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Fixed Point Theorems on ℵ-Extended Fuzzy Bipolar b-Metric Spaces

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Abstract. In this paper, we introduce the context of ℵ-Extended fuzzy bipolar b-metric space and prove fixed point theorem. Some of the well-known results in the literature are expanded and generalized by our research. Additionally, we presented applications to integral equation and fractional differential equation.

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1. Introduction

The concept of continuous triangular norm was first developed by Schweizer and Sklar [1] in 1960. Following that, Zadeh [2] presents the fuzzy set theory in 1965. Using the concept of fuzziness and the continuous t-norm, Kramosil and Michalek [3] created the fuzzy metric space in 1975.The fuzzy approach to distance is predicated on the idea that the distance—which we must estimate or determine—between any two locations need not necessarily be represented by a precise number,

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but rather is a fuzzy idea. George and Veeramani [4] revised the fuzzy metric spaces definition in 1994. The authors of Karamosil and Michalek [3], Grabeic [5] extends the well-known fixed point theorem of Banach to fuzzy metric spaces. Gregori and Sapena [6] then extended the fuzzy banach contraction theorem, in the sense of George and Veeramani's [4], to fuzzy metric space. Mutlu and Gurdal [7] generalized bipolar metric spaces, which offer a new framework for calculating the distance between objects in two different sets.The fuzzy bipolar metric space was established by Bartwal, Dimri, and Prasad [8]. Sezen [9] demonstrated controlled fuzzy metric spaces and some associated fixed point outcomes recently. For more about fuzzy metric space see ([10–14]).

2. Preliminaries

We offer the following fundamental definitions, lemmas, and propositions. Here seq means sequence and bi-seq means bisequence and $\mathcal{UFP}$ means unique fixed point and $\mathcal{FPT}$ means Fixed point theorem.

Definition 2.1. *[4] Let Δ be a nonempty set. An triple $(\Delta, \breve{w}, *)$ is said to be a fuzzy metric space if $\breve{w}$ is a fuzzy set on $\Delta^2 \times (0, \infty)$ and $*$ is a continuous $\breve{a}$-norm satisfies for all $\breve{e}, \breve{r}, \breve{q} \in \Delta$ and $\breve{a}, \varsigma > 0$;*

(1) $\breve{w}(\breve{e}, \breve{r}, \breve{a}) > 0$;
(2) $\breve{w}(\breve{e}, \breve{r}, \breve{a}) = 1$ *iff* $\breve{e} = \breve{r}$;
(3) $\breve{w}(\breve{e}, \breve{r}, \breve{a}) = \breve{w}(\breve{r}, \breve{e}, \breve{a})$;
(4) $\breve{w}(\breve{e}, \breve{q}, \breve{a} + \varsigma) \geq \breve{w}(\breve{e}, \breve{r}, \breve{a}) * \breve{w}(\breve{r}, \breve{q}, \varsigma)$;
(5) $\breve{w}(\breve{e}, \breve{r}, .) : (0, \infty) \longrightarrow (0, 1]$ *is continuous.*

Definition 2.2. *[8] Let Ψ and Δ be two non-void sets. A quadruple $(\Psi, \Delta, \breve{w}_{\breve{b}}, *)$ is called as fuzzy bipolar metric space, where $*$ is continuous $\breve{a}$-norm and $\breve{w}_{\breve{b}}$ is a fuzzy set on $\Psi \times \Delta \times (0, \infty)$, satisfies for all $\breve{a}, \varsigma, \breve{u} > 0$:*

(1) $\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}) > 0 \ \forall \ (\breve{e}, \breve{r}) \in \Psi \times \Delta$;
(2) $\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}) = 1$ *iff* $\breve{e} = \breve{r} \ \forall \ \breve{e} \in \Psi$ *and* $\breve{r} \in \breve{e}$;
(3) $\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}) = \breve{w}_{\breve{b}}(\breve{r}, \breve{e}, \breve{a}) \ \forall \ \breve{e}, \breve{r} \in \Psi \cap \Delta$;
(4) $\breve{w}_{\breve{b}}(\breve{e}_1, \breve{r}_2, \breve{a} + \varsigma + \breve{u}) \geq \breve{w}_{\breve{b}}(\breve{e}_1, \breve{r}_1, \breve{a}) * \breve{w}_{\breve{b}}(\breve{e}_2, \breve{r}_1, \varsigma) * \breve{w}_{\breve{b}}(\breve{e}_2, \breve{r}_2, \breve{u})$ *for all* $\breve{e}_1, \breve{e}_2 \in \Psi$ *and* $\breve{r}_1, \breve{r}_2 \in \Delta$;
(5) $\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, .) : [0, \infty) \longrightarrow [0, 1]$ *is left continuous;*
(6) $\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, .)$ *is non-decreasing for all* $\breve{e} \in \Psi$ *and* $\breve{r} \in \Delta$.

