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A NEW ENTROPY FORMULA AND GRADIENT ESTIMATES FOR THE LINEAR HEAT EQUATION ON STATIC MANIFOLD

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ABSTRACT. In this paper we prove a new monotonicity formula for the heat equation via a generalized family of entropy functionals. This family of entropy formulas generalizes both Perelman's entropy for evolving metric and Ni's entropy on static manifold. We show that this entropy satisfies a pointwise differential inequality for heat kernel. The consequences of which are various gradient and Harnack estimates for all positive solutions to the heat equation on compact manifold.

1. Introduction and Preliminaries

We study the heat equation defined on a compact Riemannian manifold M with static metric g

$$\Big(\frac{\partial}{\partial t} - \Delta_g\Big)u(x,t) = 0, \tag{1.1}$$

where Δ_g is the usual Laplace-Beltrami operator acting on functions in space with respect to metric g. Throughout, M will be taken to be a closed manifold (i.e., compact without boundary) except when otherwise indicated. Most of our calculations are done in local coordinates, where $\{x^i\}$ is fixed in a neighbourhood of every point $x \in M$. The Riemannian metric $g(x)$ at any point $x \in M$ is a bilinear symmetric positive definite matrix denoted in local coordinates by

$$g_{ij} = ds^2 = g_{ij}dx^i dx^j$$

The Laplace-Beltrami operator acting on a smooth function f on M is defined as the product of divergence and gradient of f written as

$$\Delta_g f := div\ grad\ f = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\Big(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x^j}f\Big),$$

where $|g| = \det(g_{ij})$ and the inverse metric $g^{ij} = (g_{ij})^{-1}$. By the above we note that

$$(grad\ f)^i = (\nabla f)^i = g^{ij}\frac{\partial}{\partial x^j}f \quad and \quad divF = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}(\sqrt{|g|}F^i).$$

Also we have the metric norm

$$|\nabla f|_g^2 = g^{ij}\nabla_i f\nabla_j f$$

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and application of Cauchy-Schwarz inequality on the expression

$$\Delta f = g^{ij}\nabla_i\nabla_j f = trHessf$$

yields the following inequality

$$(Hessf)^2 \geq \frac{1}{n}(\Delta f)^2.$$

The Riemann structure allows us to define Riemannian volume measure $dV(x)$ on M

$$dV(x) = \sqrt{|g_{ij}(x)|}dx^i.$$

By the divergence theorem we have the following integration by parts formulas for functions $f, h \in C^2(M)$

$$\int_M f\Delta_g h\ dV = -\int_M \langle\nabla f, \nabla h\rangle_g dV = \int_M \Delta_g f\ hdV.$$

For any smooth function f on M, we have the Bochner identity defined as

$$\Delta(|\nabla f|^2) = 2|\nabla\nabla f|^2 + 2\langle\nabla f, \nabla\Delta f\rangle + 2Rc(\nabla f, \nabla f),$$

where Rc is the Ricci curvature of M whose tensor components will be written in local coordinates as R_{ij}. We switch between coordinates to allow calculations to be explicit and we write in local coordinates $\nabla f = f_i$, $\nabla\nabla f = \nabla_i\nabla_j f = f_{ij}$ and $\frac{\partial}{\partial x^i} = \partial_i$. Also we write time derivative $\frac{\partial}{\partial t}f = \partial_t f = f_t$. We adopt summation convention with repeated indices summed up.

Any function $0 < u \in C^\infty(M \times [0,T])$ which satisfies (1.1) is called a positive solution. If u tends to a dirac-delta δ-function as t goes to zero, u will be called the heat kernel, that is the unique minimal positive solution on M. We are interested in the behaviours of all positive solutions, in particular, the heat kernel. We derive gradient estimates and differential Harnack inequalities via the monotone property of a new family of entropy functionals. It is well known that entropy monotonicity formulas are closely related to the gradient estimate for the heat equation. The importance of gradient estimates as well as those of Harnack inequalities can not be overemphasised in the fields of Differential geometry and Analysis among their numerous applications. Differential Harnack inequalities are used to study the behaviours of solutions to the heat equation in space-time. Li and Yau's paper [15] can be said to mark the beginning of rigorous applications of these concepts. They derived gradient estimates for positive solutions to the heat operator defined on a complete manifold with static metrics, from which they obtained Harnack inequalities. These inequalities were in turn used to establish various lower and upper bounds on the heat kernel. Precisely, Li and Yau's results for static metrics are the following;

Theorem A (Li-Yau [15]). *Let (M, g) be an n-dimensional complete Riemannian manifold. Suppose there exist some nonnegative constant k such that the Ricci curvature $R_{ij}(g) \geq -k$. Let $u \in C^{2,1}(M \times [0,T])$ be any smooth positive solution to the heat equation (1.1) in the geodesic ball $\mathcal{B}_{2\rho} \times [0,T]$. Then, the following estimate holds*

$$\sup_{x\in\mathcal{B}_\rho}\Big\{\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u}\Big\} \leq \frac{n\alpha^2}{2t} + \frac{C\alpha^2}{\rho^2}\Big(\frac{\alpha^2}{\alpha^2-1} + \sqrt{k}\rho\Big) + \frac{n\alpha^2 k}{2(\alpha-1)}. \tag{1.2}$$

for all $(x,t) \in \mathcal{B}_{2\rho,T}$, $t > 0$ *and some constants* C *depending only on* n *and* $\alpha > 1$. *Moreover, the following estimate*

$$\sup_{x\in\mathcal{B}_\rho}\left\{\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u}\right\} \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2 k}{2(\alpha-1)} \tag{1.3}$$

holds for complete noncompact manifold by letting $\rho \to \infty$. The above results have been improved by Davies [7, Section 5.3] as follows

$$\sup_{x\in\mathcal{B}_\rho}\left\{\frac{|\nabla u|^2}{u^2} - \alpha\frac{u_t}{u}\right\} \leq \frac{n\alpha^2}{2t} + \frac{n\alpha^2 k}{4(\alpha-1)}. \tag{1.4}$$

As $\alpha \to 1$, the second terms in both (1.3) and (1.4) blow up and we obtain a sharp estimate

$$\frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \leq \frac{n}{2t}.$$

Note that α can be chosen as a constant function of time only such in a way that it goes to 1 as $t \to 0$, see for instance Hamilton [11], Huang, Huang and Li [12] and Li and Xu [14].

Li and Yau derived their gradient estimates using the maximum principle, but by now it is known how to use monotonicity formulas derived from classical entropies of Shannon (from statistical thermodynamics) and Fisher's information (from information theory). Let $u > 0$ be a positive solution to (1.1) with the normalization condition $\int_M u dV(x) = 1$, then, the classical Shannon entropy is defined by

$$\mathcal{S}_0(u(t)) = \int_M u(x,t)\log u(x,t) dV(x) \tag{1.5}$$

and the Fisher information defined by

$$\mathcal{F}_0(u(t)) = \int_M \frac{|\nabla u(x,t)|^2}{u(x,t)} dV(x). \tag{1.6}$$

