Int. J. Anal. Appl. (2024), 22:204

International Journal of Analysis and Applications

On the Non-Linearity of $\mathcal{S}$-Type Variable Exponent Absolutely Summable New Difference Sequence Space

Arafa O. Mustafa[1,*], Salah H. Alshabhi[2], Mohammed N. Alshehri[3], OM Kalthum S. K. Mohamed[2], Mustafa M. Mohammed[2], Runda A. A. Bashir[2], Sakeena E. M. Hamed[1], Awad A. Bakery[2]

[1]*University of Jeddah, College of Business at Khulis, Jeddah, Saudi Arabia*

[2]*University of Jeddah, College of Science and Arts at Khulis, Department of Mathematics, Jeddah, Saudi Arabia*

[3]*Department of Mathematics, College of Arts and Sciences, Najran University, Najran, Saudi Arabia.*

[*]*Corresponding author:* arafaomustafa2020@yahoo.com, 04220355@uj.edu.sa

Abstract. This article presents the domain of general quantum difference in Nakano sequence space. Some topological and geometric behavior, the multiplication mappings defined on it, and the spectrum of mapping ideals constructed by this space and $s-$numbers have been introduced. Existing results are constructed by controlling the general quantum difference and power of this new space, which is a major strength.

1. Notations

(1): $\mathbb{Z}$ and R: The set of integers and real numbers, respectively.
(2): $\mathbb{Z}^+$: The set of non-negative integers.
(3): $\mathfrak{C}$: The space of all complex numbers.
(4): $\mathcal{G}$, $\mathcal{V}$, $\mathcal{G}_0$, and $\mathcal{V}_0$: Any infinite dimensional Banach spaces.
(5): $\mathrm{L}|_{\mathcal{G}}^{\mathcal{V}}$: The space of all bounded linear mappings from $\mathcal{G}$ into $\mathcal{V}$.
(6): $\mathfrak{T}|_{\mathcal{G}}^{\mathcal{V}}$: The space of finite rank mappings from $\mathcal{G}$ into $\mathcal{V}$.
(7): $\mathfrak{R}|_{\mathcal{G}}^{\mathcal{V}}$: The space of all approximable mappings from $\mathcal{G}$ into $\mathcal{V}$.
(8): $\mathfrak{M}|_{\mathcal{G}}^{\mathcal{V}}$: The space of all compact mappings from $\mathcal{G}$ into $\mathcal{V}$.
(9): $\mathrm{L}|_{\mathcal{G}}$, $\mathfrak{T}|_{\mathcal{G}}$, $\mathfrak{R}|_{\mathcal{G}}$, and $\mathfrak{M}|_{\mathcal{G}}$: The space of all bounded linear, finite rank, approximable, compact mappings from $\mathcal{G}$ into itself, respectively.

Received: Sep. 23, 2024.

2020 *Mathematics Subject Classification.* 46A45, 46C05, 46E30.

Key words and phrases. $s-$numbers; pre-quasi ideal; general quantum difference; multiplication mapping; simple space.

https://doi.org/10.28924/2291-8639-22-2024-204
ISSN: 2291-8639

(10): L, $\mathfrak{T}$, $\mathfrak{R}$, and $\mathfrak{M}$: The ideal of all bounded linear, finite rank, approximable and compact mappings between any arbitrary Banach spaces, respectively.

(11): $(0,\infty)^{\mathbb{Z}^+}$: The space of all sequences of positive reals.

(12): ℓ_∞, c_0, and ℓ_m: The space of bounded, null, and m-absolutely summable sequences of complex numbers, respectively.

(13): $\nearrow$: The space of all monotonic increasing sequences of positive reals.

(14): $\mathfrak{C}^{\mathbb{Z}^+}$: The space of all sequences of complex numbers.

(15): $[y]$: The integral part of y.

(16): $e_x := (0,0,...,1,0,0,...)$ as 1 lies at the x^{th} coordinate.

(17): I_x: The identity mapping on the x-dimensional Hilbert space ℓ_2^x.

(18): θ: The zero vector of the linear space of sequences $\mathcal{Q}$.

(19): I: The space of each sequences with finite non-zero coordinates.

(20): $\mathfrak{I}$: The space of all sets with a finite number of elements.

(21): $(Range(U))^c$: The complement of $Range(U)$.

(22): Ω: A Banach space of one dimension.

(23): $s_x(H)$: The x-th s-number [4].

(24): $d_x(H)$: The x-th Kolmogorov number, where $d_x(H) := \inf_{dim\ J\leq x} \sup_{\|\lambda\|\leq 1} \inf_{\beta\in J} \|H\lambda-\beta\|$.

(25): $\alpha_x(H)$: The x-th approximation number, where
$\alpha_x(H) := \inf\Big\{ \|H-Z\| : Z\in \mathrm{L}|_{\mathcal{G}}^{\mathcal{V}} \text{ and rank}(Z)\leq x\Big\}$.

(26): See [5]:

$\mathrm{L}_{\mathcal{Q}}^{s} := \Big\{\mathrm{L}_{\mathcal{Q}}^{s}|_{\mathcal{G}}^{\mathcal{V}}\Big\}$, where $\mathrm{L}_{\mathcal{Q}}^{s}|_{\mathcal{G}}^{\mathcal{V}} := \Big\{H\in \mathrm{L}|_{\mathcal{G}}^{\mathcal{V}} : ((s_x(H))_{x=0}^{\infty}\in\mathcal{Q}\Big\}$.

$\mathrm{L}_{\mathcal{Q}}^{\alpha} := \Big\{\mathrm{L}_{\mathcal{Q}}^{\alpha}|_{\mathcal{G}}^{\mathcal{V}}\Big\}$, where $\mathrm{L}_{\mathcal{Q}}^{\alpha}|_{\mathcal{G}}^{\mathcal{V}} := \Big\{H\in \mathrm{L}|_{\mathcal{G}}^{\mathcal{V}} : ((\alpha_x(H))_{x=0}^{\infty}\in\mathcal{Q}\Big\}$.

$\mathrm{L}_{\mathcal{Q}}^{d} := \Big\{\mathrm{L}_{\mathcal{Q}}^{d}|_{\mathcal{G}}^{\mathcal{V}}\Big\}$, where $\mathrm{L}_{\mathcal{Q}}^{d}|_{\mathcal{G}}^{\mathcal{V}} := \Big\{H\in \mathrm{L}|_{\mathcal{G}}^{\mathcal{V}} : ((d_x(H))_{x=0}^{\infty}\in\mathcal{Q}\Big\}$.

$\left(\mathrm{L}_{\mathcal{Q}}^{s}\right)^{\rho} := \Big\{\left(\mathrm{L}_{\mathcal{Q}}^{s}\right)^{\rho}|_{\mathcal{G}}^{\mathcal{V}}\Big\}$, where

$\left(\mathrm{L}_{\mathcal{Q}}^{s}\right)^{\rho}|_{\mathcal{G}}^{\mathcal{V}} := \Big\{Y\in \mathrm{L}|_{\mathcal{G}}^{\mathcal{V}} : ((\rho_x(Y))_{x=0}^{\infty}\in\mathcal{Q} \text{ and } \|Y-\rho_x(Y)I\| \text{ is not invertible, for all } x\in\mathbb{Z}^+\Big\}$.

2. Introduction

Suppose that $\varphi = (\varphi_h)$ is a strictly increasing, where $\varphi : \mathbb{Z}^+ \to \mathbb{Z}$, the general quantum difference (**g.c.d**) ∇_φ is defined in [6] by

$$\nabla_\varphi\lambda_h = \begin{cases} \frac{\lambda_{\varphi_h}-\lambda_{h-1}}{\varphi_h-h+1}, & \varphi_h\neq h-1,\\ 0, & \text{otherwise.}\end{cases}$$

Note that if $\varphi_h = h$, then $\nabla_\varphi\lambda_h = \nabla\lambda_h = \lambda_h-\lambda_{h-1}$, where $\lambda_h = 0$ for $h<0$, is the backward difference defined by Kizmaz [7]. The concept of variable exponent function spaces has been carefully developed, drawing upon the boundedness of the Hardy-Littlewood maximal mapping. That discusses its image processing, difference equations, and approximation theory applications.

The theory of L^{s}_{Q} introduced and investigated by Pietsch [1]. In [3], he offered and studied the behaviours of $\mathrm{L}^{\alpha}_{\ell_b}$. Makarov and Faried [8], explained that if $y > x > 0$, one has $\mathrm{L}^{\alpha}_{\ell_x}|^{\mathcal{V}}_{\mathcal{G}} \subsetneqq \mathrm{L}^{\alpha}_{\ell_y}|^{\mathcal{V}}_{\mathcal{G}} \subsetneqq \mathrm{L}|^{\mathcal{V}}_{\mathcal{G}}$. Yaying et al. [14], defined and studied the domain of b-Cesàro matrix in ℓ_η, for every $b \in (0,1]$ and $1 \leq \eta \leq \infty$. They explained the **q BI** of it for $b \in (0,1]$ and $1 < \eta < \infty$. They offered its Schauder basis, $\alpha-$, $\beta-$, and $\gamma-$ duals and represented some matrix classes connected to it. The space $\mathfrak{M}|^{\mathcal{V}}_{\mathcal{G}}$ examined by many authors for different sequence spaces for that see [9,10]. Komal et al. [15], which defines multiplication mappings on Cesàro sequence spaces under the Luxemburg norm. İlkhan et al. [16] discussed multiplication mappings on Cesàro second-order function spaces. The non-absolute type sequence spaces are a generalization of the corresponding absolute type. Hence, there exists a great interest in studying them. Numerous authors have recently clarified some non-absolute type sequence spaces and contributed new intriguing works to the literature. Consider Mursaleen and Noman [12,13], as well as Mursaleen and Başar [11], and Roopaei et al. [17]. Also, for a complete background on the multiplication operators and pre-quasi operator ideals, see [19].

