Int. J. Anal. Appl. (2025), 23:23

International Journal of Analysis and Applications

A p-Laplacian Elliptic System with Strongly Coupled Critical Terms and Concave-Convex Nonlinearities

Mohamed Hatimi*, **Rachid Echarghaoui**

Faculty of Sciences, Ibn Tofail University, Kenitra, Morocco

**Corresponding author:* mohamed.hatimi@uit.ac.ma

Abstract. The main purpose of this paper is to establish some results on positive solutions for a p-Laplacian elliptic system with strongly coupled critical terms and concave nonlinearities. With the technique of variational method, namely Nehari manifold and Palais-Smale condition we show that there are at least two nontrivial solutions for our problem.

1. Introduction and Main Results

In this paper we consider the existence and multiplicity of nonnegative solutions to

$$\begin{cases} -\Delta_p u = \dfrac{\eta_1\alpha_1}{p^*}|u|^{\alpha_1-2}|v|^{\beta_1}u + \dfrac{\eta_2\alpha_2}{p^*}|u|^{\alpha_2-2}|v|^{\beta_2}u + \lambda\dfrac{|u|^{q-2}u}{|x|^\gamma}, & x \in \Omega, \\ -\Delta_p v = \dfrac{\eta_1\beta_1}{p^*}|u|^{\alpha_1}|v|^{\beta_1-2}v + \dfrac{\eta_2\beta_2}{p^*}|u|^{\alpha_2}|v|^{\beta_2-2}v + \mu\dfrac{|v|^{q-2}v}{|x|^\gamma}, & x \in \Omega, \\ u = v = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is an open bounded set of $\mathbb{R}^N$ with Lipschitz boundary such that $0 \in \Omega$, $\eta_1, \eta_2, \lambda, \mu$, are positive parameters, $p^* := \frac{pN}{N-p}$ denotes the critical Sobolev exponent and $\Delta_p = div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacien operator. The coefficients $\alpha_1, \alpha_2, \beta_1$, and $\beta_2 > 1$, satisfy $\alpha_1 + \beta_1 = p^*$, $\alpha_2 + \beta_2 = p^*$.

Set $\eta_1 = \eta_2 = 1$, $\alpha_1 = \alpha_2 = \alpha$, $\beta_1 = \beta_2 = \beta$, and $\gamma = 0$. Then problem (1.1) becomes the following elliptic system with concave–convex nonlinearities:

$$\begin{cases} -\Delta_p u = \lambda|u|^{q-2}u + \dfrac{2\alpha}{\alpha+\beta}|u|^{\alpha-2}u|v|^{\beta} & \text{in } \Omega, \\ -\Delta_p v = \mu|v|^{q-2}v + \dfrac{2\beta}{\alpha+\beta}|u|^{\alpha}|v|^{\beta-2}v & \text{in } \Omega, \\ u = v = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

Received: Sep. 23, 2024.

2020 *Mathematics Subject Classification.* 35J50, 35J60, 35J92.

Key words and phrases. p-Laplacian; positive solutions; elliptic system; strongly coupled critical terms; Nehari manifold; variational method.

https://doi.org/10.28924/2291-8639-23-2025-23

ISSN: 2291-8639

Tsing-San Hsu [8] proved the existence of multiple positive solutions to (1.2). There are other multiplicity results for critical elliptic systems involving concave-convex nonlinearities. For example, see [1]. Following this work, the existence of possibly multiple solutions of systems involving the strongly-coupled critical terms has been extensively investigated. See, for example, [4,5,7,10–12]. In the literature [14], the authors studied the problem (1.1) for the case when $p = 2$. They obtained the existence and multiplicity of positive solutions using the fibering method and the technique of Nehari manifold decomposition. To the best of our knowledge, there are no results concerning the existence and multiplicity of positive solutions for (1.1).

We assume that

$$(\mathcal{H}_1): \quad 1 < q < p < N \quad \text{and} \quad 0 \le \gamma < N + q - \frac{qN}{p}.$$

$$(\mathcal{H}_2): \quad 1 < q < p < N \quad \text{and} \quad N - \frac{(N-p)q}{(p-1)^2} \le \gamma < N + q - \frac{qN}{p}.$$

Our main results are

Theorem 1.1. *Assume $(\mathcal{H}_1)$ holds. Then for any $(\lambda, \mu) \in \mathcal{S}_{\Theta_1}$ system (1.1) has a positive ground state solution, where $\mathcal{S}_{\Theta_1}$ is defined by (2.9).*

Theorem 1.2. *Assume $(\mathcal{H}_2)$ holds. Then there exists $\Lambda > 0$ such that for any $(\lambda, \mu) \in \mathcal{S}_\Lambda$, system (1.1) admits at least two positive solutions, one of which is a positive ground state solution.*

This paper is divided into four sections. The properties of the Nehari manifold are provided in the next section and the formulation of the variational method. The proof of Theorem 1.1 is in the third section, and, finally, by Palais-Smale condition, we prove in the last section Theorem 1.2.

2. Analysis of Fibering Maps

We define the functional space as follows:

$$X := W^{1,p}(\Omega)$$

the usual Sobolev space endowed with the norm

$$\|u\|_X := \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}.$$

The space Z is defined as $Z = W_0^{1,p}(\Omega) := \{u \in X : u = 0 \text{ on } \partial\Omega\}$, endowed with the norm

$$\|u\|_Z := \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

We set $E := Z \times Z$, with the norm

$$\|(u,v)\| := \left(\|u\|_Z^p + \|v\|_Z^p\right)^{\frac{1}{p}} = \left(\int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla v|^p dx\right)^{\frac{1}{p}}.$$

We say that $(u,v) \in E$ is a weak solution of problem (1.1) if $(u,v) \in E$, one has

$$\begin{aligned}
&\int_\Omega \left(|\nabla u|^{p-2}\nabla u\phi + |\nabla v|^{p-2}\nabla v\psi\right) dx \\
&\quad = \int_\Omega \left(\frac{\eta_1\alpha_1}{p^*}|u|^{\alpha_1-2}|v|^{\beta_1}u\phi + \frac{\eta_2\alpha_2}{p^*}|u|^{\alpha_2-2}|v|^{\beta_2}u\phi\right) dx \\
&\quad + \int_\Omega \left(\frac{\eta_1\beta_1}{p^*}|u|^{\alpha_1}|v|^{\beta_1-2}v\psi + \frac{\eta_2\beta_2}{p^*}|u|^{\alpha_2}|v|^{\beta_2-2}v\psi\right) dx \\
&\quad + \int_\Omega \left(\lambda\frac{|u|^{q-2}u}{|x|^\gamma}\phi + \mu\frac{|v|^{q-2}v}{|x|^\gamma}\psi\right) dx \quad \text{for all } (\phi,\psi) \in E.
\end{aligned} \tag{2.1}$$

Thus, the corresponding energy functional of (1.1) is defined by

$$\mathcal{J}_{\lambda,\mu}(u,v) = \frac{1}{p}\int_\Omega \left(|\nabla u|^p + |\nabla v|^p\right) dx - \frac{1}{p^*}Q(u,v) - \frac{1}{q}K_{\lambda,\mu}(u,v), \tag{2.2}$$

where

$$Q(u,v) := \int_\Omega \left(\eta_1|u|^{\alpha_1}|v|^{\beta_1} + \eta_2|u|^{\alpha_2}|v|^{\beta_2}\right) dx$$

and

$$K_{\lambda,\mu}(u,v) := \int_\Omega \left(\lambda\frac{|u|^q}{|x|^\gamma} + \mu\frac{|v|^q}{|x|^\gamma}\right) dx.$$

We can clearly observe that $\mathcal{J}_{\lambda,\mu} \in C^1(E,\mathbb{R})$ and

$$\begin{aligned}
\left\langle \mathcal{J}'_{\lambda,\mu}(u,v),(\phi,\psi)\right\rangle &= \int_\Omega \left(|\nabla u|^{p-2}\nabla u\phi + |\nabla v|^{p-2}\nabla v\psi\right) dx \\
&- \int_\Omega \left(\frac{\eta_1\alpha_1}{p^*}|u|^{\alpha_1-2}|v|^{\beta_1}u\phi + \frac{\eta_2\alpha_2}{p^*}|u|^{\alpha_2-2}|v|^{\beta_2}u\phi\right) dx \\
&- \int_\Omega \left(\frac{\eta_1\beta_1}{p^*}|u|^{\alpha_1}|v|^{\beta_1-2}v\psi + \frac{\eta_2\beta_2}{p^*}|u|^{\alpha_2}|v|^{\beta_2-2}v\psi\right) dx \\
&- \int_\Omega \left(\lambda\frac{|u|^{q-2}u}{|x|^\gamma}\phi + \mu\frac{|v|^{q-2}v}{|x|^\gamma}\psi\right) dx.
\end{aligned}$$

Let S denote the best Sobolev embedding constant defined by

$$S := \inf_{u\in Z\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx}{\left(\int_\Omega |u|^{p^*} dx\right)^{p/p^*}} \tag{2.3}$$

and

$$\begin{aligned}
S_{\eta,\alpha,\beta} &:= \inf_{(u,v)\in E\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx + \int_\Omega |\nabla v|^p dx}{\left(\int_\Omega \left(\eta_1|u|^{\alpha_1}|v|^{\beta_1} + \eta_2|u|^{\alpha_2}|v|^{\beta_2}\right) dx\right)^{p/p^*}} \\
&= \inf_{(u,v)\in E\setminus\{0\}} \|(u,v)\|^p \left(\int_\Omega \left(\eta_1|u|^{\alpha_1}|v|^{\beta_1} + \eta_2|u|^{\alpha_2}|v|^{\beta_2}\right) dx\right)^{-p/p^*},
\end{aligned} \tag{2.4}$$

Then it is easy to get that

$$\int_\Omega \left(\eta_1|u|^{\alpha_1}|v|^{\beta_1} + \eta_2|u|^{\alpha_2}|v|^{\beta_2}\right) dx \le (S_{\eta,\alpha,\beta})^{-p^*/p}\|(u,v)\|^{p^*}. \tag{2.5}$$

