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Exploring Novel Fixed Point Solutions in Boundary Value Problems

Ahmad Aloqaily[1,2], **Nizar Souayah**[3,*], **Salma Haque**[1], **Nabil Mlaiki**[1]

[1]*Department of Mathematics and Sciences, Prince Sultan University, Riyadh, 11586, Saudi Arabia*

[2]*School of computer, Data and Mathematical Sciences, Western Sydney University, Sydney 2150, Australia*

[3]*Department of Natural Sciences, Community College Al-Riyadh, King Saud University, Riyadh, Saudi Arabia*

[*]*Corresponding author:* nsouayah@ksu.edu.sa

Abstract. Within this manuscript, we present an innovative concept of contraction, building upon the foundation laid by Jleli and Samet. Subsequently, we introduce the concept of θ-contractions. Leveraging these novel ideas, we formulate a series of fresh fixed-point theorems applicable to spaces utilizing the Controlled Branciari metric. Notably, our approach integrates and consolidates diverse fixed-point outcomes, eliminating the necessity for the Hausdorff assumption. To illustrate the practicality of our findings, we provide examples and applications to boundary value problems associated with fourth-order differential equations.

1. Introduction

The exploration of metric spaces unfolds into a myriad of generalizations, each accentuating the importance of the terms and conditions that shape their definitions. These generalizations stem from the relaxation of one or more of the three fundamental axioms governing metrics, namely self-distance, symmetry, and the triangle inequality. A rich body of literature delves into various extensions of metric spaces, encompassing b-metric spaces, fuzzy-metric spaces, G-metric spaces, symmetric spaces, semi-metric spaces, conic-metric spaces, and more.

Within this expansive landscape, hybrid contractions have become a focal point of investigation, particularly in abstract spaces defined by the Branciari distance. The Branciari distance [1], [2], [3], [27], [37], [43], [45] and the Branciari b-distance [5], [6], [7], [37], [27], [46] represent axiomatic modifications of the triangle inequality, introducing a quadruple inequality

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(quadrilateral inequality multiplied by some constant) in lieu of the standard metric. While there exist similarities between the Branciari distance, Branciari b-distance, and the standard metric, their topologies exhibit notable distinctions.

The literature has often referred to these specialized spaces as rectangular metric spaces or generalized metric spaces, showcasing distinctive fixed-point properties. Eminent researchers have contributed diverse fixed-point results by manipulating and considering variations in metric-like axioms, making this field one of the most captivating areas of contemporary research.

Extended generalizations, including discussions on rectangular metric-like spaces, have given rise to discontinuous maps, adding a layer of complexity to the exploration. Remarkably intriguing fixed-point results have emerged from these generalizations, promising applicability in diverse problem-solving scenarios.

In our investigation, we introduce a new notation for θ-contraction within the extended Branciari metric-like spaces, thereby establishing a fixed-point theorem that contributes novel and compelling results to this domain. To facilitate comprehension, we introduce the abbreviations Branciari metric space (BMS) and Branciari contraction (BC) at the outset, aiming to simplify our discourse as we navigate through this intricate realm of metric space generalizations and fixed-point theorems.

Definition 1.1. *[24] Let $\mathcal{S}$ be a non-empty set and $b : \mathcal{S}^2 \longrightarrow [0, \infty)$ be a mapping satisfying the following conditions for all $\alpha, \nu \in \mathcal{S}$ and $\delta \neq \vartheta \in \mathcal{S} \setminus \{\alpha, \nu\}$:*

$$\begin{aligned} &(B1) \quad b(\alpha,\nu) = 0 \text{ if and only if } \alpha = \nu \text{(selfdistance condition / indistancy)} \\ &(B2) \quad b(\alpha,\nu) = b(\nu,\alpha) \text{(symmetry condition)} \\ &(B3) \quad b(\alpha,\nu) \le b(\alpha,\delta) + b(\delta,\vartheta) + b(\vartheta,\nu) \text{ (quadrilateral inequality condition).} \end{aligned} \tag{1.1}$$

The mapping b is called a Branciari distance and the $(\mathcal{S}, b)$ pair is called a Branciari distance space as well, shortened by "BDS".

The Branciari distance space is alternatively denoted by terms such as "rectangular metric space," "Branciari metric space," or "generalized metric space" in various scholarly sources. A comprehensive extension of these spaces is detailed in [4], offering further insights into their properties.

Simultaneously, the literature abounds with diverse extensions of the metric concept, often referred to as "generalized metrics". Noteworthy examples include works by authors such as [19], [20], [21], [23], [22] , [29], [38] , [30], [35], [36], [26], [34], [28], [47], [32], [40], [42], [43], [44], [41], [45], [39]. To obviate potential confusion, it is recommended to adhere to the specific term "Branciari distance" for clarity and precision.

In the same context, Jleli and Samet [2] have proposed the concept of Θ-contraction within the framework of Bianciari distance space. This innovative concept serves to extend existing fixed-point theorems. For the sake of completeness, let us revisit the definition of Θ-contraction:

Definition 1.2. *Let Θ be a set of all non-decreasing, continuous functions where, $\theta : (0,\infty) \to (1,\infty)$ satisfying these conditions:*

(*i*) *for every sequence $\{x_n\} \subset (0,\infty)$, $\lim_{n\to\infty} \theta(x_n) = 1 \Leftrightarrow \lim_{n\to\infty} x_n = 0^+$;*

(*ii*) *there exists such $q \in (0,1)$, $\ell \in (0,\infty]$ such that $\lim_{s\to 0^+} \dfrac{\theta(x)-1}{x^q} = \ell$.*

The definition above was applied and revised to various fixed point results. (see refer. [5]- [11])

Now, we recall the notion introduced by Kamran *et al.* [13], for extended b-metric space.

Definition 1.3. *[13] Let S be a non-empty set and $\omega : S^2 \to [1,\infty)$ be a mapping. The function $\rho_e : S^2 \to [0,\infty)$ is called an extended b-metric as long as it satisfies the following assumptions:*

(1) $\rho_e(u,\nu) = 0 \Leftrightarrow u = \nu$,

(2) $\rho_e(u,\nu) = \rho_e(\nu,u)$,

(3) $\rho_e(u,\nu) \leq \omega(u,\nu)[\rho_e(u,\tau) + \rho_e(\tau,\nu)]$,

for all $u,\nu,\tau \in S$. Then, (X,ρ_e) is called an extended b-metric space.

Definition 1.4. *Let S be a non-empty set and $\omega : S^2 \to [1,\infty)$ a mapping. A function $\sigma_e : S^2 \to [0,\infty)$ is called an extended Branciari b-distance if it satisfies for all $\alpha,\nu \in \mathcal{S}$ and all distinct $\delta,\vartheta \in \mathcal{S}\setminus\{\alpha,\nu\}$:*

(1) $\sigma_e(\alpha,\nu) = 0$ *iff* $\alpha = \nu$,

(2) $\sigma_e(\alpha,\nu) = \sigma_e(\nu,\alpha)$,

(3) $\sigma_e(\alpha,\nu) \leq \omega(\alpha,\nu)[\sigma_e(\alpha,\delta) + \sigma_e(\delta,\vartheta) + \sigma_e(\vartheta,\nu)]$,

The pair (S,σ_e) is called a Branciari b-distance space.

In this manuscript, our goal is to introduce a new concept called controlled rectangular (Branciari) metric-like space.

