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Generalized Nonlinear Variational Inequality Problems with Random Variation

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Abstract. This text deals with exploring random solutions for generalized nonlinear variational inequality problems. By leveraging the Fan-KKM theorem and Aumann's measurable selection theorem, we can establish the existence and uniqueness of random solution sets, given the conditions of monotonicity and convexity. Additionally, we use Minty's lemma to demonstrate the compactness and convexity of the random solution sets.

1. Introduction

Random fixed point theorems are generalizations of the fixed point theorem that consider the role of randomness. They are important in the theory of random equations, similar to how fixed point theorems are crucial in deterministic equations. Several authors, including Cho *et al.* [1], Hans [2], Itoh [3], Salahuddin [4], Spaeek [5], and Tsokos [6], have proved random fixed point theorems for contraction mappings in Polish spaces. Moreover, Tsokos has presented a random fixed point theorem of Schauder type in a probability measurable space of the random solution sets.

Random variational inequality problems are a type of variational inequality problems that take into account the uncertainties that are usually present in practical scenarios. They are a useful tool in studying different types of forecasting problems and stochastic control problems [7]. Researchers are currently focusing on the solvability and convexity of two-step stochastic programming, as well as the convergence of the average approximation for two-step random variational inequality problems. Ren *et al.* [8] have demonstrated a class of theorems for one-dimensional variational inequality problems with Yamada-Watanabe-type conditions on the coefficients. Random variational inequality problems are similar to random complementarity problems, and therefore, the

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relevant properties of their solutions are usually discussed in the context of theoretical research on stochastic complementarity problems. Zhang and Huang [9] have established a class of generalized set-valued random quasi-complementarity problems and have proved the existence of their solutions as well as the convergence of random sequences generated by a random iterative algorithm.

In order to solve variational inequality problems in real Hilbert spaces, Pakkaranang [10] recently introduced a dual inertial Tseng's extragradient method that makes use of Lipschitz continuous operators and pseudomonotone. In their study, Wairojjana et al. [11] proposed two algorithms of the inertial extragradient type to solve convex pseudomonotone variational inequalities with fixed point problems. They also developed an iterative sequence that, under appropriate assumptions, showed strong convergence to the common solution of the variational inequality and fixed point problem.

Based on recent research [12–21], we investigate a class of generalized nonlinear random variational inequality problems. Our research focusses on establishing the existence of random solutions for these problems. We intend to provide a comprehensive analysis and theoretical foundation for the existence of random solutions to these complex inequalities in a stochastic context by drawing on techniques and results from the cited literature.

In this paper, we assume that (Ω, Σ) is a measurable space consisting of a set Ω and a σ-algebra Σ of a subset of Ω. Let C be a nonempty subset of a Banach space $\mathbb{X}$, and $\mathbb{X}^\star$ be its dual space. Assume that $\langle \cdot, \cdot \rangle$ represents the dual pairing of $\mathbb{X}$ and $\mathbb{X}^*$, and $\|\cdot\|$ represents the norm in $\mathbb{X}$. Let $\mathscr{D} : \Omega \times C \to \mathbb{X}^\star$ be the random mapping, and $\varphi : \Omega \times C \times C \to (\infty, +\infty]$ be the random functional.

We now demonstrate the generalized nonlinear random variational inequality problem, finding $t \in \Omega, x(t) \in C$ such that

$$\langle \mathscr{D}(t, x(t)), y - x(t) \rangle + \varphi(t, y, x(t)) - \varphi(t, x(t), x(t)) \geq 0, \forall y \in C. \tag{1.1}$$

Our main objective in this article is to find a measurable selection $\gamma : \Omega \to C$ for (1.1), such that

$$\langle \mathscr{D}(t, \gamma(t)), y - \gamma(t) \rangle + \varphi(t, y, \gamma(t)) - \varphi(t, \gamma(t), \gamma(t)) \geq 0, \ \forall y \in C, t \in \Omega. \tag{1.2}$$

We note that if $\varphi(t, x(t), x(t)) = \varphi(t, x(t))$, then (1.1) reduces to the following random variational inequality problem for finding $t \in \Omega$ and $x(t) \in C$ such that

$$\langle \mathscr{D}(t, x(t)), y - x(t) \rangle + \varphi(t, y) - \varphi(t, x(t)) \geq 0, \forall y \in C, \tag{1.3}$$

and for finding a measurable selection $\gamma : \Omega \to C$ for (1.3) such that

$$\langle \mathscr{D}(t, \gamma(t)), y - \gamma(t) \rangle + \varphi(t, y) - \varphi(t, \gamma(t)) \geq 0, \ \forall y \in C. \tag{1.4}$$

Again, if $\varphi(t, x(t)) \equiv 0$, then (1.3) reduces to the following random variational inequality problem for finding $t \in \Omega$ and $x(t) \in C$ such that

$$\langle \mathscr{D}(t, x(t)), y - x(t) \rangle \geq 0, \forall y \in C, \tag{1.5}$$

and for finding a measurable selection $\gamma : \Omega \to C$ for (1.5) such that

$$\langle \mathscr{D}(t,\gamma(t)), y-\gamma(t)\rangle \geq 0,\ \forall y \in C. \tag{1.6}$$

2. Preliminaries

In this section, we present some basic concepts and assumptions associated with the multivalued mapping and the fixed point theorem. These concepts will be helpful for our main result.