This is achieved if and only if $\Omega = \mathbb{R}^N$ by the function (see [2])

$$U_\varepsilon(x) := C_{N,p}\left(\frac{\varepsilon^{\frac{1}{p-1}}}{\varepsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}}\right)^{\frac{N-p}{p}}, \quad \varepsilon > 0. \tag{2.6}$$

is an extremal function for the minimization problem (2.3), that is, it is a positive solution to the following problem

$$-\Delta_p u = |u|^{p^*-1}, \quad \text{in } \mathbb{R}^N.$$

Moreover,

$$\int_\Omega |\nabla U_\varepsilon|^p dx = \int_{\mathbb{R}^N} |U_\varepsilon|^{p^*} dx = S^{\frac{N}{p}}.$$

Let $R_0 > 0$ be a constant such that $\Omega \subset B(0,R_0)$, where $B(0,R_0) = \{x \in \mathbb{R}^N : |x| < R_0\}$. By Hölder's inequality and (2.3), for all $(u,v) \in E$ and $1 < q < p, 0 \le \gamma < N + q - \frac{qN}{p}$, we get

$$\begin{aligned}
\int_\Omega \frac{u^q}{|x|^\gamma} dx &\le \left(\int_\Omega |u|^{q\cdot\frac{p^*}{q}} dx\right)^{\frac{q}{p^*}} \left(\int_\Omega \left(\frac{1}{|x|^\gamma}\right)^{\frac{p^*}{p^*-q}} dx\right)^{\frac{p^*-q}{p^*}} \\
&\le S^{-\frac{q}{p}}\|u\|_Z^q \left(\int_{B(0,R_0)} \left(\frac{1}{|x|^\gamma}\right)^{\frac{p^*}{p^*-q}} dx\right)^{\frac{p^*-q}{p^*}} \\
&\le S^{-\frac{q}{p}}\|u\|_Z^q \left(\int_0^{R_0} \frac{r^{N-1}}{|r|^{\frac{p^*\gamma}{p^*-q}}} dr\right)^{\frac{p^*-q}{p^*}} \\
&= S^{-\frac{q}{p}}\|u\|_Z^q \left(\frac{pN - qN + pq}{pN\left(N - \gamma - \frac{qN}{p} + q\right)}\right)^{\frac{p^*-q}{p^*}} R_0^{N-\gamma-\frac{qN}{p}+q},
\end{aligned} \tag{2.7}$$

$$\int_\Omega \frac{v^q}{|x|^\gamma} dx \le S^{-\frac{q}{p}}\|v\|_Z^q \left(\frac{pN - qN + pq}{pN\left(N - \gamma - \frac{qN}{p} + q\right)}\right)^{\frac{p^*-q}{p^*}} R_0^{N-\gamma-\frac{qN}{p}+q}. \tag{2.8}$$

Set

$$\begin{aligned}
\Theta &:= \left(\frac{pN - qN + pq}{pN\left(N - \gamma - \frac{qN}{p} + q\right)}\right)^{\frac{p^*-q}{p^*}} R_0^{N-\gamma-\frac{qN}{p}+q} S^{-\frac{q}{p}}, \\
\Theta_1 &:= \left[\frac{p^* - p}{\Theta\,(p^* - q)}\right]^{\frac{p}{p-q}} \left(\frac{p-q}{p^*-q}\right)^{\frac{N-p}{p}} \left(S_{\eta,\alpha,\beta}\right)^{\frac{N}{p}}, \\
\mathcal{S}_\Theta &:= \{(\lambda,\mu) \in \mathbb{R}^2\backslash\{(0,0)\} : 0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \Theta\}.
\end{aligned} \tag{2.9}$$

We consider the Nehari manifold

$$\mathcal{N}_{\lambda,\mu} := \{(u,v) \in E\backslash\{(0,0)\} : \left\langle \mathcal{J}'_{\lambda,\mu}(u,v), (u,v) \right\rangle = 0\}.$$

Thus, $(u,v) \in \mathcal{N}_{\lambda,\mu}$ if and only if

$$\|(u,v)\|^p - Q(u,v) - K_{\lambda,\mu}(u,v) = 0. \tag{2.10}$$

Let $z = (u,v)$, then $\|z\|_E = \|(u,v)\| = \left(\|u\|_Z^p + \|v\|_Z^p\right)^{\frac{1}{p}}$. Note that $\mathcal{N}_{\lambda,\mu}$ contains every nonzero solution of (1.1). Define $\Phi(z) := \left\langle \mathcal{J}'_{\lambda,\mu}(z), z \right\rangle$, then for all $z = (u,v) \in \mathcal{N}_{\lambda,\mu}$, we have

$$\begin{aligned} \langle \Phi'(z), z \rangle &= p\|z\|_E^p - p^* Q(z) - qK_{\lambda,\mu}(z) \\ &= (p-q)\|z\|_E^p - (p^* - q)\, Q(z) \\ &= (p - p^*)\, \|z\|_E^p + (p^* - q)\, K_{\lambda,\mu}(z). \end{aligned} \tag{2.11}$$

Now, following the approach in [13], we divide $\mathcal{N}_{\lambda,\mu}$ into three separate parts:

$$\begin{aligned} \mathcal{N}^+_{\lambda,\mu} &:= \left\{ z \in \mathcal{N}_{\lambda,\mu} : \langle \Phi'(z), z \rangle > 0 \right\}, \\ \mathcal{N}^0_{\lambda,\mu} &:= \left\{ z \in \mathcal{N}_{\lambda,\mu} : \langle \Phi'(z), z \rangle = 0 \right\}, \\ \mathcal{N}^-_{\lambda,\mu} &:= \left\{ z \in \mathcal{N}_{\lambda,\mu} : \langle \Phi'(z), z \rangle < 0 \right\}. \end{aligned} \tag{2.12}$$

To present our main result, we will outline some important properties of $\mathcal{N}^+_{\lambda,\mu}$, $\mathcal{N}^0_{\lambda,\mu}$ and $\mathcal{N}^-_{\lambda,\mu}$.

Lemma 2.1. *The functional $\mathcal{J}_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$.*

Proof. If $z \in \mathcal{N}_{\lambda,\mu}$. From (2.10), (2.7) and (2.8), by applying the Hölder inequality, we obtain

$$\begin{aligned} \mathcal{J}_{\lambda,\mu}(z) &= \left(\frac{1}{p} - \frac{1}{p^*}\right)\|z\|_E^p - \left(\frac{1}{q} - \frac{1}{p^*}\right)K_{\lambda,\mu}(z) \\ &\geq \frac{1}{N}\|z\|_E^p - \left(\frac{1}{q} - \frac{1}{p^*}\right)\left(\lambda\|u\|_Z^q + \mu\|v\|_Z^q\right)\Theta \\ &\geq \frac{1}{N}\|z\|_E^p - \left(\frac{1}{q} - \frac{1}{p^*}\right)\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}} \|z\|_E^q \Theta, \end{aligned} \tag{2.13}$$

where Θ is as in (2.9). Thus, $\mathcal{J}_{\lambda,\mu}$ is coercive and bounded from below on $\mathcal{N}_{\lambda,\mu}$. The proof is complete. □

Lemma 2.2. *Suppose that $z_0 \in E$ is a local minimizer of $\mathcal{J}_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$ and $z_0 \notin \mathcal{N}^0_{\lambda,\mu}$, then z_0 is a critical point of the $\mathcal{J}_{\lambda,\mu}$.*

Proof. If $z_0 \in E$ is a local minimizer of $\mathcal{J}_{\lambda,\mu}$ on $\mathcal{N}_{\lambda,\mu}$, then $\mathcal{J}_{\lambda,\mu}(z_0) = \min_{z \in \mathcal{N}_{\lambda,\mu}} \mathcal{J}_{\lambda,\mu}(z)$ and (2.11) holds. By applying the theory of Lagrange multipliers, we can assert that there exists $\theta \in \mathbb{R}$ such that $\mathcal{J}'_{\lambda,\mu}(z_0) = \theta\Phi'(z_0)$. As $z_0 \in \mathcal{N}_{\lambda,\mu}$, we get

$$0 = \left\langle \mathcal{J}'_{\lambda,\mu}(z_0), z_0 \right\rangle = \theta \left\langle \Phi'(z_0), z_0 \right\rangle.$$

As $(\lambda,\mu)\in\mathcal{S}_{\Theta_1}$ and $z_0\notin\mathcal{N}^0_{\lambda,\mu}$. Then, $\langle\Phi'(z_0),z_0\rangle\neq 0$. Consequently, $\theta=0$ and $\mathcal{J}'_{\lambda,\mu}(z_0)=0$ in E^{-1}. □

When $(\lambda,\mu)\in\mathcal{S}_{\Theta_1}$, we will prove that $\mathcal{N}^{\pm}_{\lambda,\mu}\neq\emptyset$ and $\mathcal{N}^0_{\lambda,\mu}=\emptyset$.

Lemma 2.3. *Assume that $(\lambda,\mu)\in\mathcal{S}_{\Theta_1}$, then for every $z\in E$ with $Q(z)>0$, there exist unique $0<t^+<\bar{t}_{max}<t^-$ such that $t^+z\in\mathcal{N}^+_{\lambda,\mu}$, $t^-z\in\mathcal{N}^-_{\lambda,\mu}$ and*

$$\mathcal{J}_{\lambda,\mu}\left(t^+z\right)=\inf_{0\le t\le\bar{t}_{\max}}\mathcal{J}_{\lambda,\mu}(tz),\quad \mathcal{J}_{\lambda,\mu}\left(t^-z\right)=\sup_{t\ge\bar{t}_{\max}}\mathcal{J}_{\lambda,\mu}(tz),$$

that is, $\mathcal{N}^{\pm}_{\lambda,\mu}\neq\emptyset$;

Proof. For every $z\in E$ where $Q(z)>0$, and for any $t\ge 0$, we have

$$\left\langle\mathcal{J}'_{\lambda,\mu}(tz),tz\right\rangle=t^p\|z\|_E^p-t^{p^*}Q(z)-t^qK_{\lambda,\mu}(z).$$

We define $g,h:\mathbb{R}^+\to\mathbb{R}$ by

$$\begin{aligned}g(t)&:=t^{p-q}\|z\|_E^p-t^{p^*-q}Q(z)-K_{\lambda,\mu}(z),\\ h(t)&:=t^{p-q}\|z\|_E^p-t^{p^*-q}Q(z).\end{aligned}$$