Definition 1.5. *Let S be a non-empty set and $\omega :\ S^4 \to [1,\infty)$ a mapping. The function $B_{dist} : S\times S \to [0,\infty)$ is called a controlled Branciari metric like space(CBMLS) if it satisfies:*

(1) $B_{dist}(\alpha,\nu) = 0$ *implies* $\alpha = \nu$;

(2) $B_{dist}(\alpha,\nu) = B_{dist}(\nu,\alpha)$;

(3) $B_{dist}(\alpha,\nu) \leq \omega(\alpha,\nu,\delta,\vartheta)[B_{dist}(\alpha,\delta) + B_{dist}(\delta,\vartheta) + B_{dist}(\vartheta,\nu)]$,

for all $\alpha \neq \nu \neq \delta \neq \vartheta \in \mathcal{S}$ and all distinct $\delta,\vartheta \in \mathcal{S}\setminus\{\alpha,\nu\}$. The pair (S,B_{dist}) is called a controlled Branciari metric like space.

Example 1.1. *Suppose $S = l_r$, where $1 \leq r < \infty$ is defined as*

$$l_r = \left\{(\alpha_t)_{t\geq 1} \subseteq \mathbb{R} : \sum_{t=1}^{\infty} |\alpha_t|^r < \infty\right\}.$$

$B_{dist} : S\times S \to \mathbb{R}^+$ *as*

$$B_{dist}(\alpha,\nu) = \left(\sum_{t=1}^{\infty} |\alpha_t - \nu_t|^r\right)^{\frac{1}{r}} \text{ for all } \alpha,\nu \in S.$$

As well as $\omega : S^4 \to [1,\infty)$ *is defined by* $\omega(\alpha,\nu,\delta,\vartheta) = 2^{\frac{1}{r}}$ *for all* $\alpha,\nu \in S$. *Then* B_{dist} *satisfies all conditions for controlled Branciari metric like space.*

Example 1.2. *Suppose* $S = [0,1]$, *and let* $B_{dist} : S \times S \to \mathbb{R}$ *the mapping defined by* $B_{dist}(\alpha,\nu) = |\alpha - \nu|^2$ *where* $\omega(\alpha,\nu,\delta,\vartheta) = 7(\alpha+\nu+\delta+\vartheta)+5$ *then* (S, B_{dist}) *is a controlled Branciari metric like space. Then, we prove only the extended quadrilateral inequality where it is easy to show and verify the other conditions.*

$$\begin{aligned}
B_{dist}(\alpha,\nu) &= |\alpha-\nu|^2\\
&= |\alpha-\tau+\tau-\varpi+\varpi-\nu|^2\\
&\leq |\alpha-\tau|^2+|\tau-\varpi|^2+|\varpi-\nu|^2+2|\alpha-\tau||\tau-\varpi|\\
&+2|\tau-\varpi||\varpi-\nu|+2|\varpi-\nu||\alpha-\tau|\\
&\leq (7(\alpha+\nu+\delta+\vartheta)+5)[|\alpha-z|^2+|\tau-\varpi|^2+|\varpi-\nu|^2]\\
&= \omega(\alpha,\nu,\delta,\vartheta)[B_{dist}(\alpha,\tau)+B_{dist}(\tau,\varpi)+B_{dist}(\varpi,\nu)].
\end{aligned}$$

This leads to $B_{dist}(\alpha,\nu) \leq \omega(\alpha,\nu,\delta,\vartheta)[B_{dist}(\alpha,\tau)+B_{dist}(\tau,\varpi)+B_{dist}(\varpi,\nu)]$.

Example 1.3. *Suppose* $S = E \cup F$, *define* $E = \{\frac{1}{n}, n \in \mathbb{N}^*\}$ *and* F *is the set of positive integers. We define* $B_{dist} : S^2 \to [0,\infty)$ *by*

$$B_{dist}(u,v) = \begin{cases} 0, & \text{if } u = v\\ 3\alpha, & \text{if } u,v \in E\\ \dfrac{\alpha}{3} & \text{otherwise} \end{cases} \tag{1.2}$$

where $\omega(u,v,c,d) = \max\{u,v,c,d\} + 3\alpha$ *and* $\alpha > 0$.
Then (S, B_{dist}) *is a controlled Branciari metric like space.*

Example 1.4. *Assume* $S = [-1,1]$. *We define* $B_{dist} : S^2 \to [0,\infty)$ *by* $B_{dist}(\upsilon,\tau) = |\upsilon - \tau|$ *and* $\omega(\upsilon,\tau,c,d) = \max\{\upsilon,\tau,c,d\}+2$. *Then* (S, B_{dist}) *is a controlled Branciari metric like space.*

Example 1.5. *Let* $S = \mathcal{F}[0,1]$ *such that* $\mathcal{F}[0,1]$ *represents the space for all continuous functions defined in a closed interval* $[0,1]$. *We define* $B_{dist} : S \times S \to [0,\infty)$ *where* $B_{dist}(f,g) = |f-g|^2$ *and* $\omega : S^4 \to [1,\infty)$ *by* $\omega(f,g,h_1,h_2) = 5f+5f+3$. *Then* (S, B_{dist}) *is a controlled Branciari metric like space. Indeed, let us prove the triangle inequality since the two first conditions are trivial. Let* $f,g,h_1,h_2 \in S$, *we have*

$$\begin{aligned}
B_{dist}(f,g) &= |f-g|^2\\
&= |f-g-h_1+h_1-h_2+h_2|^2\\
&\leq |f-h_1|^2+|h_1-h_2|^2+|h_2-g|^2\\
&\leq (5f+5g+3)\Big(|f-h_1|^2+|h_1-h_2|^2+|h_2-g|^2\Big)\\
&= \omega(f,g,h_1,h_2)\Big(B_{dist}(f,h_1)+B_{dist}(h_1,h_2)+B_{dist}(g,h_2)\Big).
\end{aligned}$$

Remark 1.1. *If $\omega(\alpha,\nu,\delta,\vartheta) = s$ where $s \geq 1$, we obtain the definition of Branciari metric space. If $s = 1$ we get a standard Branciari distance. On the other hand, it is well known, b-metric doesn't have to be continuous. Accordingly, it is not necessarily for extended Branciari b-distance to be continuous as well. However, we speculate in this manuscript for every extended Branciari b-distance to be continuous.*

Definition 1.6. *Let S be a non-empty set endowed with a controlled Branciari metric like space B_{dist}, a sequence $\{\alpha_n\}$ in S is called*

(1) *Convergent to α if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ where $B_{dist}(\alpha_n, \alpha) < \epsilon$, for all $n \geq N$. For this appointed case, we mark down $\lim_{n\to\infty} \alpha_n = \alpha$.*
(2) *Cauchy if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $B_{dist}(\alpha_m, \alpha_n) < \epsilon$, for all $m, n \geq N$.*
(3) *A B_{dist}-metric space (S, B_{dist}) is complete if every Cauchy sequence in S is convergent.*

The rest of the manuscript is organized as follows:

- In the next section, we introduced many topics using Θ-BC, Ćirić-Reich-Rus type, Θ-BC and interpolative-Θ-BC. Depending on the previous new contractions, we generated and proved a significant fixed point theorems on the determination of extended Branciari *b*-distance spaces. Also, Back up applications and example are presented using different sequences
- Finally, in the third section, as an application, we presented a solution to a boundary value problem of a fourth-order differential equation

2. Main Results

Let's start by defining the following set of mappings that will be need later.

Definition 2.1. *We denote by Θ a new class of families of mappings $\Theta : [0, +\infty) \to [0, +\infty)$ satisfying these conditions:*

(Θ_1) *Θ is an upper semicontinuous mapping from the right;*
(Θ_2) *$\Theta(\alpha) < \alpha$ for all $\alpha \in (0, +\infty)$;*
(Θ_3) *$\Theta(0) = 0$.*

Let us begin by introducing the concept of Θ-BC.