Assume $\mathbb{X}$ is a Hausdorff topological vector space and (Ω, Σ) is a measurable space. Assume $\mathscr{B}(\mathbb{X})$ is the σ-algebra of all Borel subsets of $\mathbb{X}$, $CB(\mathbb{X})$ is the family of all nonempty closed convex subsets of $\mathbb{X}$, and $\Sigma \times \mathscr{B}(\mathbb{X})$ is the family of all measurable sets in $\Omega \times \mathbb{X}$.
A mapping $\mathcal{K} : \Omega \to 2^{\mathbb{X}}$ is $(\Sigma, \mathscr{B}(\mathbb{X}))$-measurable, if for any open set $B \subseteq \mathscr{B}(\mathbb{X})$,

$$\mathcal{K}^{-1}(B) = \{t \in \Omega, \gamma(t) \cap B \neq \emptyset\} \in \Sigma.$$

A mapping $\mathcal{K} : \Omega \times \mathbb{X} \to 2^{\mathbb{X}}$ is measurable, if $\mathcal{K}(\cdot, y) : \Omega \to 2^{\mathbb{X}}$ is measurable for any $y \in \mathbb{X}$. A mapping $\gamma : \Omega \to \mathbb{X}$ is a random fixed point of a measurable mapping $\mathcal{K} : \Omega \times \mathbb{X} \to 2^{\mathbb{X}}$, if it is measurable and

$$\gamma(t) \in \mathcal{K}(t, \gamma(t)).$$

Let C be a nonempty subset of a Hausdorff topological space $\mathbb{X}$, and $\mathcal{K} : C \to 2^{\mathbb{X}}$ be a multivalued mapping. For a finite set $\{\ell_1, \ell_2, \cdots, \ell_n\} \subset C$, there is a finite subset $\{v_1, v_2, \cdots, v_n\} \subset \mathbb{X}$ such that for any subset $I \subset \{1, \cdots, n\}$,

$$co\{v_i : i \in I\} \subset \bigcup_{i\in I} \mathcal{K}(\ell_i).$$

Then $\mathcal{K}$ is a generalized KKM mapping studied in [22].

Theorem 2.1. *(Fan-KKM theorem, [23]) Assume $\emptyset \neq C$ is a subset of a Hausdorff topological space $\mathbb{X}$. If the KKM-mapping $\mathcal{K} : C \to 2^{\mathbb{X}}$ is closed for each $\ell \in C$, and $\mathcal{K}(\ell_0)$ is compact for $\ell_0 \in C$, then*

$$\bigcap_{\ell\in C} \mathcal{K}(\ell) \neq \emptyset.$$

Definition 2.1. *[24] The Hausdorff topological space $\mathbb{X}$ is:*

(i) *a Polish space if $\mathbb{X}$ is separable and metrizable by a complete metric;*
(ii) *a Suslin space if $\mathbb{X}$ is a Hausdorff topological space and a continuous image of a Polish space.*

Lemma 2.1. *(Aumann's measurable selection, [25]) Let $(\Omega, \Sigma, \mathcal{P})$ be a Hausdorff topological space, and $\mathbb{X}$ be a separable Hilbert space. Then there is a measurable mapping $\mathcal{G} : \Omega \to 2^{\mathbb{X}}$ such that*

$$graph(\mathcal{G}) = \{(t, x(t)) \in \Omega \times \mathbb{X} : x(t) \in \mathcal{G}(t)\} \in \Sigma \times \mathscr{B}(\mathbb{X}).$$

If $\mathcal{G}(\cdot)$ is measurable and has a measurable selection $\gamma : \Omega \to C$, then

$$\gamma(t) \in \mathcal{G}(t)$$

is valid for all $t \in \Omega$.

Definition 2.2. *[26] The bifunction $\varphi\colon C \times C \to \mathbb{R} \cup \{+\infty\}$ is skew-symmetric if and only if*

$$\varphi(x,x) - \varphi(x,y) - \varphi(y,x) + \varphi(y,y) \geq 0, \ \forall x,y \in C. \tag{2.1}$$

If the skew-symmetric function $\varphi(\cdot,\cdot)$ is bilinear, then

$$\varphi(x,x) - \varphi(x,y) - \varphi(y,x) + \varphi(y,y) = \varphi(x-y, x-y) \geq 0, \ \forall x,y \in C. \tag{2.2}$$

3. Main results

In this section, we discuss the characteristics of random solution sets for (1.1).