Definition 2.3. *Let Ψ and Δ be two non-void sets. Let $\breve{v}, \aleph, \wp : \Psi \times \Delta \to [1, \infty)$ be three distinct functions.. A quadruple $(\Psi, \Delta, \breve{w}_{\breve{b}}, *)$ is called a $\aleph$-Extended fuzzy bipolar b-metric space($\aleph EF_{\mathsf{b}}BMS$), where $*$ is continuous $\breve{a}$-norm and $\breve{w}_{\breve{b}}$ is a fuzzy set on $\Psi \times \Delta \times (0, \infty)$, satisfies for all $\breve{a}, \varsigma, \breve{u} > 0$:*

(1) $\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}) > 0 \ \forall \ (\breve{e}, \breve{r}) \in \Psi \times \Delta$;
(2) $\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}) = 1$ *iff* $\breve{e} = \breve{r} \ \forall \ \breve{e} \in \Psi$ *and* $\breve{r} \in \Delta$;
(3) $\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}) = \breve{w}_{\breve{b}}(\breve{r}, \breve{e}, \breve{a}) \ \forall \ \breve{e}, \breve{r} \in \Psi \cap \Delta$;

(4) $\check{w}_{\check{b}}(\check{e}_1, \check{r}_2, \check{v}(\check{e}_1, \check{r}_2)\check{a} + \aleph(\check{e}_1, \check{r}_2)\varsigma + \wp(\check{e}_1, \check{r}_2)\check{u}) \geq \check{w}_{\check{b}}(\check{e}_1, \check{r}_1, \check{a})$ $* \check{w}_{\check{b}}(\check{e}_2, \check{r}_1, \varsigma) * \check{w}_{\check{b}}(\check{e}_2, \check{r}_2, \check{u})$ *for all* $\check{e}_1, \check{e}_2 \in \Psi$ *and* $\check{r}_1, \check{r}_2 \in \Delta$;

(5) $\check{w}_{\check{b}}(\check{e}, \check{r}, .) : [0, \infty) \longrightarrow [0, 1]$ *is left continuous;*

(6) $\check{w}_{\check{b}}(\check{e}, \check{r}, .)$ *is non-decreasing for all* $\check{e} \in \Psi$ *and* $\check{r} \in \Delta$.

Remark 2.1. *Taking* $\check{v}(\check{e}_1, \check{r}_2) = \aleph((\check{e}_1, \check{r}_2) = \wp(\check{e}_1, \check{r}_2) = \mathsf{b}$, *for all* $\mathsf{b} \geq 1$, *then we derive that fuzzy bipolar* b*-metric space [15].*

Example 2.1. *Let* $\Psi = \{1, 2, 3, 4\}$, $\Delta = \{2, 4, 5, 6\}$ *and* $\check{v}, \aleph, \wp : \Psi \times \Delta \to [1, \infty)$ *be three mapping defined as* $\check{v}(\check{e}, \check{r}) = \check{e} + \check{r} + 1, \aleph(\check{e}, \check{r}) = \check{e}^2 + \check{r} + 1$ *and* $\wp(\check{e}, \check{r}) = \check{e}^2 + \check{r} - 1$. *Define* $\check{w}_{\check{b}} : \Psi \times \Delta \times (0, \infty) \to [0, 1]$ *defined by*

$$\check{w}_{\check{b}}(\check{e}, \check{r}, \check{a}) = \frac{\min\{\check{e}, \check{r}\} + \check{a}}{\max\{\check{e}, \check{r}\} + \check{a}},$$

for all $\check{e} \in \Psi$ *and* $\check{r} \in \Delta$. *Then* $(\Psi, \Delta, \check{w}_{\check{b}}, \star)$ *is a* $\aleph EF_{\mathsf{b}}BMS$ *with the continuous* $\check{a}$*-norm* $\star$ *such that* $\sigma * \check{b} = \sigma\check{b}$. *Conditions 1 to 3 and 5, 6 be easily verify we only prove 4. Let* $\check{e}_1 = 1, \check{r}_2 = 4, \check{r}_1 = 2$ *and* $\check{e}_2 = 3$. *Then*