A straightforward computation shows that

$$\frac{d}{dt}\mathcal{S}_0(u(t)) = -\int_M |\nabla \log u(x,t)|^2 u(x,t) dV(x) = -\mathcal{F}_0(u(t))$$

and

$$\begin{aligned}\frac{d^2}{dt^2}\mathcal{S}_0(u(t)) &= -\frac{d}{dt}\mathcal{F}_0(u(t))\\ &= 2\int_M \Big(|Hess \log u|^2 + Rc(\nabla \log u, \nabla \log u)\Big) u dV(x),\end{aligned}$$

where Rc is the Ricci curvature of M. We now define normalised versions of $\mathcal{S}_0$ and $\mathcal{F}_0$ by

$$\begin{aligned}\mathcal{S}(u(t)) &:= \mathcal{S}_0(u(t)) + \frac{n}{2}\log(4\pi t) + \frac{n}{2}\\ &= \int_M \Big(\log u + \frac{n}{2}\log(4\pi t) + \frac{n}{2}\Big) u dV(x)\\ \mathcal{F}(u(t)) &:= t\mathcal{F}_0(u(t)) - \frac{n}{2}\\ &= \int_M \Big(t|\nabla \log u|^2 - \frac{n}{2}\Big) u dV(x).\end{aligned}$$

Here, the normalisation is done so that the entropies remain identically zero for all time when u is the heat kernel. It easily shown that $\mathcal{S}$ and $\mathcal{F}$ are identically zero on $M = \mathbb{R}^n$, the Euclidean space, for

$$u = H(x, y, t) = (4\pi t)^{-\frac{n}{2}} \exp\Big(- \frac{|x-y|^2}{4t}\Big).$$

By the above calculation, Shannon entropy $\mathcal{S}_0$ for a positive solution to the heat equation on static manifold is seen to be monotone decreasing while its derivative is monotone nondecreasing on the condition that the Ricci curvature of M is non-negative. Thus, the Shannon entropy is convex in this case. We can now define another entropy $\mathcal{W}(u,t)$ based on the above

$$\mathcal{W}(u,t) = \mathcal{F}(u,t) - \mathcal{S}(u,t) = -\frac{d}{dt}\Big(t\mathcal{S}(u,t)\Big). \tag{1.7}$$

Obviously, the entropy $\mathcal{W}(u,t)$ reads

$$\mathcal{W}(u,t) = \int_M \Big(t\frac{|\nabla u|^2}{u^2} - \log u - \frac{n}{2}\log(4\pi t) - n\Big) u dV(x).$$

Let $u = (4\pi t)^{-\frac{n}{2}} e^{-f}$ be a positive solution to the heat equation, where f is a smooth function. Here we have

$$f = -\log u - \frac{n}{2}\log(4\pi t), \quad \int_M (4\pi t)^{-\frac{n}{2}} e^{-f} = 1,$$

$$\mathcal{W}(f,t) = \int_M (t|\nabla f|^2 + f - n)(4\pi t)^{-\frac{n}{2}} e^{-f} dV(x) \tag{1.8}$$

and

$$\frac{d}{dt}\mathcal{W} = -\frac{d}{dt}\Big(t\mathcal{S}\Big) = -2t\int_M \Big(\Big|\nabla\nabla f - \frac{1}{2t}g\Big|^2 + Rc(\nabla f, \nabla f)\Big)\frac{e^{-f}}{(4\pi t)^{-\frac{n}{2}}}.$$

This is exactly Ni's result in [17] which states that $\mathcal{W}(f,t)$ is monotone nonincreasing on a closed manifold with nonnegative Ricci curvature. In the case the manifold is Ricci flat this is indeed Perelman's entropy monotonicity formula [20] on a metric evolving by the Ricci flow.

Notice that by application of integration by parts $\mathcal{F}(u(t))$ can be written as

$$\mathcal{F}(u(t)) = \int_M -\Big(t\Delta \log u + \frac{n}{2}\Big) u dV(x). \tag{1.9}$$

This has a surprising connection to the Li-Yau gradient estimate in Theorem A above. Clearly, the quantity under the integral is equivalent to the Harnack quantity of Li-Yau

$$-\Big(t\Delta\log u + \frac{n}{2}\Big)u = -\Big(\frac{\Delta u}{u} - \frac{|\nabla u|^2}{u^2} + \frac{n}{2}\Big)u.$$

Li-Yau gradient estimate [15] says $\mathcal{F}(u) \le 0$ when $Rc \ge 0$, which implies

$$\frac{u_t}{u} - \frac{\nabla u|^2}{u^2} + \frac{n}{2} \ge 0.$$

This is in turn equivalent to

$$t\Delta f - \frac{n}{2} \le 0, \tag{1.10}$$

which can be viewed as a generalized Laplacian comparison theorem. Indeed, the Laplacian comparison theorem on M is a consequence of (1.10) by applying inequality to the heat kernel and letting t tends to zero. One can also see that $\lim_{t\to 0}\mathcal{S}(u(t)) = 0$ for the heat kernel and hence $\mathcal{S}(u(t))$ is monotone increasing on nonnegative Ricci curvature manifold. Therefore, we have $\mathcal{W}(f,t) \geq 0$ for the heat kernel for some $t > 0$ if and only if M is isometric to $\mathbb{R}^n$. Note that on $\mathbb{R}^n$ we have $f = |x|^2/4t$. Lei Ni also showed that these results hold for all complete manifolds with $Rc \geq 0$. Let M be a complete Riemannian manifold with nonnegative Ricci curvature, then at $t = 1/2$, $\mathcal{W} \geq 0$ holds on M if and only if M is isometric to $\mathbb{R}^n$, (See also Weissler [22]). This is indeed equivalent to Gross logarithmic Sobolev inequalities [9] on $\mathbb{R}^n$. Thus, there is a strong relation between the log-Sobolev inequality and the geometry of the manifold which was originally discovered by Bakry, Concordet and Ledoux [3] (see also [4]). That is,

$$\int_{\mathbb{R}^n}\Big(\frac{1}{2}|\nabla f|^2 + f - n\Big)\frac{e^{-f}}{(4\pi t)^{-\frac{n}{2}}} \geq 0 \tag{1.11}$$

implies

$$\int_{\mathbb{R}^n} u\log u\ dx \leq \frac{n}{2}\log\Big(\frac{1}{2n\pi e}\int_{\mathbb{R}^n}\frac{|\nabla u|^2}{u}\Big) \tag{1.12}$$

with equality on any Gaussian with $\int_{\mathbb{R}^n} ud\mu$. To get (1.12) from (1.11) one uses the monotone property of $\mathcal{W}$ on $\mathbb{R}^n$ and asymptotic behaviour of the positive solution to the heat equation, noting that solution on $\mathbb{R}^n$ converges after rescaling at infinity to constant multiples of the usual Gaussian. The remarkable papers [17] and [18] have shown a desirable interpolation between entropy formula of Ni on static manifolds and that of Perelman [20] on evolving manifolds. The new $\mathcal{W}_\epsilon(f,t)$ discussed in this paper (see section 2) is an example of such a family of entropies connecting both Ni's and Perelman's entropies. We demonstrated this in [2, Chapter 3] and have applied it on manifold evolving by the Ricci-harmonic map flow in [1]. We remark that estimates and bounds on parabolic equations behave in similar way whether the metric is static or moving. This can be justified by the fact that heat diffusion on a bounded geometry with either static or evolving metric behaves like heat diffusion in Euclidean space, many a times, their estimates even coincide.

In this paper however, we prove the monotonicity formulas for a family of entropy functionals $\mathcal{W}_\epsilon(f,t)$ and discuss some of its analytic and geometric consequences. The plan of the rest of the paper is as follows: In Section 2 we introduce a new family of entropy functionals and prove its monotonicity for a positive solution to the heat equation. The monotonicity derived here is used in Section 3 to derive pointwise differential Harnack inequalities and gradient estimates for the heat equation. As a consequence we obtain Harnack estimates for the fundamental solution which also holds for all positive solutions in Section 4. We give Li-Yau-Hamilton type gradient estimates for bounded solutions in the last section.

2. A new entropy monotonicity formula

We emphasize that the volume is kept fixed throughout the time of evolution for the heat equation on a closed n-dimensional manifold (M,g). We also impose the condition of nonnegativity on the Ricci curvature of the underlying manifold M.