Theorem 3.1. *Let C be a nonempty closed and convex subset of a Suslin space $\mathbb{X}$, and $\mathbb{X}^*$ be its dual space. Assume $\varphi : \Omega \times C \times C \to (-\infty, +\infty]$ is a convex and lower semicontinuous random functional, and $\mathscr{D} : \Omega \times C \to \mathbb{X}^*$ is a continuous random mapping. Suppose the following assumptions hold:*

(i) *the map $\mathcal{K}\colon \Omega \times C \to 2^C$ satisfies*

$$\mathcal{K}(t,y) = \{t \in \Omega, x(t) \in C : \langle \mathscr{D}(t,x(t)), y - x(t)\rangle + \varphi(t,y,x(t)) - \varphi(t,x(t),x(t)) \geq 0\};$$

(ii) *there is a compact subset $C' \subset \mathbb{X}$ and $\ell \in C \cap C'$ such that*

$$\langle \mathscr{D}(t,x(t)), \ell - x(t)\rangle + \varphi(t,\ell,x(t)) - \varphi(t,x(t),x(t)) < 0, \forall\, x \in C \setminus C';$$

(iii) *$\langle \mathscr{D}(t,x(t)), y - x(t)\rangle + \varphi(t,y,x(t))$ is quasiconvex and upper semicontinuous at $y \in C$;*

(iv) *$\mathbb{Y}$ is a finite-dimensional subspace of $\mathbb{X}$. For any finite-dimensional section $\mathscr{N} = C \cap \mathbb{Y}$ and any net $\{x_\alpha(t)\} \subset C \cap C'$ with $\{x_\alpha(t)\} \to x(t) \in \mathscr{N}$, one has*

$$\langle \mathscr{D}(t,x_\alpha(t)), y - x_\alpha(t)\rangle + \varphi(t,y,x_\alpha(t)) \geq \varphi(t,x_\alpha(t),x_\alpha(t)), \forall y \in \mathscr{N}, \tag{3.1}$$

it follows that

$$\langle \mathscr{D}(t,x(t)), y - x(t)\rangle + \varphi(t,y,x(t)) \geq \varphi(t,x(t),x(t)), \forall\, y \in \mathscr{N}; \tag{3.2}$$

(v) *$\varphi(t,x(t),x(t)) - \langle \mathscr{D}(t,x(t)), y - x(t)\rangle$ is lower semicontinuous at $x \in C$.*

Then (1.1) *provides a random solution set.*

Proof. There are three parts to proving the existence of a random solution for (1.1):

First part. In this part, we will first demonstrate that

(a) $\mathcal{K}$ possesses a measurable image,

(b) $\mathcal{K}$ is a KKM-mapping,

(c) $\bigcap_{y \in C} \mathcal{K}(t,y) \neq \emptyset$, for $t \in \Omega$.

First, we establish the measurability of $\mathcal{K}$.

(a) Let B denote an open subset of C. Assume there is a sequence $\{y_n(t)\}$ in C. For each $t \in \Omega, x(t) \in C$, we have

$$\begin{aligned}\mathcal{K}^{-1}(B) &= \{(t, x(t)) \in \Omega \times C \colon \mathcal{K}(t, y(t)) \cap B \neq \emptyset\} \\ &= \bigcap \{(t, x(t)) \in \Omega \times C \colon \mathcal{K}(t, y_n(t)) \cap B \neq \emptyset\} \\ &= \left\{(t, x(t)) \in \Omega \times C \colon \bigcap_{x(t) \in B} \mathcal{K}(t, y_n(t)) \neq \emptyset\right\} \in \Sigma. \end{aligned} \tag{3.3}$$

Thus, $\mathcal{K}$ is a measurable.

Based on Theorem 3.5 in [27], $graph(\mathcal{K})$ is $\Sigma \times \mathscr{B}$-measurable, implying that $\mathcal{K}$ has a measurable image.

(b) We demonstrate that $\mathcal{K}$ is a KKM-mapping. Assuming $\mathcal{K}$ is not a KKM-mapping, there exists a finite set $\{y_1(t), y_2(t), \cdots, y_n(t)\} \subset C$ such that

$$co(y_1(t), y_2(t), \cdots, y_n(t)) \not\subset \bigcup_{i=1}^{n} \mathcal{K}(t, y_i(t)).$$