$$\begin{aligned}\check{w}_{\check{b}}(1, 4, \check{v}(1, 4)\check{a} + \aleph(1, 4)\varsigma + \wp(1, 4)\check{u}) &= \frac{\min\{1, 4\} + 6\check{a} + 6\varsigma + 4\check{u}}{\max\{1, 4\} + 6\check{a} + 6\varsigma + 4\check{u}} \\ &\geq \frac{6 + 2\varsigma + 6\check{a} + 3\check{a}\varsigma + 2\check{u} + 2\check{a}\check{u} + \check{a}\varsigma\check{u}}{24 + 8\varsigma + 12\check{a} + 4\check{a}\varsigma + 6\check{u} + 2\varsigma\check{u} + 3\varsigma\check{u} + \check{a}\varsigma\check{u}} \\ &\geq \check{w}_{\check{b}}(1, 2, \check{a}) \star \check{w}_{\check{b}}(3, 2, \varsigma) \star \check{w}_{\check{b}}(3, 4, \check{u}).\end{aligned}$$

Similarly, the remaining conditions can be proved. Hence $(\Psi, \Delta, \check{w}_{\check{b}}, \star)$ *is a* $\aleph EF_{\mathsf{b}}BMS$.

Example 2.2. *We replace product* $\check{a}$*-norm in Example 2.1 by minimum* $\check{a}$*-norm, then* $(\Psi, \Delta, \check{w}_{\check{b}}, \star)$ *is not a* $\aleph EF_{\mathsf{b}}BMS$. *For instance, let* $\check{e}_1 = 1, \check{r}_2 = 4, \check{r}_1 = 2$, $\check{e}_2 = 3$ *and* $\check{a} = 0.02, \varsigma = 0.03, \check{u} = 0.04$ *with* $\check{v}, \aleph, \wp : \Psi \times \Delta \to [1, \infty)$ *be three mapping defined as* $\check{v}(\check{e}, \check{r}) = \check{e} + \check{r} + 1, \aleph(\check{e}, \check{r}) = \check{e}^2 + \check{r} + 1$ *and* $\wp(\check{e}, \check{r}) = \check{e}^2 + \check{r} - 1$, *then*

$$\check{w}_{\check{b}}(1, 4, 0.12 + 0.18 + 0.16) = \frac{1 + 0.46}{4 + 0.46} = 0.32735,$$

and

$$\check{w}_{\check{b}}(1, 2, 0.02) = \frac{1 + 0.02}{2 + 0.02} = 1.009,$$

$$\check{w}_{\check{b}}(3, 2, 0.03) = \frac{2 + 0.03}{3 + 0.03} = 0.6699,$$

$$\check{w}_{\check{b}}(3, 4, 0.04) = \frac{3 + 0.04}{4 + 0.04} = 0.7524.$$

Clearly,

$$\begin{aligned}\check{w}_{\check{b}}(1, 4, 0.12 + 0.18 + 0.16) \ngeq \check{w}_{\check{b}}(1, 2, 0.02) \star \check{w}_{\check{b}}(3, 2, 0.03) \\ \star \check{w}_{\check{b}}(3, 4, 0.04).\end{aligned}$$

Hence $(\Psi, \Delta, \check{w}_{\check{b}}, \star)$ *is not a* $\aleph EF_{\mathsf{b}}BMS$ *with minimum* $\check{a}$*-norm.*

Definition 2.4. *Let $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$ be a $\aleph EF_{\mathsf{b}}BMS$. The points belong to $\Psi, \varDelta$ and $\Psi \cap \varDelta$ is said to be Left, Right and Central points respectively.*

Lemma 2.1. *Let $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$ be a $\aleph EF_{\mathsf{b}}BMS$ implies that*

$$\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \upsilon\breve{a}) \geq \breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a})$$

for $\breve{e} \in \Psi, \breve{r} \in \varDelta$ and $\upsilon \in (0, 1)$. Then $\breve{e} = \breve{r}$.