Clearly, we obtain $h(0)=0$, and $h(t)\to-\infty$ as $t\to\infty$. Sinse

$$h'(t)=t^{p-q-1}\left[(p-q)\|z\|_E^p-(p^*-q)\,t^{p^*-p}Q(z)\right],\quad\text{for all } t>0,$$

there exists a unique $\bar{t}_{\max}>0$ such that $h(t)$ achieves its maximum at $\bar{t}_{\max}>0$, increasing for $t\in[0;\bar{t}_{\max})$ and decreasing for $t\in(\bar{t}_{\max};1)$. solving $h'(t)=0$, we obtain

$$\bar{t}_{\max}=\left[\frac{(p-q)\|z\|_E^p}{(p^*-q)\,Q(z)}\right]^{\frac{1}{p^*-p}}>0.$$

Moreover,

$$h(\bar{t}_{\max})=\left[\frac{(p-q)\|z\|_E^p}{(p^*-q)\,Q(z)}\right]^{\frac{p-q}{p^*-p}}\frac{p^*-p}{p^*-q}\|z\|_E^p.$$

Then from, (2.7), (2.8) and (2.9) by the Holder inequality, we obtain

$$\begin{aligned}g(\bar{t}_{\max})&=h(\bar{t}_{\max})-K_{\lambda,\mu}(u,v)\\&=\left[\frac{(p-q)\|z\|_E^p}{(p^*-q)\,Q(z)}\right]^{\frac{p-q}{p^*-p}}\frac{p^*-p}{p^*-q}\|z\|_E^p-\int_\Omega\left(\lambda\frac{u^q}{|x|^\gamma}+\mu\frac{v^q}{|x|^\gamma}\right)dx\\&\ge\left[\frac{(p-q)\|z\|_E^p}{(p^*-q)\,\|z\|_E^{p^*}\left(S_{\eta,\alpha,\beta}\right)^{-\frac{p^*}{p}}}\right]^{\frac{p-q}{p^*-p}}\frac{p^*-p}{p^*-q}\|z\|_E^p-\left(\lambda\|u\|_Z^q+\mu\|v\|_Z^q\right)\Theta\\&\ge\left(\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-p}}\left(S_{\eta,\alpha,\beta}\right)^{\frac{p^*(p-q)}{p(p^*-p)}}\frac{p^*-p}{p^*-q}\|z\|_E^q-\left(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\|z\|_E^q\Theta\\&>0,\end{aligned}\tag{2.14}$$

where Θ is as in (2.9) and the last inequality valid for all $(\lambda,\mu)\in\mathcal{S}_{\Theta_1}$. Consequently, there exist t^+ and t^- such that

$$g\left(t^+\right)=g\left(t^-\right) \text{ and } \quad g'\left(t^+\right)>0>g'\left(t^-\right),$$

for $0<t^+<\bar{t}_{\max}<t^-$. We have $t^+z\in\mathcal{N}^+_{\lambda,\mu}$, $t^-z\in\mathcal{N}^-_{\lambda,\mu}$ and

$$\mathcal{J}_{\lambda,\mu}\left(t^-z\right)\geq\mathcal{J}_{\lambda,\mu}(tz)\geq\mathcal{J}_{\lambda,\mu}\left(t^+z\right),$$

for each $t\in[t^+,t^-]$, and $\mathcal{J}_{\lambda,\mu}\left(t^+z\right)\leq\mathcal{J}_{\lambda,\mu}(tz)$ for each $t\in[0,t^+]$. Thus

$$\mathcal{J}_{\lambda,\mu}\left(t^+z\right)=\inf_{0\leq t\leq\bar{t}_{\max}}\mathcal{J}_{\lambda,\mu}(tz),\quad \mathcal{J}_{\lambda,\mu}\left(t^-z\right)=\sup_{t\geq\bar{t}_{\max}}\mathcal{J}_{\lambda,\mu}(tz).$$

□

Lemma 2.4. *For $(\lambda,\mu)\in\mathcal{S}_{\Theta_1}$, we have $\mathcal{N}^0_{\lambda,\mu}=\emptyset$ and $\mathcal{N}^-_{\lambda,\mu}$ is a closed set.*

Proof. From Lemma 2.3 we have that there exist exactly two numbers t^+ and t^- such that $0<t^+<t^-$ and $g\left(t^+\right)=g\left(t^-\right)=0$. Furthermore, $g'\left(t^+\right)>0>g'\left(t^-\right)$, If, by contradiction, $z\in\mathcal{N}^0_{\lambda,\mu}$, then we have that $g(1)=0$ with $g'(1)=0$. Then, either $t^+=1$ or $t^-=1$. In turn, either $g'(1)>0$ or $g'(1)<0$, which is a contradiction. Thus, $\mathcal{N}^0_{\lambda,\mu}=\emptyset$ for all $(\lambda,\mu)\in\mathcal{S}_{\Theta_1}$.
Finally, we demonstrate that $\mathcal{N}^-_{\lambda,\mu}$ is a closed set for all $(\lambda,\mu)\in\mathcal{S}_{\Theta_1}$. Assume that $\{z_n\}\subset\mathcal{N}^-_{\lambda,\mu}$ such that $z_n\to z$ in E as $n\to+\infty$, then we must demonstrate that $z\in\mathcal{N}^-_{\lambda,\mu}$. As $z_n\in\mathcal{N}^-_{\lambda,\mu}$, from the definition of $\mathcal{N}^-_{\lambda,\mu}$, one has

$$(p-q)\left\|z_n\right\|^p_E-\left(p^*-q\right)Q\left(z_n\right)<0. \tag{2.15}$$

Consequently, as $z_n\to z$ in E as $n\to+\infty$, it follows from (2.15) that

$$(p-q)\|z\|^p_E-\left(p^*-q\right)Q(z)\leq 0,$$

thus $z\in\mathcal{N}^-_{\lambda,\mu}\cup\mathcal{N}^0_{\lambda,\mu}$, then $z\in\mathcal{N}^-_{\lambda,\mu}$ because $\mathcal{N}^0_{\lambda,\mu}=\emptyset$ for all $(\lambda,\mu)\in\mathcal{S}_{\Theta_1}$. Therefore, $\mathcal{N}^-_{\lambda,\mu}$ is a closed set in E for all $(\lambda,\mu)\in\mathcal{S}_{\Theta_1}$. □

Lemma 2.5. *For each $z\in E$ such that $K_{\lambda,\mu}(z)>0$, if $(\lambda,\mu)\in\mathcal{S}_{\Theta_1}$, where $\mathcal{S}_{\Theta_1}$ is defined as in (2.9), then there exist t^+,t^- with $0<t^+<t_{\max}<t^-$ such that $t^+z\in\mathcal{N}^+_{\lambda,\mu}$ and $t^-z\in\mathcal{N}^-_{\lambda,\mu}$. We have*

$$t_{\max}=\left[\frac{\left(p^*-q\right)K_{\lambda,\mu}(z)}{\left(p^*-p\right)\|z\|^p_E}\right]^{\frac{1}{p-q}}>0,$$

$$\mathcal{J}_{\lambda,\mu}\left(t^+z\right)=\inf_{0\leq t\leq t_{\max}}\mathcal{J}_{\lambda,\mu}(tz),\quad \mathcal{J}_{\lambda,\mu}\left(t^-z\right)=\sup_{t\geq t_{\max}}\mathcal{J}_{\lambda,\mu}(tz).$$

Proof. The proof is almost the same as that Lemma 2.3 and is omitted here. □

Since $\mathcal{N}_{\lambda,\mu}=\mathcal{N}^+_{\lambda,\mu}\cup\mathcal{N}^-_{\lambda,\mu}$, then from Lemma 2.1 and Lemma 2.3, the following quantities are well defined

$$\theta_{\lambda,\mu}=\inf_{z\in\mathcal{N}_{\lambda,\mu}}\mathcal{J}_{\lambda,\mu}(z),\quad \theta^+_{\lambda,\mu}=\inf_{z\in\mathcal{N}^+_{\lambda,\mu}}\mathcal{J}_{\lambda,\mu}(z),\quad \theta^-_{\lambda,\mu}=\inf_{z\in\mathcal{N}^-_{\lambda,\mu}}\mathcal{J}_{\lambda,\mu}(z).$$

Lemma 2.6.

(i) if $(\lambda,\mu)\in\mathcal{S}_{\Theta_1}$*, then* $\theta_{\lambda,\mu}\leq\theta^+_{\lambda,\mu}<0$*;*
(ii) if $(\lambda,\mu)\in\mathcal{S}_{\left(\frac{q}{p}\right)^{\frac{p}{p-q}}\Theta_1}$*, then there exists a positive constant*

$$d_0=d_0(\lambda,\mu,p,q,N,S_{\eta,\alpha,\beta},\Theta),$$

such that $\theta^-_{\lambda,\mu}>d_0$*.*

Proof. (i) For $z=(u,v)\in\mathcal{N}^+_{\lambda,\mu}$. By (2.10), (2.11) and (2.12), we have

$$\frac{p-q}{p^*-q}\|z\|^p_E>Q(z). \tag{2.16}$$

Based on (2.10) and (2.16), we have

$$\begin{aligned}\mathcal{J}_{\lambda,\mu}(z)&=\left(\frac{1}{p}-\frac{1}{q}\right)\|z\|^p_E+\left(\frac{1}{q}-\frac{1}{p^*}\right)Q(z)\\&<\left[\left(\frac{1}{p}-\frac{1}{q}\right)+\left(\frac{1}{q}-\frac{1}{p^*}\right)\frac{p-q}{p^*-q}\right]\|z\|^p_E\\&=-\frac{p-q}{qN}\|z\|^p_E<0,\end{aligned}$$

Therefore, using the definition of $\theta_{\lambda,\mu}$ and $\theta^+_{\lambda,\mu}$, we can deduce that $\theta_{\lambda,\mu}\leq\theta^+_{\lambda,\mu}<0$.
(ii) Suppose that $(\lambda,\mu)\in\mathcal{S}_{\left(\frac{q}{p}\right)^{\frac{p}{p-q}}\Theta_1}$ and $z=(u,v)\in\mathcal{N}^-_{\lambda,\mu}$. By (2.9), (2.11) and (2.12), one has

$$\frac{p-q}{p^*-q}\|z\|^p_E<Q(z)\leq S^{-\frac{p^*}{p}}_{\eta,\alpha,\beta}\|z\|^{p^*}_E,$$

which implies that

$$\|z\|_E>\left(\frac{p-q}{p^*-q}\right)^{\frac{1}{p^*-p}}S^{\frac{p^*}{p(p^*-p)}}_{\eta,\alpha,\beta}. \tag{2.17}$$

Based on (2.13) and (2.17), we can deduce that

$$\mathcal{J}_{\lambda,\mu}(z)\geq\|z\|^q_E\left[\frac{1}{N}\|z\|^{p-q}_E-\left(\frac{p^*-q}{p^*q}\right)\left(\lambda^{\frac{p}{p-q}}+\mu^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\Theta\right]\geq d_0,$$

where $d_0=d_0\left(\lambda,\mu,q,p,N,S_{\eta,\alpha,\beta},\Theta\right)$ is a positive constant. □

3. Proof of Theorem 1.1

First, we introduce the following definitions related to the $(PS)_c$-sequence.