Therefore, $\bar{y}(t) \in co(y_1(t), y_2(t), \cdots, y_n(t))$, $\bar{y}(t) = \sum_{i=1}^{n} \lambda_i y_i(t)$ ($\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$, $i = 1, 2, \cdots, n$), and

$$\bar{y}(t) \notin \bigcup_{i=1}^{n} \mathcal{K}(t, y_i(t)).$$

Thus,

$$\bar{y}(t) \notin \mathcal{K}(t, y_i(t)).$$

Hence,

$$\langle \mathscr{D}(t, \bar{y}(t)), y_i(t) - \bar{y}(t)\rangle + \varphi(t, y_i(t), \bar{y}(t)) - \varphi(t, \bar{y}(t), \bar{y}(t)) < 0 \tag{3.4}$$

and

$$\varphi(t, y_i(t), \bar{y}(t)) < +\infty, \text{ for } i = 1, 2, \cdots, n.$$

Based on condition (iii), we have

$$\langle \mathscr{D}(t, x(t)), y - x(t)\rangle + \varphi(t, y, x(t))$$

is quasiconvex, and

$$\{t \in \Omega, y(t) \in C : \langle \mathscr{D}(t, \bar{y}(t)), y - \bar{y}(t)\rangle + \varphi(t, y, \bar{y}(t)) - \varphi(t, \bar{y}(t), \bar{y}(t)) < 0\} \tag{3.5}$$

is a convex set. Thus, (3.4) and (3.5) lead to

$$\bar{y}(t) = \sum_{i=1}^{n} \lambda_i y_i(t) \in \{t \in \Omega, y \in C : \langle \mathscr{D}(t, \bar{y}(t)), y - \bar{y}(t)\rangle + \varphi(t, y, \bar{y}(t)) - \varphi(t, \bar{y}(t), \bar{y}(t)) < 0\}.$$

Therefore,

$$\langle \mathscr{D}(t, \bar{y}(t)), \bar{y}(t) - \bar{y}(t)\rangle + \varphi(t, \bar{y}(t), \bar{y}(t)) - \varphi(t, \bar{y}(t), \bar{y}(t)) < 0,$$

while

$$\langle \mathscr{D}(t,\bar{y}(t)), \bar{y}(t)-\bar{y}(t)\rangle + \varphi(t,\bar{y}(t),\bar{y}(t)) - \varphi(t,\bar{y}(t),\bar{y}(t)) = 0.$$

This produces a contradiction.

Hence, $\mathcal{K}$ is a KKM-mapping.

(c) For every $t \in \Omega$, $y(t) \in C$, the set $\mathcal{K}(t, y(t))$ is weakly closed in $\mathbb{X}$. Indeed, taking a sequence $\{x_\alpha(t)\} \subset \mathcal{K}(t, y(t))$ with $\{x_\alpha(t)\} \to x(t)$, one has

$$\langle \mathscr{D}(t,y(t)), y-x_\alpha(t)\rangle + \varphi(t,y,x_\alpha(t)) - \varphi(t,x_\alpha(t),x_\alpha(t)) \geq 0.$$

Using conditions (iv) and (v), we obtain

$$\langle \mathscr{D}(t,y(t)), y-x(t)\rangle + \varphi(t,y,x(t)) \geq \varphi(t,x(t),x(t)).$$

Therefore,

$$\langle \mathscr{D}(t,y(t)), y-x(t)\rangle + \varphi(t,y,x(t)) - \varphi(t,x(t),x(t)) \geq 0,$$

which suggests that

$$x(t) \in \mathcal{K}(t,y(t)), \forall t \in \Omega.$$

Due to condition (ii), there must be $x(t) \in \mathcal{K}(t, y(t))$ for all $x(t) \in C \setminus C'$. However, if there exists $x(t)$ such that

$$x(t) \in \mathcal{K}(t,y(t)),$$

imply that $x(t) \in C$ and $x(t) \in C'$. Therefore,

$$\mathcal{K}(t,y(t)) \subset C'$$

because $\mathcal{K}(t, y(t)) \subset C$. Therefore, $\mathcal{K}(t, y(t))$ is compact. Based on Theorem 2.1, we have

$$\bigcap_{y(t)\in C} \mathcal{K}(t,y(t)) \neq \emptyset.$$

Second part. Let $\mathcal{G}\colon \Omega \to 2^C$ be a measurable map such that

$$\mathcal{G}(t) = \bigcap_{y(t)\in C} \mathcal{K}(t,y(t)).$$

Consider the dense subset $\{y_i\}_{i=1}^{\infty}$ in C. It ought to be demonstrated that

$$\bigcap_{y(t)\in C} \mathcal{K}(t,y(t)) = \bigcap_{i=1}^{\infty} \mathcal{K}(t,y_i(t)). \tag{3.6}$$

Therefore,

$$\bigcap_{y(t)\in C} \mathcal{K}(t,y(t)) \subset \bigcap_{i=1}^{\infty} \mathcal{K}(t,y_i(t)).$$

Thus, all we have to do is acquire

$$\bigcap_{i=1}^{\infty}\mathcal{K}(t,y_i(t))\subset\bigcap_{y(t)\in C}\mathcal{K}(t,y(t)).$$