Proof. We know

$$\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \upsilon\breve{a}) \geq \breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}) \text{ for } \breve{a} > 0. \tag{2.1}$$

Since $\upsilon\breve{a} < \breve{a}$ for all $\breve{a} > 0$ and $\upsilon \in (0, 1)$, by 6 we have

$$\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \upsilon\breve{a}) \leq \breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}). \tag{2.2}$$

From (2.1) and (2.2) and definition of $\aleph$EF_{b}BMS, we get $\breve{e} = \breve{r}$. □

Definition 2.5. *Let $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$ be a $\aleph EF_{\mathsf{b}}BMS$. A seq $\{\breve{e}_\ell\} \in \Psi$ converges to a right point $\breve{r}$ iff for every $\epsilon > 0$ and $\breve{a} > 0$, we can find that $\ell_0 \in \mathbb{N}$ implies that $\breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}, \breve{a}) \to 1$ as $\ell \to \infty$ $\forall$ $\ell \geq \ell_0$. Similarly, a right seq $\{\breve{r}_\ell\}$ converges to a left point $\breve{e}$ iff for every $\epsilon > 0$ and $\breve{a} > 0$, we can find that $\ell_0 \in \mathbb{N}$ implies that $\breve{w}_{\breve{b}}(\breve{e}, \breve{r}_\ell, \breve{a}) \to 1$ as $\ell \to \infty$ $\forall$ $\ell \geq \ell_0$.*

Definition 2.6. *Let $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$ be a $\aleph EF_{\mathsf{b}}BMS$ then:*

(1) *The $(\breve{e}_\ell, \breve{r}_\ell) \in \Psi \times \varDelta$ are reffered as bi-seq on $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$.*
(2) *Suppose $\breve{e}_\ell$ and $\breve{r}_\ell$ are converges, the seq $(\breve{e}_\ell, \breve{r}_\ell) \in \Psi \times \varDelta$ are called as convergent seq. Suppose $\breve{e}_\ell$ and $\breve{r}_\ell$ are converges to some center point, bi-seq $(\breve{e}_\ell, \breve{r}_\ell)$ is called as biconvergent seq.*
(3) *A bi-seq $(\breve{e}_\ell, \breve{r}_\ell)$ on $\aleph EF_{\mathsf{b}}BMS$ $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$ are called as Cauchy bi-seq if for each $\epsilon > 0$, we can find that $\ell_0 \in \mathbb{N}$ implies that $\breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\varrho, \breve{a}) \to 1$ as $\ell, \varrho \to \infty$ for all $\breve{a} > 0$, $\ell, \varrho \geq \ell_0(\ell, \varrho \in \mathbb{N})$.*

Definition 2.7. *The fuzzy $\aleph EF_{\mathsf{b}}BMS$ $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$ is called a complete if every Cauchy bi-seq in $\Psi \times \varDelta$ is convergent in it.*

Proposition 2.1. *In a $\aleph EF_{\mathsf{b}}BMS$, every convergent Cauchy bi-seq is biconvergent.*

Proof. Let $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$ be a $\aleph$EF_{b}BMS and a bi-seq $(\breve{e}_\ell, \breve{r}_\ell) \in \Psi \times \varDelta$ implies that $\{\breve{e}_\ell\} \to \breve{r} \in \varDelta$ and $\{\breve{r}_\ell\} \to \breve{e} \in \Psi$. Since $(\breve{e}_\ell, \breve{r}_\ell)$ is convergent Cauchy bi-seq, so for all $\breve{a} > 0$ we have

$$\breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\varrho, \breve{a}) \to 1 \text{ as } \ell \to \infty,$$

which indicates

$$\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}) = 1 \text{ for all } \breve{a} > 0.$$

Hence by 2 bi-seq $(\breve{e}_\ell, \breve{r}_\ell)$ is biconvergent. □

Proposition 2.2. *In a $\aleph EF_{\mathsf{b}}BMS$, every biconvergent bi-seq is a Cauchy bi-seq.*