Definition 2.2. *Let (S, B_{dist}) be a controlled Branciari metric like space and $T : X \to X$ a mapping. T is called a Θ-Branciari contraction as long as there exists a function $\theta \in \Theta$ where*

$$\theta(B_{dist}(T\alpha, T\nu)) \leq [\theta(B_{dist}(\alpha, \nu))]^r \ \text{ if } \ B_{dist}(T\alpha, T\nu) \neq 0 \text{ for } \alpha, \nu \in S.$$

Moreover, $r \in (0, 1)$ where $\sup_{m\geq 1} \lim_{n\to\infty} \omega(\alpha_n, , \alpha_{n+1}, \alpha_{n+2}, \alpha_m) < \frac{1}{r}$, and $x_n = T^n\alpha_0$ for $\alpha_0 \in S$.

Theorem 2.1. *Let (S, B_{dist}) be a complete controlled Branciari metric like space and $T : X \to X$ a Θ-BC. Then T has a unique fixed point in S.*

Proof. For an arbitrary point $\alpha_0 \in S$ we develop an iterative sequence $\{\alpha_n\}_n$ as follows:

$$\alpha_n = T^n \alpha_0 \text{ for every } n \in \mathbb{N}.$$

Let $T^{n_*}\alpha = T^{n_*+1}\alpha$ for some $n_* \in \mathbb{N}$, then $T^{n_*}\alpha$ is clearly a fixed point of T.
Therefore, without loss of generality, we might suppose that $B_{dist}(T^n\alpha, T^{n+1}\alpha) > 0$ for every $n \in \mathbb{N}$
From Definition 2.2 we deduce that

$$\begin{aligned}\theta(B_{dist}(\alpha_n, \alpha_{n+1})) &= \theta(B_{dist}(T\alpha_{n-1}, T\alpha_n)) \\ &\leq [\theta(B_{dist}(\alpha_{n-1}, \alpha_n))]^r \\ &\leq [\theta(B_{dist}(\alpha_{n-2}, \alpha_{n-1}))]^{r^2}.\end{aligned}$$

similarly, we obtain that

$$\theta(B_{dist}(\alpha_n, \alpha_{n+1})) \leq [\theta(B_{dist}(\alpha_0, \alpha_1))]^{r^n}. \tag{2.1}$$

Consistently, we deduce that

$$1 < \theta(B_{dist}(\alpha_n, \alpha_{n+1})) \leq [\theta(B_{dist}(\alpha_0, \alpha_1))]^{r^n} \text{for all } n \in \mathbb{N}. \tag{2.2}$$

Assuming $n \to \infty$ in (2.2), then $\theta(B_{dist}(\alpha_n, \alpha_{n+1})) \to 1$ as $n \to \infty$.
From (Θ_1), we have

$$\lim_{n\to\infty} B_{dist}(\alpha_n, \alpha_{n+1}) = 0. \tag{2.3}$$

As well as, it is easily to deduce that

$$\lim_{n\to\infty} B_{dist}(\alpha_n, \alpha_{n+2}) = 0. \tag{2.4}$$

From (Θ_2), there exist $q \in (0,1)$ and $l \in (0,\infty]$ where

$$\lim_{n\to\infty} \frac{\theta(B_{dist}(\alpha_n, \alpha_{n+1})) - 1}{[B_{dist}(\alpha_n, \alpha_{n+1})]^q} = l. \tag{2.5}$$

Assume $l < \infty$. For this case, suppose $B = \dfrac{l}{2} > 0$. Using (**??**), we pick $n_0 \in \mathbb{N}$ where

$$\left| \frac{\theta(B_{dist}(\alpha_n, \alpha_{n+1})) - 1}{[B_{dist}(\alpha_n, \alpha_{n+1})]^q} - l \right| \leq B$$

for all $n \geq n_0$.
This implies to $\left| \dfrac{\theta(B_{dist}(\alpha_n, \alpha_{n+1})) - 1}{[B_{dist}(\alpha_n, \alpha_{n+1})]^q} \right| \geq l - B = B$ for all $n \geq n_0$.
Hence, we derive that

$$n[B_{dist}(\alpha_n, \alpha_{n+1})^q] \leq n\Big[\frac{\theta(B_{dist}(\alpha_n, \alpha_{n+1})) - 1}{B}\Big] \text{ for all } n \geq n_0.$$

Assume that $l = \infty$ and $B > 0$ be an arbitrary positive number. Using the definition of the limit, we find $n_0 \in \mathbb{N}$ where

$$\frac{\theta(B_{dist}(\alpha_n, \alpha_{n+1})) - 1}{[B_{dist}(\alpha_n, \alpha_{n+1})]^q} \geq B \text{ for all } n \geq n_0.$$

This leads to

$$n[B_{dist}(\alpha_n, \alpha_{n+1})]^q \leq n\Big[\frac{\theta(B_{dist}(\alpha_n, \alpha_{n+1})) - 1}{B}\Big] \text{ for all } n \geq n_0.$$

Thus, in all cases, there exist $\frac{1}{B} > 0$ and $n_0 \in \mathbb{N}$ such that

$$n[B_{dist}(\alpha_n, \alpha_{n+1})]^q \leq n\Big[\frac{\theta(B_{dist}(\alpha_n, \alpha_{n+1})) - 1}{B}\Big] \text{ for all } n \geq n_0.$$

Recall equation (2.1), we obtain

$$[B_{dist}(\alpha_n, \alpha_{n+1})]^q \leq [\theta(B_{dist}(\alpha_0, \alpha_1))]^{r^n} - 1 \text{ for all } n \geq n_0.$$

Now, by assuming that $n \to \infty$ in the aforementioned inequality, we deduce that

$$\lim_{n\to\infty} n[B_{dist}(\alpha_n, \alpha_{n+1})]^q = 0.$$

As a result, there exist $n_1 \in \mathbb{N}$ where

$$B_{dist}(\alpha_n, \alpha_{n+1}) \leq \frac{1}{n^{\frac{1}{q}}} \text{ for all } n \geq n_1. \tag{2.6}$$

Let $N = \max\{n_0, n_1\}$. Due to the modified triangle inequality, we end up to the following two cases:

For every $n \geq 1$, we consider the following two cases.

Case 1: Suppose $\alpha_n = \alpha_m$ for some integers $n \neq m$. Then, if for $m > n$ we get $T^{m-n}(\alpha_n) = \alpha_n$. Choose $\nu = \alpha_n$ and $p = m - n$ which means, $T^p\nu = \nu$, that is ν is a periodic point of T. So, $B_{dist}(\nu, T\nu) = B_{dist}(T^p\nu, T^{p+1}\nu)$. Thus, we can easily deduce the above argument as $B_{dist}(\nu, T\nu) = 0$, so $\nu = T\nu$, that is ν is a fixed point for T.