Let $u = u(x,t)$ be a positive solution to the heat equation

$$\Box u = \Big(\frac{\partial}{\partial t} - \Delta\Big)u(x,t) = 0. \tag{2.1}$$

Let $f : M \times (0,T] \to \mathbb{R}$ be smoothly defined as $u = (4\pi t)^{-\frac{n}{2}} e^{-f}$ with normalization condition $\int_M u(x,t)dV(x) = 1$. We introduce a generalized family of entropy by

$$\mathcal{W}_\epsilon(f,t) = \int_M \Big[\frac{\epsilon^2 t}{4\pi}|\nabla f|^2 + f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{n\epsilon^2}{4\pi}\Big]\frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}dV(x), \tag{2.2}$$

where $0 < \epsilon^2 \leq 4\pi$. We remark that if $\epsilon^2 = 4\pi$, we recover the Perelman's entropy as in the special case considered by Ni in [17]. From this entropy formula we later derive the corresponding differential inequality and gradient estimate for the fundamental solution, which in fact, holds for all positive solutions to the heat equation. The same entropy is used by the author in his Phd thesis [2] to examine the surprising relation that exists between the entropy formula for heat equation and the conjugate heat equation under the Ricci flow. We have also used its monotonicity properties combined with some Sobolev-type inequalities to derive sharp upper bound for conjugate heat kernel along Ricci-harmonic map heat flow in [1].

Lemma 2.1. *Let $u = (4\pi t)^{-\frac{n}{2}} e^{-f}$ be a positive solution to the heat equation $\Box u = 0$ on a closed Riemannian manifold M. Then*

$$(\partial_t - \Delta)|\nabla f|^2 = -2f_{ij}^2 - 2\langle \nabla f, \nabla|\nabla f|^2\rangle - 2R_{ij}f_i f_j \tag{2.3}$$

and

$$(\partial_t - \Delta)(\Delta f) = -2f_{ij}^2 - 2\langle \nabla f, \nabla|\nabla f|^2\rangle - 2\langle \nabla f, \nabla \partial_t f\rangle - 2R_{ij}f_i f_j. \tag{2.4}$$

Moreover, if $w = 2\Delta f - |\nabla f|^2$, then

$$(\partial_t - \Delta)w = -2f_{ij}^2 - 2R_{ij}f_i f_j - 2\langle \nabla w, \nabla f\rangle. \tag{2.5}$$

Proof. Since $u = (4\pi t)^{-\frac{n}{2}} e^{-f}$, $f = -\log u - \frac{n}{2}\log(4\pi t)$ and $\frac{\partial}{\partial t} f = \Delta f - |\nabla f|^2 - \frac{n}{2t}$.

(1)

$$\frac{\partial}{\partial t}|\nabla f|^2 = \frac{\partial}{\partial t}(g^{ij}\partial_i f \partial_j f) = 2g^{ij}\partial_i f \partial_j f \partial_t f = 2\langle \nabla f, \nabla \partial_t f\rangle. \tag{2.6}$$

By Bochner identity

$$\begin{aligned}
\Delta(|\nabla f|^2) &= 2f_{ij}^2 + 2\langle \nabla f, \nabla \Delta f\rangle + 2R_{ij}f_i f_j \\
&= 2f_{ij}^2 + 2\langle \nabla f, \nabla(\partial_t f + |\nabla f|^2\rangle + 2R_{ij}f_i f_j \\
&= 2f_{ij}^2 + 2\langle \nabla f, \nabla|\nabla f|^2\rangle + 2\langle \nabla f, \nabla \partial_t f\rangle + 2R_{ij}f_i f_j.
\end{aligned}$$

Adding the last equality to (2.6) proves (2.3).

(2)

$$\begin{aligned}
(\partial_t - \Delta)(\Delta f) &= \Delta(\Delta f - |\nabla f|^2) - \Delta(\Delta f) = -\Delta|\nabla f|^2 \\
&= -2f_{ij}^2 - 2\langle \nabla f, \nabla(\partial_t f + |\nabla f|^2\rangle - 2R_{ij}f_i f_j \\
&= -2f_{ij}^2 - 2\langle \nabla f, \nabla|\nabla f|^2\rangle - \partial_t(|\nabla f|^2) - 2R_{ij}f_i f_j.
\end{aligned}$$

(3)

$$\begin{aligned}
(\partial_t - \Delta)w &= 2(\partial_t - \Delta)\Delta f - (\partial_t - \Delta)|\nabla f|^2 \\
&= -2f_{ij}^2 - 2R_{ij}f_i f_j - 2\langle \nabla f, \nabla(|\nabla f|^2 + 2\partial_t f).\rangle
\end{aligned}$$

This ends the proof of the lemma. □

We are now set to establish the monotone property of the $W_\epsilon(f,t)$-entropy. By the monotonicity formula for this entropy functional, we will derive gradient estimates and the corresponding differential Harnack inequalities for the fundamental solution to the heat equation on a static manifold.

Proposition 2.2. *Let M be any closed manifold and $u = (4\pi t)^{-\frac{n}{2}}e^{-f}$ be any positive solution to the heat equation $\Box u = (\partial_t - \Delta)u = 0$ on $M \times (0.T]$. Denoting*

$$P_\epsilon = \frac{\epsilon^2 t}{4\pi}\Big(2\Delta f - |\nabla f|^2\Big) + f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{n\epsilon^2}{4\pi}, \tag{2.7}$$

where $0 < \epsilon^2 \leq 4\pi$. Then

$$(\partial_t - \Delta)P_\epsilon \leq -\frac{\epsilon^2 t}{2\pi}\Big(\Big|f_{ij} - \frac{\sqrt{\pi}}{\epsilon t}g_{ij}\Big|^2 + R_{ij}f_if_j\Big) - 2\langle\nabla P_\epsilon, \nabla f\rangle - \Big(1 - \frac{\epsilon^2}{4\pi}\Big)|\nabla f|^2. \tag{2.8}$$

Proof. Here we write

$$\widetilde{P_\epsilon} = \frac{\epsilon^2 t}{4\pi}w + \widetilde{f} + \frac{n}{2}\ln\Big(\frac{1}{\epsilon^2 t}\Big) - \frac{n\epsilon^2}{4\pi}.$$

Since $f = -\ln u - \frac{n}{2}\ln(4\pi t)$, taking $u = e^{-\widetilde{f}}$ implies $f = \widetilde{f} - \frac{n}{2}\ln(4\pi t)$. we notice also that $\nabla\widetilde{f} = \nabla f$, $\Delta\widetilde{f} = \Delta f$ and $\widetilde{f}_{ij} = f_{ij}$, then $(\partial_t - \Delta)\widetilde{f} = -|\nabla\widetilde{f}|^2 - \frac{n}{2t}$. Now by direct differentiation and application of Lemma 2.1, we have the following computation

$$\begin{aligned}(\partial_t - \Delta)P_\epsilon &= \frac{\epsilon^2 t}{4\pi}(\partial_t - \Delta)w + \frac{\epsilon^2}{4\pi}w + (\partial_t - \Delta)\widetilde{f} + \frac{\partial}{\partial t}\Big(\frac{n}{2}\ln\Big(\frac{1}{\epsilon^2 t}\Big) - \frac{n\epsilon^2}{4\pi}\Big)\\ &= \frac{\epsilon^2 t}{4\pi}\Big(-2f_{ij}^2 - 2R_{ij}f_if_j - 2\langle\nabla w, \nabla f\rangle\Big) + \frac{\epsilon^2}{4\pi}(2\Delta f - |\nabla f|^2) - |\nabla f|^2 - \frac{n}{2t}\\ &= \frac{\epsilon^2 t}{4\pi}\Big(-2f_{ij}^2 - \frac{2\pi}{\epsilon^2}\frac{n}{t^2} - 2R_{ij}f_if_j\Big) + \frac{\epsilon^2}{4\pi}(2\Delta f - |\nabla f|^2) - 2\langle\frac{\epsilon^2 t}{4\pi}\nabla w, \nabla f\rangle - |\nabla f|^2.\end{aligned}$$

