}}{(2k+1)^2} \\ &= \sum_{k \in \mathbb{N}^*} \frac{(-1)^k}{(2k)^2} - i \sum_{k \in \mathbb{N}} \frac{(-1)^k}{(2k+1)^2}. \end{aligned}$$

Then

$$\begin{aligned} |\langle u^1, u^2 \rangle| &= \sqrt{\left(\sum_{k \in \mathbb{N}^*} \frac{(-1)^k}{(2k)^2} \right)^2 + \left(\sum_{k \in \mathbb{N}} \frac{(-1)^k}{(2k+1)^2} \right)^2} \\ &\leq \sqrt{\frac{1}{16} + 1}. \end{aligned}$$

It follows that

$$|\langle u^1, u^2 \rangle| < \|u^1\|^2 = \|u^2\|^2 = \frac{\pi^2}{6}.$$

This means that u is frame.

We will see in the counterexample below that the condition of Theorem 2.1 is not satisfied and hence it is not a frame.

Counterexample 2.1. *Let $H = \mathbb{C}^2$ and $u = (u^1, u^2) = \{(\frac{e^{i\pi n}}{n}, \frac{-e^{i\pi n}}{n})\}_{n\in\mathbb{N}} \in l^2(H)$. We have*

$$\|u^1\|^2 = \|u^2\|^2 = \sum_{n\in\mathbb{N}} \frac{1}{n^2} = \frac{\pi^2}{6},$$

and

$$|\langle u^1, u^2\rangle| = |\sum_{n\in\mathbb{N}} \frac{-1}{n^2}| = \frac{\pi^2}{6}.$$

So

$$min(\|u^1\|^2, \|u^2\|^2) = \frac{\pi^2}{6}$$

and

$$\sum_{1=i<j}^{j=p} |\langle u^i, u^j\rangle| = |\langle u^1, u^2\rangle| = \frac{\pi^2}{6};$$

We notice that the condition of Theorem 2.1 is not satisfied.
For $x = (z, z) \in \mathbb{C}^2$

$$\begin{aligned}\sum_{n\in\mathbb{N}} |\langle u_n, x\rangle_H|^2 &= \|u^1\|^2|z|^2 + \|u^2\|^2|z|^2 + 2Re\langle u^1, u^2\rangle z\bar{z}\\ &= \frac{\pi^2}{6}(2|z|^2) - 2\frac{\pi^2}{6}z\bar{z}\\ &= \frac{\pi^2}{3}(|z|^2 - |z|^2) = 0.\end{aligned}$$

This implies that u is not a frame.

3. Frames for $l^2(H)$

Definition 3.1. *Let H be a complex p-dimentional Hilbert space. A sequence $u \in l^2(H)$ is called a frame for $l^2(H)$ if there exists $A; B > 0$ such that:*

$$A\|v\| \leq \sum_{(n,m)\in\in\mathbb{N}^2} |\langle u_n, v_m\rangle|^2 \leq B\|v\|^2, \ \forall v \in l^2(H)$$

The following theorem tells us that frames for $l^2(H)$ are frames for H.

Theorem 3.1. *Let $u \in l^2(H)$.*
u is a frame for H iff $\exists\, A, B > 0$ such that:

$$A\|v\|^2 \leq \sum_{n,m\in\mathbb{N}} |\langle u_n, v_m\rangle|^2 \leq B\|v\|^2, \forall v \in l^2(H)$$

Proof. ($\Rightarrow$): Suppose that $\sum_{n\in\mathbb{N}} |\langle u_n, x\rangle|^2 > 0, \forall x \in H\setminus\{0\}$. Then

$$\sum_{n,m\in\mathbb{N}} |\langle u_n, v_m\rangle|^2 = \sum_{m\in\mathbb{N}} \left(\sum_{n\in\mathbb{N}} |\langle u_n, v_m\rangle|^2\right),$$

so

$$A\sum_{m\in\mathbb{N}} \|v_m\|_H^2 \leq \sum_{n,m\in\mathbb{N}} |\langle u_n, v_m\rangle|^2 \leq B\sum_{m\in\mathbb{N}} \|v_m\|_H^2,$$

hence

$$A\|v\|^2_{l^2(H)} \leq \sum_{n,m\in\mathbb{N}} |\langle u_n, v_m\rangle|^2 \leq B\|v\|^2_{l^2(H)}.$$

($\Leftarrow$): Suppose that $\sum_{n,m\in\mathbb{N}} |\langle u_n, v_m\rangle|^2 > 0, \forall v \in l^2(H)\setminus\{0\}$.