Case 2: Let $T^n\alpha \neq T^m\alpha$ for all integers $n \neq m$. Let $n < m$ be two natural numbers, to prove that $\{\alpha_n\}$ is a Cauchy sequence, we want to consider the following two subcases:

Subcase 1: Assume the assumption that $n - m$ is odd, this leads to $B_{dist}(\alpha_n, \alpha_m)$ converges to 0 where $n, m \to \infty$. To show that case we might assume that $m = n + 2p + 1$. Therefore,

$$\begin{aligned}
&B_{dist}(\alpha_n, \alpha_{n+2p+1}) \\
&\leq w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1})[B_{dist}(\alpha_n, \alpha_{n+1}) + B_{dist}(\alpha_{n+1}, \alpha_{n+2}) + B_{dist}(\alpha_{n+2}, \alpha_{n+2p+1})] \\
&\leq w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1})B_{dist}(\alpha_n, \alpha_{n+1}) + w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1}). \\
&B_{dist}(\alpha_{n+1}, \alpha_{n+2}) + w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1})w(\alpha_{n+2}, \alpha_{n+3}, \alpha_{n+4}, \alpha_{n+2p+1}). \\
&[B_{dist}(\alpha_{n+2}, \alpha_{n+3}) + B_{dist}(\alpha_{n+3}, \alpha_{n+4}) + B_{dist}(\alpha_{n+4}, \alpha_{n+2p+1})] \\
&\leq w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1})B_{dist}(\alpha_n, \alpha_{n+1}) + w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1})\times \\
&B_{dist}(\alpha_{n+1}, \alpha_{n+2}) + w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1})w(\alpha_{n+2}, \alpha_{n+3}, \alpha_{n+4}, \alpha_{n+2p+1})\times \\
&B_{dist}(\alpha_{n+2}, \alpha_{n+3}) + w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1})w(\alpha_{n+2}, \alpha_{n+3}, \alpha_{n+4}, \alpha_{n+2p+1})\times \\
&B_{dist}(\alpha_{n+3}, \alpha_{n+4}) + w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1})w(\alpha_{n+2}, \alpha_{n+3}, \alpha_{n+4}, \alpha_{n+2p+1})\times \\
&B_{dist}(\alpha_{n+4}, \alpha_{n+2p+1}) \\
&\leq \cdots
\end{aligned}$$

$$\leq w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1}) B_{dist}(\alpha_n, \alpha_{n+1}) + w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1}) \times$$
$$B_{dist}(\alpha_{n+1}, \alpha_{n+2}) + w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1}) w(\alpha_{n+2}, \alpha_{n+3}, \alpha_{n+4}, \alpha_{n+2p+1}) \times$$
$$B_{dist}(\alpha_{n+2}, \alpha_{n+3}) + w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1}) w(\alpha_{n+2}, \alpha_{n+3}, \alpha_{n+4}, \alpha_{n+2p+1}) \times$$
$$B_{dist}(\alpha_{n+3}, \alpha_{n+4}) + \cdots + w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p+1}) w(\alpha_{n+2}, \alpha_{n+3}, \alpha_{n+4}, \alpha_{n+2p+1})$$
$$\cdots w(\alpha_{n+2p-2}, \alpha_{n+2p-1}, \alpha_{n+2p}, \alpha_{n+2p+1}) B_{dist}(\alpha_{n+2p}, \alpha_{n+2p+1}).$$

which gives us

$$B_{dist}(\alpha_n, \alpha_m) \leq \sum_{j=n}^{n+m-1} B_{dist}(\alpha_j, \alpha_{j+1}) \prod_{i=n}^{n+m-1} \omega(\alpha_i, \alpha_{i+1}, \alpha_{i+2}, \alpha_m).$$

Since $\sup_{m\geq 1} \lim_{n\to\infty} \omega(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_m) < \frac{1}{r}$, we have

$$B_{dist}(\alpha_n, \alpha_m) \leq \sum_{j=n}^{n+m-1} B_{dist}(\alpha_j, \alpha_{j+1}) \prod_{i=n}^{n+m-1} \omega(x_i, \alpha_{n+1}, \alpha_{n+2}, \alpha_m) \leq \frac{1}{r} \sum_{j=n}^{\infty} \frac{1}{j^{\frac{1}{q}}}.$$

Consequently, $B_{dist}(\alpha_n, \alpha_m)$ converges to 0 where $n, m \to \infty$ and $\frac{1}{q} > 1$.

Subcase 2: Suppose that $n - m$ is even. Then $B_{dist}(\alpha_n, \alpha_m)$ converges to 0 as long as $n, m \to \infty$. To show this, we might assume that $m = n + 2p$. Then,

$$B_{dist}(\alpha_n, \alpha_{n+2p}) \leq w(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_{n+2p}) [B_{dist}(\alpha_n, \alpha_{n-2}) + B_{dist}(\alpha_{n-2}, \alpha_{n+2p+1}) + B_{dist}(\alpha_{n+2p}, \alpha_{n+2p+1})]$$

Therefore, by fact that $\sup_{m\geq 1} \lim_{n\to\infty} \omega(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_m) < \frac{1}{r}$, subcase 1, (2.3) and (2.4) we can easily deduce that $B_{dist}(\alpha_n, \alpha_m)$ converges to 0 where $n, m \to \infty$, which imply that the sequence $\{\alpha_n\}$ is a Cauchy sequence in S .

Since (S, B_{dist}) is a complete controlled Branciari metric like space, there exists a point η in S where $\{\alpha_n\}$ converges to η.

In the following, we shall indicate that T is continuous.
Assume that $T\alpha \neq Ty$, then from (2.2), we deduce

$$\begin{aligned} \ln[\theta B_{dist}(T\alpha, T\nu)] &\leq r \ln[\theta B_{dist}(\alpha, \nu)] \\ &\leq \ln[\theta B_{dist}(\alpha, \nu)]. \end{aligned}$$

While θ is a non-decreasing, from the aforementioned observation we derive $B_{dist}(T\alpha, T\nu) \leq B_{dist}(\alpha, \nu)$ for all distinct $\alpha, \nu \in S$.
From this investigation we deduce that, $B_{dist}(\alpha_{n+1}, T\lambda) = B_{dist}(T\alpha_n, T\lambda) \leq B_{dist}(\alpha_n, \lambda)$ for every $n \in \mathbb{N}$. Assume $n \to \infty$ for the above inequality, then we get $\alpha_{n+1} \to T\lambda$. Using the rectangle inequality, we get

$$B_{dist}(\lambda, T\lambda) \leq \omega(\lambda, T\lambda, \alpha_n, \alpha_{n+1})[B_{dist}(\lambda, \alpha_n) + B_{dist}(\alpha_n, \alpha_{n+1}) + B_{dist}(\alpha_{n+1}, T\lambda)].$$

Take the limit as $n \to \infty$ and using (2.6) and the Definition 1.1 of (2), we get $B_{dist}(\lambda, T\lambda) = 0$ that one implies that $T\lambda = \lambda$, which is a contradiction to the assumption of T does not have a periodic point. Accordingly, suppose that λ is a periodic point for T with period q.
Assume that the set for fixed points of T be empty. Since $B_{dist}(\tau, T\tau) > 0$ for all $\tau \in S$ and $B_{dist}(\tau, T^q\tau) = 0$ for $q > 1$.
Using Definition 2.1, we get

$$\begin{aligned} \theta(B_{dist}(\tau, T\tau)) &= \theta(B_{dist}(T^q\tau, T^{q+1}\tau)) \\ &\leq [\theta(B_{dist}(\tau, T\tau))]^{r^q} \\ &< \theta(B_{dist}(\tau, T\tau)), \end{aligned}$$

which gives to a contradiction. However, there exists a point $\lambda \in S$ where $T\lambda = \lambda$.
Assume f does have another fixed point ζ where $\lambda \neq \zeta$. Then it is clearly $B_{dist}(\lambda, \zeta) = B_{dist}(f\lambda, f\zeta) \neq 0$.
Next, using the condition (2.2), to get,

$$\begin{aligned} \theta(B_{dist}(\lambda, \zeta)) = \theta(B_{dist}(T\lambda, T\zeta)) &= \theta(B_{dist}(T^q\lambda, T^q\zeta)) \\ &\leq [\theta(B_{dist}(\lambda, \zeta))]^{r^q} \\ &< \theta(B_{dist}(\lambda, \zeta)), \text{ a contradiction.} \end{aligned}$$