Notice that

$$2\langle\frac{\epsilon^2 t}{4\pi}\nabla w, \nabla f\rangle = 2\langle(\nabla P_\epsilon - \widetilde{f}), \nabla f\rangle = 2\langle\nabla P_\epsilon, \nabla f\rangle - 2|\nabla f|^2$$

and

$$(2\Delta f - |\nabla f|^2) = (2\partial_t f + |\nabla f|^2)$$

Then we have

$$\begin{aligned}(\partial_t - \Delta)P_\epsilon &\leq -2\frac{\epsilon^2 t}{4\pi}\Big(f_{ij}^2 + \frac{\pi}{\epsilon^2}\frac{n}{t^2} - \frac{2\sqrt{\pi}}{\epsilon t}\Delta f + R_{ij}f_if_j\Big) - 2\langle\nabla P_\epsilon, \nabla f\rangle + \frac{\epsilon^2}{4\pi}|\nabla f|^2 - |\nabla f|^2\\ &= -\frac{2\epsilon^2 t}{4\pi}\Big(\Big|f_{ij} - \frac{\sqrt{\pi}}{\epsilon t}g_{ij}\Big|^2 + R_{ij}f_if_j\Big) - 2\langle\nabla P_\epsilon, \nabla f\rangle - \Big(1 - \frac{\epsilon^2}{4\pi}\Big)|\nabla f|^2.\end{aligned}$$

□

Theorem 2.3. *Let M be a closed Riemannian manifold. Assume that $u = (4\pi t)^{-\frac{n}{2}}e^{-f}$ is a positive solution to the heat equation $(\partial_t - \Delta)u = 0$, then, we have the following monotonicity formula for $\mathcal{W}_\epsilon(f,t)$ defined in (2.2)*

(2.9)

$$\frac{d}{dt}\mathcal{W}_\epsilon(f,t) = -\int_M\left[\frac{\epsilon^2 t}{2\pi}\left(\Big|f_{ij} - \frac{\sqrt{\pi}}{\epsilon t}g_{ij}\Big|^2 + R_{ij}f_if_j\right) + \Big(1 - \frac{\epsilon^2}{4\pi}\Big)|\nabla f|^2\right]\frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}}dV(x)$$

with (f, t) *satisfying*

$$\int_M \frac{e^{-f}}{(4\pi t)^{\frac{n}{2}}} dV(x) = 1 \tag{2.10}$$

and $0 < \epsilon^2 \leq 4\pi$.

Proof. Combining Proposition 2.2 with the fact that $\Box u = 0$ and $u\nabla f = -\nabla u$, we have

$$\begin{aligned}(\partial_t - \Delta)(P_\epsilon u) &= (\partial_t - \Delta)P_\epsilon \cdot u + P_\epsilon(\partial_t - \Delta)u - 2\langle \nabla P_\epsilon, \nabla u\rangle \\ &= -\frac{\epsilon^2 t}{2\pi}\Big(\Big|f_{ij} - \frac{\sqrt{\pi}}{\epsilon t}g_{ij}\Big|^2 + R_{ij}f_if_j\Big)u - 2\langle \nabla P_\epsilon, \nabla f\rangle u \\ &\qquad - \Big(1 - \frac{\epsilon^2}{4\pi}\Big)|\nabla f|^2 u - 2\langle \nabla P_\epsilon, \nabla u\rangle.\end{aligned}$$

Integrating over M, we have

$$\begin{aligned}\int_M P_\epsilon u dV(x) &= \int_M \Big[\frac{\epsilon^2 t}{4\pi}\Big(2\Delta f - |\nabla f|^2\Big) + f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{n\epsilon^2}{4\pi}\Big] u dV(x) \\ &= \int_M \Big[\frac{\epsilon^2 t}{4\pi}|\nabla f|^2 + f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{n\epsilon^2}{4\pi}\Big] u dV(x) \\ &\qquad + \frac{2\epsilon^2 t}{4\pi}\int_M (\Delta f - |\nabla f|^2) u dV(x) \\ &= \mathcal{W}_\epsilon(f, t),\end{aligned}$$

in the sense that the second integral in the RHS vanishes on a closed manifold since $(\Delta f - |\nabla f|^2)u = -\Delta u$. Therefore

$$\begin{aligned}\frac{d}{dt}\mathcal{W}_\epsilon(f, t) &= \frac{\partial}{\partial t}\int_M P_\epsilon u \ dV(x) \\ &= \int_M \Big(\frac{d}{dt}P_\epsilon \ u + P_\epsilon \frac{\partial}{\partial t} u\Big) dV(x) \\ &= \int_M \Big[(\partial_t - \Delta)P_\epsilon \ u + P_\epsilon(\partial_t - \Delta)u\Big] dV(x) \\ &= \int_M (\partial_t - \Delta)P_\epsilon \ u dV(x),\end{aligned}$$

where we have used integration by parts and $\Box u = 0$. Using the evolution $(\partial_t - \Delta)P_\epsilon$ from Proposition 2.2, we get the desired result. Moreover, if the manifold has nonnegative Ricci curvature, i.e, $R_{ij} \geq 0$, it becomes obvious from (2.9) that $d\mathcal{W}_\epsilon/dt \leq 0$. □

We remark that Kuang and Zhang [13] have a result in this direction, it is stated as follows; Let M be a closed Riemannian manifold with nonnegative Ricci curvature. Let u be the fundamental solution to the heat equation with $f = -\ln u - \frac{n}{2}\ln(4\pi t)$, we have

$$t(\alpha\Delta f - |\nabla f|^2) + f - \alpha\frac{n}{2} \leq 0 \tag{2.11}$$

for any constant $\alpha \geq 1$. Indeed, if $\alpha = 2$, this is exactly the differential inequality

$$t(2\Delta f - |\nabla f|^2) + f - n \leq 0$$

proved in [17]. Dividing through by $\alpha \cdot t$, with $\alpha \geq 1$ and $t \geq 0$, we obtain

$$\Delta f - \frac{|\nabla f|^2}{\alpha} + \frac{f}{\alpha t} - \frac{n}{2t} \leq 0$$

as $t \to \infty$, which is precisely the Li-Yau gradient estimate. For $\alpha > 2$, the gradient estimate is an interpolation of Perelman's estimate and Li-Yau estimate. For $1 \leq \alpha \leq 2$, it is considered in [13]. In Euclidean space $\mathbb{R}^n$, if u is the fundamental solution to the heat equation then (2.11) becomes an equality.

3. Gradient Estimates for Heat Equation on Static Manifold

The monotonicity formula in the last section may be viewed as a local version of the Perelman's $\mathcal{W}$-entropy formula in [20]. In what follows, we want to show that the local entropy satisfies a pointwise differential inequality for the heat kernel. We have the following fashioned after [17, Theorem 1.2] with the proof follows from the argument of [16, Proposition 3.6].