For every $x \in H$, let $v \in l^2(H)$ such that $v^i_j = x^i\delta_{1j}$, (δ is the chroniker symbol), ie $\forall i \in \{1, ..., p\}$, $v^i = (x^i, 0, 0, 0, ...)$.

Then:

$$\sum_{n,m\in\mathbb{N}} |\langle u_n, v_m\rangle|^2 = \sum_{n\in\mathbb{N}} |\langle u_n, x\rangle|^2 > 0.$$

□

Proposition 3.1. *The operator $T : l^2(H) \longrightarrow l^2_{\mathbb{N}^2}(\mathbb{C})$ defined by*

$$T\{v_m\}_{m\in\mathbb{N}} = \{\langle u_n, v_m\rangle\}_{(n,m)\in\mathbb{N}^2}$$

is linear and bounded.

Proof. It is clear that T is linear.

$$\begin{aligned}
\|T\{v_m\}_{m\in\mathbb{N}}\| &= \sum_{n,m\in\mathbb{N}} |\langle u_n, v_m\rangle_H|^2 \\
&\leq \sum_{n,m\in\mathbb{N}} \|u_n\|^2_H \|v_m\|^2_H \\
&\leq \sum_{n\in\mathbb{N}} \left(\|u_n\|^2_H \sum_{m\in\mathbb{N}} \|v_m\|^2_H \right) \\
&\leq \sum_{n\in\mathbb{N}} \left(\|u_n\|^2_H \|v\|^2_{l^2(H)} \right) \\
&\leq \|u\|^2_{l^2(H)} \|v\|^2_{l^2(H)}.
\end{aligned}$$

□

Definition 3.2. *The operator T defined as follows:*

$$\begin{aligned}
T : l^2(H) &\longrightarrow l^2_{\mathbb{N}^2}(\mathbb{C}) \\
\{v_m\}_{m\in\mathbb{N}} &\longrightarrow \{\langle u_n, v_m\rangle\}_{(n,m)\in\mathbb{N}^2}
\end{aligned}$$

is the analysis operator

Corollary 3.1. *The adjoint of T is*

$$\begin{aligned}
T^* : l^2_{\mathbb{N}^2}(\mathbb{C}) &\longrightarrow l^2(H) \\
\{\lambda_{n,m}\}_{n,m\in\mathbb{N}^2} &\longrightarrow \{\sum_{n\in\mathbb{N}} \lambda_{n,m} u_n\}_{m\in\mathbb{N}}
\end{aligned}$$

Proof.

$$\begin{aligned}\langle T^*\{\lambda_{n,m}\}_{(n,m)\in\mathbb{N}^2},\{v_m\}_{m\in\mathbb{N}}\rangle_{l^2(H)} &= \langle\{\lambda_{n,m}\}_{(n,m)\in\mathbb{N}^2},T\{v_m\}_{m\in\mathbb{N}}\rangle_{l^2_{\mathbb{N}^2}(\mathbb{C})}\\ &= \langle\{\lambda_{n,m}\}_{(n,m)\in\mathbb{N}^2},\{\langle u_n,v_m\rangle\}_{(n,m)\in\mathbb{N}^2}\rangle_{l^2_{\mathbb{N}^2}(\mathbb{C})}\\ &= \sum_{(n,m)\in\mathbb{N}^2}\lambda_{n,m}\overline{\langle v_m,u_n\rangle_H}\\ &= \sum_{m\in\mathbb{N}}\langle\sum_{n\in\mathbb{N}}\lambda_{n,m}u_n,v_m\rangle_H.\end{aligned}$$

Hence

$$T^*\{\lambda_{n,m}\}_{n,m\in\mathbb{N}^2}=\{\sum_{n\in\mathbb{N}}\lambda_{n,m}u_n\}_{m\in\mathbb{N}}.$$

□

The operator $S=T^*T:l^2(H)\longrightarrow l^2(H)$ defined by

$$S(\{v_m\}_{m\in\mathbb{N}})=\{\sum_{n\in\mathbb{N}}\langle u_n,v_m\rangle u_n\}_{m\in\mathbb{N}}$$

is the frame operator of u.