Definition 3.1. *Let* $c\in\mathbb{R}$*,* E *be a Banach space and* $\mathcal{J}_{\lambda,\mu}\in C^1(E,\mathbb{R})$*.*
(i) $\{z_n\}$ *is a* $(PS)_c$*-sequence in* E *for* $\mathcal{J}_{\lambda,\mu}$ *if* $\mathcal{J}_{\lambda,\mu}(z_n)=c+o(1)$ *and* $\mathcal{J}'_{\lambda,\mu}(z_n)=o(1)$ *strongly in* E^{-1} *as* $n\to\infty$*.*
(ii) We say that $\mathcal{J}_{\lambda,\mu}$ *satisfies the* $(PS)_c$ *condition if any* $(PS)_c$*-sequence* $\{z_n\}$ *for* $\mathcal{J}_{\lambda,\mu}$ *admits a convergent subsequence in* E*.*

Lemma 3.1.
(i) If $(\lambda,\mu)\in\mathcal{S}_{\Theta_1}$*, then there exists a* $(PS)_{\theta_{\lambda,\mu}}$*-sequence* $\{z_n\}\subset\mathcal{N}_{\lambda,\mu}$ *in* E *for* $\mathcal{J}_{\lambda,\mu}$ *,*

(ii) If $(\lambda,\mu) \in \mathcal{S}_{\left(\frac{q}{p}\right)^{\frac{p}{p-q}}\Theta_1}$*, then there exists a* $(PS)_{\theta^-_{\lambda,\mu}}$*-sequence* $\{z_n\} \subset \mathcal{N}^-_{\lambda,\mu}$ *in* E *for* $\mathcal{J}_{\lambda,\mu}$*.*

Proof. The proof is similar to the one in [13]. □

Now, we will demonstrate the existence of a local minimum for $\mathcal{J}_{\lambda,\mu}$ on $\mathcal{N}^+_{\lambda,\mu}$.

Proposition 3.1. *If* $(\mathcal{H}_1)$ *holds and* $(\lambda,\mu) \in \mathcal{S}_{\Theta_1}$*. Then* $\mathcal{J}_{\lambda,\mu}$ *has a minimizer* $z_1 = (u_1, v_1) \in \mathcal{N}^+_{\lambda,\mu}$ *and satisfies the following:*
(i) $\mathcal{J}_{\lambda,\mu}(z_1) = \theta_{\lambda,\mu} = \theta^+_{\lambda,\mu} < 0$*;*
(ii) z_1 *is a positive solution of system (1.1).*

Proof. By Lemma 3.1, there exists a $(PS)_{\theta_{\lambda,\mu}}$-sequence $\{z_n\} = \{(u_n, v_n)\} \subset \mathcal{N}_{\lambda,\mu}$ of $\mathcal{J}_{\lambda,\mu}$ such that

$$\mathcal{J}_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu} + o_n(1), \quad \mathcal{J}'_{\lambda,\mu}(z_n) = o_n(1). \tag{3.1}$$

Combining with Lemma 2.1, we can conclude that the sequence $\{z_n\}$ is bounded in E. After passing to a subsequence (still denoted by $\{z_n\}$), we can find $z_1 = (u_1, v_1) \in E$ such that

$$\begin{cases} u_n \rightharpoonup u_1, \quad v_n \rightharpoonup v_1, & \text{weakly in } Z, \\ u_n \to u_1, \quad v_n \to v_1, & \text{strongly in } L^r(\Omega)\ (1 \le r < p^*), \\ u_n(x) \to u_1(x), \quad v_n(x) \to v_1(x), & \text{a.e. in } \Omega. \end{cases} \tag{3.2}$$

From (3.1), we have $\left\langle \mathcal{J}'_{\lambda,\mu}(z_n), \varphi \right\rangle \to 0$ as $n \to \infty$ for all $\varphi \in E$. By (3.1) and (3.2), it is easy to see that z_1 is a solution of system (1.1). Because $\{z_n\} \subset \mathcal{N}_{\lambda,\mu}$, we deduce that

$$K_{\lambda,\mu}(z_n) = -\frac{p^* q}{p^* - q}\mathcal{J}_{\lambda,\mu}(z_n) + \frac{q(p^* - p)}{p(p^* - q)}\|z_n\|_E^p \tag{3.3}$$

Taking $n \to \infty$ in (3.3), by (3.1), (3.2) and the fact $\theta_{\lambda,\mu} < 0$, we obtain

$$K_{\lambda,\mu}(z_1) \ge -\frac{p^* q}{p^* - q}\theta_{\lambda,\mu} > 0.$$

Therefore, $z_1 \in \mathcal{N}_{\lambda,\mu}$ is a nontrivial solution of system (1.1). Next, we prove that $z_n \to z_1$ strongly in E and $\mathcal{J}_{\lambda,\mu}(z_1) = \theta_{\lambda,\mu}$. Similar to (2.7) and (2.8), for some $q < r < p^*$, by the Hölder inequality, one gets,

$$\begin{aligned} K_{\lambda,\mu}(z_n) = &\int_\Omega \left(\lambda \frac{|u_n|^q}{|x|^\gamma} + \mu \frac{|v_n|^q}{|x|^\gamma}\right) dx \\ \le &\lambda \left(\int_\Omega |u_n|^{q\cdot\frac{r}{q}} dx\right)^{\frac{q}{r}} \left(\int_\Omega \left(\frac{1}{|x|^\gamma}\right)^{\frac{r}{r-q}} dx\right)^{\frac{r-q}{r}} \\ &+ \mu \left(\int_\Omega |v_n|^{q\cdot\frac{r}{q}} dx\right)^{\frac{q}{r}} \left(\int_\Omega \left(\frac{1}{|x|^\gamma}\right)^{\frac{r}{r-q}} dx\right)^{\frac{r-q}{r}} \\ \le &C|u_n|^{\frac{q}{r}} + \tilde{C}|v_n|^{\frac{q}{r}}, \end{aligned}$$

where $C, \tilde{C} > 0$ are constants. By (3.2) and the Lebesgue dominated convergence theorem, we have

$$\lim_{n\to\infty} K_{\lambda,\mu}(z_n) = K_{\lambda,\mu}(z_1). \tag{3.4}$$

Noting $z_1 \in \mathcal{N}_{\lambda,\mu}$ and applying the Fatou lemma and (3.4), one has

$$\begin{aligned}\theta_{\lambda,\mu} \le \mathcal{J}_{\lambda,\mu}(z_1) &= \frac{1}{N}\|z_1\|_E^p - \frac{p^*-q}{p^*q}K_{\lambda,\mu}(z_1)\\ &\le \liminf_{n\to\infty}\left(\frac{1}{N}\|z_n\|_E^p - \frac{p^*-q}{p^*q}K_{\lambda,\mu}(z_n)\right)\\ &= \liminf_{n\to\infty}\mathcal{J}_{\lambda,\mu}(z_n) = \theta_{\lambda,\mu},\end{aligned}$$

which implies that $\mathcal{J}_{\lambda,\mu}(z_1) = \theta_{\lambda,\mu}$ and $\lim_{n\to\infty}\|z_n\|_E^p = \|z_1\|_E^p$. Combining with (3.2), $z_n \rightharpoonup z_1$ as $n \to \infty$ in E, it shows that $z_n \to z_1$. Moreover, we have $z_1 \in \mathcal{N}^+_{\lambda,\mu}$. Otherwise, if $z_1 \in \mathcal{N}^-_{\lambda,\mu}$, then by Lemma 2.3 there exist unique $t_0^\pm$such that $t_0^\pm z_1 \in \mathcal{N}^\pm_{\lambda,\mu}$ and $t_0^+ < t_0^- = 1$. Because of

$$\frac{d}{dt}\mathcal{J}_{\lambda,\mu}\left(t_0^+ z_1\right) = 0, \quad \frac{d^2}{dt^2}\mathcal{J}_{\lambda,\mu}\left(t_0^+ z_1\right) > 0$$

there exists $\bar{t} \in \left(t_0^+, t_0^-\right)$such that $\mathcal{J}_{\lambda,\mu}\left(t_0^+ z_1\right) < \mathcal{J}_{\lambda,\mu}(\bar{t}z_1)$. Again by Lemma 2.3, we have

$$\mathcal{J}_{\lambda,\mu}\left(t_0^+ z_1\right) < \mathcal{J}_{\lambda,\mu}(\bar{t}z_1) \le \mathcal{J}_{\lambda,\mu}\left(t_0^- z_1\right) = \mathcal{J}_{\lambda,\mu}(z_1)$$

which is a contradiction. Thus, by Lemma 2.6(i),$\mathcal{J}_{\lambda,\mu}(z_1) = \theta_{\lambda,\mu}$, and $z_1 \in \mathcal{N}^+_{\lambda,\mu}$. Consequently, we get that $\mathcal{J}_{\lambda,\mu}(z_1) = \theta_{\lambda,\mu} = \theta^+_{\lambda,\mu} < 0$. We conclude by proving that z_1 is a positive solution of the system (1.1). Specifically, $u_1 \not\equiv 0, v_1 \not\equiv 0$. Without loss of generality, let's assume that $v_1 \equiv 0$. Then, since u_1 is a nontrivial nonnegative solution of the system

$$\begin{cases} -\Delta_p u = \lambda\frac{|u|^{q-2}u}{|x|^\gamma}, & \text{in } \Omega,\\ u = 0, & \text{on } \partial\Omega.\end{cases}$$