Lemma 2.1. *[21] Suppose that $\eta_h > 0$ and $\varpi_h, \sigma_h \in \mathfrak{C}$, for all $h \in \mathbb{Z}^+$, and $\hbar = \max\{1, \sup_h \eta_h\}$, one has*

$$|\varpi_h + \sigma_h|^{\eta_h} \leq 2^{\hbar-1}\left(|\varpi_h|^{\eta_h} + |\sigma_h|^{\eta_h}\right). \tag{2.1}$$

The solution of discrete dynamical systems is found inside a designated sequence space. There is significant enthusiasm in the field of mathematics for creating novel sequence spaces. Considering that the verification of numerous fixed-point theorems in a specific space necessitates either enlarging the space or extending the self-mapping that operates within it, both of these alternatives are feasible. In this article, we define and discuss several inclusion relations for the domain of **g.c.d** in Nakano sequence space (**Nss**), $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ in Section 3. In Section 4, we offer the sufficient setups on $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ under definite function ϕ to form pre-modular private sequence space (**p-m**$\mathfrak{pss}$). So it is a pre-quasi normed private sequence space (**p-q N** $\mathfrak{pss}$). In Section 5, we give some topological and geometric behaviors of the multiplication operators defined on $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$. In Section 6, first, we offer the sufficient conditions (not necessary) on $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ such that $\overline{\mathfrak{T}} = \mathrm{L}^{s}_{\left(\ell(\nabla_\varphi, \eta)\right)_\phi}$. That represents the non-linearity of $s-$ type $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ spaces (see Rhoades [22]). Second, what are the setups of $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ to generate $\mathrm{L}^{s}_{\left(\ell(\nabla_\varphi, \eta)\right)_\phi}$ pre-quasi Banach ideal (**p-q BI**)? Third, we present the sufficient conditions on $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ for which $\mathrm{L}^{\alpha}_{\left(\ell(\nabla_\varphi, \eta)\right)_\phi}$ is strictly contained for different ϕ and powers. We offer the conditions for which $\mathrm{L}^{\alpha}_{\left(\ell(\nabla_\varphi, \eta)\right)_\phi}$ and $\mathrm{L}^{s}_{\left(\ell(\nabla_\varphi, \eta)\right)_\phi}$ are minimum and simple, respectively. Next we examine the sufficient settings on $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ with $\left(\mathrm{L}^{s}_{\left(\ell(\nabla_\varphi, \eta)\right)_\phi}\right)^\rho = \mathrm{L}^{s}_{\left(\ell(\nabla_\varphi, \eta)\right)_\phi}$. Lastly, we explain our conclusion in Section 7.

3. General quantum difference in Nakano sequence space

This section provides a discussion on the definition of $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$, along with an exploration of possible inclusion relations.

Definition 3.1. *Assume $(\eta_x) \in (0,\infty)^{\mathbb{Z}^+}$ and ∇_φ is an absolutely non-decreasing.*
$\left(\ell(\nabla_\varphi,\eta)\right)_{\phi_1} := \left\{\lambda = (\lambda_y) \in \mathfrak{C}^{\mathbb{Z}^+} : \phi_1(\rho\lambda) < \infty, \text{ for some } \rho > 0\right\}$, *where* $\phi_1(\lambda) = \sum_{y=0}^{\infty} \left|\nabla_\varphi|\lambda_y|\right|^{\eta_y}$.

Theorem 3.1. *Let $(\eta_x) \in (0,\infty)^{\mathbb{Z}^+}$, we have*

$$\left(\ell(\nabla_\varphi,\eta)\right)_{\phi_1} \subset \left(\ell(\nabla_\varphi,\eta)\right)_{\phi_2},$$

where $\phi_2(\lambda) = \sum_{y=0}^{\infty} \left|\nabla_\varphi\lambda_y\right|^{\eta_y}$.

Proof. We have

$$\begin{aligned}
\left(\ell(\nabla_\varphi,\eta)\right)_{\phi_1} &= \left\{\lambda = (\lambda_y) \in \mathfrak{C}^{\mathbb{Z}^+} : \phi_1(\rho\lambda) < \infty, \text{ for some } \rho > 0\right\} \\
&= \left\{\lambda = (\lambda_y) \in \mathfrak{C}^{\mathbb{Z}^+} : \sum_{y=0}^{\infty} \left|\rho\nabla_\varphi\lambda_y\right|^{\eta_y} \le \sum_{y=0}^{\infty} \left|\rho\nabla_\varphi|\lambda_y|\right|^{\eta_y} < \infty, \text{ for some } \rho > 0\right\} \\
&\subset \left\{\lambda = (\lambda_y) \in \mathfrak{C}^{\mathbb{Z}^+} : \phi_2(\rho\lambda) < \infty, \text{ for some } \rho > 0\right\} = \left(\ell(\nabla_\varphi,\eta)\right)_{\phi_2}.
\end{aligned}$$

□

Theorem 3.2. *Let $(\eta_x) \in (0,\infty)^{\mathbb{Z}^+} \cap \ell_\infty$, one has*

$$\left(\ell(\nabla_\varphi,\eta)\right)_{\phi} = \left\{\lambda = (\lambda_y) \in \mathfrak{C}^{\mathbb{Z}^+} : \phi(\rho\lambda) < \infty, \text{ for any } \rho > 0\right\},$$

where $\phi(\lambda) = \sum_{y=0}^{\infty} \left|\nabla_\varphi\lambda_y\right|^{\eta_y}$.

Proof. Suppose that $(\eta_x) \in (0,\infty)^{\mathbb{Z}^+} \cap \ell_\infty$, we have

$$\begin{aligned}
\left(\ell(\nabla_\varphi,\eta)\right)_{\phi} &= \left\{\lambda = (\lambda_y) \in \mathfrak{C}^{\mathbb{Z}^+} : \phi(\rho\lambda) < \infty, \text{ for some } \rho > 0\right\} \\
&= \left\{\lambda = (\lambda_y) \in \mathfrak{C}^{\mathbb{Z}^+} : \sum_{y=0}^{\infty} \left(\left|\nabla_\varphi\rho\lambda_y\right|\right)^{\eta_y} < \infty, \text{ for some } \rho > 0\right\} \\
&= \left\{\lambda = (\lambda_y) \in \mathfrak{C}^{\mathbb{Z}^+} : \inf_y \rho^{\eta_y} \sum_{y=0}^{\infty} \left(\left|\nabla_\varphi\lambda_y\right|\right)^{\eta_y} < \infty, \text{ for some } \rho > 0\right\} \\
&= \left\{\lambda = (\lambda_y) \in \mathfrak{C}^{\mathbb{Z}^+} : \sum_{y=0}^{\infty} \left(\left|\nabla_\varphi\lambda_y\right|\right)^{\eta_y} < \infty\right\} \\
&= \left\{\lambda = (\lambda_y) \in \mathfrak{C}^{\mathbb{Z}^+} : \phi(\rho\lambda) < \infty, \text{ for any } \rho > 0\right\}.
\end{aligned}$$

□

Theorem 3.3. *If $(\eta_m) \in (0,\infty)^{\mathbb{Z}^+}$ and $\varphi_m \neq m-1$, for all $m \in \mathbb{Z}^+$, one has $\left(\ell(\nabla_\varphi,\eta)\right)_{\phi}$ is a non-absolute type, where* $\phi(\lambda) = \sum_{m=0}^{\infty} \left|\nabla_\varphi\lambda_m\right|^{\eta_m}$.

Proof. Suppose that without loss of generality that $\varphi_x = x-2$, for all $x \in \mathbb{Z}^+$. One has

$$\phi\Big(1,-1,0,0,0,\dots\Big) = \sum_{y=0}^{\infty}\left|\lambda_{y-2}-\lambda_{y-1}\right|^{\eta_y} = 2+2^{\eta_2} \neq 2 = \sum_{y=0}^{\infty}\left||\lambda|_{y-2}-|\lambda|_{y-1}\right|^{\eta_y} = \phi\Big(1,1,0,0,0,\dots\Big).$$

If $\varphi_x = x$, we have

$$\phi\Big(1,-1,0,0,0,\dots\Big) = \sum_{y=0}^{\infty}\left|\lambda_{y}-\lambda_{y-1}\right|^{\eta_y} = 2+2^{\eta_1} \neq 2 = \sum_{y=0}^{\infty}\left||\lambda|_{y}-|\lambda|_{y-1}\right|^{\eta_y} = \phi\Big(1,1,0,0,0,\dots\Big).$$

That completes the proof. □

Definition 3.2. *Let* $(\eta_x) \in (0,\infty)^{\mathbb{Z}^+}$.

$$(\ell(\nabla,\eta))_\phi := \Big\{\lambda = (\lambda_x) \in \mathfrak{C}^{\mathbb{Z}^+} : \phi(\rho\lambda) < \infty, \text{ for some } \rho > 0\Big\}, \text{ where } \phi(\lambda) = \sum_{y=0}^{\infty}\left|\lambda_y - \lambda_{y-1}\right|^{\eta_y}.$$

Theorem 3.4. *Assume* $(\eta_x) \in (0,\infty)^{\mathbb{Z}^+} \cap \ell_\infty$ *and* $\varphi_x > x$*, for all* $x \in \mathbb{Z}^+$*, then*

$$(\ell(\nabla,\eta))_\phi \subsetneqq \Big(\ell(\nabla_\varphi,\eta)\Big)_\phi.$$

Proof. Let $\lambda \in (\ell(\nabla,\eta))_\phi$, since

$$\sum_{y=0}^{\infty}\left|\frac{\lambda_{\varphi_y}-\lambda_{y-1}}{\varphi_y - y + 1}\right|^{\eta_y} \leq \sum_{y=0}^{\infty}\left|\lambda_y - \lambda_{y-1}\right|^{\eta_y} < \infty.$$

Therefore, $\lambda \in \Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$. If we choose $\lambda = (1,0,1,0,\dots)$, then $\lambda \notin (\ell(\nabla,\eta))_\phi$ and $\lambda \in \Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$, where $\varphi_x = x+1$. □

Theorem 3.5. *Assume* $(\eta_x) \in \nearrow \cap \ell_\infty$ *with* $\eta_0 > 0$ *and* $\varphi_x < x-1$*, for all* $x \in \mathbb{Z}^+$*, then*

$$(\ell(\nabla,\eta))_\phi \subsetneqq \Big(\ell(\nabla_\varphi,\eta)\Big)_\phi.$$

Proof. Let $\lambda \in (\ell(\nabla,\eta))_\phi$, since

$$\sum_{y=0}^{\infty}\left|\frac{\lambda_{\varphi_y}-\lambda_{y-1}}{\varphi_y - y + 1}\right|^{\eta_y} \leq 2^{\hbar-1}\left(\sum_{y=0}^{\infty}\left|\lambda_y - \lambda_{y-1}\right|^{\eta_y} + \sum_{m=\varphi(0)+1}^{\infty}\left|\lambda_m - \lambda_{m-1}\right|^{\eta_{\varphi^{-1}(m-1)}}\right) \leq 2^{\hbar}\sum_{y=0}^{\infty}\left|\lambda_y - \lambda_{y-1}\right|^{\eta_y} < \infty.$$

Therefore, $\lambda \in \Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$. □

4. Pre-modular-$\Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$ spaces

We investigate here the sufficient conditions on $\Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$ with definite function ϕ to be **p-m**$\mathfrak{pss}$, where $\phi(\lambda) = \sum_{y=0}^{\infty}\left|\nabla_\varphi|\lambda_y|\right|^{\eta_y}$, for all $\lambda \in \Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$. This implies that $\Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$ is a **p-q N** $\mathfrak{pss}$.