We have

$$\langle Sv,v\rangle=\sum_{(n,m)\in\mathbb{N}^2}|\langle u_n,v_m\rangle|^2,\quad\forall v\in l^2(H).$$

Then

$$AI\le S\le BI.$$

This means that S is a bounded, positive and invertible operator.

We state here a lemma which we use in the following theorem.

Lemma 3.1. *[5](Lemma 2.5.1)*
Let H, K be Hilbert spaces, and suppose that $U:K\longrightarrow H$ is a bounded operator with closed range $\mathcal{R}_U$. Then there exists a bounded operator $U^\dagger:H\longrightarrow K$ for which

$$UU^\dagger x=x,\forall x\in\mathcal{R}_U. \tag{3.1}$$

The operator $U^\dagger$ is called the pseudo-inverse of U.

Proposition 3.2. *u is a frame for $l^2(H)$ if and only if*

$$T^*:l^2_{\mathbb{N}^2}(\mathbb{C})\longrightarrow l^2(H)$$

is a well defined bounded and surjective operator.

Proof. If u is a frame, then S is inversible, so T^* is surjective.
Convesely, suppose T^* be well defined, bounded and onto. We have

$$T(v)=\{\langle u_n,v_m\rangle\}_{(n,m)\in\mathbb{N}^2},\quad\forall v\in l^2(H),$$

then

$$\|T(v)\|^2 = \sum_{(n,m)\in\mathbb{N}^2} |\langle u_n, v_m\rangle|^2 \leq B\|v\|^2, \quad \forall v \in l^2(H).$$

T is onto, then there exists an operator $(T^*)^\dagger : l^2(H) \longrightarrow l^2_{\mathbb{N}^2}(\mathbb{C})$ (the pseudo inverse of T^*), such that $T^*(T^*)^\dagger v = v, \quad \forall v \in l^2(H),(\mathcal{R}_{T^*} = H)$,
then $T^*(T^\dagger)^* v = v, \quad \forall v \in l^2(H)$. Thus, $T^\dagger T v = v, \quad \forall v \in l^2(H)$. It follows that

$$\|v\|^2 \leq \|T^\dagger\|^2 \|Tv\|^2, \quad \forall v \in l^2(H),$$

so

$$\frac{1}{\|T^\dagger\|^2}\|v\|^2 \leq \sum_{(n,m)\in\mathbb{N}^2} |\langle u_n, v_m\rangle|^2 \leq B\|v\|^2, \quad \forall v \in l^2(H).$$

□

Proposition 3.3. *If $u \in l^2(H)$ is a frame for $l^2(H)$ with lower and upper bounds A and B respectively, such that $\langle u^i, u^j\rangle = 0; \forall i \neq j$, then $A_{op} = min_{1\leq i\leq p}\|u^i\|^2$ and $B_{op} = max_{1\leq i\leq p}\|u^i\|^2$.*

Proof. By an analogous proof of theorem (2.2), we have

$$\begin{aligned}\sum_{(n,m)\in\mathbb{N}^2} |\langle u_n, v_m\rangle|^2 &= \sum_{i,j=1}^{p} \langle u^i, u^j\rangle_{l^2(\mathbb{C})} \langle v^j, v^i\rangle_{l^2(\mathbb{C})} \\ &\geq \left(min(\|u^1\|^2, ..., \|u^p\|^2) - \sum_{1=i<j}^{j=p} |\langle u^i, u^j\rangle| \right) \|v\|^2.\end{aligned}$$

Since $\langle u^i, u^j\rangle = 0; \forall i \neq j$, we have:

$$\forall v \in l^2(H), \quad \sum_{(n,m)\in\mathbb{N}^2} |\langle u_n, v_m\rangle|^2 \geq \left(min(\|u^1\|^2, ..., \|u^p\|^2) \right) \|v\|^2.$$

let $\|u^{i_0}\|^2 = min(\|u^1\|^2, ..., \|u^p\|^2)$, then $\sum_{(n,m)\in\mathbb{N}^2} |\langle u_n, v_m\rangle|^2 \geq \|u^{i_0}\|^2\|v\|^2$,
with: $v = (a, ..., 0, v^{i_0}, 0, ..., 0)$ we obtain; $\sum_{(n,m)\in\mathbb{N}2} |\langle u_n, v_m\rangle|^2 = \|u^{i_0}\|^2\|v\|^2$,
so $A_{op} = \|u^{i_0}\|^2$, and we can get that: $B_{op} = max_{1\leq i\leq p}\|u^i\|^2$ in the same way. □