Proof. Let $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$ be a $\aleph$EF_{b}BMS and bi-seq $(\breve{e}_{\ell}, \breve{r}_{\varrho}) \in \Psi \times \varDelta$ converges to a point $\breve{e}_0 \in \Psi \cap \varDelta$ for all $\ell, \varrho \in \mathbb{N}$ and $\breve{a} > 0$, by 4, we have

$$\breve{w}_{\breve{b}}(\breve{e}_{\ell}, \breve{r}_{\varrho}, \breve{a}) \geq \breve{w}_{\breve{b}}(\breve{e}_{\ell}, \breve{e}_0, \frac{\breve{a}}{3\breve{v}(\breve{e}_{\ell}, \breve{e}_0)}) * \breve{w}_{\breve{b}}(\breve{e}_0, \breve{e}_0, \frac{\breve{a}}{3\breve{v}(\breve{e}_0, \breve{e}_0)})$$
$$* \breve{w}_{\breve{b}}(\breve{e}_0, \breve{r}_{\varrho}, \frac{\breve{a}}{3\breve{v}(\breve{e}_0, \breve{r}_{\varrho})})$$

as $\ell, \varrho \to \infty$, we get

$$\breve{w}_{\breve{b}}(\breve{e}_{\ell}, \breve{r}_{\varrho}, \breve{a}) \geq 1 \text{ for all } \breve{a} > 0.$$

Which indicates $\breve{w}_{\breve{b}}(\breve{e}_{\ell}, \breve{r}_{\varrho}, \breve{a}) \to 1 \; \forall \; \breve{a} > 0$. Hence, $(\breve{e}_{\ell}, \breve{r}_{\ell})$ is a Cauchy bi-seq. □

Lemma 2.2. *Let $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$ be a $\aleph$EF_{b}BMS and $\chi \in \Psi \cap \varDelta$ is a limit of a seq then it is a unique limit of the seq.*

Proof. consider $\{\breve{e}_{\ell}\} \in \Psi$ be a seq. Assume that $\{\breve{e}_{\ell}\} \to \breve{r} \in \varDelta$ and $\{\breve{e}_{\ell}\} \to \chi \in \Psi \cap \varDelta$, then $\forall \; \breve{a}, \varsigma, \breve{u} > 0$, defined as

$$\breve{w}_{\breve{b}}(\chi, \breve{r}, \breve{a} + \varsigma + \breve{u}) \geq \breve{w}_{\breve{b}}(\chi, \chi, \frac{\breve{a}}{\breve{v}(\chi, \breve{r})}) * \breve{w}_{\breve{b}}(\breve{e}_{\ell}, \chi, \frac{\varsigma}{\aleph(\chi, \breve{r})}) * \breve{w}_{\breve{b}}(\breve{e}_{\ell}, \breve{r}, \frac{\breve{u}}{\wp(\chi, \breve{r})})$$

as $\ell \to \infty$, we get

$$\breve{w}_{\breve{b}}(\chi, \breve{r}, \breve{a} + \varsigma + \breve{u}) \geq 1,$$

which indicates $\chi = \breve{r}$, i.e., seq $\{\breve{e}_{\ell}\}$ have a unique limit. □

Definition 2.8. *A point $\breve{e} \in \Psi \cap \varDelta$ is said to be $\mathcal{FP}$ for the mapping Φ on $\breve{e} \in \Psi \cap \varDelta$ such that $\breve{e} = \Phi\breve{e}$.*

Motivated by Sezen [9], we prove $\mathcal{FPT}$ on $\aleph$EF_{b}BMS with an application.

3. Main Result

We demonstrate the extension of several well-known $\mathcal{FPT}$ to $\aleph$EF_{b}BMS in this section.

Theorem 3.1. *Let $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$ be a complete $\aleph$EF_{b}BMS with three functions $\breve{v}, \aleph, \wp : \Psi \times \varDelta \to [1, \infty)$ such that*

$$\lim_{\breve{a} \to \infty} \breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}) = 1 \; \forall \; \breve{e} \in \Psi, \breve{r} \in \varDelta. \tag{3.1}$$

Let $\Phi : \Psi \cup \varDelta \to \Psi \cup \varDelta$ be map as follows:

(i) $\Phi(\Psi) \subseteq \Psi$ *and* $\Phi(\varDelta) \subseteq \varDelta$;

(ii) $\breve{w}_{\breve{b}}(\Phi(\breve{e}), \Phi(\breve{r}), \upsilon\breve{a}) \geq \breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}) \; \forall \; \breve{e} \in \Psi, \breve{r} \in \varDelta$ *and* $\breve{a} > 0$, *where* $\upsilon \in (0, 1)$.

Also, assume that for every $\breve{e} \in \Psi$, we deduce

$$\lim_{\ell \to \infty} \breve{v}(\breve{e}_{\ell}, \breve{r}) \text{ and } \lim_{\ell \to \infty} \breve{v}(\breve{r}, \breve{e}_{\ell}),$$

Then Φ has a $\mathcal{UFP}$.