Definition 4.1. *[20] The space Q is called a $\mathfrak{pss}$, when it satisfies the next settings:*

(1): $e_h \in Q$*, so that* $h \in \mathbb{Z}^+$*,*

(2): *If* $f = (f_x) \in \mathfrak{C}^{\mathbb{Z}^+}$*,* $|g| = (|g_x|) \in Q$ *and* $|f_x| \leq |g_x|$*, with* $x \in \mathbb{Z}^+$*, then* $|f| \in Q$*, i.e.,* Q *is solid,*

(3): $\left(\left|f_{[\frac{y}{2}]}\right|\right)_{y=0}^{\infty} \in \mathcal{Q}$, *if* $\left(\left|f_y\right|\right)_{y=0}^{\infty} \in \mathcal{Q}$.

Theorem 4.1. *[20] If $\mathcal{Q}$ is a* $\mathfrak{pss}$, *then* $\mathrm{L}^s_{\mathcal{Q}}$ *is an Operators' ideal (**OI**).*

Definition 4.2. *[20] A subspace of the* $\mathfrak{pss}$-$\mathcal{Q}$ *is named a **p-m**$\mathfrak{pss}$, if we have an operator* $\mu : \mathcal{Q} \to [0, \infty)$ *that satisfies the next settings:*

- **(i):** *Suppose that* $\delta \in \mathcal{Q}$, $\delta = \theta \Longleftrightarrow \mu(|\delta|) = 0$, *and* $\mu(\delta) \geq 0$,
- **(ii):** *for* $\delta \in \mathcal{Q}$ *and* $\tau \in \mathfrak{C}$, *there are* $E_0 \geq 1$ *with* $\mu(\tau\delta) \leq |\tau| E_0 \mu(\delta)$,
- **(iii):** *there are* $G_0 \geq 1$ *such that* $\mu(\gamma + \varepsilon) \leq G_0(\mu(\gamma) + \mu(\varepsilon))$, *with* $\gamma, \varepsilon \in \mathcal{Q}$,
- **(iv):** *if* $x \in \mathbb{Z}^+$, $|\lambda_x| \leq |\beta_x|$, *we have* $\mu((|\lambda_x|)) \leq \mu((|\beta_x|))$,
- **(v):** *the inequality,* $\mu((|\lambda_x|)) \leq \mu((|\lambda_{[\frac{x}{2}]}|)) \leq D_0 \mu((|\lambda_x|))$ *holds, for* $D_0 \geq 1$,
- **(vi):** $\bar{\mathrm{I}} = \mathcal{Q}_\mu$,
- **(vii):** *one has* $\varpi > 0$ *such that* $\mu(\lambda, 0, 0, 0, ...) \geq \varpi |\lambda| \mu(1, 0, 0, 0, ...)$, *where* $\lambda \in \mathfrak{C}$.

Definition 4.3. *[20] Suppose that* μ *satisfies the parts (i)-(iii) of Definition 4.2, then the* $\mathfrak{pss}$ $\mathcal{Q}_\mu$ *is said to be a **p-q N** $\mathfrak{pss}$. If the space* $\mathcal{Q}$ *is complete under* μ, *hence* $\mathcal{Q}_\mu$ *is called a pre-quasi Banach private sequence space (**p-q B** $\mathfrak{pss}$).*

Theorem 4.2. *[20] All **p-m**$\mathfrak{pss}$ $\mathcal{Q}_\mu$ is a **p-q N** $\mathfrak{pss}$.*

Theorem 4.3. *The space* $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ *is a* $\mathfrak{pss}$, *when the next parts are verified:*

- **(f1):** *If* $(\eta_x) \in \nearrow \cap \ell_\infty$.
- **(f2):** $|\varphi(x) - x + 1| \geq 1$, *with* $x \in \mathbb{Z}^+$.
- **(f3):** *Assume that* $|\lambda_x| \leq |\beta_x|$, *with* $x \in \mathbb{Z}^+$, *then* $\left|\nabla_\varphi |\lambda_x|\right| \leq \left|\nabla_\varphi |\beta_x|\right|$.

Proof. (1-i) For $\beta, \lambda \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$. We obtain

$$\sum_{y=0}^{\infty} \left|\nabla_\varphi |\lambda_y + \beta_y|\right|^{\eta_y} \leq 2^{\hbar - 1} \left(\sum_{y=0}^{\infty} \left|\nabla_\varphi |\lambda_y|\right|^{\eta_y} + \sum_{y=0}^{\infty} \left|\nabla_\varphi |\beta_y|\right|^{\eta_y} \right) < \infty,$$

hence, $\lambda + \beta \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$.

(1-ii) If $\rho \in \mathfrak{C}$, $\lambda \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$ and as $(\eta_x) \in \nearrow \cap \ell_\infty$, we have

$$\sum_{y=0}^{\infty} \left|\rho \nabla_\varphi |\lambda_y|\right|^{\eta_y} \leq \sup_y |\rho|^{\eta_y} \left|\nabla_\varphi |\lambda_y|\right|^{\eta_y} < \infty.$$

Then, $\rho\lambda \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$. From conditions (1-i) and (1-ii), hence $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ is a linear space.
For $(\eta_y) \in \nearrow \cap \ell_\infty$, $\eta_0 > 0$, since $e_b \in (\ell(\eta))_\phi$ with $b \in \mathbb{Z}^+$, and

$$(\ell(\eta))_\phi \subsetneqq (\ell(\nabla, \eta))_\phi \subsetneqq \left(\ell(\nabla_\varphi, \eta)\right)_\phi.$$

Therefore, $e_b \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$, for every $b \in \mathbb{Z}^+$.
(2) Suppose that $|\lambda_b| \leq |\beta_b|$, with $b \in \mathbb{Z}^+$ and $|\beta| \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$. One has

$$\sum_{y=0}^{\infty} \left|\nabla_\varphi |\lambda_y|\right|^{\eta_y} \leq \sum_{y=0}^{\infty} \left|\nabla_\varphi |\beta_y|\right|^{\eta_y} < \infty,$$

so $|\lambda| \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$.
(3) Let $(|\lambda_y|) \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$, with $(\eta_y) \in \nearrow \cap \ell_\infty$, we have

$$\sum_{h=0}^{\infty} \left|\nabla_\varphi |\lambda_{[\frac{h}{2}]}|\right|^{\eta_h} = \sum_{h=0}^{\infty} \left|\nabla_\varphi |\lambda_h|\right|^{\eta_{2h}} + \sum_{h=0}^{\infty} \left|\nabla_\varphi |\lambda_h|\right|^{\eta_{2h+1}} \leq 2\sum_{h=0}^{\infty} \left|\nabla_\varphi |\lambda_h|\right|^{\eta_h} < \infty,$$

therefore $(|\lambda_{[\frac{y}{2}]}|) \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$. □

Because of Theorem 4.1, we conclude the next theorem.

Theorem 4.4. *Suppose that the parts of Theorem 4.3 are verified, one has* $\mathrm{L}^s_{\left(\ell(\nabla_\varphi, \eta)\right)_\phi}$ *is an* ***OI****.*

Theorem 4.5. *Suppose that the conditions of Theorem 4.3 are verified, then the space* $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ *is a* ***p-m***$\mathfrak{pss}$*.*

Proof. (i) Definitely, $\phi(\lambda) = \sum_{y=0}^{\infty} \left|\nabla_\varphi |\lambda_y|\right|^{\eta_y} \geq 0$ and $\phi(|\lambda|) = 0 \Leftrightarrow \lambda = \theta$.
(ii) Let $\rho \in \mathfrak{C}$, $\lambda \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$ and as $(\eta_x) \in \nearrow \cap \ell_\infty$, we have

$$\phi(\rho\lambda) = \sum_{y=0}^{\infty} \left|\rho\nabla_\varphi |\lambda_y|\right|^{\eta_y} \leq \sup_y |\rho|^{\eta_y} \left|\nabla_\varphi |\lambda_y|\right|^{\eta_y} \leq E_0 |\rho| \phi(\lambda) < \infty,$$

where $E_0 = \max\left\{1, \sup_x |\rho|^{\eta_x - 1}\right\} \geq 1$.
(iii) Assume $\lambda, \beta \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$. One gets

$$\phi(\lambda + \beta) = \sum_{y=0}^{\infty} \left|\nabla_\varphi |\lambda_y + \beta_y|\right|^{\eta_y} \leq 2^{\hbar - 1}\left(\sum_{y=0}^{\infty} \left|\nabla_\varphi |\lambda_y|\right|^{\eta_y} + \sum_{y=0}^{\infty} \left|\nabla_\varphi |\beta_y|\right|^{\eta_y}\right) = 2^{\hbar-1}(\phi(\lambda) + \phi(\beta)) < \infty,$$

hence, the inequality $\phi(\lambda + \beta) \leq 2^{\hbar-1}(\phi(\lambda) + \phi(\beta))$ is satisfied.
(iv) Suppose that $|\lambda_b| \leq |\beta_b|$, with $b \in \mathbb{Z}^+$ and $|\beta| \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$. One has

$$\phi((|\lambda_b|)) = \sum_{b=0}^{\infty} \left|\nabla_\varphi |\lambda_b|\right|^{\eta_b} \leq \sum_{b=0}^{\infty} \left|\nabla_\varphi |\beta_b|\right|^{\eta_b} = \phi((|\beta_b|)).$$

(v) Let $(|\lambda_y|) \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$, with $(\eta_y) \in \nearrow \cap \ell_\infty$, we have

$$\phi((|\lambda_{[\frac{y}{2}]}|)) = \sum_{y=0}^{\infty} \left|\nabla_\varphi |\lambda_{[\frac{y}{2}]}|\right|^{\eta_y} = \sum_{y=0}^{\infty} \left|\nabla_\varphi |\lambda_y|\right|^{\eta_{2y}} + \sum_{y=0}^{\infty} \left|\nabla_\varphi |\lambda_y|\right|^{\eta_{2y+1}} \leq 2\sum_{y=0}^{\infty} \left|\nabla_\varphi |\lambda_y|\right|^{\eta_y} \leq D_0 \phi((|\lambda_y|)).$$

(vi) It is obvious that $\bar{\mathrm{I}} = \ell(\nabla_\varphi, \eta)$.
(vii) There are $0 < \varpi \leq \sup_x |\lambda|^{\eta_x - 1}$ such that $\phi(\lambda, 0, 0, 0, ...) \geq \varpi|\lambda|\phi(1, 0, 0, 0, ...)$, for each $\lambda \neq 0$ and $\varpi > 0$, if $\lambda = 0$. □