Corollary 3.2. *Let $\{u_n\}_{n\in\mathbb{N}} \in l^2(H)$ a frame for $l^2(H)$, such that $\langle u^i, u^j\rangle = 0; \forall i \neq j$ and $\|u^i\| = \|u^j\|, \forall i, j \in \{1, 2, ..., p\}$, then u is a tight frame.*
If furthermore $\|u^i\| = 1, \forall i \in \{1, 2, ..., p\}$, then u is a normalized tight frame.

Proof. If $\langle u^i, u^j\rangle = 0; \forall i \neq j$, then:

$$\sum_{(n,m)\in\mathbb{N}^2} |\langle u_n, v_m\rangle|^2 = \sum_{i=1}^{p} \|u^i\|^2\|v^i\|^2, \ \forall v \in l^2(H)$$

so:

$$\sum_{(n,m)\in\mathbb{N}^2} |\langle u_n, v_m\rangle|^2 = \|u^1\|^2 \left(\sum_{i=1}^{p} \|v^i\|^2 \right) = \|u^1\|^2.\|v\|^2$$

then, u is a tight frame , with $A = B = \|u^1\|^2$. □

Example 3.1. *Let $z \in \mathbb{C}$ such that $|z| = 1$.*
Let $u \in l^2(\mathbb{C}^3)$ defined as follows:

$$u^1 = \left(\frac{z}{\sqrt{1\times 2}}, 0, 0, \frac{z}{\sqrt{2\times 3}}, 0, 0, \frac{z}{\sqrt{3\times 4}}, 0, 0, \frac{z}{\sqrt{4\times 5}}, 0, ...\right)$$

$$u^2 = \left(0, \frac{z}{\sqrt{1\times 2}}, 0, 0, \frac{z}{\sqrt{2\times 3}}, 0, 0, \frac{z}{\sqrt{3\times 4}}, 0, 0, \frac{z}{\sqrt{4\times 5}}, 0, ...\right)$$

$$u^3 = \left(0, 0, \frac{z}{\sqrt{1\times 2}}, 0, 0, \frac{z}{\sqrt{2\times 3}}, 0, 0, \frac{z}{\sqrt{3\times 4}}, 0, 0, \frac{z}{\sqrt{4\times 5}}, 0, ...\right)$$

ie

$$u^i_{3k+i} = \frac{z}{\sqrt{(k+1)(k+2)}} \quad and \quad u^i_{3k+j} = 0\,(\forall i \neq j).$$

We have $\langle u^i, u^j \rangle = 0$; $\forall i \neq j$ and for every $i \in \{1, 2, 3\}$:

$$\|u^i\|^2 = \sum_{n=1}^{\infty} |\frac{z}{\sqrt{n(n+1)}}|^2 = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1,$$

then u is a normalized tight frame.

4. Construction of frames for $l^2(H)$

Let us recall the essential facts about fusion frames. Our references are: [2] and [3]

Definition 4.1. *(Def 3.1 in [2]) Let H be a Hilbert space and I be some index set, let $\{v_i\}_{i\in I}$ a family of weights, i.e, $v_i > 0$ for all $i \in I$ and $\{W_i\}_{i\in I}$ be a family of closed subspaces of H. $\{(W_i, v_i)\}_{i\in I}$ is said to be a fusion frame or a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H if there exist constants $0 < A \leq B < \infty$ such that*

$$A\|x\|^2 \leq \sum_{i\in I} v_i^2 \|\pi_{W_i}(x)\|^2 \leq B\|x\|^2 \quad \forall x \in H$$

where P_{W_i} denotes the orthogonal projection onto W_i, for each $i \in I$. The fusion frame $W = \{(W_i, v_i)\}_{i\in I}$ is called tight if $A = B$ and Parseval if $A = B = 1$. If $v_i = v$ for all $i \in I$, then W is called v-uniform. Moreover, W is called an orthonormal fusion basis for H if $H = \oplus_{i\in I} H_i$. If $W = \{(W_i, v_i)\}_{i\in I}$ possesses an upper fusion frame bound but not necessarily a lower bound, we call it a Bessel fusion sequence with Bessel fusion bound B. The normalized version of W is obtained when we choose $v_i = 1$ for all $i \in I$.