On the contrary, we presume that

$$\bigcap_{i=1}^{\infty}\mathcal{K}(t,y_i(t))\not\subset\bigcap_{y(t)\in C}\mathcal{K}(t,y(t)),$$

then there is a random selection

$$x_0(t)\in\bigcap_{i=1}^{\infty}\mathcal{K}(t,y_i(t))$$

and

$$x_0(t)\notin\bigcap_{y(t)\in C}\mathcal{K}(t,y(t)).$$

Therefore, there exists $y_0(t)\in C$ such that

$$x_0(t)\notin \mathcal{C}(t,y_0(t)),$$

that is

$$\langle\mathscr{D}(t,x_0(t)),y_0(t)-x_0(t)\rangle+\varphi(t,y_0(t),x_0(t))-\varphi(t,x_0(t),x_0(t))<0. \tag{3.7}$$

There exists $\{y_{n_j}(t)\}\subset\{y_i(t)\}$ such that

$$\{y_{n_j}(t)\}\to\{y_0(t)\}$$

where $\{y_i(t)\}_{i=1}^{\infty}$ is a countable dense subset of C. Therefore,

$$x_0(t)\in\bigcap_{i=1}^{\infty}\mathcal{K}(t,y_i(t))\subset\bigcap_{j=1}^{\infty}\mathcal{K}(t,y_{n_j}(t)).$$

Thus, we have

$$\langle\mathscr{D}(t,x_0(t)),y_{n_j}(t)-x_0(t)\rangle+\varphi(t,y_{n_j}(t),x_0(t))-\varphi(t,x_0(t),x_0(t))\geq 0,\forall j\geq 1.$$

Again, from condition (iii), we have

$$\langle\mathscr{D}(t,x(t)),y-x(t)\rangle+\varphi(t,y,x(t))$$

is upper semicontinuous. Thus

$$\begin{aligned}\langle\mathscr{D}(t,x_0(t)),&y_0(t)-x_0(t)\rangle+\varphi(t,y_0(t),x_0(t))\\&\geq\overline{\lim_{j\to\infty}}\left\{\langle\mathscr{D}(t,x_0(t)),y_{n_j}(t)-x_0(t)\rangle+\varphi(t,y_{n_j}(t),x_0(t))\right\}\\&\geq\varphi(t,x_0(t),x_0(t)).\end{aligned} \tag{3.8}$$

Hence, from (3.8), we have

$$\langle\mathscr{D}(t,x_0(t)),y_0(t)-x_0(t)\rangle+\varphi(t,y_0(t),x_0(t))-\varphi(t,x_0(t),x_0(t))\geq 0. \tag{3.9}$$

This leads to the contradiction. Hence,

$$x_0(t) \notin \mathcal{K}(t, y_0(t)).$$

This implies that

$$\bigcap_{i=1}^{\infty} \mathcal{K}(t, y_i(t)) \subset \bigcap_{y(t)\in C} \mathcal{K}(t, y(t)).$$

Thus

$$\bigcap_{y(t)\in C} \mathcal{K}(t, y(t)) = \bigcap_{i=1}^{\infty} \mathcal{K}(t, y(t)). \tag{3.10}$$

Third part. Consider the mapping $\mathcal{G}\colon \Omega \to 2^C$ such that

$$\mathcal{G}(t) = \bigcap_{y(t)\in C} \mathcal{K}(t, y(t)).$$

This implies that

$$\begin{aligned} graph(\mathcal{G}) &= \left\{(t, x(t))\colon x(t) \in \mathcal{G}(t) = \bigcap_{i=1}^{\infty} \mathcal{K}(t, y_i(t))\right\} \\ &= \bigcap_{i=1}^{\infty} \{(t, x(t))\colon x(t) \in \mathcal{K}(t, y_i(t))\} \in \Sigma \times \mathscr{B}(C). \end{aligned} \tag{3.11}$$

Using Lemma 2.1, find a measurable mapping $\mathcal{G} : \Omega \to C$, such that

$$\mathcal{G}(t) \in \bigcap_{y(t)\in C} \mathcal{K}(t, y(t)).$$

Then there exists a measurable selection $\gamma\colon \Omega \to C$ such that

$$\gamma(t) \in \mathcal{G}(t).$$

Hence (1.1) has a random solution set.

□

Corollary 3.1. *Let $\emptyset \neq C$ be a compact convex subset of a Suslin space $\mathbb{X}$, and $\mathbb{X}^*$ be its dual space. Assume the random functional $\varphi : \Omega \times C \times C \to (-\infty, +\infty]$ is convex and lower semicontinuous, whereas a random mapping $\mathscr{D} : \Omega \times C \to \mathbb{X}^*$ is continuous. If the following assumptions hold:*

(i) *A mapping $\mathcal{K} : \Omega \times C \to 2^C$ fulfils*

$$\mathcal{K}(t, y(t)) = \{t \in \Omega, x(t) \in C : \langle \mathscr{D}(t, x(t)), y(t) - x(t)\rangle + \varphi(t, y, x(t)) - \varphi(t, x(t), x(t)) \geq 0\},$$

then, Theorem 3.1 implies that $\mathcal{K}$ has a measurable graph.