Theorem 4.6. *The space $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ is a **p-q B** $\mathfrak{pss}$, if the conditions of Theorem 4.3 are satisfied.*

Proof. According to Theorem 4.5, the space $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ is a **p-m**$\mathfrak{pss}$. By Theorem 4.2, $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ is a **p-q N** $\mathfrak{pss}$. Suppose that $\lambda^a = (\lambda^a_x)_{x=0}^\infty$ is a Cauchy sequence in $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$, one has for all $\varepsilon \in (0, 1)$, there exists $a_0 \in \mathbb{Z}^+$ with $a, b \geq a_0$, hence

$$\phi(\lambda^a - \lambda^b) = \sum_{y=0}^{\infty} \left|\nabla_\varphi|\lambda^a_y - \lambda^b_y|\right|^{\eta_y} < \varepsilon^\hbar.$$

That gives $\left|\nabla_\varphi|\lambda^a_y - \lambda^b_y|\right| < \varepsilon$. So, $(\nabla_\varphi|\lambda^b_y|)$ is a Cauchy sequence in $\mathfrak{C}$, for fixed $y \in \mathbb{Z}^+$, which gives $\lim_{b\to\infty} \nabla_\varphi|\lambda^b_y| = \nabla_\varphi|\lambda^0_y|$, for fixed $y \in \mathbb{Z}^+$. Then, $\phi(\lambda^a - \lambda^0) < \varepsilon^\hbar$, for all $a \geq a_0$. More, to explain that $\lambda^0 \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$, one gets $\phi(\lambda^0) \leq 2^{\hbar-1}(\phi(\lambda^a - \lambda^0) + \phi(\lambda^a)) < \infty$, then $\lambda^0 \in \left(\ell(\nabla_\varphi, \eta)\right)_\phi$. Which implies that $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ is a **p-q B** $\mathfrak{pss}$. □

Theorem 4.7. *[18] When $s-$ type $\boldsymbol{Q}_\mu := \left\{\delta = (s_x(V)) \in \mathrm{R}^{\mathbb{Z}^+} : V \in \mathrm{L}|^{\mathcal{V}}_{\mathcal{G}} \text{ and } \mu(\delta) < \infty\right\}$. Assume $\mathrm{L}^s_{\boldsymbol{Q}_\mu}$ is an **OI**, one has the following:*

i. $\mathrm{I} \subset s-$ *type* $\boldsymbol{Q}_\mu$.
ii. *Let $(s_x(V_1))_{x=0}^\infty \in s-$ type $\boldsymbol{Q}_\mu$ and $(s_x(V_2))_{x=0}^\infty \in s-$ type $\boldsymbol{Q}_\mu$, so $(s_x(V_1 + V_2))_{x=0}^\infty \in s-$ type $\boldsymbol{Q}_\mu$.*
iii. *Suppose that $\varepsilon \in \mathfrak{C}$ and $(s_x(V))_{x=0}^\infty \in s-$ type $\boldsymbol{Q}_\mu$, one has $|\varepsilon| \, (s_x(V))_{x=0}^\infty \in s-$ type $\boldsymbol{Q}_\mu$.*
iv. *The sequence space $\boldsymbol{Q}_\mu$ is solid. i.e., if $(s_x(Y))_{x=0}^\infty \in s-$ type $\boldsymbol{Q}_\mu$ and $s_x(V) \leq s_x(Y)$, for any $x \in \mathbb{Z}^+$ and $V, Y \in \mathrm{L}|^{\mathcal{V}}_{\mathcal{G}}$, so $(s_x(V))_{x=0}^\infty \in s-$ type $\boldsymbol{Q}_\mu$.*

By Theorem 4.7, we have the next behavior of the $s-$ type $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$.

Theorem 4.8. *For $s-$ type $\left(\ell(\nabla_\varphi, \eta)\right)_\phi := \left\{\delta = (s_x(V)) \in \mathrm{R}^{\mathbb{Z}^+} : V \in \mathrm{L}|^{\mathcal{V}}_{\mathcal{G}} \text{ and } \phi(\delta) < \infty\right\}$. The following parts are established:*

i. $s-$ *type* $\left(\ell(\nabla_\varphi, \eta)\right)_\phi \supset \mathrm{I}$.
ii. *Assume that $(s_x(V_1))_{x=0}^\infty \in s-$ type $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ and $(s_x(V_2))_{x=0}^\infty \in s-$ type $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$, hence $(s_x(V_1 + V_2))_{x=0}^\infty \in s-$ type $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$.*
iii. *Suppose that $r \in \mathfrak{C}$ and $(s_x(V))_{x=0}^\infty \in s-$ type $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$, then $|r| \, (s_x(V))_{x=0}^\infty \in s-$ type $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$.*
iv. $s-$ *type* $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ *is a solid.*

Theorem 4.9. *Suppose that the conditions of Theorem 4.3 are verified, one has $\mathrm{L}^s_{\left(\ell(\nabla_\varphi, \eta)\right)_\phi}$ is not **OI**.*

Proof. For $\eta_y = 2$, $\phi(y) = y$, for all $y \in \mathbb{Z}^+$, $g = (1,1,1,\dots)$ and $m = (1,0,1,0,\dots)$. Clearly, $|m_y| \leq |g_y|$, for all $y \in \mathbb{Z}^+$ and $g \in s-$ type $\left(\ell(\nabla_\varphi,\eta)\right)_\phi$. But $m \notin s-$ type $\left(\ell(\nabla_\varphi,\eta)\right)_\phi$. Hence the $s-$ type $\left(\ell(\nabla_\varphi,\eta)\right)_\phi$ is not solid. Given Theorem 4.7, we conclude that $L^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$ is not **OI**. □

5. Multiplication mappings on $\left(\ell(\nabla_\varphi,\eta)\right)_\phi$

We discuss some topological and geometric structures of the multiplication mapping defined on the space $\left(\ell(\nabla_\varphi,\eta)\right)_\phi$ under definite function ϕ, where $\phi(\lambda) = \sum_{x=0}^{\infty} \left|\nabla_\varphi|\lambda_y|\right|^{\eta_x}$, for all $\lambda \in \left(\ell(\nabla_\varphi,\eta)\right)_\phi$.

Definition 5.1. *[20] If $\mathcal{Q}_\mu$ is a **p-q N** pss and $\vartheta = (\vartheta_x) \in \mathfrak{C}^{\mathbb{Z}^+}$. The operator $T_\vartheta : \mathcal{Q}_\mu \to \mathcal{Q}_\mu$ is said to be a multiplication on $\mathcal{Q}_\mu$, if $T_\vartheta\lambda = \left(\vartheta_x\lambda_x\right) \in \mathcal{Q}_\mu$, with $\lambda \in \mathcal{Q}_\mu$. Assume $T_\vartheta \in L(\mathcal{Q}_\mu)$, then the multiplication operator is called created by ϑ.*

Theorem 5.1. *Fixing $\vartheta \in \mathfrak{C}^{\mathbb{Z}^+}$ and the conditions of Theorem 4.3 are satisfied, one has*

$$\vartheta \in \ell_\infty \Longleftrightarrow T_\vartheta \in L(\left(\ell(\nabla_\varphi,\eta)\right)_\phi).$$

Proof. Suppose that $\vartheta \in \ell_\infty$. Hence, there exists $\nu > 0$ so that $|\vartheta_x| \leq \nu$, for every $x \in \mathbb{Z}^+$. Assume $\lambda \in \left(\ell(\nabla_\varphi,\eta)\right)_\phi$, one gets

$$\begin{aligned}\phi(T_\vartheta\lambda) = \phi(\vartheta\lambda) &= \sum_{h=0}^{\infty}\left(\left|\nabla_\varphi|\vartheta_h\lambda_h|\right|\right)^{\eta_h} \leq \sum_{h=0}^{\infty}\left(\left|\nu\nabla_\varphi|\lambda_h|\right|\right)^{\eta_h} \leq \sup_h \nu^{\eta_h}\sum_{h=0}^{\infty}\left(\left|\nabla_\varphi|\lambda_h|\right|\right)^{\eta_h} \\ &= \sup_h \nu^{\eta_h}\phi(\lambda).\end{aligned}$$

Hence, $T_\vartheta \in L(\left(\ell(\nabla_\varphi,\eta)\right)_\phi)$.

After that, when $T_\vartheta \in L(\left(\ell(\nabla_\varphi,\eta)\right)_\phi)$ and $\vartheta \notin \ell_\infty$. So for every $x \in \mathbb{Z}^+$, there are $b \in \mathbb{Z}^+$ with $|\vartheta_x| > b$. We have

$$\phi(T_\vartheta e_x) = \phi(\vartheta e_x) = \sum_{y=0}^{\infty}\left(\left|\nabla_\varphi|\vartheta_y(e_x)_y|\right|\right)^{\eta_y} > \sum_{y=0}^{\infty}\left(\left|b\nabla_\varphi|(e_x)_y|\right|\right)^{\eta_y} > b^{\eta_0}\phi(e_x).$$

That gives, $T_\vartheta \notin L(\left(\ell(\nabla_\varphi,\eta)\right)_\phi)$. So $\vartheta \in \ell_\infty$. □

Theorem 5.2. *Assume that $\vartheta \in \mathfrak{C}^{\mathbb{Z}^+}$ and $\left(\ell(\nabla_\varphi,\eta)\right)_\phi$ is a **p-q N** pss.*

$$|\vartheta_x| = 1, \text{ for any } x \in \mathbb{Z}^+ \Leftrightarrow T_\vartheta \text{ is an isometry.}$$

Proof. For any $\lambda \in \left(\ell(\nabla_\varphi,\eta)\right)_\phi$, we have

$$\phi(T_\vartheta\lambda) = \phi(\vartheta\lambda) = \sum_{k=0}^{\infty}\left(\left|\nabla_\varphi|\vartheta_k\lambda_k|\right|\right)^{\eta_k} = \sum_{k=0}^{\infty}\left(\left|\nabla_\varphi|\lambda_k|\right|\right)^{\eta_k} = \phi(\lambda),$$

Hence, T_ϑ is an isometry.