By the standard regularity theory, we have $u_1 > 0$ in Ω and

$$\left\|(u_1, 0)\right\|^p = K_{\lambda,\mu}(u_1, 0) > 0.$$

Moreover, we may choose $\omega \in Z\backslash\{0\}$ such that

$$\|(0,\omega)\|^p = K_{\lambda,\mu}(0,\omega) > 0.$$

Now,

$$K_{\lambda,\mu}(u_1,\omega) = K_{\lambda,\mu}(u_1, 0) + K_{\lambda,\mu}(0,\omega) > 0.$$

Consequently, by Lemma 2.5 there is a unique $0 < t^+ < t_{\max}$ such that $(t^+u_1, t^+\omega) \in \mathcal{N}^+_{\lambda,\mu}$. Moreover,

$$t_{\max} = \left[\frac{(p^*-q)K_{\lambda,\mu}(u_1,\omega)}{(p^*-p)\left\|(u_1,\omega)\right\|^p}\right]^{\frac{1}{p-q}} = \left(\frac{p^*-q}{p^*-p}\right)^{\frac{1}{p-q}} > 1$$

and

$$\mathcal{J}_{\lambda,\mu}\left(t^+u_1, t^+\omega\right) = \inf_{0\le t\le t_{\max}}\mathcal{J}_{\lambda,\mu}(tu_1, t\omega)$$

This implies

$$\theta^+_{\lambda,\mu} \le \mathcal{J}_{\lambda,\mu}\left(t^+u_1, t^+\omega\right) \le \mathcal{J}_{\lambda,\mu}\left(u_1,\omega\right) < \mathcal{J}_{\lambda,\mu}\left(u_1,0\right) = \theta^+_{\lambda,\mu},$$

which is a contradiction. Finally, by Lemma 2.2 and the strong maximum principle, we deduce that $u_1, v_1 > 0$ in Ω and z_1 is a positive solution of system (1.1). □

Proof of Theorem 1.1. By Proposition 3.1, we get that for all $\lambda,\mu > 0$ and $(\lambda,\mu) \in \mathcal{S}_{\Theta_1}$, (1.1) has a positive solution $z_1 \in \mathcal{N}^+_{\lambda,\mu}$. □

4. Proof of Theorem 1.2

To the existence of a second positive solution for the system (1.1), we must impose a stronger condition. In this section, we will first find the range of c where $(PS)_c$ condition holds for $\mathcal{J}_{\lambda,\mu}$.

Lemma 4.1. *Assume that $\{z_n\} \subset E$ is a $(PS)_c$-sequence for $\mathcal{J}_{\lambda,\mu}$ and $z_n \rightharpoonup z$ in E, then z is a critical point of $\mathcal{J}_{\lambda,\mu}$, and there exists a positive constant C_0 such that*

$$\mathcal{J}_{\lambda,\mu}(z) \ge -C_0\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right), \tag{4.1}$$

where

$$C_0 = \frac{p-q}{p}\left[\left(\frac{pN-qN+pq}{p^2}\right)\Theta\right]^{\frac{p}{p-q}}.$$

Proof. Let $z_n = (u_n, v_n)$ and $z = (u,v)$. If $\{z_n\}$ is a $(PS)_c$-sequence for $\mathcal{J}_{\lambda,\mu}$ such that

$$\mathcal{J}_{\lambda,\mu}\left(z_n\right) = c + o_n(1), \quad \mathcal{J}'_{\lambda,\mu}\left(z_n\right) = o_n(1). \tag{4.2}$$

We claim that the sequence $\{z_n\}$ is bounded in E. Indeed, for sufficiently large n, we have

$$\begin{aligned} c + o(1) + \|z_n\|_E &\ge \mathcal{J}_{\lambda,\mu}\left(z_n\right) - \frac{1}{p^*}\left\langle \mathcal{J}'_{\lambda,\mu}\left(z_n\right), z_n\right\rangle \\ &= \left(\frac{1}{p} - \frac{1}{p^*}\right)\|z_n\|^p_E - \left(\frac{1}{q} - \frac{1}{p^*}\right)K_{\lambda,\mu}\left(z_n\right) \\ &\ge \frac{1}{N}\|z_n\|^p_E - \left(\frac{1}{q} - \frac{1}{p^*}\right)\left(\lambda\|u_n\|^q_Z + \mu\|v_n\|^q_Z\right)\Theta \\ &\ge \frac{1}{N}\|z_n\|^p_E - \left(\frac{1}{q} - \frac{1}{p^*}\right)\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)^{\frac{p-q}{p}}\|z_n\|^q_E\,\Theta, \end{aligned}$$

The given statement implies that the sequence z_n is bounded in E. Therefore, our claim is true. Passing to a subsequence (still denoted by $\{z_n\}$), there exists $z = (u,v) \in \mathcal{J}_{\lambda,\mu}$ such that $z_n \to z$ in E and

$$\begin{cases} u_n \rightharpoonup u, \quad v_n \rightharpoonup v, & \text{weakly in } Z, \\ u_n \to u, \quad v_n \to v, & \text{strongly in } L^r(\Omega) \quad (1 \le r < p^*), \\ u_n(x) \to u(x), \quad v_n(x) \to v(x), & \text{a.e. in } \Omega. \end{cases} \tag{4.3}$$

By taking $\varphi = (\phi_1, \phi_2) \in E$. Combining with (2.7), (2.8) and (4.3), one gets

$$\lim_{n\to\infty}\int_\Omega \frac{|u_n|^{q-2}u_n}{|x|^\gamma}\phi_1 dx = \int_\Omega \frac{|u|^{q-2}u}{|x|^\gamma}\phi_1 dx, \lim_{n\to\infty}\int_\Omega \frac{|v_n|^{q-2}v_n}{|x|^\gamma}\phi_2 dx = \int_\Omega \frac{|v|^{q-2}v}{|x|^\gamma}\phi_2 dx. \tag{4.4}$$

Since $\left\{|u_n|^{\alpha_i-2}|v_n|^{\beta_i}u_n\right\}$ and $\left\{|u_n|^{\alpha_i}|v_n|^{\beta_i-2}v_n\right\}$ for $i = 1,2$ are uniformly bounded in $\left(L^{p^*}(\Omega)\right)'$ and converge to $|u|^{\alpha_i-2}|v|^{\beta_i}u$ and $|u|^{\alpha_i}|v|^{\beta_i-2}v$ respectively, we can get that

$$|u_n|^{\alpha_i-2}|v_n|^{\beta_i}u_n \rightharpoonup |u|^{\alpha_i-2}|v|^{\beta_i}u, \quad |u_n|^{\alpha_i}|v_n|^{\beta_i-2}v_n \rightharpoonup |u|^{\alpha_i}|v|^{\beta_i-2}v$$

weakly in $\left(L^{p^*}(\Omega)\right)' \times \left(L^{p^*}(\Omega)\right)'$ for $i = 1,2$ as $n \to \infty$. Thus, it is concluded from (4.2) and (4.4) that

$$\lim_{n\to\infty} \left\langle \mathcal{J}'_{\lambda,\mu}(z_n), \varphi\right\rangle = \left\langle \mathcal{J}'_{\lambda,\mu}(z), \varphi\right\rangle = 0 \tag{4.5}$$

In particular, choosing $\varphi = z$ in (4.5), one get $\left\langle \mathcal{J}'_{\lambda,\mu}(z), z\right\rangle = 0$ and (2.10) is true. Consequently,

$$\mathcal{J}_{\lambda,\mu}(z) = \left(\frac{1}{p} - \frac{1}{p^*}\right)\|z\|_E^p - \left(\frac{1}{q} - \frac{1}{p^*}\right)K_{\lambda,\mu}(z). \tag{4.6}$$

Combining (2.7), (2.8) and the Young inequality, we have

$$\begin{aligned}
K_{\lambda,\mu}(u,v) \leq& \left(\lambda\|u\|_Z^q + \mu\|v\|_Z^q\right)\Theta \\
=&\left(\left[\frac{p}{q}\frac{1}{N}\left(\frac{1}{q}-\frac{1}{p^*}\right)^{-1}\right]^{\frac{q}{p}}\|u\|_Z^q\right)\left(\left[\frac{p}{q}\frac{1}{N}\left(\frac{1}{q}-\frac{1}{p^*}\right)^{-1}\right]^{-\frac{q}{p}}\lambda\Theta\right) \\
&+\left(\left[\frac{p}{q}\frac{1}{N}\left(\frac{1}{q}-\frac{1}{p^*}\right)^{-1}\right]^{\frac{q}{p}}\|v\|_Z^q\right)\left(\left[\frac{p}{q}\frac{1}{N}\left(\frac{1}{q}-\frac{1}{p^*}\right)^{-1}\right]^{-\frac{q}{p}}\mu\Theta\right) \\
\leq& \frac{1}{N}\left(\frac{1}{q}-\frac{1}{p^*}\right)^{-1}\left(\|u\|_Z^p + \|v\|_Z^p\right) + \widehat{C}\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right) \\
=&\frac{1}{N}\left(\frac{1}{q}-\frac{1}{p^*}\right)^{-1}\|(u,v)\|^p + \widehat{C}\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)
\end{aligned} \tag{4.7}$$

with

$$\widehat{C} = \frac{p-q}{p}\left(\left[\frac{p}{q}\frac{1}{N}\left(\frac{1}{q}-\frac{1}{p^*}\right)^{-1}\right]^{-\frac{q}{p}}\Theta\right)^{\frac{p}{p-q}} = \frac{p-q}{p}\left[\left(\frac{pN-qN+pq}{p^2}\right)^{\frac{q}{p}}\Theta\right]^{\frac{p}{p-q}}.$$

Then (4.1) follows from (4.6) and (4.7) with $C_0 = \left(\frac{1}{q} - \frac{1}{p^*}\right)\widehat{C}$. □

Lemma 4.2. *Suppose that $(\mathcal{H}_1)$ holds, then $\mathcal{J}_{\lambda,\mu}$ satisfies the $(PS)_c$ condition in E, with c satisfying*

$$\infty < c < c_{\lambda,\mu} = \frac{1}{N}S_{\eta,\alpha,\beta}^{\frac{N}{p}} - C_0\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)$$

where C_0 is in Lemma 4.1.