Theorem 4.1. *(Thm 3.2 in [2]) For each $i \in I$ let $v_i > 0$ and let $\{f_{ij}\}_{j\in J_i}$ be a frame sequence in H with frame bounds A_i and B_i. Define $W_i = span_{j\in J_i}\{f_{ij}\}$ for all $i \in I$ and choose an orthonormal basis $\{e_{ij}\}_{j\in J_i}$ for each subspace W_i. Suppose that $0 < A = inf_{i\in I} A_i \leq B = sup_{i\in I} B_i < \infty$. The following conditions are equivalent.*

(i) *$\{v_i f_{ij}\}_{i\in I, j\in J_i}$ is a frame for H.*
(ii) *$\{v_i e_{ij}\}_{i\in I, j\in J_i}$ is a frame for H.*
(iii) *$\{W_i\}_{i\in i}$ is a frame of subspaces with respect to $\{v_i\}_{i\in I}$ for H .*

In the following lemma, we will give an orthonormal basis of $l^2(H)$.

Lemma 4.1. *Let $\{a_i\}_{0\le i\le p-1}$ be an orthonormal basis of H. The family $\{e_n\}_{n\in\mathbb{N}}$ defined by: for each $n\in\mathbb{N}$ such that $n = kp+i$ (for some $k\in\mathbb{N}$ and $0\le i\le p-1$);*

$$e_n := e_k^i = \{\delta_{kj}a_i\}_{j\in\mathbb{N}}$$

is an orthonormal basis of $l^2(H)$.

Proof. Let $n = jp+i$ and $n' = kp+l$, if $n \neq n'$ then $(i,j)\neq(k,l)$ so $\langle e_n, e_{n'}\rangle = \langle e_j^i, e_l^k\rangle = 0$? This implies that $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal family.
Let $u\in l^2(H)$

$$\begin{aligned}\sum_{n\in\mathbb{N}} |\langle u, e_n\rangle_{l^2(H)}|^2 &= \sum_{k\in\mathbb{N}}\sum_{i=0}^{p-1} |\langle u, e_k^i\rangle_{l^2(H)}|^2\\ &= \sum_{k\in\mathbb{N}}\sum_{i=0}^{p-1} |u_k^i|_{\mathbb{C}}^2\\ &= \sum_{k\in\mathbb{N}} \|u_k\|_H^2\\ &= \|u\|_{l^2(H)}^2\end{aligned}$$

so $\{e_n\}_{n\in\mathbb{N}}$ satisfy Parseval's identity, it is then an orthonormal basis of $l^2(H)$. □

Theorem 4.2. *Let H be a complex finite-dimentional Hilbert space ($dimH = p$).*
Then, there exist $\{W_i\}_{i\in I}$ a family of closed subspaces in $l^2(H)$ and $\{\upsilon_i\}_{i\in I}$ a family of weights, i.e, $\upsilon_i > 0$ for all $i\in I$ such that, $\{(W_i,\upsilon_i)\}_{i\in I}$ is a fusion frame for H.

Proof. Let $\{e_i\}_{i\in\{1,\dots,p\}}$ be an orthonormal basis of H, we set $V_i = span\{e_i\}$ and let $W_i = l^2(V_i)$, (we can identify $l^2(V_i)$ with $l^2(\mathbb{C})$).
Let $\{\upsilon_i\}_{i\in I}$ be a family of weights., the set I in this case is $I = \{1,\dots,p\}$.
Let $u\in l^2(H)$, then

$$\sum_{i\in I}\upsilon_i^2\|\pi_{W_i}u\|^2 = \sum_{i=1}^{p}\upsilon_i^2\|u_i\|^2.$$

Then

$$min_{i\in\{1,\dots,p\}}(\upsilon_i^2).\|u\|^2 \le \sum_{i\in I}\upsilon_i^2\|\pi_{W_i}u\|^2 \le max_{i\in\{1,\dots,p\}}(\upsilon_i^2).\|u\|^2.$$

Hence, $\{(W_i,\upsilon_i)\}_{i\in I}$ is a fusion frame with fusion frame bounds $A := min_{i\in\{1,\dots,p\}}(\upsilon_i^2)$ and $B := max_{i\in\{1,\dots,p\}}(\upsilon_i^2)$. □

Remark 4.1. *If $\upsilon_i = \upsilon$ for all $i\in\{1,\dots,p\}$ then $\{(W_i,\upsilon)\}_{i\in I}$ is a υ-tight fusion frame.*
If $\upsilon_i = 1$ for all $i\in\{1,\dots,p\}$ then

$$\sum_{i\in I}\upsilon_i^2\|\pi_{W_i}u\|^2 = \|u_i\|^2.$$

$\{(W_i,1)\}_{i\in I}$ is a Parseval fusion frame, it is also an orthonormal fusion basis because $l^2(H) = \oplus_{i\in\{1,\dots,p\}}W_i$.