Let T_ϑ be an isometry and $|\vartheta_x| < 1$, for some $x = x_0$. We get

$$\phi(T_\vartheta e_{x_0}) = \phi(\vartheta e_{x_0}) = \sum_{k=0}^{\infty} \left(\left|\nabla_\varphi|\vartheta_k(e_{x_0})_k|\right|\right)^{\eta_k} < \sum_{k=0}^{\infty} \left(\left|\nabla_\varphi|(e_{x_0})_k|\right|\right)^{\eta_k} = \phi(e_{x_0}).$$

Also when $|\vartheta_{x_0}| > 1$, Obviously, $\phi(T_\vartheta e_{x_0}) > \phi(e_{x_0})$. So, $|\vartheta_x| = 1$, for all $x \in \mathbb{Z}^+$. □

Theorem 5.3. *Assume $\vartheta \in \mathfrak{C}^{\mathbb{Z}^+}$ and the conditions of Theorem 4.3 are satisfied. Then $T_\vartheta \in \mathfrak{R}(\left(\ell(\nabla_\varphi, \eta)\right)_\phi)$, if and only if, $(\vartheta_b)_{b=0}^{\infty} \in c_0$.*

Proof. Suppose that $T_\vartheta \in \mathfrak{R}(\left(\ell(\nabla_\varphi, \eta)\right)_\phi)$, one gets $T_\vartheta \in \mathfrak{M}(\left(\ell(\nabla_\varphi, \eta)\right)_\phi)$. If $\lim_{x\to\infty} \vartheta_x \neq 0$. Hence, one has $\varrho > 0$ with $K_\varrho = \{x \in \mathbb{Z}^+ : |\vartheta_x| \geq \varrho\} \not\subseteq \mathfrak{I}$. Let $\{\alpha_x\}_{x\in\mathbb{Z}^+} \subset K_\varrho$. Hence, $\{e_{\alpha_x} : \alpha_x \in K_\varrho\} \in \ell_\infty$ is an infinite set in $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$. For $\alpha_a, \alpha_b \in K_\varrho$, one obtains

$$\begin{aligned}\phi(T_\vartheta e_{\alpha_a} - T_\vartheta e_{\alpha_b}) = \phi(\vartheta e_{\alpha_a} - \vartheta e_{\alpha_b}) &= \sum_{k=0}^{\infty} \left(\left|\nabla_\varphi|\vartheta_k((e_{\alpha_a})_k - (e_{\alpha_b})_k)|\right|\right)^{\eta_k} \\ &\geq \sum_{k=0}^{\infty} \left(\varrho\left|\nabla_\varphi|((e_{\alpha_a})_k - (e_{\alpha_b})_k)|\right|\right)^{\eta_k} \geq \inf_k \varrho^{\eta_k}\phi(e_{\alpha_a} - e_{\alpha_b}).\end{aligned}$$

Therefore, $\{e_{\alpha_b} : \alpha_b \in K_\varrho\} \in \ell_\infty$, which cannot have a convergent subsequence under T_ϑ. So $T_\vartheta \notin \mathfrak{M}(\left(\ell(\nabla_\varphi, \eta)\right)_\phi)$. Hence $T_\vartheta \notin \mathfrak{R}(\left(\ell(\nabla_\varphi, \eta)\right)_\phi)$, so explains a contradiction. So, $\lim_{x\to\infty} \vartheta_x = 0$. On the contrary, assume $\lim_{x\to\infty} \vartheta_x = 0$. Then for all $\varrho > 0$, one has $K_\varrho = \{x \in \mathbb{Z}^+ : |\vartheta_x| \geq \varrho\} \subset \mathfrak{I}$. Then, for every $\varrho > 0$, we obtain $\dim\left(\left(\left(\ell(\nabla_\varphi, \eta)\right)_\phi\right)_{K_\varrho}\right) = \dim\left(\mathfrak{C}^{K_\varrho}\right) < \infty$. So $T_\vartheta \in \mathfrak{T}\left(\left(\left(\ell(\nabla_\varphi, \eta)\right)_\phi\right)_{K_\varrho}\right)$. Suppose that $\vartheta_a \in \mathfrak{C}^{\mathbb{Z}^+}$, for every $a \in \mathbb{Z}^+$, where

$$(\vartheta_a)_b = \begin{cases} \vartheta_b, & b \in K_{\frac{1}{a+1}}, \\ 0, & \text{otherwise.} \end{cases}$$

Obviously, $L_{\vartheta_a} \in \mathfrak{T}\left(\left(\left(\ell(\nabla_\varphi, \eta)\right)_\phi\right)_{B_{\frac{1}{a+1}}}\right)$ such as $\dim\left(\left(\left(\ell(\nabla_\varphi, \eta)\right)_\phi\right)_{B_{\frac{1}{a+1}}}\right) < \infty$, for all $a \in \mathbb{Z}^+$. Since $(\eta_x) \in \nearrow \cap \ell_\infty$ with $\eta_0 > 0$, we get

$$\begin{aligned}\phi((T_\vartheta - L_{\vartheta_a})\lambda) = \mu\left(\left((\vartheta_b - (\vartheta_a)_b)\lambda_b\right)_{b=0}^{\infty}\right) &= \sum_{b=0}^{\infty} \left(\left|\nabla_\varphi|(\vartheta_b - (\vartheta_a)_b)\lambda_b|\right|\right)^{\eta_b} \\ &= \sum_{b=0, b\in K_{\frac{1}{a+1}}}^{\infty} \left(\left|\nabla_\varphi|(\vartheta_b - (\vartheta_a)_b)\lambda_b|\right|\right)^{\eta_b} + \sum_{b=0, b\notin K_{\frac{1}{a+1}}}^{\infty} \left(\left|\nabla_\varphi|(\vartheta_b - (\vartheta_a)_b)\lambda_b|\right|\right)^{\eta_b} \\ &= \sum_{b=0, b\notin K_{\frac{1}{a+1}}}^{\infty} \left(\left|\nabla_\varphi|\vartheta_b\lambda_b|\right|\right)^{\eta_b} \leq \frac{1}{(a+1)^{\eta_0}} \sum_{b=0, b\notin K_{\frac{1}{a+1}}}^{\infty} \left(\left|\nabla_\varphi|\lambda_b|\right|\right)^{\eta_b} \\ &< \frac{1}{(a+1)^{\eta_0}} \sum_{b=0}^{\infty} \left(\left|\nabla_\varphi|\lambda_b|\right|\right)^{\eta_b} = \frac{1}{(a+1)^{\eta_0}}\phi(\lambda).\end{aligned}$$

Hence, $\|T_\vartheta - L_{\vartheta_a}\| \le \frac{1}{(a+1)^{\eta_0}}$. Which explains $T_\vartheta \in \mathfrak{R}(\left(\ell(\nabla_\varphi,\eta)\right)_\phi)$. □

Theorem 5.4. *Assume that* $\vartheta \in \mathfrak{C}^{\mathbb{Z}^+}$ *and the conditions of Theorem 4.3 are verified. Then* $T_\vartheta \in \mathfrak{M}(\left(\ell(\nabla_\varphi,\eta)\right)_\phi)$*, if and only if,* $(\vartheta_x)_{x=0}^\infty \in c_0$.

Proof. Clearly, as $\mathfrak{R}(\left(\ell(\nabla_\varphi,\eta)\right)_\phi) \subsetneqq \mathfrak{M}(\left(\ell(\nabla_\varphi,\eta)\right)_\phi)$. □

Corollary 5.1. *Let the conditions of Theorem 4.3 be satisfied, then* $\mathfrak{M}(\left(\ell(\nabla_\varphi,\eta)\right)_\phi) \subsetneqq \mathrm{L}(\left(\ell(\nabla_\varphi,\eta)\right)_\phi)$.

Proof. Since the multiplication mapping I on $\left(\ell(\nabla_\varphi,\eta)\right)_\phi$ is created by $\vartheta = (1,1,\ldots)$. Which gives $I \notin \mathfrak{M}(\left(\ell(\nabla_\varphi,\eta)\right)_\phi)$ and $I \in \mathrm{L}(\left(\ell(\nabla_\varphi,\eta)\right)_\phi)$. □

Theorem 5.5. *If* $T_\vartheta \in \mathrm{L}(\left(\ell(\nabla_\varphi,\eta)\right)_\phi)$*, where the space* $\left(\ell(\nabla_\varphi,\eta)\right)_\phi$ *is a* ***p-q B*** $\mathfrak{pss}$*. Then, there exists* $p>0$ *and* $q>0$ *under* $p < |\vartheta_x| < q$*, for every* $x \in (\ker(\vartheta))^c$*, if and only if, Range*(T_ϑ) *is closed.*

Proof. Suppose that enough conditions are verified. Hence, there exists $\varepsilon > 0$ so that $|\vartheta_x| \ge \varepsilon$, for all $x \in (\ker(\vartheta))^c$. To prove the space $Range(T_\vartheta)$ is closed, assume g is a limit point of $Range(T_\vartheta)$. One has $T_\vartheta\delta_x \in \left(\ell(\nabla_\varphi,\eta)\right)_\phi$, for every $x \in \mathbb{Z}^+$ so that $\lim_{x\to\infty} T_\vartheta\delta_x = g$. The sequence $T_\vartheta\delta_x$ is a Cauchy sequence. For $(\eta_x) \in \nearrow \cap \ell_\infty$, we have

$$\begin{aligned}\phi(T_\vartheta\delta_l - T_\vartheta\delta_m) &= \sum_{h=0}^{\infty}\left(\left|\nabla_\varphi|(\vartheta_h(\delta_l)_h - \vartheta_h(\delta_m)_h)|\right|\right)^{\eta_h}\\ &= \sum_{h=0,h\in(\ker(\vartheta))^c}^{\infty}\left(\left|\nabla_\varphi|(\vartheta_h(\delta_l)_h - \vartheta_h(\delta_m)_h)|\right|\right)^{\eta_h} + \sum_{h=0,h\notin(\ker(\vartheta))^c}^{\infty}\left(\left|\nabla_\varphi|(\vartheta_h(\delta_l)_h - \vartheta_h(\delta_m)_h)|\right|\right)^{\eta_h}\\ &\ge \sum_{h=0,h\in(\ker(\vartheta))^c}^{\infty}\left(\left|\nabla_\varphi|(\vartheta_h(\delta_l)_h - \vartheta_h(\delta_m)_h)|\right|\right)^{\eta_h} = \sum_{h=0}^{\infty}\left(\left|\nabla_\varphi|(\vartheta_h(w_l)_h - \vartheta_h(w_m)_h)|\right|\right)^{\eta_h}\\ &> \sum_{h=0}^{\infty}\left(\zeta_x\left|\sum_{h=0}^{x}\varepsilon((w_l)_h - (w_m)_h)\right|\right)^{\eta_h} \ge \inf_h \varepsilon^{\eta_h}\mu\Big(w_l - w_m\Big),\end{aligned}$$

where

$$(w_l)_x = \begin{cases}(\delta_l)_x, & x \in (\ker(\vartheta))^c,\\ 0, & x \notin (\ker(\vartheta))^c.\end{cases}$$