Proof. Let $\{z_n\} \subset E$ be a $(PS)_c$-sequence satisfying $\mathcal{J}_{\lambda,\mu}(z_n) = c + o(1)$ and $\mathcal{J}'_{\lambda,\mu}(z_n) = o(1)$, where $z_n = (u_n, v_n)$. Similarly to Lemma 4.1, the sequence z_n is bounded in E. Additionally, we can derive (3.2) for some $z = (u,v) \in E$. Set $\widetilde{u}_n = u_n - u, \widetilde{v}_n = v_n - v$ and $\widetilde{z}_n = (\widetilde{u}_n, \widetilde{v}_n)$. From Brézis-Lieb's lemma [3], it follows that

$$\left\|\widetilde{z}_n\right\|_E^p = \|z_n\|_E^p - \|z\|_E^p + o(1) \tag{4.8}$$

and by Lemma 2.3 in [6] one has

$$\int_\Omega |\widetilde{u}_n|^{\alpha_i}|\widetilde{v}_n|^{\beta_i}dx = \int_\Omega |u_n|^{\alpha_i}|v_n|^{\beta_i}dx - \int_\Omega |u|^{\alpha_i}|v|^{\beta_i}dx + o(1), \quad i = 1, 2. \tag{4.9}$$

Consequently, from (3.4), one gets

$$\left\|\widetilde{z}_n\right\|_E^p + \|z\|_E^p - Q\left(\widetilde{z}_n\right) - Q(z) - K_{\lambda,\mu}(z) = o(1)$$

and

$$\lim_{n\to\infty}\left\langle \mathcal{J}'_{\lambda,\mu}\left(z_n\right), z\right\rangle = \|z\|_E^p - Q(z) - K_{\lambda,\mu}(z) = 0. \tag{4.10}$$

Since $\mathcal{J}_{\lambda,\mu}\left(z_n\right) = c + o(1), \mathcal{J}'_{\lambda,\mu}\left(z_n\right) = o(1)$ and by (4.8) to (4.10), we can deduce that

$$\frac{1}{p}\left\|\widetilde{z}_n\right\|_E^p - \frac{1}{p^*}Q\left(\widetilde{z}_n\right) = c - \mathcal{J}_{\lambda,\mu}(z) + o(1) \tag{4.11}$$

and

$$\left\|\widetilde{z}_n\right\|_E^p - Q\left(\widetilde{z}_n\right) = o(1)$$

Now, we can assume that

$$\lim_{n\to\infty}\left\|\widetilde{z}_n\right\|_E^p = \lim_{n\to\infty} Q\left(\widetilde{z}_n\right) = l. \tag{4.12}$$

If $l = 0$, the proof is complete. For $l > 0$, it follows from (4.12) and the definition of $S_{\eta,\alpha,\beta}$ that

$$\left\|\widetilde{z}_n\right\|_E^p \geq S_{\eta,\alpha,\beta} Q^{\frac{p}{p^*}}\left(\widetilde{z}_n\right)$$

which means that

$$l \geq S_{\eta,\alpha,\beta}^{\frac{N}{p}}. \tag{4.13}$$

From (4.10) to (4.13) and Lemma 4.1, we have

$$c = \left(\frac{1}{p} - \frac{1}{p^*}\right)l + \mathcal{J}_{\lambda,\mu}(z) \geq c_{\lambda,\mu},$$

which contradicts the definition of c. Therefore, $l = 0$ and $z_n \to z$ strongly in E. □

Next, we will demonstrate the existence of a local minimum for $\mathcal{J}_{\lambda,\mu}$ on $\mathcal{N}^-_{\lambda,\mu}$, thereby obtaining a second positive solution for the system (1.1)
We define

$$f(\tau) := \frac{1+\tau^p}{\left(\eta_1\tau^{\beta_1} + \eta_2\tau^{\beta_2}\right)^{\frac{p}{p^*}}}, \quad \tau > 0. \tag{4.14}$$

Since f is continuous on $(0,\infty)$ such that $\lim_{\tau\to 0^+} f(\tau) = \lim_{\tau\to+\infty} f(\tau) = +\infty$, then there exists $\tau_0 > 0$ such that

$$f\left(\tau_0\right) := \min_{\tau>0} f(\tau) > 0. \tag{4.15}$$

Lemma 4.3. *Suppose that $N > p$ and $0 < \eta_i < \infty$ $(i = 1, 2)$, then*
(i) $S_{\eta,\alpha,\beta} = f\left(\tau_0\right) S$;
(ii) $S_{\eta,\alpha,\beta}$ has the minimizers $\left(U_\varepsilon(x), \tau_{min}U_\varepsilon(x)\right)$, $\forall\varepsilon > 0$, where $U_\varepsilon(x)$ are defined as in (2.6).

Proof. i) Suppose $w \in Z\backslash\{0\}$. Choosing $(u,v)=(w,\tau_0 w)$ in (2.4) we have

$$\frac{1+\tau_0^p}{\left(\eta_1\tau_0^{\beta_1}+\eta_2\tau_0^{\beta_2}\right)^{\frac{p}{p^*}}}\frac{\int_\Omega |\nabla w|^p dx}{\left(\int_\Omega |w|^{p^*}dx\right)^{p/p^*}} \geq S_{\eta,\alpha,\beta}. \tag{4.16}$$

Taking the infimum as $w \in Z\backslash\{0\}$ in (4.16), we have

$$f(\tau_0)\, S \geq S_{\eta,\alpha,\beta}. \tag{4.17}$$

Let $\{(u_n,v_n)\} \subset E$ be a minimizing sequence of $S_{\eta,\alpha,\beta}$ and define $z_n = s_n v_n$, where

$$s_n = \left(\left(\int_\Omega |v_n|^{p^*}dx\right)^{-1}\int_\Omega |u_n|^{p^*}dx\right)^{\frac{1}{p^*}}.$$

Then

$$\int_\Omega |z_n|^{p^*}dx = \int_\Omega |u_n|^{p^*}dx \tag{4.18}$$

From the Young inequality and (4.17) it follows that

$$\begin{aligned}\int_\Omega |u_n|^{\alpha_i}|z_n|^{\beta_i}dx &\leq \frac{\alpha_i}{p^*}\int_\Omega |u_n|^{p^*}dx + \frac{\beta_i}{p^*}\int_\Omega |z_n|^{p^*}dx \\ &= \int_\Omega |u_n|^{p^*}dx = \int_\Omega |z_n|^{p^*}dx, \quad i=1,2.\end{aligned} \tag{4.19}$$

Consequently,

$$\begin{aligned}\frac{\int_\Omega |\nabla u_n|^p dx + \int_\Omega |\nabla v_n|^p dx}{\left(\int_\Omega \left(\eta_1|u_n|^{\alpha_1}|v_n|^{\beta_1}+\eta_2|u_n|^{\alpha_2}|v_n|^{\beta_2}\right)dx\right)^{p/p^*}} &\geq \frac{\int_\Omega |\nabla u_n|^p dx}{\left(\left(\eta_1 s_n^{-\beta_1}+\eta_2 s_n^{-\beta_2}\right)\int_\Omega |u_n|^{p^*}dx\right)^{\frac{p}{p^*}}} \\ &\quad + \frac{s_n^{-p}\int_\Omega |\nabla z_n|^p dx}{\left(\left(\eta_1 s_n^{-\beta_1}+\eta_2 s_n^{-\beta_2}\right)\int_\Omega |z_n|^{p^*}dx\right)^{\frac{p}{p^*}}} \\ &\geq f\left(s_n^{-1}\right)S \\ &\geq f(\tau_0)\, S.\end{aligned}$$

As $n \to \infty$ we have

$$S_{\eta,\alpha,\beta} \geq f(\tau_0)\, S,$$

which together with (4.17) implies that

$$S_{\eta,\alpha,\beta} = f(\tau_0)\, S.$$

ii) By (2.4) and (4.15), $S_{\eta,\alpha,\beta}$ has the minimizers $(U_\varepsilon(x), \tau_0 U_\varepsilon(x))$. □

Lemma 4.4. *There exist a nonnegative function $\tilde{z} \in E\backslash\{0\}$ and a positive constant Λ_* such that for all $(\lambda,\mu) \in \mathcal{S}_{\Lambda_*}$ we have*

$$\sup_{t\geq 0} \mathcal{J}_{\lambda,\mu}(t\tilde{z}) < c_{\lambda,\mu}.$$

where $c_{\lambda,\mu}$ is the constant defined in Lemma (4.2). In particular, $\theta^-_{\lambda,\mu} < c_{\lambda,\mu}$, for all $(\lambda,\mu) \in \mathcal{S}_{\Lambda_*}$.

Proof. Since $0 \in \Omega$, there exists $\rho_0 > 0$ such that $B(0,\rho_0) \subset \Omega$.
Also, let us introduce a cut-off function $\psi \in C_0^\infty(\Omega)$ such that $\psi(x) = 1$ for $|x| < \frac{\rho_0}{2}$, $\psi(x) = 0$ for $|x| > \rho_0$, $0 \leq \psi(x) \leq 1$ for $\frac{\rho_0}{2} \leq |x| \leq \rho_0$ and $|\nabla\psi| \leq C_1$.
Define

$$u_\varepsilon(x) := \varepsilon^{-\frac{N-p}{p}}\psi(x)U\left(\frac{x}{\varepsilon}\right) = \frac{\varepsilon^{\frac{N-p}{p(p-1)}}}{\left(\varepsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}}\psi(x),$$

where

$$U(x) := \frac{1}{\left(1+|x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}}. \tag{4.20}$$

Set $z_\varepsilon = (u_\varepsilon, \tau_0 u_\varepsilon)$, where $\varepsilon > 0$ small enough. For any $t \geq 0$, we denote

$$\begin{aligned}
\Phi_\varepsilon(t) &= \mathcal{J}_{\lambda,\mu}(tz_\varepsilon) \\
&= \mathcal{J}_{\lambda,\mu}(tu_\varepsilon, t\tau_0 u_\varepsilon) \\
&= \frac{t^p}{p}\left(1+\tau_0^p\right)\|u_\varepsilon\|_Z^p - \frac{t^{p^*}}{p^*}\left(\eta_1\tau_0^{\beta_1} + \eta_2\tau_0^{\beta_2}\right)\int_\Omega u_\varepsilon^{p^*}dx - \left(\lambda+\mu\tau_0^q\right)\frac{t^q}{q}\int_\Omega \frac{u_\varepsilon^q}{|x|^\gamma}dx \\
&= \Phi_{\varepsilon,1}(t) - \left(\lambda+\mu\tau_0^q\right)\Phi_{\varepsilon,2}(t).
\end{aligned}$$