Theorem 3.1. *Let M be a closed manifold with nonnegative Ricci curvature and $H(x,y,t) = H = (4\pi t)^{-\frac{n}{2}} e^{-f}$ be the heat kernel, where H tends to a δ-function as $t \to 0$ and satisfies $\int_M H dV(x) = 1$. Then for all $t > 0$, we have*

$$P_\epsilon = \frac{\epsilon^2 t}{4\pi}\Big(2\Delta f - |\nabla f|^2\Big) + f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{n\epsilon^2}{4\pi} \leq 0. \tag{3.1}$$

Proof. Let h be any compactly supported smooth function for all $t_0 > 0$. Suppose $h(\cdot, t)$ is a positive solution to the backward heat equation $(\partial_t + \Delta)h = 0$, (This is Perelman's argument in [20, Corollary 9.3]), then, it is clear that $\frac{\partial}{\partial t}\int_M HhdV = 0$ and we have by direct calculation that

$$\begin{aligned}
\frac{\partial}{\partial t}\int_M hP_\epsilon H dV(x) &= \int_M \Big[\partial_t h(P_\epsilon H) + h\partial_t(P_\epsilon H)\Big] dV(x)\\
&= \int_M \Big[(\partial_t + \Delta)h(P_\epsilon H) + h(\partial_t - \Delta)P_\epsilon H)\Big] dV(x)\\
&= \int_M h(\partial_t - \Delta)P_\epsilon H dV(x)\\
&\leq 0.
\end{aligned}$$

The inequality is due to Theorem 2.3 since $R_{ij} \geq 0$. We are left to showing that for everywhere positive function $h(\cdot, t)$, the limit of $\int_M hP_\epsilon HdV(x)$ is nonpositive as $t \to 0$. We assume the claim apriori (i.e, $\lim_{t\to 0}\int_M hP_\epsilon HdV = 0$) and conclude the result.

For completeness, we devote the next effort to justifying the claim

$$\lim_{t\to 0}\int_M hP_\epsilon HdV \leq 0. \tag{3.2}$$

Our argument follows from [16], for detail see [17, 19, 20], the calculation in [13] is also similar. If H tends to a dirac δ-function, say at a point $p \in M$, for $t \to 0$, then f satisfies $f(x,t) \to \frac{d^2(p,\,x)}{4t}$. This is in relation to l- length of Perelman. This yields

$$\lim_{t\to 0}\int_M fhHdV \leq \limsup_{t\to 0}\int_M \frac{d^2(p,x)}{4t} hHdV = \frac{n}{2}h(p,0). \tag{3.3}$$

Meanwhile, by the strong Maximum principle we have $h(x,0)>0$ and $\lim_{t\to 0}\int_M hHdV = h(x,0)$, hence by scaling argument, we assume that $h(x,0)=1$. All these will soon become clearer. Rewriting P_ϵ and using integrating by parts methods we have

$$\int_M P_\epsilon hHdV = \int_M \frac{\epsilon^2 t}{4\pi}(|\nabla f|^2 - \frac{n}{2t})hHdV - \frac{\epsilon^2 t}{2\pi}\int_M \langle \nabla f, \nabla h\rangle HdV + \int_M fHhdV + \frac{n}{2}\Big[\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{\epsilon^2}{4\pi}\Big]\int_M Hh\ dV.$$

Though, the H appearing in the last equation is actually the heat kernel on an evolving manifold in Ni's result [19] while h satisfies the forward heat equation, his argument still holds in our case, we only need the asymptotic behaviour of heat kernel on a fixed metric. We should also note that since $h(\cdot,t_0)$ is compactly supported and by strong maximum principle we have $h(\cdot,t_0), |\nabla h(\cdot,t_0)|$ and $|\Delta h(\cdot,t_0)|$ bounded on M. This implies that there exists a bounded solution $h(\cdot,t_0)$.

It turns out that we need to show that there exists a constant $B\geq 0$ which may depend on the geometry of the underlying manifold and independent of t as $t\to 0$, such that $\int_M P_\epsilon hHdV \leq B(n)$.

Now we claim that the first two terms on the right hand side of the last equation vanish as $t\to 0$, we can see this in the following argument. By integration by parts and the fact that $\nabla H = -H\nabla f$, we have

$$-t\int_M \langle \nabla f, \nabla h\rangle HdV = t\int_M \langle \nabla H, \nabla h\rangle dV = -t\int_M H\Delta h dV$$

is bounded since $|\Delta h|$ is bounded as stated earlier. Thus, the second term is bounded and goes to zero as $t\to 0$. We need a bound of Li-Yau type to obtain a bound for the first term $|\nabla f|^2$. See Lemma 3.2 below for the statement of the result ([5]) see also [6, Corollary 16.23] and Souplet and Zhang [21]). By this we have for the heat kernel in the present case that

$$t\int_M |\nabla f|^2 \leq 2\Big(B(n,\delta) + \frac{d^2(x,y)}{(4-\delta)t}\Big), \tag{3.4}$$

which is also clearly seen to be bounded from above as $t\to 0$ by the justification of asymptotic behaviour of the heat kernel. We have now reduced the analysis to

$$\lim_{t\to 0}\int_M P_\epsilon hHdV \leq \limsup_{t\to 0}\int_M \Big(f + \frac{nq}{2}\Big)hHdV, \tag{3.5}$$

where $q = \ln(\frac{4\pi}{\epsilon^2}) - \frac{\epsilon^2}{4\pi}$. For simplicity, we can choose ϵ such that $\epsilon^2 \to 4\pi$ as $t\to 0$ so that the whole problem is reduced to finding

$$\lim_{t\to 0}\int_M \Big(f - \frac{n}{2}\Big)hHdV. \tag{3.6}$$

Using the asymptotic behaviour of the heat kernel, i.e, $f\approx \frac{d^2}{4t}$ as $t\to 0$. Recall (Cf. [8, 19]) as $t\to 0$

$$H(x,y,t)\sim (4\pi t)^{-\frac{n}{2}}\exp\Big(\frac{d^2(x,y)}{4t}\Big)\sum_{j=o}^{\infty} u_j(x,y,t)t^j := w_k(x,y,t)$$

where $d^2(x,y)$ is the distance function and $w_k(x,y,t)$ satisfies uniformly for all $x,y\in M$

$$w_k(x,y,t) = O\Big(t^{k+1-\frac{n}{2}}\exp\Big(\frac{\delta d^2(x,y)}{4t}\Big)\Big)$$

and δ is just a number depending only on the geometry of (M, g). The function can be chosen such that $u_0(x, y, 0) = 1$. Though, the above asymptotic result does not require any curvature assumption, a result due to Cheeger and Yau [5] states that on manifold with nonnegative Ricci curvature (which is our case), the heat kernel satisfies

$$H(x, y, t) \geq (4\pi t)^{-\frac{n}{2}} \exp\Big(\frac{d^2(x, y)}{4t}\Big)$$

which implies

$$f(x, t) \leq \frac{d^2(x, y)}{4t}.$$

Therefore

$$\begin{aligned}\lim_{t\to 0}\int_M \Big(f - \frac{n}{2}\Big)hHdV &\leq \lim_{t\to 0}\int_M \Big(\frac{d^2(x, y)}{4t} - \frac{n}{2}\Big)h(y, t)H(x, y, t)dV(y)\\ &= \lim_{t\to 0}\int_M \Big(\frac{d^2(x, y)}{4t} - \frac{n}{2}\Big)\frac{e^{-d^2(x,y)/4t}}{(4\pi t)^{\frac{n}{2}}}H(y, t)dV(y).\end{aligned}$$

It is easy to see that for any $\delta > 0$, the integration of the above integrand in the domain $d(x, y) \leq \delta$ converges to zero exponentially fast. Therefore

$$\lim_{t\to 0}\int_M \Big(f - \frac{n}{2}\Big)hHdV \leq \lim_{t\to 0}\int_{d(x,y)\leq\delta} \Big(\frac{d^2(x, y)}{4t} - \frac{n}{2}\Big)\frac{e^{\frac{-d^2(x,y)}{4t}}}{(4\pi t)^{\frac{n}{2}}}h(y, t)dV(y). \tag{3.7}$$

Whenever δ is chosen sufficiently small, $d(x, y)$ is asymptotically sufficiently close to the Euclidean distance. By standard approximation, we have

$$\lim_{t\to 0}\int_M \Big(f - \frac{n}{2}\Big)hHdV \leq \lim_{t\to 0}\int_{d(x,y)\leq\delta} \Big(\frac{|x-y|^2}{4t} - \frac{n}{2}\Big)\frac{e^{\frac{-|x-y|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}h_p(y)dV(y), \tag{3.8}$$

where h_p is the pullback of $h(\cdot, 0)$ to the Euclidean space from the region $d(x, y) \leq \delta$.