That gives, $\{w_l\}$ is a Cauchy sequence in $\left(\ell(\nabla_\varphi,\eta)\right)_\phi$. Since $\left(\ell(\nabla_\varphi,\eta)\right)_\phi$ is complete. So, there exists $\delta \in \left(\ell(\nabla_\varphi,\eta)\right)_\phi$ under $\lim_{x\to\infty} w_x = \delta$. Since $T_\vartheta \in \mathrm{L}(\left(\ell(\nabla_\varphi,\eta)\right)_\phi)$, we have $\lim_{x\to\infty} T_\vartheta w_x = T_\vartheta\delta$. But $\lim_{x\to\infty} T_\vartheta w_x = \lim_{x\to\infty} T_\vartheta\delta_x = g$. Then $T_\vartheta\delta = g$. Therefore, $g \in Range(T_\vartheta)$. So $Range(T_\vartheta)$ is closed. After, suppose that the necessity condition is satisfied. Hence, there exists $\varepsilon > 0$ so that $\phi(T_\vartheta\delta) \ge \varepsilon\phi(\delta)$ and $\delta \in \left(\left(\ell(\nabla_\varphi,\eta)\right)_\phi\right)_{(\ker(\vartheta))^c}$. If $K = \left\{x \in (\ker(\vartheta))^c : |\vartheta_x| < \varepsilon\right\} \neq \emptyset$, hence for $a_0 \in K$, one gets

$$\phi(T_\vartheta e_{l_0}) = \mu\Big(\Big(\vartheta_m(e_{l_0})_m\Big)_{m=0}^{\infty}\Big) = \sum_{m=0}^{\infty}\left(\left|\nabla_\varphi|\vartheta_m(e_{l_0})_m|\right|\right)^{\eta_m} < \sum_{y=0}^{\infty}\left(\left|\nabla_\varphi|(e_{l_0})_m\varepsilon|\right|\right)^{\eta_m} \le \sup_m \varepsilon^{\eta_m}\phi(e_{l_0}),$$

that is a contradiction. Hence $K = \phi$, we have $|\vartheta_x| \geq \varepsilon$, with $x \in (\ker(\vartheta))^c$. □

Theorem 5.6. *Let $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ be a **p-q B** $\mathfrak{pss}$ and $\vartheta \in \mathfrak{C}^{\mathbb{Z}^+}$. Hence, we have $p > 0$ and $q > 0$ under $p < |\vartheta_x| < q$ and $x \in \mathbb{Z}^+$, if and only if, $T_\vartheta \in \mathrm{L}(\left(\ell(\nabla_\varphi, \eta)\right)_\phi)$ is invertible.*

Proof. Suppose that $\kappa \in \mathfrak{C}^{\mathbb{Z}^+}$ and $\kappa_x = \frac{1}{\vartheta_x}$. Given Theorem 5.1, hence $T_\vartheta,\ L_\kappa \in \mathrm{L}|_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$. So $T_\vartheta.L_\kappa = L_\kappa.T_\vartheta = I$. Therefore, $L_\kappa = L_\vartheta^{-1}$. Next, let T_ϑ is invertible. So $Range(T_\vartheta) = \left(\left(\ell(\nabla_\varphi, \eta)\right)_\phi\right)_{\mathbb{Z}^+}$. Therefore, $Range(T_\vartheta)$ is closed. By Theorem 5.5, one obtains $p > 0$ then $|\vartheta_x| \geq p$, for all $x \in (\ker(\vartheta))^c$. Hence $\ker(\vartheta) = \emptyset$, if $\vartheta_{x_0} = 0$, where $x_0 \in \mathbb{Z}^+$, which explains $e_{x_0} \in \ker(T_\vartheta)$, so there is a contradiction, since $\ker(T_\vartheta)$ is trivial. Hence, $|\vartheta_x| \geq p$, for all $x \in \mathbb{Z}^+$. As $T_\vartheta \in \ell_\infty$. By Theorem 5.1, one gets $q > 0$ with $|\vartheta_x| \leq q$, for all $x \in \mathbb{Z}^+$. So, $p \leq |\vartheta_x| \leq q$ such that $x \in \mathbb{Z}^+$. □

Definition 5.2. *[23] An operator $T \in \mathrm{L}|_{Q}$ is said to be Fredholm, if $\dim(Range(T))^c < \infty$, $\dim(\ker(T)) < \infty$ and $Range(T)$ is closed.*

Theorem 5.7. *If $T_\vartheta \in \mathrm{L}(\left(\ell(\nabla_\varphi, \eta)\right)_\phi)$, where $\left(\ell(\nabla_\varphi, \eta)\right)_\phi$ is a **p-q B** $\mathfrak{pss}$. Then T_ϑ is Fredholm operator, if and only if, (a) $\ker(\vartheta) \subsetneq \mathbb{Z}^+$ is a finite and (b) $|\vartheta_x| \geq \varrho$, with $x \in (\ker(\vartheta))^c$.*

Proof. Assume that the parts (a) and (b) are verified. Given Theorem 5.5, the part (b) gives that $Range(T_\vartheta)$ is closed. The part (a) implies that $\dim(\ker(T_\vartheta)) < \infty$ and $\dim((Range(T_\vartheta))^c) < \infty$. So, T_ϑ is Fredholm. Assume T_ϑ is Fredholm operator, let $\ker(\vartheta) \subsetneq \mathbb{Z}^+$ be an infinite, so $e_z \in \ker(T_\vartheta)$, for all $z \in \ker(\vartheta)$. As e_z's are linearly independent, we have that $\dim(\ker(T_\vartheta)) = \infty$, this explains a contradiction. Therefore, $\ker(\vartheta) \subsetneq \mathbb{Z}^+$ must be finite. The part (b) comes from Theorem 5.5. □

6. Behaviour of the operators' ideals

Firstly, we shall revisit the fundamental principles of **OIs**.

Definition 6.1. *[24] A class $\mathbb{H} \subseteq \mathrm{L}$ is named an **OI**, if all vector $\mathbb{H}|_{\mathcal{G}}^{\mathcal{V}} = \mathbb{H} \cap \mathrm{L}|_{\mathcal{G}}^{\mathcal{V}}$ satisfies the following conditions:*

- **(i):** $I_\Omega \in \mathbb{H}$.
- **(ii):** $\mathbb{H}|_{\mathcal{G}}^{\mathcal{V}}$ *is a linear space on* $\mathfrak{C}$.
- **(iii):** *Suppose that $N \in \mathrm{L}|_{\mathcal{G}_0}^{\mathcal{G}}$, $M \in \mathbb{H}|_{\mathcal{G}}^{\mathcal{V}}$ and $P \in \mathrm{L}|_{\mathcal{V}}^{\mathcal{V}_0}$, then $PMN \in \mathbb{H}|_{\mathcal{G}_0}^{\mathcal{V}_0}$.*

Definition 6.2. *[5] A mapping $\Lambda : \mathbb{H} \to [0, \infty)$ is called a pre-quasi norm on the **OI** $\mathbb{H}$, if it verifies the next conditions:*

- **(1):** *If $M \in \mathbb{H}|_{\mathcal{G}}^{\mathcal{V}}$, $\Lambda(M) \geq 0$ and $\Lambda(M) = 0 \iff M = 0$,*
- **(2):** *there are $E_0 \geq 1$ so that $\Lambda(\kappa X) \leq E_0|\kappa|\Lambda(X)$, with $X \in \mathbb{H}|_{\mathcal{G}}^{\mathcal{V}}$ and $\kappa \in \mathfrak{C}$,*
- **(3):** *one has $G_0 \geq 1$ so that $\Lambda(Z_1 + Z_2) \leq G_0[\Lambda(Z_1) + \Lambda(Z_2)]$, for all $Z_1, Z_2 \in \mathbb{H}|_{\mathcal{G}}^{\mathcal{V}}$,*
- **(4):** *we have $D_0 \geq 1$, when $N \in \mathrm{L}|_{\mathcal{G}_0}^{\mathcal{G}}$, $M \in \mathbb{H}|_{\mathcal{G}}^{\mathcal{V}}$ and $P \in \mathrm{L}_{\mathcal{V}}^{\mathcal{V}_0}$, then $\Lambda(PMN) \leq D_0\ \|P\|\Lambda(M)\ \|N\|$.*

Theorem 6.1. *[20] Assume $(Q)_\mu$ is a **p-m**$\mathfrak{pss}$, then Λ is a pre-quasi norm on $\mathrm{L}^s_{(Q)_\mu}$, where $\Lambda(Y) = \mu(s_x(Y))_{x=0}^\infty$, with $Y \in \mathrm{L}^s_{(Q)_\mu}|_{\mathcal{G}}^{\mathcal{V}}$.*

Theorem 6.2. *[5] A* ***p-q N*** *on the ideal* $\mathbb{H}$*, if it is* ***q N*** *on* $\mathbb{H}$*.*

Definition 6.3. *[3] A Banach space* Q *is called simple if* $\mathrm{L}|_Q$ *includes a unique non-trivial closed ideal.*

Theorem 6.3. *[3] We have*

$$\mathfrak{T}|_{\mathcal{G}} \subsetneqq \mathfrak{R}|_{\mathcal{G}} \subsetneqq \mathfrak{M}|_{\mathcal{G}} \subsetneqq \mathrm{L}|_{\mathcal{G}}.$$

In this section. First, we present the sufficient conditions (not necessary) on $\Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$ under $\overline{\mathfrak{T}} = \mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$. This provides an explanation regarding the non-linear nature of $s-$ type $\Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$ spaces (see Rhoades [22]). Second, for which conditions on $\Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$, is $\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$ **p-q BI**? Third, we offer the sufficient conditions on $\Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$ under $\mathrm{L}^\alpha_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$ to be strictly contained for different ϕ and powers. We offer the conditions for which $\mathrm{L}^\alpha_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$ is minimum. Also, we examine the setups when the class $\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$ is simple. We give fourthly the sufficient settings on $\Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$ for which L with the sequence of eigenvalues in $\Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$ identical with $\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$.