(ii) *$\langle \mathscr{D}(t, x(t)), y(t) - x(t)\rangle + \varphi(t, y, x(t))$ is quasiconvex and upper semicontinuous for $y \in C$.*

(iii) *$\varphi(t, x(t), x(t)) - \langle \mathscr{D}(t, x(t)), y(t) - x(t)\rangle$ is lower semicontinuous for $x(t) \in C$.*

Then, (1.1) *has a random solution set.*

Proof. Assume that $C' = C$. Then $C \setminus C'$ because C is compact. There exists $\ell \in C$ such that $\mathcal{K}(t, \ell)$ is compact for which

$$\bigcap_{y(t)\in C} \mathcal{K}(t, y(t)) \neq \emptyset.$$

Therefore, we need to demonstrate that condition (iv) in Theorem 3.1 holds. For any $t \in \Omega$, $y(t) \in \mathcal{N} \subset C$, and a random sequence $\{x_\alpha(t)\} \to x(t) \in \mathcal{N}$, we have

$$\varphi(t, x_\alpha(t), x_\alpha(t)) - \langle \mathscr{D}(t, x_\alpha(t)), y(t) - x_\alpha(t)\rangle \leq \varphi(t, y, x_\alpha(t)),\ \forall\ y(t) \in \mathcal{N}, t \in \Omega.$$

It implies that

$$\begin{aligned}\varphi(t, x(t), x(t)) - \langle \mathscr{D}(t, x(t)), y(t) - x(t)\rangle &\leq \underline{\lim_{\alpha}} \{\varphi(t, x_\alpha(t), x_\alpha(t)) - \langle \mathscr{D}(t, x_\alpha 9t)), y(t) - x_\alpha(t)\rangle\} \\ &\leq \varphi(t, y(t), x(t)). \end{aligned} \tag{3.12}$$

This implies that condition (iv) of Theorem 3.1 is true.
Thus, the solution to (1.1) exists. □

Theorem 3.2. *Let $\emptyset \neq C$ be a closed convex subset of a Suslin space $\mathbb{X}$, and $\mathbb{X}^*$ be the dual space. Assume a convex and lower semicontinuous random functional $\varphi : \Omega \times C \times C \to (-\infty, +\infty]$ and a continuous and strictly monotonic random mapping $\mathscr{D} : \Omega \times C \to \mathbb{X}^*$. If conditions (i)-(ii) and (iv)-(v) in Theorem 3.1, along with the following condition:*

(iii)′ *$\langle \mathscr{D}(t, x(t)), y(t) - x(t)\rangle + \varphi(t, y(t), x(t))$ is strictly convex and upper semicontinuous about $t \in \Omega$, $y(t) \in C$, are met.*
Then (1.1) *has an unique random solution in $C \bigcap C'$..*

Proof. By contradiction, let's assume that for $t \in \Omega$, $x_1(t), x_2(t) \in C \cap C'$ are two distinct solutions, such that

$$\langle \mathscr{D}(t, x_1(t)), y(t) - x_1(t)\rangle + \varphi(t, y(t), x_1(t)) - \varphi(t, x_1(t), x_1(t)) \geq 0,\ \forall t \in \Omega, \forall y(t) \in C, \tag{3.13}$$

and

$$\langle \mathscr{D}(t, x_2(t)), y(t) - x_2(t)\rangle + \varphi(t, y(t), x_2(t)) - \varphi(t, x_2(t), x_2(t)) \geq 0,\ \forall t \in \Omega, \forall y(t) \in C. \tag{3.14}$$

With $y(t) = x_2(t)$ in (3.13) and $y(t) = x_1(t)$ in (3.14), we get

$$\langle \mathscr{D}(t, x_1(t)), x_2(t) - x_1(t)\rangle + \varphi(t, x_2(t), x_1(t)) - \varphi(t, x_1(t), x_1(t)) \geq 0\ \forall t \in \Omega, \forall y(t) \in C, \tag{3.15}$$

and

$$\langle \mathscr{D}(t, x_2(t)), x_1(t) - x_2(t)\rangle + \varphi(t, x_2(t), x_1(t)) - \varphi(t, x_2(t), x_2(t)) \geq 0\ \forall t \in \Omega, \forall y(t) \in C. \tag{3.16}$$

Adding (3.15) and (3.16), we have

$$\begin{aligned}&\langle \mathscr{D}(t, x_1(t)), x_2(t) - x_1(t)\rangle - \langle \mathscr{D}(t, x_2(t)), x_2(t) - x_1(t)\rangle + \varphi(t, x_1(t), x_2(t)) \\ &- \varphi(t, x_2(t), x_2(t)) + \varphi(t, x_2(t), x_1(t)) - \varphi(t, x_1(t), x_1(t)) \geq 0. \end{aligned} \tag{3.17}$$