Splitting the last integrand as in [13] we are left with

$$\begin{aligned}\lim_{t\to 0}\int_M \Big(f - \frac{n}{2}\Big)hHdV &\leq h_p(x)\lim_{t\to 0}\int_{\mathbb{R}^n} \Big(\frac{|x-y|^2}{4t} - \frac{n}{2}\Big)\frac{e^{\frac{-|x-y|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}dV(y)\\ &= h_p(\cdot)\lim_{t\to 0}\int_{\mathbb{R}^n} \Big(\frac{|y|^2}{4t}\frac{e^{-\frac{|y|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}\Big)dV(y) - \frac{n}{2}h_p(\cdot).\end{aligned}$$

The last equality is due to convolution properties of the heat kernel. Lastly we show that the RHS evaluates to 0. Recall, using standard Gauss integral, that

$$\begin{aligned}\int_{\mathbb{R}^n} |y|^2 e^{-\alpha|y|^2}d\mathbf{y} &= n\Big(\int_{-\infty}^{\infty} y^2 e^{-\alpha y^2}dy\Big)\Big(\int_{-\infty}^{\infty} e^{-\alpha y^2}dy\Big)^{n-1}\\ &= \frac{n}{2}\sqrt{\frac{\pi}{\alpha^3}}\cdot\Big(\sqrt{\frac{\pi}{\alpha}}\Big)^{n-1} = \frac{n}{2\alpha}\Big(\sqrt{\frac{\pi}{\alpha}}\Big)^n,\end{aligned}$$

so that we have

$$\begin{aligned}\int_{\mathbb{R}^n} \Big(\frac{|y|^2}{4t}\frac{e^{-\frac{|y|^2}{4t}}}{(4\pi t)^{\frac{n}{2}}}\Big)dV(y) &= \frac{1}{(4\pi t)^{\frac{n}{2}}}\cdot\frac{n}{4t}\Big(\int_{-\infty}^{\infty} y^2 e^{-\frac{1}{4t}y^2}\mathbf{dy}\Big)\Big(\int_{-\infty}^{\infty} \mathbf{e^{-\frac{1}{4t}y^2}dy}\Big)^{\mathbf{n-1}}\\ &= \frac{n}{2},\end{aligned}$$

by taking $\alpha = 1/4t$ in the above. We can then conclude the claim. □

Lemma 3.2. *On a complete Riemannian Manifold (M, g) with nonnegative Ricci curvature, the following estimate holds for the gradient of the heat kernel $H(x, y, t)$ and all $\delta > 0$,*

$$\frac{|\nabla H|^2}{H} \leq \frac{2H}{t}\Big(B(n,\delta) + \frac{d^2(x,y)}{(4-\delta)t}\Big) \tag{3.9}$$

for all x, y in M and $t > 0$.

4. Harnack Estimates for the Heat Kernel

The following differential Harnack quantity for linear heat equation on static manifold follows immediately as an application of the reuslts in the last subsection.

Corollary 4.1. *Let M be a closed manifold with curvature bounded from below by $Rc \geq 0$. Then we have*

$$\frac{\epsilon^2 t}{4\pi}\Big(2\Delta f - |\nabla f|^2\Big) + f + \frac{n}{2}\Big(\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{\epsilon^2}{2\pi}\Big) \leq 0, \tag{4.1}$$

where $f = -\ln(4\pi t)^{\frac{n}{2}} H$ and H is the positive minimal solution to the heat equation

$$\Big(\frac{\partial}{\partial t} - \Delta_x\Big)H(x, y, t) = 0.$$

Remark 4.2. *Note that the quantity $2\Delta f - |\nabla f|^2$ can be expressed as $\frac{|\nabla u|^2}{u^2} - \frac{2u_t}{u}$ in terms of u, which is similar to Li-Yau gradient estimate* [15] *on a manifold with nonnegative Ricci curvature, $\frac{u_t}{u} - \frac{|\nabla u|^2}{u^2} + \frac{n}{2t} \geq 0$. This is equivalent to the differential Harnack inequality $2t\Delta f \leq n$, where $f = -\ln(4\pi t)^{\frac{n}{2}} u$, which can be regarded as a generalized Laplacian comparison theorem in space for Heat kernel on M.*

However, we have from (4.1) that

$$\begin{aligned} f &\leq \frac{n}{2}\Big[\frac{\epsilon^2}{2\pi} - \ln\frac{4\pi}{\epsilon^2}\Big] - \frac{\epsilon^2 t}{4\pi}(2\Delta f - |\nabla f|^2) \\ &\leq \frac{n}{2}\Big[\frac{\epsilon^2}{2\pi} - \ln\frac{4\pi}{\epsilon^2}\Big] - \frac{\epsilon^2 n}{8\pi} = \frac{n}{2}\Big[\frac{\epsilon^2}{4\pi} - \ln\frac{4\pi}{\epsilon^2}\Big]. \end{aligned}$$

Define

$$Q(x,t) = \frac{\epsilon^2}{\pi} t f(x,t) \tag{4.2}$$

$$(\partial_t - \Delta)Q(x,t) = \frac{\epsilon^2}{\pi} f(x,t) + \frac{\epsilon^2}{\pi} t(\partial_t - \Delta) f \leq \frac{n\epsilon^2}{2\pi}\Big[\frac{\epsilon^2}{4\pi} - \ln\frac{4\pi}{\epsilon^2}\Big]. \tag{4.3}$$

Still as $\epsilon = 2\sqrt{\pi}$ we recover Ni's generalized Laplacian. From Corollary 4.1, we have the differential Harnack inequality as follows

$$\frac{\epsilon^2 t}{4\pi}\Big(2\Delta f - |\nabla f|^2\Big) + f + \frac{n}{2}\Big(\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{\epsilon^2}{2\pi}\Big) \leq 0.$$

Multiplying through by $-\frac{2\pi}{\epsilon^2 t}$, we have

$$\begin{aligned} -\Delta f + \frac{1}{2}|\nabla f|^2 - \frac{2\pi}{\epsilon^2 t} f - \frac{n\pi}{\epsilon^2 t}\Big(\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{\epsilon^2}{2\pi}\Big) \geq 0 \\ -\Delta f + \frac{1}{2}|\nabla f|^2 - \frac{2\pi}{\epsilon^2 t} f + \frac{n}{2t} - \frac{n\pi}{\epsilon^2 t}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) \geq 0. \end{aligned}$$

Recall that $(\partial_t - \Delta)H = 0$ implies $\Delta f = \partial_t f + |\nabla f|^2 + \frac{n}{2t}$, then we have

$$-\partial_t f - \frac{1}{2}|\nabla f|^2 - \frac{2\pi}{\epsilon^2 t} f \geq \frac{n\pi}{\epsilon^2 t} \ln\Big(\frac{4\pi}{\epsilon^2}\Big)$$

$$\begin{aligned}\partial_t f + \frac{1}{2}|\nabla f|^2 &\leq -\frac{2\pi}{\epsilon^2 t} f - \frac{n\pi}{\epsilon^2 t} \ln\Big(\frac{4\pi}{\epsilon^2}\Big)\\ &= -\frac{2\pi}{\epsilon^2 t}\Big(f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big)\Big).\end{aligned}$$