Theorem 4.3. *Let $\{W_i\}_{i\in I}$ a family of closed subspaces in $l^2(H)$ and $\{v_i\}_{i\in I}$ a family of weights, i.e, $v_i > 0$ for all $i \in I$ such that $\{(W_i, v_i)\}_{i\in I}$ is a fusion frame for H. If $\{u_{ij}\}_{j\in J_i}$ is a frame for W_i for each $i \in I$, then $\{v_i u_{ij}\}_{i\in I, j\in J_i}$ is frame for $l^2(H)$.*

Proof. This follows immediately from Theorem 4.2 above and Theorem 3.2 in [2]. □

Example 4.1. *Let $\{a_i\}_{0\le i\le p-1}$ be an orthonormal basis of H, and $V_i = span\{a_i\}$. by lemma 4.1 $\{e^i_j\}_{j\in\mathbb{N}}$ is an orthonormal basis of $W_i = l^2(V_i)$, let $\{v_i\}_{i\in\{0,...p-1\}}$ be any family of wights.*
$\{v_i e^i_n\}_{i\in\{0,...,p-1\},n\in\mathbb{N}}$ is a frame for $l^2(H)$. Indeed, we set $(v_0 e^0_n, ..., v_{p-1} e^{p-1}_n) := ve_n$.
Let $f \in l^2(H)$

$$\begin{aligned}\sum_{(n,m)\in\in\mathbb{N}^2} |\langle ve_n, f_m\rangle|^2 &= \sum_{(n,m)\in\in\mathbb{N}^2} \sum_{i=0}^{p-1} |\langle v_i e^i_n, f^i_m\rangle|^2 \\ &= \sum_{i=0}^{p-1} v_i^2 \sum_{(n,m)\in\in\mathbb{N}^2} |\langle e^i_n, f^i_m\rangle|^2 \\ &= \sum_{i=0}^{p-1} v_i^2 \sum_{n\in\mathbb{N}} |f^i_n|^2 \\ &= \sum_{i=0}^{p-1} v_i^2 \|f^i\|^2.\end{aligned}$$

Then

$$A\|f\|^2 \le \sum_{(n,m)\in\in\mathbb{N}^2} |\langle ve_n, f_m\rangle|^2 \le B\|f\|^2$$

with $A := min_{i\in\{1,...,p\}}(v_i^2)$ and $B := max_{i\in\{1,...,p\}}(v_i^2)$.

Example 4.2. *Let $f = (f^1, f^2, f^3)$ be the sequence in $l^2(\mathbb{C}^3)$ defined by:*

$$\{(f^1_n, f^2_n, f^3_n)\}_{n\in\mathbb{N}} = \{(\frac{e^{i\alpha\pi n}}{n}, \frac{e^{i\beta\pi n}}{n}, \frac{e^{i\gamma\pi n}}{n})\}_{n\in\mathbb{N}}$$

such that $\alpha + \beta \neq 2k$ and $\beta + \gamma \neq 2k$ for every $k \in \mathbb{Z}$.
Let $\{e_i\}_{1\le i\le 3}$ be the standard orthonormal basis of $\mathbb{C}^3$. We set $V_1 = span\{e_1, e_2\}$, $V_2 = span\{e_2, e_3\}$, let $W_i = l^2(V_i), i \in \{1, 2\}$

$$\|f^1\|^2 = \|f^2\|^2 = \|f^3\|^2 = \frac{\pi^2}{6}.$$

$$|\langle f^1, f^2\rangle| = |\sum_{n\in\mathbb{N}} \frac{1}{n^2} e^{i\pi(\alpha+\beta)n}| < \sum_{n\in\mathbb{N}} \frac{1}{n^2} = \|f^1\|^2 = \|f^2\|^2$$