Based on Definition 2.1 and strictly monotonicity of $\mathscr{D}(t, x(t))$, the equation (3.17) implies that

$$\langle \mathscr{D}(t, x_2(t)), x_2(t) - x_1(t)\rangle - \langle \mathscr{D}(t, x_1(t)), x_2(t) - x_1(t)\rangle = 0. \tag{3.18}$$

For any $t \in \Omega, x(t) \in C$,

$$\langle \mathscr{D}(t, x(t)), y(t) - x(t)\rangle + \varphi(t, y(t), x(t))$$

is strictly convex . Assuming $y(t) = \frac{1}{2}(x_1(t) + x_2(t))$ in (3.13), we get

$$\begin{aligned} \varphi(t, \frac{1}{2}(x_1(t) + x_2(t)), x_1(t)) - \quad & \varphi(t, x_1(t), x_1(t)) + \langle \mathscr{D}(t, x_1(t)), \frac{1}{2}(x_1(t) + x_2(t)) - x_1(t)\rangle \\ & > \varphi\left(t, \frac{1}{2}(x_1(t) + x_2(t)), x_1(t)\right) + \langle \mathscr{D}(t, x_1(t)), \frac{1}{2}(x_2(t) - x_1(t))\rangle \\ & \geq \varphi(t, x_1(t), x_1(t)). \end{aligned} \tag{3.19}$$

This implies that

$$\langle \mathscr{D}(t, x_1(t)), x_2(t) - x_1(t)\rangle - \varphi(t, x_1(t), x_1(t)) + \varphi(t, x_2(t), x_1(t)) > 0. \tag{3.20}$$

Again, using $y(t) = \frac{1}{2}(x_1(t) + x_2(t))$ in (3.14) and following a similar procedure as in (3.19), we obtain

$$\langle \mathscr{D}(t, x_2(t)), x_1(t) - x_2(t)\rangle - \varphi(t, x_2(t), x_2(t)) + \varphi(t, x_1(t), x_2(t)) > 0. \tag{3.21}$$

Adding (3.20) and (3.21), and using the Definition 2.2, we get

$$\langle \mathscr{D}(t, x_1(t)), x_2(t) - x_1(t)\rangle + \langle \mathscr{D}(t, x_2(t)), x_1(t) - x_2(t)\rangle > 0. \tag{3.22}$$

This implies that

$$\langle \mathscr{D}(t, x_2(t)), x_2(t) - x_1(t)\rangle - \langle \mathscr{D}(t, x_1(t)), x_2(t) - x_1(t)\rangle < 0. \tag{3.23}$$

This produces a contradiction. Therefore, the uniqueness of random solution for (1.1) is proven. □

We will now discuss the compactness and convexity of the random solution set for equation (1.1).

Theorem 3.3. *Let $\emptyset \neq C$ be a closed convex subset of a Suslin space $\mathbb{X}$, and $\mathbb{X}^*$ be the dual space. Assume that a random functional $\varphi : \Omega \times C \times C \to (-\infty, +\infty]$ is convex, and a random mapping $\mathscr{D} : \Omega \times C \to \mathbb{X}^*$ is monotone and continuous. If conditions (i)-(ii) and (iv)-(v) in Theorem 3.1 are met, along with the following conditions:*

(iii)′ *$\langle \mathscr{D}(t, x(t)), y(t) - x(t)\rangle + \varphi(t, y(t), x(t))$ is convex and upper semicontinuous in $y(t) \in C$;*

(vi) *$\langle \mathscr{D}(t, x(t)), y(t) - x(t)\rangle$ and $\varphi(t, y(t), x(t))$ are lower semicontinuous at $y(t) \in C$.*

Then the random solution set of (1.1) *is compact and convex in $C \bigcap C'$.*

Proof. First, we state Minty's lemma [28] as follows:

Let $\mathbb{X}$ be a Hausdorff topological vector space with a closed convex subset C. Suppose that the random functional φ and the random mapping $\mathscr{D}$ satisfy the following conditions:

(i) $\varphi\colon \Omega \times C \times C \to (-\infty, +\infty]$ is lower semicontinuous at $y(t) \in C$.

(ii) $\mathscr{D}\colon \Omega \times C \to \mathbb{X}^*$ is monotone, semicontinuous, and lower semicontinuous at $y(t) \in C$.