Therefore, $\lambda = \zeta$. This leads to , T has a unique fixed point in S. ◻

Example 2.1. *Consider the following sequence:*

$$\begin{aligned} \tau_1 &= 1 \times 2 \\ \tau_2 &= 1 \times 2 + 2 \times 3 \\ \tau_3 &= 1 \times 2 + 2 \times 3 + 3 \times 4 \\ \tau_4 &= 1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 \\ \tau_n &= 1 \times 2 + 2 \times 3 + 3 \times 4 + 4 \times 5 + ... + n(n+1) = \frac{n(n+1)(n+2)}{3}. \end{aligned} \tag{2.7}$$

Suppose that $\alpha = \{\tau_n : n \in \mathbb{N}\}$. Define $B_{dist} : S \times S \to [0, \infty)$ as $B_{dist}(\alpha, \nu) = |\alpha - \nu|^2$, $\omega : S \times S \to [1, \infty)$ as $\omega(\alpha, \nu) = 4\alpha + 2\nu + 3$. Thus, (S, B_{dist}) is a complete controlled Branciari metric like space.
Set out the mapping $T : S \to S$ as $T(\tau_1) = \tau_1$, $T(\tau_n) = \tau_{n-1}$ for all $n \geq 2$. To prove that T be a Θ-BC with $\theta(t) = e^t$.
While, $\theta(B_{dist}(T\alpha, T\nu)) \leq [\theta(B_{dist}(\alpha, \nu))]^r$ which yields $e^{(B_{dist}(T\alpha, T\nu))} \leq [e^{(B_{dist}(\alpha, \nu))}]^r$. Taking the log for both sides to get,

$$B_{dist}(T\alpha, T\nu) \leq rB_{dist}(\alpha, \nu).$$

However, to show that T is a Θ-BC, it is enough to show the aforementioned equation.

Case-1: *For $n = 1$, $m > 2$, then*

$$\begin{aligned} B_{dist}(T\tau_1, T\tau_m) &= B_{dist}(\tau_1, \tau_{m-1}) \\ &= \left|\frac{m(m-1)(m+1)-6}{3}\right|^2. \end{aligned}$$

$$B_{dist}(\tau_1, \tau_m) = \left|\frac{m(m+1)(m+2)-6}{3}\right|^2.$$

Then, consider that,

$$\begin{aligned} \frac{B_{dist}(T\tau_1, T\tau_m)}{B_{dist}(\tau_1, \tau_m)} &= \left|\frac{m(m-1)(m+1)-6}{m(m+1)(m+2)-6}\right|^2 \\ &< r \;\; where \;\; r \in (0,1). \end{aligned}$$

Case-2: *For $m > n > 1$, we have*

$$\begin{aligned} B_{dist}(T\tau_n, T\tau_m) &= B_{dist}(\tau_{n-1}, \tau_{m-1}) \\ &= B_{dist}\left(\frac{(n-1)n(n+1)}{3}, \frac{(m-1)m(m+1)}{3}\right) \\ &= \left|\frac{(n-1)n(n+1)}{3} - \frac{(m-1)m(m+1)}{3}\right|^2 \\ &= \left|\frac{n^3-n}{3} - \frac{m^3-m}{3}\right|^2 \\ &= \left|\frac{n^3-m^3-(n-m)}{3}\right|^2 \\ &= \left|\frac{(n-m)(n^2+nm+m^2)-(n-m)}{3}\right|^2 \\ &= \left|\frac{(n-m)(n^2+nm+m^2-1)}{3}\right|^2. \end{aligned}$$

$$\begin{aligned} B_{dist}(\tau_n, \tau_m) &= B_{dist}\left(\frac{n(n+1)(n+2)}{3}, \frac{m(m+1)(m+2)}{3}\right) \\ &= \left|\frac{n(n+1)(n+2)}{3} - \frac{m(m+1)(m+2)}{3}\right|^2 \\ &= \left|\frac{n^3+3n^2+2n}{3} - \frac{m^3+3m^2+2m}{3}\right|^2 \\ &= \left|\frac{n^3-m^3-3(n^2-m^2)+2(n-m)}{3}\right|^2 \\ &= \left|\frac{(n-m)(n^2+nm+m^2)+3(n+m)+2}{3}\right|^2. \end{aligned}$$

Consider,

$$\frac{B_{dist}(T\tau_n, T\tau_m)}{B_{dist}(\tau_n, \tau_m)} = \left|\frac{n^2 + nm + m^2 - 1}{n^2 + nm + m^2 + 3(n+m) + 2}\right|^2$$
$$< r, \;\; where \;\; r \in (0,1)$$

We deduce that T fulfill Θ-BC with $\theta(t) = e^t$. Therefore from Theorem 2.1, T has a unique fixed point τ_1.

If we take $\omega(\alpha, \nu, \delta, \vartheta) = b > 1$ in Theorem 2.1, we get the following corollary.

Corollary 2.1. *Suppose that T is a self-map on a complete controlled Branciari metric like space (S, d). There exists $\vartheta \in \Theta$, $r \in (0,1)$ where*

$$\vartheta(d(T\alpha, T\nu)) \leq [\vartheta(d(\alpha, \nu))]^r \text{ if } \; d(T\alpha, T\nu) \neq 0 \; for \;\; \alpha, \nu \in S.$$

Therefore, T has a unique fixed point in S.

Also, we can establish the following corollary by assuming that $\omega(\alpha, \nu, \delta, \vartheta) = 1$ in theorem 2.1.

Corollary 2.2. *Suppose that T is a self-map on a complete Branciari distance space. If there exist $\delta \in \Theta$ and $r \in (0,1)$ where*

$$\delta(d(T\alpha, T\nu)) \leq [\delta(d(\alpha, \nu))]^r \; as \; well \; as \;\; d(T\alpha, T\nu) \neq 0 \; for \;\; \alpha, \nu \in S.$$

Then T seize a unique fixed point in S.

Definition 2.3. *Suppose that (S, B_{dist}) is a B_{dist}-metric space. The mapping $f : S \to S$ is called Ćirić-Reich-Rus type Θ-BC noted CRR-Θ-Branciari contraction, if there exists a function as $\theta \in \Theta$ and a non-negative real number $r < 1$ such that*

$$\theta(B_{dist}(f\alpha, f\nu)) \leq [M_{f,\theta}(\alpha, \nu)]^r, \tag{2.8}$$

for all $\alpha, \nu \in S$, where

$$M_{f,\theta}(\alpha, \nu) := \max\{\theta(B_{dist}(\alpha, \nu)), \theta(B_{dist}(\nu, f\nu))\},$$

where $\limsup_{n,m\to\infty} \omega(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_m) < \frac{1}{r}$, $\alpha_n = f^n x_0$ for $\alpha_0 \in S$ and $r \in (0,1)$.

Theorem 2.2. *Let (S, B_{dist}) be a complete controlled Branciari metric like space and $f : S \to S$ is a CRR-Θ-BC. Therefore, f has a unique fixed point in S.*

Proof. According to Theorem 2.1, we develope an iterative sequence $\{\alpha_n\}_{0^\infty}$ beside starting an arbitrary point $\alpha_0 \in S$ as:

$$\alpha_n = f^n \alpha_0 \text{ for every } n \in \mathbb{N}.$$

Without loss of generality, suppose that $B_{dist}(f^n\alpha, f^{n+1}\alpha) > 0$ for all $n \in \mathbb{N}$. Indeed, if $f^{n_*}\alpha = f^{n_*+1}x$ for some $n_* \in \mathbb{N}$, then $f^{n_*}\alpha$ will be a fixed point of T.