Notice that $\Phi_\varepsilon(0) = 0$, $\lim_{t\to+\infty}\Phi_\varepsilon(t) = -\infty$, and $\lim_{t\to 0^+}\Phi_\varepsilon(t) = 0$ uniformly for all ε. If $\inf_{0<\varepsilon\leq 1}\sup_{t\geq 0}\Phi_\varepsilon(t) \leq 0$ then $\mathcal{J}_{\lambda,\mu}(tz_\varepsilon) \leq 0 < c_{\lambda,\mu}$, for any $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \frac{S_{\eta,\alpha,\beta}^{\frac{N}{p}}}{NC_0}$. Thus, for any $0 < \lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} < \frac{S_{\eta,\alpha,\beta}^{\frac{N}{p}}}{NC_0}$, one obtains

$$\sup_{t\geq 0}\mathcal{J}_{\lambda,\mu}(tz_\varepsilon) \leq c_{\lambda,\mu}.$$

On the other hand, if $\inf_{0<\varepsilon\leq 1}\sup_{t\geq 0}\Phi_\varepsilon(t) > 0$, then $\sup_{t\geq 0}\Phi_\varepsilon(t) > 0$ and it attains for some $t_\varepsilon > 0$. So, there exist two constants $t_1, t_2 > 0$ such that $t_1 < t_\varepsilon < t_2$.
Step 1. We show that

$$\Phi_{\varepsilon,1}(t) \leq \frac{1}{N}S_{\eta,\alpha,\beta}^{\frac{N}{p}} + C_2\varepsilon^{\frac{N-p}{p-1}}.$$

From Hsu [9] (Lemma 4.3), we have the following estimates:

$$\begin{aligned}
\|\nabla u_\varepsilon\|_Z^p = \int_\Omega |\nabla u_\varepsilon|^p \,\mathrm{d}x &= S^{N/p} + O\left(\varepsilon^{\frac{N-p}{p-1}}\right) \\
\int_\Omega |u_\varepsilon(x)|^{p^*}\,\mathrm{d}x &= S^{N/p} + O\left(\varepsilon^{\frac{N}{p-1}}\right)
\end{aligned} \tag{4.21}$$

as $\varepsilon \to 0$.

Note that $\Phi_{\varepsilon,1}$ is increasing in $(0, t_{\max})$ and decreasing in $(t_{\max}, \infty)$, where $t_{\max}$ satisfies $\Phi'_{\varepsilon,1}(t_{\max}) = 0$, one has

$$t_{\max} = \left[\frac{\left(1+\tau_0^p\right) \|u_\varepsilon\|_Z^p}{\left(\eta_1 \tau_0^{\beta_1} + \eta_2 \tau_0^{\beta_2}\right) \int_\Omega u_\varepsilon^{p^*} dx} \right]^{\frac{N-p}{p^2}},$$

Then, from 4.21 and Lemma 4.3, we conclude that

$$\begin{aligned} \Phi_{\varepsilon,1}(t) &\leq \Phi_{\varepsilon,1}(t_{\max}) \\ &\leq \frac{1}{N} \left[\frac{\left(1+\tau_0^p\right) \|u_\varepsilon\|_Z^p}{\left(\left(\eta_1 \tau_0^{\beta_1} + \eta_2 \tau_0^{\beta_2}\right) \int_\Omega u_\varepsilon^{p^*} dx\right)^{\frac{p}{p^*}}} \right]^{\frac{N}{p}} \\ &\leq \frac{1}{N} \left[f(\tau_0) \frac{S^{\frac{N}{p}} + O\left(\varepsilon^{\frac{N-p}{p-1}}\right)}{\left(S^{\frac{N}{p}} + O\left(\varepsilon^{\frac{N}{p-1}}\right)\right)^{\frac{p}{p^*}}} \right]^{\frac{N}{p}} \\ &\leq \frac{1}{N} \left[f(\tau_0) S\right]^{\frac{N}{p}} + C_2 \varepsilon^{\frac{N-p}{p-1}} \\ &= \frac{1}{N} S_{\eta,\alpha,\beta}^{\frac{N}{p}} + C_2 \varepsilon^{\frac{N-p}{p-1}}. \end{aligned} \tag{4.22}$$

Step 2. Now, we estimate $\Phi_{\varepsilon,2}(t_\varepsilon)$ and we claim that if we set $\varepsilon = \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)^{\frac{p-1}{N-p}}$, then there exists $\Lambda_* > 0$ such that for all $(\lambda, \mu) \in \mathcal{S}_{\Lambda_*}$ we have $\sup\limits_{t \geq 0} \mathcal{J}_{\lambda,\mu}(t\tilde{z}) < c_{\lambda,\mu}$.

$$\begin{aligned} \Phi_{\varepsilon,2}(t_\varepsilon) &= \frac{t_\varepsilon^q}{q} \int_\Omega \frac{u_\varepsilon^q}{|x|^\gamma} dx \\ &= \frac{t_\varepsilon^q}{q} \int_\Omega \frac{\psi^q(x) \varepsilon^{\frac{(N-p)q}{p(p-1)}}}{|x|^\gamma \left(\varepsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}\right)^{\frac{(N-p)q}{p(p-1)}}} dx \\ &\geq \frac{t_1^q}{q} \int_{|x| \leq \frac{\rho_0}{2}} \frac{\varepsilon^{\frac{(N-p)q}{p(p-1)}}}{|x|^\gamma \left(\varepsilon^{\frac{p}{p-1}} + |x|^{\frac{p}{p-1}}\right)^{\frac{(N-p)q}{p(p-1)}}} dx \\ &= \frac{t_1^q}{q} \int_0^{\frac{\rho_0}{2}} \frac{\varepsilon^{\frac{(N-p)q}{p(p-1)}} r^{N-1}}{|r|^\gamma \varepsilon^{\frac{(N-p)q}{(p-1)^2}} \left[1 + \left(\frac{r}{\varepsilon}\right)^{\frac{p}{p-1}}\right]^{\frac{(N-p)q}{p(p-1)}}} dr \end{aligned} \tag{4.23}$$

$$
\begin{aligned}
&= \frac{t_1^q}{q}\varepsilon^{N-\gamma+\frac{q}{(p-1)^2}-\frac{qN}{p(p-1)^2}} \int_0^{\frac{\rho_0}{2\varepsilon}} \frac{r^{N-1}}{r^\gamma\left(1+r^{\frac{p}{p-1}}\right)^{\frac{(N-p)q}{p(p-1)}}}dr \\
&= \frac{t_1^q}{q}\varepsilon^{N-\gamma+\frac{q}{(p-1)^2}-\frac{qN}{p(p-1)^2}} \int_0^{1} \frac{r^{N-1}}{r^\gamma\left(1+r^{\frac{p}{p-1}}\right)^{\frac{(N-p)q}{p(p-1)}}}dr \\
&\quad + \frac{t_1^q}{q}\varepsilon^{N-\gamma+\frac{q}{(p-1)^2}-\frac{qN}{p(p-1)^2}} \int_1^{\frac{\rho_0}{2\varepsilon}} \frac{r^{N-1}}{r^\gamma\left(1+r^{\frac{p}{p-1}}\right)^{\frac{(N-p)q}{p(p-1)}}}dr.
\end{aligned}
\tag{4.24}
$$

From (4.24), we get

$$
\Phi_{\varepsilon,2}\left(t_\varepsilon\right) = \frac{t_\varepsilon^q}{q}\int_\Omega \frac{u_\varepsilon^q}{|x|^\gamma}dx \geq \begin{cases} C_3\varepsilon^{N-\gamma+\frac{q}{(p-1)^2}-\frac{qN}{p(p-1)^2}}, & \gamma > N-\frac{(N-p)q}{(p-1)^2}, \\ C_4\varepsilon^{\frac{qN}{p(p-1)}-\frac{q}{p-1}}|\ln\varepsilon|, & \gamma = N-\frac{(N-p)q}{(p-1)^2}, \\ C_5\varepsilon^{\frac{qN}{p(p-1)}-\frac{q}{p-1}}, & \gamma < N-\frac{(N-p)q}{(p-1)^2}, \end{cases}
$$

where $C_i > 0 (i = 3,4,5)$ are positive constants (C_i independent of ε). The case of $\gamma > N - \frac{(N-p)q}{(p-1)^2}$, combining (4.22) with (4.24), one has

$$
\begin{aligned}
\sup_{t\geq 0}\mathcal{J}_{\lambda,\mu}\left(tz_\varepsilon\right) &= \Phi_\varepsilon\left(t_\varepsilon\right) \\
&= \Phi_{\varepsilon,1}\left(t_\varepsilon\right) - \left(\lambda+\mu\tau_0^q\right)\Phi_{\varepsilon,2}\left(t_\varepsilon\right) \\
&\leq \frac{1}{N}S_{\eta,\alpha,\beta}^{\frac{N}{p}} + C_2\varepsilon^{\frac{N-p}{p-1}} - C_3\left(\lambda+\mu\tau_0^q\right)\varepsilon^{N-\gamma+\frac{q}{(p-1)^2}-\frac{qN}{p(p-1)^2}}.
\end{aligned}
$$

Let $\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} = \varepsilon^{\frac{N-p}{p-1}}$, that is, $\varepsilon = \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)^{\frac{p-1}{N-p}}$, then we can choose $\delta_1 > 0$ such that for all $(\lambda,\mu) \in \mathcal{S}_{\delta_1}$ we have

$$
\begin{aligned}
&C_2\varepsilon^{\frac{N-p}{p-1}} - C_3\left(\lambda+\mu\tau_0^q\right)\varepsilon^{N-\gamma+\frac{q}{(p-1)^2}-\frac{qN}{p(p-1)^2}} \\
=&C_2\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right) - C_3\left(\lambda+\mu\tau_0^q\right)\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)^{\frac{p(p-1)^2N-p(p-1)^2\gamma+pq-qN}{p(p-1)^2(N-p)}} \\
<&-C_0\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right),
\end{aligned}
$$