Then, for any $t \in \Omega, x(t) \in C$,

$$\langle \mathscr{D}(t, x(t)), y(t) - x(t) \rangle + \varphi(t, y(t), x(t))$$

is convex at $y(t)$, then there exists $\bar{x}(t) \in C$ such that

$$\mathcal{K}(t, y(t)) = \{t \in \Omega, \bar{x}(t) \in C : \langle \mathscr{D}(t, \bar{x}(t)), y(t) - \bar{x}(t) \rangle + \varphi(t, y(t), \bar{x}(t)) - \varphi(t, \bar{x}(t), \bar{x}(t)) \geq 0\}$$

and

$$\mathcal{H}(t, y(t)) = \{t \in \Omega, \bar{x}(t) \in C : \langle \mathscr{D}(t, y(t)), \bar{x}(t) - y(t) \rangle + \varphi(t, \bar{x}(t), y(t)) - \varphi(t, y(t), y(t)) \leq 0\}$$

coincide.

From the above, we will focus on the random solution sets depending on the compactness and convexity of the set. Let $\mathcal{S}$ be the solution set of (1.1) in $C \cap C'$. Then,

$$\bar{x}(t) \in \mathcal{S} = \bigcap_{y(t) \in C} \mathcal{K}(t, y(t)) \neq \emptyset.$$

For any $t \in \Omega, y(t) \in C$, let

$$\mathcal{H}(t, y(t)) = \{t \in \Omega, \bar{x}(t) \in C : \langle \mathscr{D}(t, y(t)), \bar{x}(t) - y(t) \rangle + \varphi(t, \bar{x}(t), y(t)) - \varphi(t, y(t), y(t)) \leq 0\}. \tag{3.24}$$

Based on Minty's lemma, we have

$$\bigcap_{y(t) \in C} \mathcal{K}(t, y(t)) = \bigcap_{y(t) \in C} \mathcal{H}(t, y(t)).$$

Now, we can derive the following from conditions (iii)′ and (vi) in Theorem 3.3,

$$\langle \mathscr{D}(t, y(t)), \bar{x}(t) - y(t) \rangle + \varphi(t, \bar{x}(t), \bar{x}(t))$$

is convex and lower semicontinuous. Since $\mathcal{H}(t, y(t))$ is closed and convex, then

$$\mathcal{S} = \bigcap_{y(t) \in C} \mathcal{H}(t, y(t))$$

is also closed convex in $\mathbb{X}$.

Thus,

$$\bigcap_{y(t) \in C} \mathcal{H}(t, y(t)) = \overline{\bigcap_{y(t) \in C} \mathcal{K}(t, y(t))} \subset \overline{\mathcal{K}(t, \ell)}. \tag{3.25}$$

Using the argument of contradiction, let us assume that there exists $x(t) \in \mathcal{K}(t, \ell)$ and $x(t) \notin C'$, such that

$$x(t) \in C \setminus C'.$$

Then from condition (ii) in Theorem 3.1,

$$\langle \mathscr{D}(t, x(t)), \ell - x(t) \rangle + \varphi(t, \ell, x(t)) - \varphi(t, x(t), x(t)) < 0,$$

which conflicts with the statement $x(t) \in \mathcal{K}(t, \ell)$. Hence

$$\mathcal{K}(t, \ell) \subset C'.$$

Since C' is a compact and closed subset. From (3.25), we get

$$\overline{\mathcal{K}(t,\ell)} \subset C'.$$

Therefore, $\mathcal{S}$ is a compact convex set in C, which means that $\mathcal{S}$ is compact and convex in $C \cap C'$. □

4. Conclusion

In this article, we investigate a novel generalized nonlinear variational inequality problem that incorporates random variations. We begin by examining the existence of a random solution set for the equation (1.1) within a Suslin space $\mathbb{X}$, which includes a closed convex subset C. This analysis is based on the principles of the Fan-KKM theorem and Aumann's measurable selection theorem.

We then illustrate the existence of solution sets in a Suslin space $\mathbb{X}$ with a compact convex subset C, as well as in a separable reflexive Banach space $\mathbb{X}$ that contains a closed convex subset C. We establish the uniqueness of the random solution for the equation (1.1), demonstrating that this uniqueness is closely associated to the strict convexity and strict monotonicity of the mapping $\mathscr{D}$.

To address general infinite-dimensional problems, we explore compactness and convexity using Minty's lemma. The results of this study represent a generalization of Browder's nonlinear variational inequality problems in uncertain environments. Our findings suggest that solutions can exist in a reflexive Banach space by appropriately relaxing the conditions on the subset C.

In addition to investigating the existence and uniqueness of random solutions, we also examine the compactness and convexity of these random solution sets. Theorem 3.1 generalizes Browder's nonlinear variational inequality problems for a random mapping $\mathscr{D}\colon \Omega \times C \to \mathbb{X}^*$ and a random functional $\varphi\colon \Omega \times C \times C \to (-\infty, +\infty]$, extending previous results by Chuong and Thuan [24].

Moreover, Theorem 3.1 can be utilized to derive existence theorems for various other stochastic nonlinear variational inequality problems. We also enhance the convexity condition in section (iii) of Theorem 3.2, strengthening the monotonicity condition of the random mapping $\mathscr{D}$.

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