$$|\langle f^2, f^3\rangle| = |\sum_{n\in\mathbb{N}} \frac{1}{n^2} e^{i\pi(\beta+\gamma)n}| < \sum_{n\in\mathbb{N}} \frac{1}{n^2} = \|f^2\|^2 = \|f^3\|^2$$

by Theorem 2.1 $\{(f^1_n, f^2_n)\}_{n\in\mathbb{N}}$ is frame for W_1 and $\{(f^2_n, f^3_n)\}_{n\in\mathbb{N}}$ is a frame for W_2.
Let v_1,v_2 be two weights (we assume that $v_1 \le v_2$). Let $g \in l^2(\mathbb{C}^3)$

$$\begin{aligned}v_1\|\pi_{W_1} g\|^2 + v_2\|\pi_{W_2} g\|^2 &= v_1(\|g^1\|^2 + \|g^2\|^2) + v_2(\|g^2\|^2 + \|g^3\|^2) \\ &= v_1\|g^1\|^2 + (v_1 + v_2)\|g^2\|^2 + v_2\|g^3\|^2.\end{aligned}$$

Then

$$v_1\|g\|^2 \leq \sum_{i=1}^{2} v_i\|\pi_{W_i}g\|^2 \leq (v_1+v_2)\|g\|^2.$$

This means that $(W_i, v_i)_{i=1,2}$ *is a fusion frame for* $l^2(\mathbb{C}^3)$.
Lel $f_1 = \{f_{1n}\}_{n\in\mathbb{N}} := \{(f_n^1, f_n^2)\}_{n\in\mathbb{N}}$ *and* $f_2 = \{f_{2n}\}_{n\in\mathbb{N}} := \{(f_n^2, f_n^3)\}_{n\in\mathbb{N}}$, *if* A_1, B_1 *are frame bounds of* f_1 *and* A_2, B_2 *are frame bounds of* f_2. *We set* $A = min(A_1, A_2)$ *and* $B = max(B_1, B_1)$. *For* $g = (g^1, g^2, g^3) \in l^2(\mathbb{C}^3)$

$$\sum_{n,m\in\mathbb{N}} |\langle v_1 f_{1n}, g_m\rangle|^2 + \sum_{n,m\in\mathbb{N}} |\langle v_2 f_{2n}, g_m\rangle|^2 = v_1^2 \sum_{n,m\in\mathbb{N}} |\langle f_{1n}, g_m\rangle|^2 + v_2^2 \sum_{n,m\in\mathbb{N}} |\langle f_{2n}, g_m\rangle|^2.$$

As each element $h \in W_1$ *can be writen in the form* $h = (h^1, h^2, 0)$ *as an element of* $l^2(\mathbb{C}^3)$ *and if* $h \in W_2$ *it can be writen* $h = (0, h^2, h^3)$ *and the fact that each* f_i *is a frame for* W_i, $i \in \{1, 2\}$, *we get*

$$A(\|g^1\|^2 + \|g^2\|^2) \leq \sum_{n,m\in\mathbb{N}} |\langle f_{1n}, g_m\rangle|^2 \leq B(\|g^1\|^2 + \|g^2\|^2)$$

and

$$A(\|g^2\|^3 + \|g^2\|^2) \leq \sum_{n,m\in\mathbb{N}} |\langle f_{2n}, g_m\rangle|^2 \leq B(\|g^2\|^2 + \|g^3\|^2).$$

Thus

$$Av_1^2\|g\|^2 \leq Av_1^2\|g^1\|^2 + A(v_1^2+v_2^2)\|g^2\|^2 + Av_2^2\|g^3\|^2 \leq \sum_{i=1}^{2}\sum_{n,m\in\mathbb{N}} |\langle f_{in}, g_m\rangle|^2$$

and

$$\sum_{i=1}^{2}\sum_{n,m\in\mathbb{N}} |\langle f_{in}, g_m\rangle|^2 \leq Bv_1^2\|g^1\|^2 + B(v_1^2+v_2^2)\|g^2\|^2 + Bv_2^2\|g^3\|^2 \leq B(v_1^2+v_2^2)\|g\|^2.$$

This means that $\{v_i f_{in}\}_{i\in\{1,2\},n\in\mathbb{N}}$ *is a frame for* $l^2(\mathbb{C}^3)$

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