By the Young's inequality we have on the path $\gamma(t)$, ($\gamma(t) : [t_1, t_2] \to M$ is a minimizing geodesic connecting points x_1 and x_2 such that $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$.)

$$\begin{aligned}\frac{d}{dt} f(\gamma(t), t) &= \partial_t f + \langle \nabla f, \gamma'(t)\rangle\\ &\leq \partial_t f + \frac{1}{2}|\nabla f|^2 + \frac{1}{2}|\gamma'(t)|^2\\ &= \frac{1}{2}|\gamma'(t)|^2 - \frac{2\pi}{\epsilon^2 t}\Big(f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big)\Big)\end{aligned}$$

since we have from (4.1) that

$$f \leq \frac{n}{2}\Big(\frac{\epsilon^2}{4\pi} - \ln\frac{4\pi}{\epsilon^2}\Big),$$

inserting this quantity in the above inequality gives the following Harnack Estimates

$$\frac{d}{dt} f(\gamma(t), t) \leq \frac{1}{2}|\gamma'(t)|^2 - \frac{n}{4t}. \tag{4.4}$$

After the usual integration of (4.4) and exponentiation we have the following

Corollary 4.3. *With the notation and assumption of Corollary 4.1, we have the following differential Harnack estimates*

$$\frac{u(x_2, t_2)}{u(x_1, t_1)} \leq \Big(\frac{t_1}{t_2}\Big)^{\frac{n}{4}} \exp\Big[\frac{1}{2}\int_{t_1}^{t_2} |\gamma'(t)|^2 dt\Big]. \tag{4.5}$$

Remark 4.4. *If M is a closed manifold with nonnegative Ricci curvature and $u = (4\pi t)^{-\frac{n}{2}} e^{-f}$ is the heat kernel on M. Then $\mathcal{W}_\epsilon(f, t_0) \geq 0$ for some $t_0 > 0$, if and only if M is isometric to Euclidean space $\mathbb{R}^n$. Recall that we have obtained that $\frac{d}{dt}\mathcal{W}_\epsilon(f, t) \leq 0$ and $\mathcal{W}_\epsilon(f, t) \leq 0$ which in turn imply that we must have $\mathcal{W}_\epsilon(f, t) \equiv 0$ for $0 \leq t \leq t_0$. For instance, in the case $\epsilon = 2\sqrt{\pi}$, we have*

$$|f_{ij} - \frac{1}{2t} g_{ij}|^2 = 0 \quad \textit{and} \quad f_{ij} - \frac{1}{2t} g_{ij} = 0.$$

Taking the trace of the above yields

$$t\Delta f - \frac{n}{2} = 0. \tag{4.6}$$

Because $f(x, t) \approx \widetilde{f}(x, t) = \frac{d^2(p,x)}{4t}$ for t small, we have $\lim_{t\to 0} 4tf = d^2(p, x)$. Hence (4.6) implies that

$$\Delta d^2(p, x) = 2n \tag{4.7}$$

so that we can get for the area $A_p(r)$ of $\partial B_p(r)$ and the volume $V_p(r)$ of the ball $B_p(r)$, the following quotient

$$\frac{A_p(r)}{V_p(r)} = \frac{n}{r}.$$

This shows that $V_p(r)$ is exactly the same as the volume function of Euclidean balls.

This argument shows that the Li-Yau Harnack inequality, which is equivalent to $2t\Delta f - n \leq 0$ for $u = (4\pi t)^{-\frac{n}{2}}e^{-f}$ becomes an equality if and only if the manifold M with $Rc \geq 0$ is isometric to $\mathbb{R}^n$ and u is precisely the heat kernel. If $t = \frac{1}{2}$ and $M = \mathbb{R}^n$, the inequality $\mathcal{W}_\epsilon(f, t_0) \geq 0$ for $\epsilon^2 = 4\pi$, is equivalent to

$$\int_{\mathbb{R}^n} (\frac{1}{2}|\nabla f|^2 + f - n)(2\pi)^{-\frac{n}{2}}e^{-f}dV \geq 0 \tag{4.8}$$

for all f with the condition $\int_M (2\pi)^{-\frac{n}{2}}e^{-f}dV = 1$.

The above implies a sharp (Gross) logarithmic Sobolev inequality on $\mathbb{R}^n$. For details about logarithmic-Sobolev inequalities see for instance [9, 10, 22]. In the same vein our dual entropy also yields a version of logarithmic Sobolev inequality. (This will not be discussed here).

Remark 4.5. *Note that $f_{ij} - \frac{\sqrt{\pi}}{\epsilon t}g_{ij} \geq 0 \implies \Delta f \geq \frac{n\sqrt{\pi}}{\epsilon t}$ which in turns $\implies \frac{|\nabla u|^2}{u^2} - \frac{u_t}{u} \geq \frac{n\sqrt{\pi}}{\epsilon t}$.*

It turns out that $\mathcal{W}_\epsilon(f, t)$ being finite with u being the heat kernel, also has strong topological and geometric consequences. For instance, in the case M has nonnegative curvature, it implies that M has finite fundamental group. In fact one can show that M is of maximum volume growth if and only if the entropy $\mathcal{W}_\epsilon(f, t)$ is uniformly bounded for all $t \geq 0$, where u is the heat kernel. This analogy was originally discovered in [20] for ancient solution to the Ricci flow with bounded nonnegative curvature, where Perelman claims that ancient solution to the Ricci flow with nonnegative curvature operator is κ-noncollapsed if and only if the entropy is uniformly bounded for any fundamental solution to the conjugate heat equation.

Lastly, in this subsection we make some comment to show how sharp the dual entropy for the heat equation. Recall

$$\mathcal{W}_\epsilon(f, t) = \int_M \Big[\frac{\epsilon^2 t}{4\pi}|\nabla f|^2 + f + \frac{n}{2}\ln\Big(\frac{4\pi}{\epsilon^2}\Big) - \frac{n\epsilon^2}{4\pi}\Big]HdV \tag{4.9}$$

with $f = -\ln(4\pi t)^{\frac{n}{2}}H$ and $\int_M HdV = 1$ and $0 < \epsilon^2 \leq 4\pi$.

Rewrite $\mathcal{W}_\epsilon(f, t)$ as

(4.10)

$$\mathcal{W}_\epsilon(f, t) = \frac{\epsilon^2}{4\pi}\int_M (t|\nabla f|^2 + f - n)HdV + (1 - \frac{\epsilon^2}{4\pi})\int_M fHdV + \frac{n}{2}\ln\frac{4\pi}{\epsilon^2}\int_M HdV.$$

Hence, we have the following

Proposition 4.6. *For $0 < \epsilon^2 \leq 4\pi$, $f = -\ln(4\pi t)^{\frac{n}{2}}H$ with $\int_M HdV = 1$, we have the following monotonicity formula on a manifold with nonnegative Ricci curvature;*

$$\frac{d}{dt}\mathcal{W}_\epsilon(f, t) \leq -\frac{\epsilon^2}{2\pi}t\int_M \Big(|f_{ij} - \frac{1}{2t}g_{ij}|^2 + R_{ij}f_if_j\Big)HdV. \tag{4.11}$$

Proof. The proof follows from a straight forward computation on $\mathcal{W}_\epsilon$ using the idea of [17, Theorem 1.1].