Then, for for all $(\lambda,\mu) \in \mathcal{S}_{\delta_1}$, one gets

$$
\sup_{t\geq 0}\mathcal{J}_{\lambda,\mu}\left(tz_\varepsilon\right) < \frac{1}{N}S_{\eta,\alpha,\beta}^{\frac{N}{p}} - C_0\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right).
$$

The case of $\gamma = N - \frac{(N-p)q}{(p-1)^2}$, it follows from (4.22) and (4.24) that

$$\begin{aligned}\sup_{t\geq 0} \mathcal{J}_{\lambda,\mu}(tz_\varepsilon) &= \Phi_\varepsilon(t_\varepsilon)\\ &= \Phi_{\varepsilon,1}(t_\varepsilon) - \left(\lambda + \mu\tau_0^q\right)\Phi_{\varepsilon,2}(t_\varepsilon)\\ &\leq \frac{1}{N} S_{\eta,\alpha,\beta}^{\frac{N}{p}} + C_2\varepsilon^{\frac{N-p}{p-1}} - C_4\left(\lambda + \mu\tau_0^q\right)\varepsilon^{\frac{qN}{p(p-1)} - \frac{q}{p-1}}|\ln\varepsilon|\end{aligned}$$

Let $\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}} = \varepsilon^{\frac{N-p}{p-1}}$, that is, $\varepsilon = \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)^{\frac{p-1}{N-p}}$, choosing $\delta_2 > 0$ such that for all for all $(\lambda, \mu) \in \mathcal{S}_{\delta_2}$, then one has

$$\begin{aligned}&C_2\varepsilon^{\frac{N-p}{p-1}} - C_4\left(\lambda + \mu\tau_0^q\right)\varepsilon^{\frac{qN}{p(p-1)} - \frac{q}{p-1}}|\ln\varepsilon|\\ =&C_2\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right) - C_4\left(\lambda + \mu\tau_0^q\right)\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)^{\frac{q}{p}}|\ln\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)|\\ <&-C_0\left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right).\end{aligned}$$

Consequently, for for all $(\lambda, \mu) \in \mathcal{S}_{\delta_2}$, we obtain

$$\sup_{t\geq 0} \mathcal{J}_{\lambda,\mu}(tz_\varepsilon) < c_{\lambda,\mu}.$$

If we set $\Lambda_* := \min\left\{\frac{S_{\eta,\alpha,\beta}^{\frac{N}{p}}}{NC_0}, \delta_1, \delta_2\right\} > 0$ and $\varepsilon = \left(\lambda^{\frac{p}{p-q}} + \mu^{\frac{p}{p-q}}\right)^{\frac{p-1}{N-p}}$, then for $(\lambda, \mu) \in \mathcal{S}_{\Lambda_*}$, we have

$$\sup_{t\geq 0} \mathcal{J}_{\lambda,\mu}(tz_\varepsilon) < c_{\lambda,\mu},$$

Step 3. We prove that $\theta^-_{\lambda,\mu} < c_{\lambda,\mu}$, for all $(\lambda, \mu) \in \mathcal{S}_{\Lambda_*}$.
By the definition of z_ε and u_ε, we have

$$K_{\lambda,\mu}(z_\varepsilon) > 0, \quad Q(z_\varepsilon) > 0$$

Combining this with Lemma 2.3 and Lemma 2.5, from the definition of $\theta^-_{\lambda,\mu}$, we obtain $t^-_\varepsilon z_\varepsilon \in \mathcal{N}^-_{\lambda,\mu}$ and

$$0 < \theta^-_{\lambda,\mu} \leq \mathcal{J}_{\lambda,\mu}(t^-_\varepsilon z_\varepsilon) \leq \sup_{t\geq 0} \mathcal{J}_{\lambda,\mu}(tz_\varepsilon) < c_{\lambda,\mu}.$$

for all $(\lambda, \mu) \in \mathcal{S}_{\Lambda_*}$. The proof is finalized by taking $\tilde{z} = z_\varepsilon$. □

Proposition 4.1. *If $(\mathcal{H}_2)$ holds and $(\lambda, \mu) \in \mathcal{S}_\Lambda$. Then $\mathcal{J}_{\lambda,\mu}$ has a minimizer $z_2 = (u_2, v_2) \in \mathcal{N}^-_{\lambda,\mu}$ which satisfies the following:*
(i) $\mathcal{J}_{\lambda,\mu}(z_2) = \theta^-_{\lambda,\mu}$;
(ii) z_2 is a positive solution of system (1.1),
where $\Lambda = \min\left\{\Lambda_*, \left(\frac{q}{p}\right)^{\frac{p}{p-q}}\Theta_1\right\}$

Proof. If $(\lambda,\mu) \in \mathcal{S}_{\left(\frac{q}{p}\right)^{\frac{p}{p-q}}\Theta_1}$, then by Lemma 3.1 there exists a $(PS)_{\theta^-_{\lambda,\mu}}$-sequence $\{z_n\} \subset \mathcal{N}^-_{\lambda,\mu}$ in E for $\mathcal{J}_{\lambda,\mu}$. From Lemmas 2.6(ii), 4.2 and 4.4 for $(\lambda,\mu) \in \mathcal{S}_{\Lambda_*}$, $\mathcal{J}_{\lambda,\mu}$ satisfies $(PS)_{\theta^-_{\lambda,\mu}}$condition and $\theta^-_{\lambda,\mu} \in (0, c_{\lambda,\mu})$. By Lemma 2.1 and from coercivity of $\mathcal{J}_{\lambda,\mu}$ in $\mathcal{N}_{\lambda,\mu}$, we get that $\{z_n\}$ is bounded in E. Therefore, there exists a subsequence still denoted by $\{z_n\}$ and a nontrivial solution $z_2 = (u_2, v_2) \in \mathcal{N}^-_{\lambda,\mu}$ such that $z_n \rightharpoonup z_2$ weakly in E. Finally by the same arguments as in the proof of Proposition 3.1, for all $(\lambda,\mu) \in \mathcal{S}_\Lambda$, we have that z_2 is a positive solution of (1.1). □

Proof of Theorem 1.2. By Proposition 3.1 and 4.1 we obtain that for all $\lambda, \mu > 0$ and $(\lambda,\mu) \in \mathcal{S}_\Lambda$, (1.1) has two positive solutions z_1, z_2 with $z_1 \in \mathcal{N}^+_{\lambda,\mu}$ and $z_2 \in \mathcal{N}^-_{\lambda,\mu}$. Since $\mathcal{N}^+_{\lambda,\mu} \cap \mathcal{N}^-_{\lambda,\mu} = \emptyset$, this implies that z_1 and z_2 are distinct. This completes the proof. □

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. Benmouloud, R. Echarghaoui, Si.M. Sbaï, Multiplicity of Positive Solutions for a Critical Quasilinear Elliptic System with Concave and Convex Nonlinearities, J. Math. Anal. Appl. 396 (2012), 375–385. https://doi.org/10.1016/j.jmaa.2012.05.078.

[2] L. Brasco, S. Mosconi, M. Squassina, Optimal Decay of Extremals for the Fractional Sobolev Inequality, Calc. Var. Partial Differ. Equ. 55 (2016), 23. https://doi.org/10.1007/s00526-016-0958-y.

[3] H. Brézis, E. Lieb, A Relation between Pointwise Convergence of Functions and Convergence of Functionals, Proc. Amer. Math. Soc. 88 (1983), 486–490. https://doi.org/10.1090/s0002-9939-1983-0699419-3.

[4] K. Benlhachmi, R. Echarghaoui, A. Hatimi, H. Hadad, Multiplicity of Positive Solutions for a Fractional Elliptic System With Strongly Coupled Critical Terms, Int. J. Anal. Appl. 22 (2024), 164. https://doi.org/10.28924/2291-8639-22-2024-164.

[5] R. Echarghaoui, A. Hatimi, M. Hatimi, A Fractional Elliptic System With Strongly Coupled Critical Terms and Concave-Convex Nonlinearities, Int. J. Anal. Appl. 22 (2024), 107. https://doi.org/10.28924/2291-8639-22-2024-107.

[6] P. Han, The Effect of the Domain Topology on the Number of Positive Solutions of Elliptic Systems Involving Critical Sobolev Exponents, Houston J. Math. 32 (2006), 1241-1257.

[7] T.-S. Hsu, Existence and Multiplicity of Positive Solutions to a Perturbed Singular Elliptic System Deriving from a Strongly Coupled Critical Potential, Bound. Value Probl. 2012 (2012), 116. https://doi.org/10.1186/1687-2770-2012-116.

[8] T.-S. Hsu, Multiple Positive Solutions for a Critical Quasilinear Elliptic System with Concave–Convex Nonlinearities, Nonlinear Anal.: Theory Methods Appl. 71 (2009), 2688–2698. https://doi.org/10.1016/j.na.2009.01.110.

[9] T.-S. Hsu, Multiplicity Results for p -Laplacian with Critical Nonlinearity of Concave-Convex Type and Sign-Changing Weight Functions, Abstr. Appl. Anal. 2009 (2009), 652109. https://doi.org/10.1155/2009/652109.

[10] Y. Huang, D. Kang, On the Singular Elliptic Systems Involving Multiple Critical Sobolev Exponents, Nonlinear Anal.: Theory Methods Appl. 74 (2011), 400–412. https://doi.org/10.1016/j.na.2010.08.051.

[11] D. Kang, M. Liu, L. Xu, Critical Elliptic Systems Involving Multiple Strongly–Coupled Hardy–Type Terms, Adv. Nonlinear Anal. 9 (2019), 866–881. https://doi.org/10.1515/anona-2020-0029.

[12] D. Kang, X. Liu, Singularities of Solutions to Elliptic Systems Involving Different Hardy-Type Terms, J. Math. Anal. Appl. 468 (2018), 757–765. https://doi.org/10.1016/j.jmaa.2018.08.044.

[13] G. Tarantello, On Nonhomogeneous Elliptic Equations Involving Critical Sobolev Exponent, Ann. Inst. H. Poincaré C Anal. Non Linéaire 9 (1992), 281–304. https://doi.org/10.1016/s0294-1449(16)30238-4.

[14] X. Zhou, H.-Y. Li, J.-F. Liao, Multiplicity of Positive Solutions for a Semilinear Elliptic System with Strongly Coupled Critical Terms and Concave Nonlinearities, Qual. Theory Dyn. Syst. 22 (2023), 126. https://doi.org/10.1007/s12346-023-00825-9.