6.1. **Denseness.**

Theorem 6.4. $\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}} = \overline{\mathfrak{T}|^{\mathcal{V}}_{\mathcal{G}}}$*, if the conditions of Theorem 4.3 are sufficiently verified (not necessarily).*

Proof. Since $e_x \in \Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$, for every $x \in \mathbb{Z}^+$ and $\Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$ is a linear space. Suppose that $Z \in \mathfrak{T}|^{\mathcal{V}}_{\mathcal{G}}$, then $(s_x(Z))_{x=0}^\infty \in \mathrm{I}$. Therefore, $\overline{\mathfrak{T}|^{\mathcal{V}}_{\mathcal{G}}} \subseteq \mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}}$. To prove that $\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}} \subseteq \overline{\mathfrak{T}|^{\mathcal{V}}_{\mathcal{G}}}$, let $Z \in \mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}}$, then $(s_x(Z))_{x=0}^\infty \in \Big(\ell(\nabla_\varphi,\eta)\Big)_\phi$. As $\phi(s_x(Z))_{x=0}^\infty < \infty$, assume $\rho \in (0,1)$, one obtains $x_0 \in \mathbb{Z}^+ \setminus \{0\}$ with $\phi((s_x(Z))_{x=x_0}^\infty) < \frac{\rho}{4}$. Since $s_x(Z)$ is decreasing, we have

$$\sum_{x=x_0+1}^{2x_0} \Big(|\nabla_\varphi s_{2x_0}(Z)|\Big)^{\eta_x} \le \sum_{x=x_0+1}^{2x_0} \Big(|\nabla_\varphi s_x(Z)|\Big)^{\eta_x} \le \sum_{x=x_0}^{\infty} \Big(|\nabla_\varphi s_x(Z)|\Big)^{\eta_x} < \frac{\rho}{4}. \tag{6.1}$$

Hence there exists $Y \in \mathfrak{T}_{2x_0}|^{\mathcal{V}}_{\mathcal{G}}$ so that $\mathrm{rank}(Y) \le 2x_0$ and

$$\sum_{x=2x_0+1}^{3x_0} \Big(\Big|\nabla_\varphi \|Z-Y\|\Big|\Big)^{\eta_x} \le \sum_{x=x_0+1}^{2x_0} \Big(\Big|\nabla_\varphi \|Z-Y\|\Big|\Big)^{\eta_x} < \frac{\rho}{4}, \tag{6.2}$$

since $(\eta_x) \in \nearrow \cap \ell_\infty$, we can choose

$$\sum_{x=0}^{x_0} \Big(\Big|\nabla_\varphi \|Z-Y\|\Big|\Big)^{\eta_x} < \frac{\rho}{4}. \tag{6.3}$$

By inequalities (1)-(4), hence

$$d(Z,Y)=\phi\left(s_x(Z-Y)\right)_{x=0}^{\infty}=\sum_{x=0}^{3x_0-1}\left(\left|\nabla_{\varphi}s_x(Z-Y)\right|\right)^{\eta_x}+\sum_{x=3x_0}^{\infty}\left(\left|\nabla_{\varphi}s_x(Z-Y)\right|\right)^{\eta_x}$$

$$\leq\sum_{x=0}^{3x_0}\left(\left|\nabla_{\varphi}\|Z-Y\|\right|\right)^{\eta_x}+\sum_{x=x_0}^{\infty}\left(\left|\nabla_{\varphi}s_{x+2x_0}(Z-Y)\right|\right)^{\eta_{x+2x_0}}$$

$$\leq\sum_{x=0}^{3x_0}\left(\left|\nabla_{\varphi}\|Z-Y\|\right|\right)^{\eta_x}+\sum_{x=x_0}^{\infty}\left(\left|\nabla_{\varphi}s_x(Z)\right|\right)^{\eta_x}$$

$$\leq 3\sum_{x=0}^{x_0}\left(\left|\nabla_{\varphi}\|Z-Y\|\right|\right)^{\eta_x}+\sum_{x=x_0}^{\infty}\left(\left|\nabla_{\varphi}s_x(Z)\right|\right)^{\eta_x}<\rho.$$

For the opposite direction, we have a contradiction since $I_4\in \mathrm{L}^s_{(\ell(\nabla,(-2,1,1,\ldots)))_\phi}|_{\mathcal{G}}^{\mathcal{V}}$, but $\eta_0>0$ is not verified. □

6.2. **Pre-quasi Banach ideal.**

Theorem 6.5. *The subclass* $\left(\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi},\Lambda\right)$ *is a **p-q BI**, where* $\Lambda(Y)=\phi\Big(\left(s_x(Y)\right)_{x=0}^{\infty}\Big)$*, whenever the settings of Theorem 4.3 are satisfied.*

Proof. Since $\left(\ell(\nabla_\varphi,\eta)\right)_\phi$ is a **p-m**$\mathfrak{pss}$, so from theorem 6.1, Λ is a pre-quasi norm on $\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$. Assume $(D_x)_{x\in\mathbb{Z}^+}$ is a Cauchy sequence in $\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|_{\mathcal{G}}^{\mathcal{V}}$. Since $\mathrm{L}|_{\mathcal{G}}^{\mathcal{V}}\supseteq\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|_{\mathcal{G}}^{\mathcal{V}}$, then

$$\Lambda(D_a-D_b)=\sum_{y=0}^{\infty}\left(\left|\nabla_\varphi s_y(D_a-D_b)\right|\right)^{\eta_y}\geq\left(\left|\nabla_\varphi\,\|D_a-D_b\|\right|\right)^{\eta_0},$$

so $(D_b)_{b\in\mathbb{Z}^+}$ is a Cauchy sequence in $\mathrm{L}|_{\mathcal{G}}^{\mathcal{V}}$. As $\mathrm{L}|_{\mathcal{G}}^{\mathcal{V}}$ is a Banach space, one obtains $D\in\mathrm{L}|_{\mathcal{G}}^{\mathcal{V}}$ with $\lim_{b\to\infty}\|D_b-D\|=0$. Since $\left(s_x(D_b)\right)_{x=0}^{\infty}\in\left(\ell(\nabla_\varphi,\eta)\right)_\phi$, for every $b\in\mathbb{Z}^+$. Given Definition 4.2, conditions (ii), (iii), and (v), we have

$$\Lambda(D)=\sum_{y=0}^{\infty}\left(\left|\nabla_\varphi s_y(D)\right|\right)^{\eta_x}\leq 2^{\hbar-1}\sum_{y=0}^{\infty}\left(\left|\nabla_\varphi s_{[\frac{y}{2}]}(D-D_b)\right|\right)^{\eta_y}+2^{\hbar-1}\sum_{y=0}^{\infty}\left(\left|\nabla_\varphi s_{[\frac{y}{2}]}(D_b)\right|\right)^{\eta_y}$$

$$\leq 2^{\hbar-1}\sum_{y=0}^{\infty}\left(\left|\nabla_\varphi\,\|D-D_b\|\right|\right)^{\eta_y}+2^{\hbar-1}D_0\sum_{y=0}^{\infty}\left(\left|\nabla_\varphi s_y(D_b)\right|\right)^{\eta_y}<\infty.$$

Therefore, $\left(s_x(D)\right)_{x=0}^{\infty}\in\left(\ell(\nabla_\varphi,\eta)\right)_\phi$, then $D\in\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|_{\mathcal{G}}^{\mathcal{V}}$. □

6.3. **Minimality and simplicity.**

Theorem 6.6. *Suppose that the setups of Theorem 4.3 are satisfied with* $\varphi_2(x)\geq\varphi_1(x)$ *and* $0<\eta_x^{(1)}<\eta_x^{(2)}$*, for all* $x\in\mathbb{Z}^+$*, hence*

$$\mathrm{L}^s_{\left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_\phi}|_{\mathcal{G}}^{\mathcal{V}}\subsetneqq\mathrm{L}^s_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_\phi}|_{\mathcal{G}}^{\mathcal{V}}\subsetneqq\mathrm{L}|_{\mathcal{G}}^{\mathcal{V}}.$$

Proof. Assume $Z \in L^s_{\left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}}$, hence $(s_x(Z)) \in \left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_\phi$. One gets

$$\sum_{x=0}^{\infty}\left(\left|\nabla_{\varphi_2}s_x(Z)\right|\right)^{\eta_x^{(2)}} < \sum_{x=0}^{\infty}\left(\left|\nabla_{\varphi_1}s_x(Z)\right|\right)^{\eta_x^{(1)}} < \infty,$$

then $Z \in L^s_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}}$. After, take $(s_x(Z))_{x=0}^{\infty}$ such that $\left|\nabla_{\varphi_1}s_x(Z)\right| = \frac{1}{\sqrt[\eta_x^{(1)}]{x+1}}$, one gets $Z \in L|^{\mathcal{V}}_{\mathcal{G}}$ under

$$\sum_{x=0}^{\infty}\left(\left|\nabla_{\varphi}s_x(Z)\right|\right)^{\eta_x^{(1)}} = \sum_{x=0}^{\infty}\frac{1}{x+1} = \infty,$$

and

$$\sum_{x=0}^{\infty}\left(\left|\nabla_{\varphi_2}s_x(Z)\right|\right)^{\eta_x^{(2)}} \le \sum_{x=0}^{\infty}\left(\left|\nabla_{\varphi_1}s_x(Z)\right|\right)^{\eta_x^{(2)}} = \sum_{x=0}^{\infty}\left(\frac{1}{x+1}\right)^{\frac{\eta_x^{(2)}}{\eta_x^{(1)}}} < \infty.$$

Then $Z \notin L^s_{\left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}}$ and $Z \in L^s_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}}$.

Clearly, $L^s_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}} \subset L|^{\mathcal{V}}_{\mathcal{G}}$. After, by taking $(s_x(Z))_{x=0}^{\infty}$ such that $\left|\nabla_{\varphi_2}s_z(Z)\right| = \frac{1}{\sqrt[\eta_x^{(2)}]{x+1}}$. We have $Z \in L|^{\mathcal{V}}_{\mathcal{G}}$ such that $Z \notin L^s_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}}$. □

Theorem 6.7. *Suppose that the conditions of Theorem 4.3 are satisfied, hence* $L^{\alpha}_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$ *is minimum.*