$$\frac{d}{dt}\mathcal{W}_\epsilon(f, t) = \frac{\epsilon^2}{4\pi}\frac{\partial}{\partial t}\Big(\int_M t|\nabla f|^2 + f - n\Big)HdV + (1 - \frac{\epsilon^2}{4\pi})\frac{\partial}{\partial t}\Big(\int_M fHdV\Big). \tag{4.12}$$

We are only left to justify the non-positivity of $\frac{\partial}{\partial t}\Big(\int_M fHdV\Big)$. Then we have by integration by parts

$$\begin{aligned}\frac{\partial}{\partial t}\Big(\int_M fHdV\Big) &= \int_M\Big(\frac{\partial}{\partial t}f\ H + f\frac{\partial}{\partial t}H\Big)dV\\ &= \int_M\Big(\frac{\partial}{\partial t}f\ H + f\Delta H + f(\frac{\partial}{\partial t}-\Delta)H\Big)dV\\ &= \int_M\Big(\frac{\partial}{\partial t}+\Delta\Big)fHdV\\ &= \int_M(2\Delta f - |\nabla f|^2 - \frac{n}{2t})HdV,\end{aligned}$$

where we have used the facts that $(\frac{\partial}{\partial t}-\Delta)H=0$ and $\frac{\partial}{\partial t}f = \Delta f - |\nabla f|^2 - \frac{n}{2t}$. Taking $f = -\ln((4\pi t)^{\frac{n}{2}}H$, then the integrand in the RHS of the last equality becomes

$$2\Delta f - |\nabla f|^2 - \frac{n}{2t} = \frac{|\nabla H|^2}{H^2} - \frac{2\Delta H}{H} - \frac{n}{2t} \le 0, \tag{4.13}$$

which is precisely the Li-Yau Harnack inequality since we are on nonnegative Ricci curvature manifold. Hence our claim. □

5. LYH gradient estimates for positive solutions

In the next we give useful estimates found by Hamilton [11]. He was inspired by the results of Li and Yau [15], hence the estimates are popularly referred to as Li-Yau-Hamilton (LYH) estimates. We state and prove the result for bounded solutions on a closed manifold. As an application of this LYH-type estimates we can obtain a sharp upper bound on the heat kernel.

Theorem 5.1. *Let (M,g) be a closed Riemannian manifold with $R_{ij} \ge -kg_{ij}$, where $k \ge 0$. Suppose u is a positive solution to the heat equation with $u \le M < \infty$. Then*

$$t\frac{|\nabla u|^2}{u^2} \le (1+2kt)\log\Big(\frac{M}{u}\Big). \tag{5.1}$$

Proof. Let $f = \log u$ so that $|\nabla f|^2 = |\nabla \log u|^2 = \frac{|\nabla u|^2}{u^2}$ and $\Big(\frac{\partial}{\partial t} - \Delta\Big)f = |\nabla f|^2$. Define a heat type operator

$$\mathcal{L} := \Big(\frac{\partial}{\partial t} - \Delta - \langle\nabla f, \nabla\cdot\rangle\Big).$$

The idea to this proof is to apply the heat-type operator $\mathcal{L}$ on the quantity

$$t\frac{|\nabla u|^2}{u^2} - (1+2kt)\log\Big(\frac{M}{u}\Big)$$

and then use weak maximum principle. Recall from the calculation in Lemma 2.1 and the Bochner identity that

$$\frac{\partial}{\partial t}|\nabla f|^2 = 2f_i f_{ti} \quad \& \quad \Delta|\nabla f|^2 = 2|f_{ij}|^2 + 2f_j f_{jji} + 2R_{ij}f_if_j.$$

Hence

$$\Big(\frac{\partial}{\partial t} - \Delta\Big)|\nabla f|^2 = -2|f_{ij}|^2 - 2R_{ij}f_if_j + 2\langle\nabla f, \nabla|\nabla f|^2\rangle$$

and

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)(t|\nabla f|^2) = |\nabla f|^2 + t(\frac{\partial}{\partial t}-\Delta)|\nabla f|^2$$
$$= |\nabla f|^2 - 2t|f_{ij}|^2 - 2tR_{ij}f_if_j + 2t\langle\nabla f, \nabla|\nabla f|^2\rangle.$$

Using the condition that $R_{ij} \geq -kg_{ij}$, we have

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)(t|\nabla f|^2) = (1+2kt)|\nabla f|^2 + 2\langle\nabla f, \nabla(t|\nabla f|^2)\rangle. \tag{5.2}$$

On the other hand

$$\mathcal{L}\Big((1+2kt)\log\Big(\frac{M}{u}\Big)\Big) = 2k\log\Big(\frac{M}{u}\Big) + (1+2kt)\mathcal{L}\Big(\log\Big(\frac{M}{u}\Big)\Big)$$
$$= 2k\log\Big(\frac{M}{u}\Big) + \Big(\frac{\partial}{\partial t}-\Delta\Big)\log\Big(\frac{M}{u}\Big)$$
$$- 2(1+2kt)\langle f, \nabla\log\Big(\frac{M}{u}\Big)\rangle.$$

Computing

$$\Big(\frac{\partial}{\partial t}-\Delta\Big)\log\Big(\frac{M}{u}\Big) = \Big(\frac{\partial}{\partial t}-\Delta\Big)\log M - \Big(\frac{\partial}{\partial t}-\Delta\Big)\log u$$
$$= -|\nabla f|^2$$
$$= -2\langle\nabla f, \nabla\log u\Big)\rangle + |\nabla f|^2$$
$$= 2\langle\nabla f, \nabla\log\Big(\frac{M}{u}\Big)\rangle + |\nabla f|^2.$$

Then

$$\mathcal{L}\Big((1+2kt)\log\Big(\frac{M}{u}\Big)\Big) = 2k\log\Big(\frac{M}{u}\Big) + 2\langle\nabla f, \nabla\log\Big(\frac{M}{u}\Big)\rangle + |\nabla f|^2$$
$$- 2(1+2kt)\langle f, \nabla\log\Big(\frac{M}{u}\Big)\rangle$$

$$= 2k\log\Big(\frac{M}{u}\Big) + 2\langle\nabla f, \nabla((1+2kt)\log\Big(\frac{M}{u}\Big)\rangle \tag{5.3}$$
$$+ (1+2kt)|\nabla f|^2. \tag{5.4}$$

Combining the expressions in (5.2) and (5.3) we arrive at

$$\mathcal{L}\Big(t|\nabla f|^2 - (1+2kt)\log\Big(\frac{A}{u}\Big)\Big) \leq -2k\log\Big(\frac{M}{u}\Big) \tag{5.5}$$

since $k \geq 0$ and $0 \leq \log\frac{M}{u} < \infty$. Note that at $t = 0$,

$$-\log\Big(\frac{A}{u}\Big) \leq 0 \quad and \quad t|\nabla f|^2 - (1+2kt)\log\Big(\frac{A}{u}\Big) \leq 0.$$

Hence, by the weak maximum principle we have

$$t|\nabla f|^2 - (1+2kt)\log\Big(\frac{A}{u}\Big) \leq 0$$

for all $t \geq 0$. This completes the proof. □

The above result can be extended to the case of complete noncompact manifold, although, a little more effort will be required. The idea here is to use ϵ-regularization method by supposing that the solution $u \geq \epsilon$, replacing u by $u_\epsilon = u + \epsilon$ for a sufficiently small $\epsilon > 0$ and letting ϵ go to zero after the analysis for u_ϵ is completed. An application of this result shows we can bound the maximum of a positive solution by its integral (see [11]). Furthermore, the estimate yields sharp lower and upper bounds for the fundamental solution, See [6].

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