We show that $\lim_{n\to\infty} B_{dist}(\alpha_n, \alpha_{n+1}) = 0$.
By applying the contraction condition (2.8), we get

$$\theta(B_{dist}(\alpha_{n+1}, \alpha_n)) \leq [M_{f,\theta}(\alpha_n, \alpha_{n-1})]^r. \tag{2.9}$$

Then,

$$\begin{aligned} M_{f,\theta}(\alpha_n, \alpha_{n-1}) &\leq \max\{\theta(B_{dist}(\alpha_n, \alpha_{n-1})), \theta(B_{dist}(\alpha_n, f\alpha_n)), \theta(B_{dist}(\alpha_{n-1}, f\alpha_{n-1}))\} \\ &= \max\{\theta(B_{dist}(\alpha_n, \alpha_{n-1})), \theta(B_{dist}(\alpha_n, \alpha_{n+1})), \theta(B_{dist}(\alpha_{n-1}, \alpha_n))\} \\ &\leq \max\{\theta(B_{dist}(\alpha_n, \alpha_{n-1})), \theta(B_{dist}(\alpha_n, \alpha_{n+1}))\}. \end{aligned}$$

If $M_{f,\theta}(\alpha_n, \alpha_{n-1}) = \theta(B_{dist}(\alpha_n, \alpha_{n+1}))$, then we deduce the inequality (2.9) as

$$\theta(B_{dist}(\alpha_{n+1}, \alpha_n)) \leq \theta(B_{dist}(\alpha_n, \alpha_{n+1}))^r \Leftrightarrow \ln\left(\theta(B_{dist}(\alpha_{n+1}, \alpha_n))\right) \leq r \ln\left(\theta(B_{dist}(\alpha_{n+1}, \alpha_n))\right),$$

which is a contradiction (since $r < 1$). Thus, we have $M_{f,\theta}(\alpha_n, \alpha_{n-1}) = \theta(B_{dist}(\alpha_{n-1}, \alpha_n))$. It yields from (2.9) that

$$\theta(B_{dist}(\alpha_n, \alpha_{n+1})) \leq [\theta(B_{dist}(\alpha_{n-1}, \alpha_n))]^r.$$

Recursively, we deduce to

$$\theta(B_{dist}(\alpha_n, \alpha_{n+1})) \leq [\theta(B_{dist}(\alpha_0, \alpha_1))]^{r^n}.$$

By following the relevant lines in the proof of the theorem 2.2, we reach to the conclusion that the sequence $\{\alpha_n\}$ in S is a Cauchy sequence. Regarding that (S, B_{dist}) is a complete controlled Branciari metric like space , there exists a point λ in S where $\{\alpha_n\}$ converges to λ.

Without lost of the generality, suppose $f^n\alpha \neq \lambda$ for every n (or for n large enough). Suppose $B_{dist}(\lambda, T\lambda) > 0$. Applying (2.8), we deduce that

$$\theta(B_{dist}(f\alpha_n, f\lambda)) \leq [M_{f,\theta}(\alpha_n, \lambda)]^r, \tag{2.10}$$

for all $\alpha, \nu \in S$, where

$$M_{f,\theta}(\alpha_n, \lambda) := \max\{\theta(B_{dist}(\alpha_n, \lambda)), \theta(B_{dist}(\alpha_n, f\alpha_n)), \theta(B_{dist}(\lambda, f\lambda))\}.$$

Now, take $n \to \infty$ in the above inequality, we derive that

$$\theta(B_{dist}(\lambda, f\lambda)) \leq [\theta(B_{dist}(\lambda, f\lambda))]^r < \theta(B_{dist}(\lambda, f\lambda)),$$

which is a contradiction. However, $f\lambda = \lambda$, that is, f has a fixed point in S.
Assume that $\lambda \neq \zeta$ are two fixed points for f. So, clearly $B_{dist}(\lambda, \zeta) = B_{dist}(f\lambda, f\zeta) \neq 0$.

Using the condition (2.11) we get,

$$\begin{aligned}1 < \theta(B_{dist}(\lambda,\zeta)) &= \theta(B_{dist}(f\lambda, f\zeta))\\ &\le [\max\{\theta(B_{dist}(\lambda,\zeta)), \theta(B_{dist}(\lambda, f\lambda)), \theta(B_{dist}(\zeta, f\zeta))\}]^{r}\\ &< \theta(B_{dist}(\lambda,\zeta)),\end{aligned}$$

which is a contradiction. Correspondingly, we conclude $\lambda = \zeta$. Therefore, f has a unique fixed point in S. □

Definition 2.4. *Let (S, B_{dist}) be an extended Branciari b-distance. A mapping $f : S \to S$ is called an interpolative-Θ-BC if there exists a function $\theta \in \Theta$ and a non-negative real numbers r_1, r_2, r_3 with $r_1 + r_2 + r_3 < 1$ such that*

$$\theta(B_{dist}(f\alpha, f\nu)) \le [\theta(B_{dist}(\alpha,\nu))]^{r_1}[\theta(B_{dist}(\alpha, f\alpha))]^{r_2}[\theta(B_{dist}(\nu, f\nu))]^{r_3},$$

for all $\alpha, \nu \in S$.

where $\limsup_{n,m\to\infty} \omega(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_m) < \frac{1}{r}$, $\alpha_n = f^n\alpha_0$ for $\alpha_0 \in S$ and $r \in (0,1)$.

Theorem 2.3. *Assume that (S, B_{dist}) is a controlled Branciari metric like such that B_{dist} is a continuous functional. If $f : S \to S$ is an interpolative-Θ-Branciari contraction, then f has a unique fixed point in S.*

We skip the proof since

$$[\theta(B_{dist}(\alpha,\nu))]^{r_1}[\theta(B_{dist}(\alpha, f\alpha))]^{r_2}[\theta(B_{dist}(\nu, f\nu))]^{r_3}$$

$$\le [M_{\theta,f}(\alpha,\nu)]^{r_1+r_2+r_3}.$$

Thus, it is sufficient to choose $r := r_1 + r_2 + r_3 < 1$ from Theorem 2.2 to figure the above Theorem .

Example 2.2. *Let $S = [-1, 1]$, we define $B_{dist} : S^2 \to [0,\infty)$ by $B_{dist}(x, y) = |x - y|^2$ with $B_{dist}(\frac{1}{3}, \frac{1}{3}) = 2$. Taking $\limsup_{n,m\to\infty} \omega(\alpha, \nu, \delta, \vartheta) < \frac{1}{r}$. Since $r \in [0, 1)$, we fix $r = \frac{1}{4}$*
It is not difficult to prove that it is a controlled Branciari metric like space.
Let $T : S \to S$ be a mapping defined by $T(x) = \frac{x}{2}$. Now we define $\theta : [0,\infty) \to [0,\infty)$ by $\theta(t) = e^t$.
$[\theta(B_{dist}(\alpha,\nu))]^{r_1} = (e^{(|\alpha-\nu|)^2})^{r_1}$
$[\theta(B_{dist}(\alpha, T(\alpha)))]^{r_2} = (e^{(|\alpha-\frac{\alpha}{2}|)^2})^{r_2} = (e^{\frac{\alpha^2}{4}})^{r_2} = e^{\frac{\alpha^2\times r_2}{4}}$
$[\theta(B_{dist}(\nu, T(\nu)))]^{r_3} = (e^{(|\nu-\frac{\nu}{2}|)^2})^{r_3} = (e^{\frac{\nu^2}{4}})^{r_3} = e^{\frac{\nu^2\times r_3}{4}}$
Let $r_1 = \frac{1}{2}, r_2 = \frac{1}{5}, r_3 = \frac{1}{6}$
$r_1 + r_2 + r_3 = \frac{1}{2} + \frac{1}{5} + \frac{1}{6} = \frac{13}{15} < 1.$
$[\theta(B_{dist}(\alpha,\nu))]^{r_1}[\theta(B_{dist}(\alpha, T(\alpha)))]^{r_2}[\theta(B_{dist}(\nu, T(\nu)))]^{r_3} = e^{(|\alpha-\nu|)^2)^{\frac{1}{2}}} e^{\frac{\alpha^2\frac{1}{5}}{4}} e^{\frac{\nu^2\frac{1}{6}}{4}}$
$= e^{(|\alpha-\nu|+\frac{\alpha^2}{20}+\frac{\nu^2}{24})}.$
Now taking the following cases:

Case 1: $e^{(|\alpha-\nu|)^2} > e^{\frac{\alpha^2}{4}} > e^{\frac{\nu^2}{4}}$
$e^{(|\alpha-\nu|+\frac{\alpha^2}{20}+\frac{\nu^2}{24})} < e^{((|\alpha-\nu|)^2)^{\frac{1}{2}}} e^{((|\alpha-\nu|)^2)^{\frac{1}{5}}} e^{((|\alpha-\nu|)^2)^{\frac{1}{6}}} < e^{(|\alpha-\nu|)^{1+\frac{2}{5}+\frac{2}{6}}} = e^{(|\alpha-\nu|)^{\frac{26}{15}}}$ *(1)*

Case 2*: Now* $m_{\theta,f} = max\{\theta(B_{dist}(\alpha,\nu), B_{dist}(\alpha, f_\alpha), B_{dist}(\nu, f_\nu))$.
Let $m_{\theta,f}(\alpha,\nu) = e^(|\alpha-\nu|)^2)$
$[m_{\theta,f}(\alpha,\nu)]^{r_1+r_2+r_3} = [e^(|\alpha-\nu|)^{1+\frac{2}{5}+\frac{2}{6}}] = e^(|\alpha-\nu|)^{\frac{26}{15}}$ *(2)*
From the above two cases we get
$[\theta(B_{dist}(\alpha,\nu)]^{r_1}[\theta(B_{dist}(\alpha, f_\alpha)]^{r_2}[\theta(B_{dist}(\nu, f_\nu)]^{r_3} \leq [m_{\theta,f}(\alpha,\nu)]^{r_1+r_2+r_3}$.
Similarly, we can prove that for all cases like,
$e^{\frac{\alpha^2}{4}} \geq e^{\frac{\alpha^2}{4}} \geq e^{(\alpha-2)^2}$
$e^{\frac{\nu^2}{4}} \geq e^{\frac{\alpha^2}{4}} \geq e^{(\nu-2)^2}$.
Note that all other possibilities in the above condition holds. Therefore, By Theorem 2.2 there exists an unique fixed point in S.

Remark 2.1. *In Theorem 2.3, letting* $r_2 = 0, r_3 = 0$ *then we obtain the following result.*

Theorem 2.4. *Let* f *be a self-mapping on a controlled Branciari metric like space* (S, B_{dist}) *and* $\theta \in \Theta$*. If there exists* $r_1 \in [0,1)$ *where*

$$\theta(B_{dist}(f\alpha, f\nu)) \leq [\theta(B_{dist}(\alpha,\nu))]^{r_1} \text{ for all } x, y \in S \tag{2.11}$$

such that $r \in [0,1)$ *and* $\limsup_{m,n\to\infty} \omega(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_m) < \dfrac{1}{r}$*. Then* f *has a unique fixed point in S.*

Example 2.3. *Let* $S = A \cup B$*, where* $A = \{3,4,5\}$, $B = \{\dfrac{2}{n}; n = 3,4,5,6,7\}$ *and* $B_{dist} : X \times X \to [0,\infty)$ *defined as*
$B_{dist}(x,x) = 0$ *except for* $B_{dist}(4,4) = \dfrac{1}{2}$
$B_{dist}(\frac{2}{3}, \frac{2}{4}) = B_{dist}(\frac{2}{5}), \frac{2}{6} = \frac{1}{8}$
$B_{dist}(\frac{2}{3}, \frac{2}{5}) = B_{dist}(\frac{2}{4}), \frac{2}{5} = \frac{2}{8}$
$B_{dist}(\frac{2}{3}, \frac{2}{6}) = B_{dist}(\frac{2}{5}), \frac{2}{7} = \frac{3}{8}$
$B_{dist}(\frac{2}{3}, \frac{2}{7}) = B_{dist}(\frac{2}{4}), \frac{2}{6} = \frac{7}{8}$
$B_{dist}(x,y) = |x-y|$ *otherwise*
$B_{dist}(\frac{2}{3}, \frac{2}{7}) = \frac{7}{8} > B_{dist}(\frac{2}{3}, \frac{2}{4}) + B_{dist}(\frac{2}{4}, \frac{2}{5}) + B_{dist}(\frac{2}{5}, \frac{2}{7}) = \frac{6}{8}$*. Then S is not a Branciari distance space.*
Taking $\limsup_{n,m\to\infty} \omega(\alpha,\nu,\delta,\vartheta) < \dfrac{1}{r}$*. Since* $r \in [0,1)$ *we take* $r = \frac{1}{2}$.
On the other hand, $B_{dist}(\frac{2}{3}, \frac{2}{7}) \leq \omega(\alpha,\nu,\delta,\vartheta)[B_{dist}(\frac{2}{3}, \frac{2}{4}) + B_{dist}(\frac{2}{4}, \frac{2}{5}) + B_{dist}(\frac{2}{5}, \frac{2}{7})]$
So, it is a controlled Branciari distance space.
Let $T : S \to S$ *be a mapping defined by* $T(x) = \dfrac{x}{7}$
Now, we define $\theta : [o,\infty) \to [0,\infty)$ *by* $\theta(t) = 3^{\sqrt{t}}$.
$\theta(B_{dist}(T_\alpha, T_\nu)) = \theta(|\frac{\alpha}{7} - \frac{\nu}{7}|) = 3^{\sqrt{|\frac{\alpha}{7}-\frac{\nu}{7}|}} = \theta(B_{dist}(\alpha,\nu)) = 3^{\sqrt{|\alpha-\nu|}}$.

But, $\theta(B_{dist}(T_\alpha, T_\nu)) \leq [\theta(B_{dist}(\alpha, \nu))]^r$, *where* $r = \dfrac{1}{2}$.
All the conditions of the above theorem are satisfied, then T has a fixed point in S.

In Theorem 2.3, letting $r_1 = 0, r_2 = 0, r_3 = 0$ then the above theorem can be reduced as follows.

Theorem 2.5. *Let* (S, B_{dist}) *be a controlled Branciari metric like space where* B_{dist} *is continuous function,* $\theta \in \Theta$ *and* $f : S \to S$ *is a mapping. Assume there is* $r_4 \in [0,1)$ *such that*

$$\theta(B_{dist}(f\alpha, f\nu)) \leq [\theta(B_{dist}(\alpha, f\nu) + B_{dist}(\nu, f\alpha))]^{r_4} \text{for all } \alpha, \nu \in S \tag{2.12}$$

and $r \in [0,1)$ *where* $\limsup_{m,n\to\infty} \omega(\alpha_n, \alpha_{n+1}, \alpha_{n+2}, \alpha_m) < \dfrac{1}{r}$. *Consequently,* f *possesses a unique fixed point in S.*

3. Existence of a Solution of Fourth-Order Differential Equation

We shall consider the following problem

$$\begin{cases} \eta^4(\nu) = g(\nu, \eta(\nu), \eta', \eta'', \eta''') \\ \eta(0) = \eta'(0) = \eta''(1) = \eta'''(1) = 0; \ \nu \in [0,1] \end{cases} \tag{3.1}$$

where $g : [0,1] \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ is continuous function.