Proof. We have that $(L^{\alpha}_{\ell(\nabla_\varphi,\eta)},\Lambda)$, where $\Lambda(Z) = \sum_{x=0}^{\infty}\left(\left|\nabla_\varphi\alpha_x(Z)\right|\right)^{\eta_x}$, is a **p-q BI**. Suppose that $L^{\alpha}_{\ell(\nabla_\varphi,\eta)}|^{\mathcal{V}}_{\mathcal{G}} = L|^{\mathcal{V}}_{\mathcal{G}}$, one gets $\eta > 0$ with $\Lambda(Z) \le \eta\|Z\|$, for every $Z \in L|^{\mathcal{V}}_{\mathcal{G}}$. Becouse of Dvoretzky's theorem [2], for all $b \in \mathbb{Z}^+$, we have the quotient spaces $\mathcal{G}/Y_b$ and subspaces M_b of $\mathcal{V}$ which can be transformed onto ℓ_2^b by isomorphisms V_b and X_b so that $\|V_b\|\|V_b^{-1}\| \le 2$ and $\|X_b\|\|X_b^{-1}\| \le 2$. Assume that I_b is the identity mapping on ℓ_2^b, T_b is the quotient mapping from $\mathcal{G}$ onto $\mathcal{G}/Y_b$ and J_b is the natural embedding mapping from M_b into $\mathcal{V}$. Suppose that m_z is the Bernstein numbers [1], one has

$$\begin{aligned}1 =& m_z(I_b) = m_z(X_bX_b^{-1}I_bV_bV_b^{-1}) \le \|X_b\|m_z(X_b^{-1}I_bV_b)\|V_b^{-1}\| = \|X_b\|m_z(J_bX_b^{-1}I_bV_b)\|V_b^{-1}\| \\ &\le \|X_b\|d_z(J_bX_b^{-1}I_bV_b)\|V_b^{-1}\| = \|X_b\|d_z(J_bX_b^{-1}I_bV_bT_b)\|V_b^{-1}\| \\ &\le \|X_b\|\alpha_z(J_bX_b^{-1}I_bV_bT_b)\|V_b^{-1}\|,\end{aligned}$$

for $0 \le x \le b$. Suppose that m is the greatest integer, so $\varphi(m) = 0$. Then we have

$$\begin{aligned}&\frac{x+1}{|1-m|} \le \|X_b\|\left|\nabla_\varphi\alpha_x(J_bX_b^{-1}I_bV_bT_b)\right|\|V_b^{-1}\| \Rightarrow \\ &\left(\frac{x+1}{|1-m|}\right)^{\eta_x} \le (\|X_b\|\|V_b^{-1}\|)^{\eta_x}\left(\left|\nabla_\varphi\alpha_x(J_bX_b^{-1}I_bV_bT_b)\right|\right)^{\eta_x}.\end{aligned}$$

Hence, for some $\varrho \geq 1$, one gets

$$\sum_{x=0}^{b}\left(\frac{x+1}{|1-m|}\right)^{\eta_x} \leq \varrho\|X_b\|\|V_b^{-1}\|\sum_{x=0}^{b}\left(\left|\nabla_{\varphi}\alpha_x(J_bX_b^{-1}I_bV_bT_b)\right|\right)^{\eta_x} \Rightarrow$$

$$\sum_{x=0}^{b}\left(\frac{x+1}{|1-m|}\right)^{\eta_x} \leq \varrho\|X_b\|\|V_b^{-1}\|\Lambda(J_bX_b^{-1}I_bV_bT_b) \Rightarrow$$

$$\sum_{x=0}^{b}\left(\frac{x+1}{|1-m|}\right)^{\eta_x} \leq \varrho\eta\|X_b\|\|V_b^{-1}\|\|J_bX_b^{-1}I_bV_bT_b\| \Rightarrow$$

$$\sum_{x=0}^{b}\left(\frac{x+1}{|1-m|}\right)^{\eta_x} \leq \varrho\eta\|X_b\|\|V_b^{-1}\|\|J_bX_b^{-1}\|\|I_b\|\|V_bT_b\| = \varrho\eta\|X_b\|\|V_b^{-1}\|\|X_b^{-1}\|\|I_b\|\|V_b\| \leq 4\varrho\eta.$$

So there exists a contradiction whenever $b \to \infty$. Then, $\mathcal{G}$ and $\mathcal{V}$ both cannot be infinite dimensional when $\mathrm{L}^{\alpha}_{\ell(\nabla_{\varphi},\eta)}|^{\mathcal{V}}_{\mathcal{G}} = \mathrm{L}|^{\mathcal{V}}_{\mathcal{G}}$. □

We can easily prove the following corollary as theorem 6.7.

Corollary 6.1. $\mathrm{L}^{d}_{\ell(\nabla_{\varphi},\eta)}$ *is minimum, whenever the parts of Theorem 4.3 are satisfied.*

Lemma 6.1. *[3] Suppose that $P \in \mathrm{L}|^{\mathcal{V}}_{\mathcal{G}}$ and $P \notin \Re|^{\mathcal{V}}_{\mathcal{G}}$, then $N \in \mathrm{L}|_{\mathcal{G}}$ and $M \in \mathrm{L}|_{\mathcal{V}}$ with $MPNe_x = e_x$, for all $x \in \mathbb{Z}^+$.*

Theorem 6.8. *Let the conditions of Theorem 4.3 be satisfied with $\varphi_2(x) \geq \varphi_1(x)$ and $0 < \eta_x^{(1)} < \eta_x^{(2)}$, for all $x \in \mathbb{Z}^+$, then*

$$\begin{aligned}&\mathrm{L}\Big(\mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}, \mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}\Big)\\ &= \Re\Big(\mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}, \mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}\Big).\end{aligned}$$

Proof. Suppose that $X \in \mathrm{L}\Big(\mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}, \mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}\Big)$ and
$X \notin \Re\Big(\mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}, \mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}\Big)$. From Lemma 6.1, one has
$Y \in \mathrm{L}\Big(\mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}\Big)$ and $Z \in \mathrm{L}\Big(\mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}\Big)$ with $ZXYI_b = I_b$. Hence, for every $b \in \mathbb{Z}^+$, we get

$$\begin{aligned}\|I_b\|_{\mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}} &= \sum_{x=0}^{\infty}\left(\left|\nabla_{\varphi_1}s_x(I_b)\right|\right)^{\eta_x^{(1)}}\\ &\leq \|ZXY\|\|I_b\|_{\mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}} \leq \sum_{x=0}^{\infty}\left(\left|\nabla_{\varphi_2}s_x(I_b)\right|\right)^{\eta_x^{(2)}}.\end{aligned}$$

That fails Theorem 6.6. So $X \in \Re\Big(\mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}, \mathrm{L}^{s}_{\left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_{\phi}}|^{\mathcal{V}}_{\mathcal{G}}\Big)$, □

Corollary 6.2. *Let the conditions of Theorem 4.3 be satisfied with $\varphi_2(x) \geq \varphi_1(x)$ and $0 < \eta_x^{(1)} < \eta_x^{(2)}$, for every $x \in \mathbb{Z}^+$, hence*

$$\begin{aligned} &\mathrm{L}\Big(\mathrm{L}^s_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_\phi}\Big|^{\mathcal{V}}_{\mathcal{G}}, \mathrm{L}^s_{\left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_\phi}\Big|^{\mathcal{V}}_{\mathcal{G}}\Big) \\ &= \mathfrak{M}\Big(\mathrm{L}^s_{\left(\ell(\nabla_{\varphi_2},(\eta_x^{(2)}))\right)_\phi}\Big|^{\mathcal{V}}_{\mathcal{G}}, \mathrm{L}^s_{\left(\ell(\nabla_{\varphi_1},(\eta_x^{(1)}))\right)_\phi}\Big|^{\mathcal{V}}_{\mathcal{G}}\Big). \end{aligned}$$

Proof. Since $\mathfrak{R} \subset \mathfrak{M}$, the proof follows. □

Theorem 6.9. *Suppose that the conditions of Theorem 4.3 be satisfied, hence $\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$ is simple.*

Proof. Let the closed ideal $\mathfrak{M}(\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}})$ contains $X \notin \mathfrak{R}(\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}})$. From Lemma 6.1, we have $Y, Z \in \mathrm{L}(\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}})$ such that $ZXYI_b = I_b$. Therefore, $I_{\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}}} \in \mathfrak{M}(\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}})$. Hence $\mathrm{L}(\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}}) = \mathfrak{M}(\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}})$. So, $\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$ is a simple Banach space. □

6.4. **Spectrum.**

Theorem 6.10. *Assume that the settings of Theorem 4.3 are satisfied and ∇_φ^{-1} exists and bounded linear, hence*

$$\left(\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}\right)^\rho\Big|^{\mathcal{V}}_{\mathcal{G}} = \mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}\Big|^{\mathcal{V}}_{\mathcal{G}}.$$

Proof. Let $H \in \left(\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}\right)^\rho|^{\mathcal{V}}_{\mathcal{G}}$, one has $(\rho_x(H))_{x=0}^\infty \in \left(\ell(\nabla_\varphi,\eta)\right)_\phi$ and $\|H - \rho_x(H)I\| = 0$, for every $x \in \mathbb{Z}^+$. Hence $X = \rho_x(H)I$, for every $x \in \mathbb{Z}^+$, then $s_x(H) = s_x(\rho_x(H)I) = |\rho_x(H)|$, for any $x \in \mathbb{Z}^+$. So, $(s_x(H))_{x=0}^\infty \in \left(\ell(\nabla_\varphi,\eta)\right)_\phi$, one gets $X \in \mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}}$. After that, if $X \in \mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}|^{\mathcal{V}}_{\mathcal{G}}$. Hence $(s_x(H))_{x=0}^\infty \in \left(\ell(\nabla_\varphi,\eta)\right)_\phi$. Therefore, one obtains $\sum_{x=0}^\infty \left(\left|\nabla_\varphi s_x(H)\right|\right)^{\eta_x} < \infty$. So $\lim_{x\to\infty} \nabla_\varphi s_x(H) = 0$. Since ∇_φ^{-1} exists and bounded linear, then $\lim_{x\to\infty} s_x(H) = 0$. Suppose that $\|H - s_x(H)I\|^{-1}$ exists, for every $x \in \mathbb{Z}^+$. So $\|H - s_x(H)I\|^{-1}$ exists and bounded, for every $x \in \mathbb{Z}^+$. Hence, $\lim_{x\to\infty} \|H - s_x(H)I\|^{-1} = \|X\|^{-1}$ exists and bounded. Since $\left(\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}, \Lambda\right)$ is a **p-q OI**, one has

$$I = XX^{-1} \in \mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}\Big|^{\mathcal{V}}_{\mathcal{G}} \Rightarrow (s_x(I))_{x=0}^\infty \in \ell(\nabla_\varphi,\eta) \Rightarrow \lim_{x\to\infty} s_x(I) = 0.$$

We have a contradiction, since $\lim_{x\to\infty} s_x(I) = 1$. Therefore $\|H - s_x(H)I\| = 0$, for all $x \in \mathbb{Z}^+$. That gives $X \in \left(\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}\right)^\rho|^{\mathcal{V}}_{\mathcal{G}}$. □

7. Conclusion

We offered the topological and geometric structure of the domain of **g.c.d** in **Nss**, as well as the multiplication mappings defined on it, the class $\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$, and the class $\left(\mathrm{L}^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}\right)^\rho$. This article presented a novel space of solutions for numerous difference equations, the spectrum of $L^s_{\left(\ell(\nabla_\varphi,\eta)\right)_\phi}$, and proved that closed **OIs** are certain to play an important role in the Banach lattice principle.

Acknowledgements: This work was funded by the University of Jeddah, Jeddah, Saudi Arabia, under grant No. (UJ-23-DR-128). Therefore, the authors thank the University of Jeddah for its technical and financial support.

Authors' Contributions: All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

Conflicts of Interest: The authors declare that there are no conflicts of interest regarding the publication of this paper.

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