The current undertaking revolves around the intricate problem known as the boundary value problem (BVP), a commonly used abbreviation. This mathematical challenge is instrumental in simulating complex phenomena, specifically focusing on the equilibrium state of elastic beam deformations. More precisely, it addresses scenarios characterized by one endpoint in a state of freedom while the other remains statically positioned. In the domain of mechanics, this particular BVP is recognized as the cantilever beam equation, showcasing its prevalence and importance in the mathematical landscape. The acquisition of solutions to such problems assumes a pivotal role, given their substantial significance in various mathematical applications. To tackle this task, we leverage the fixed point technique, a powerful method employed for determining solutions to the BVP.

Within the confines of this section, we delve into the exploration of the existence of a solution for the boundary value problem associated with a fourth-order differential equation. Designating the space of all continuous bounded functions defined in the interval [0,1] as S, symbolized by $S = \mathcal{F}[0,1]$, we introduce the controlled Branciari metric to measure distances within this space. The metric, expressed as $B_{dist}(f(s), g(s)) = \max_{s\in S} |f(s) - g(s)|^2$, plays a crucial role in establishing the framework for our analysis. Additionally, we introduce a mapping $\omega : S^4 \to [1, \infty)$, defined as $\omega(u_1, u_2, u_3, u_4) = 5u_1 + 5u_2 + 3$, providing an additional layer of complexity to our mathematical exploration.

With this foundational groundwork securely established, we proceed to reframe the fourth-order ordinary differential equation (BVP) in the form of a comprehensive integral expression:

$$\eta(t) = \int_0^1 \mathcal{G}(\nu, s) g(s, \eta(s), \eta'(s)) ds, \quad \eta \in \mathcal{F}[0,1]$$

Here, $\mathcal{G}(\nu, s)$ denotes Green's function of the homogeneous linear problem $\eta^4(\nu) = 0, \eta(0) = \eta'(0) = \eta''(1) = \eta'''(1) = 0$, providing explicit insight into the nature of the underlying mathematical framework.

$$\mathcal{G}(\nu, s) = \begin{cases} \frac{1}{6}\nu^2(3s-\nu), & 0 \le \nu \le s \le 1 \\ \frac{1}{6}s^2(3\nu - s), & 0 \le s \le \nu \le 1. \end{cases} \tag{3.2}$$

From (3.2), we can easily check that $\mathcal{G}(\nu, s)$ has the following properties

$$\frac{1}{3}\nu^2 s^2 \le \mathcal{G}(\nu, s) \le \frac{1}{2}\nu^2 \text{ (or } \frac{1}{2}s^2), \ \nu, s \in [0,1].$$

Theorem 3.1. *Suppose that the following conditions are hold.*

(1) $g : [0,1] \times \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ *is continuous.*

(2) *There exists* $\tau \in [1, \infty)$ *where the following condition holds for every* $\eta, y \in S$,

$$|g(s, \eta, \eta') - g(s, y, y')| \le \sqrt{20} e^{\frac{-\tau}{2}} |\eta(s) - y(s)|, \quad s \in [0,1].$$

(3) *There exists* $\eta_0 \in X$ *where for every* $\nu \in [0,1]$, *we deduce*

$$\eta_0 t \le \int_0^1 \mathcal{G}(\nu, s) g(s, \eta_0(s), \eta_0'(s)) ds.$$

Then the BVP problem has a solution in S.

Proof: Assume the mapping $f : S \to S$ is defined as

$$f(\eta)(\nu) = \int_0^1 \mathcal{G}(\nu, s) g(s, \eta(s), \eta'(s)) ds.$$

Then $\eta = f\eta$, which imply that the BVP has a unique solution.
Consider,

$$\begin{aligned}
|f(\eta)(\nu) - f(y)(\nu)|^2 &= |\int_0^1 \mathcal{G}(\nu, s) g(s, \eta(s), \eta'(s)) ds - \int_0^1 \mathcal{G}(\nu, s) g(s, y(s), y'(s)) ds|^2 \\
&\le \int_0^1 (\mathcal{G}(\nu, s))^2 |g(s, \eta(s), \eta'(s)) - g(s, y(s), y'(s)|^2 ds \\
&\le \int_0^1 \frac{1}{4} s^4 20 e^{-\tau} |\eta(s) - y(s)|^2 ds \\
&\le 20 e^{-\tau} B_{dist}(\eta, y) \int_0^1 \frac{1}{4} s^4 ds \\
&\le 20 e^{-\tau} B_{dist}(\eta, y) \frac{1}{20} \\
&= e^{-\tau} B_{dist}(\eta, y)
\end{aligned}$$

where this deduce to,

$$B_{dist}(f(\eta), f(y)) \leq e^{-\tau} B_{dist}(\eta, y)$$
$$\sqrt{B_{dist}(f(\eta), f(y))} \leq \sqrt{e^{-\tau} B_{dist}(\eta, y)}$$
$$e^{\sqrt{B_{dist}(f(\eta),f(y))}} \leq \left(e^{\sqrt{B_{dist}(\eta,y)}}\right)^{\sqrt{e^{-\tau}}}; \text{ where } e^{-\tau} < 1 \text{ as } \tau \geq 1.$$

Thus, $e^{\sqrt{B_{dist}(f(\eta),f(y))}} \leq \left(e^{\sqrt{B_{dist}(\eta,y)}}\right)^{\sqrt{r}}$ with $r = \sqrt{e^{-\tau}}$ which gives,

$$\theta(B_{dist}(f\eta, fy)) \leq [\theta(B_{dist}(\eta, y))]^r \text{ where } \theta(t) = e^{\sqrt{\nu}}.$$

Since all conditions of Theorem 2.1 are satisfied, f has a fixed point. Therefore BVP has a solution in S.

4. Conclusion

In the course of our study, we have introduced a novel concept, namely Controlled Branciari metric-like spaces. Through rigorous analysis and mathematical reasoning, we have successfully demonstrated the existence and uniqueness of fixed points for self-mapping within these spaces. Our approach is founded on the application of the θ-contraction principle, a conceptual framework pioneered by Jleli and Samet. It is noteworthy that our methodology does not rely on the restrictive assumption of Hausdorff, thereby widening the applicability of our results. The significance of our findings is further underscored by the broader implications of our work. By establishing the existence and uniqueness of fixed points in Controlled Branciari metric-like spaces, we contribute to the advancement of theoretical foundations in this mathematical domain. The introduced θ-contraction principle serves as a powerful tool, enabling a more nuanced and comprehensive understanding of self-mapping dynamics within the specified spaces. Furthermore, to highlight the practical relevance of our theoretical contributions, we have applied our results to the realm of boundary value problems associated with fourth-order differential equations. This application not only demonstrates the versatility of our findings but underscores their potential utility in addressing real-world mathematical challenges. In essence, our work extends beyond theoretical abstraction to offer valuable insights and solutions to problems with practical implications. In conclusion, our study encompasses the introduction of a new mathematical concept, the establishment of fixed-point results within Controlled Branciari metric-like spaces, and the application of these results to fourth-order differential equations. This comprehensive exploration contributes to the ongoing development of mathematical theory and its application in addressing complex problems in diverse domains.

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