Proof. Fix $\check{e}_0 \in \Psi$ and $\check{r}_0 \in \Delta$ and assume that $\Phi(\check{e}_\ell) = \check{e}_{\mathfrak{n}+1}$ and $\Phi(\check{r}_\ell) = \check{r}_{\mathfrak{n}+1}$ for all $\ell \in \mathbb{N} \cup \{0\}$. Then we get $(\check{e}_\ell, \check{r}_\ell)$ as a bi-seq on $\aleph$EF$_{\mathfrak{b}}$BMS $(\Psi, \Delta, \breve{w}_{\check{b}}, *)$. Now, we have

$$\breve{w}_{\check{b}}(\check{e}_1, \check{r}_1, \check{a}) = \breve{w}_{\check{b}}(\Phi(\check{e}_0), \Phi(\check{r}_0), \check{a}) \geq \breve{w}_{\check{b}}(\check{e}_0, \check{r}_0, \frac{\check{a}}{\upsilon})$$

for all $\check{a} > 0$ and $\ell \in \mathbb{N}$. Using induction, we get

$$\breve{w}_{\check{b}}(\check{e}_\ell, \check{r}_\ell, \check{a}) = \breve{w}_{\check{b}}(\Phi(\check{e}_{\mathfrak{n}-g}), \Phi(\check{r}_{\mathfrak{n}-g}), \check{a}) \geq \breve{w}_{\check{b}}(\check{e}_0, \check{r}_0, \frac{\check{a}}{\upsilon^\ell}) \tag{3.2}$$

and

$$\breve{w}_{\check{b}}(\check{e}_{\ell+1}, \check{r}_\ell, \check{a}) = \breve{w}_{\check{b}}(\Phi(\check{e}_\ell), \Phi(\check{r}_{\mathfrak{n}-g}), \check{a}) \geq \breve{w}_{\check{b}}(\check{e}_1, \check{r}_0, \frac{\check{a}}{\upsilon^\ell}) \tag{3.3}$$

$\forall\ \check{a} > 0$ and $\ell \in \mathbb{N}$. Consider $\ell < \varrho$, for $\ell, \varrho \in \mathbb{N}$. Then

$$\begin{aligned}
\breve{w}_{\check{b}}(\check{e}_\ell, \check{r}_\varrho, \check{a}) \geq & \breve{w}_{\check{b}}(\check{e}_\ell, \check{r}_\ell, \frac{\check{a}}{3\breve{v}(\check{e}_\ell, \check{r}_\varrho, \check{a})}) \star \breve{w}_{\check{b}}(\check{e}_{\ell+1}, \check{r}_\ell, \frac{\check{a}}{3\aleph(\check{e}_\ell, \check{r}_\varrho, \check{a})}) \\
& \star \breve{w}_{\check{b}}(\check{e}_{\ell+1}, \check{r}_\varrho, \frac{\check{a}}{3\wp(\check{e}_\ell, \check{r}_\varrho, \check{a})}) \\
& \geq \breve{w}_{\check{b}}(\check{e}_\ell, \check{r}_\ell, \frac{\check{a}}{3\breve{v}(\check{e}_\ell, \check{r}_\varrho, \check{a})}) \star \breve{w}_{\check{b}}(\check{e}_{\ell+1}, \check{r}_\ell, \frac{\check{a}}{3\aleph(\check{e}_\ell, \check{r}_\varrho, \check{a})}) \\
& \star \breve{w}_{\check{b}}(\check{e}_{\ell+1}, \check{r}_{\ell+1}, \frac{\check{a}}{3^2\wp(\check{e}_\ell, \check{r}_\varrho, \check{a})\breve{v}(\check{e}_{\ell+1}, \check{r}_\varrho)}) \\
& \star \breve{w}_{\check{b}}(\check{e}_{\ell+2}, \check{r}_{\ell+1}, \frac{\check{a}}{3^2\wp(\check{e}_\ell, \check{r}_\varrho)\aleph(\check{e}_{\ell+1}, \check{r}_\varrho)}) \\
& \star \breve{w}_{\check{b}}(\check{e}_{\ell+2}, \check{r}_\varrho, \frac{\check{a}}{3^2\wp(\check{e}_\ell, \check{r}_\varrho)\wp(\check{e}_{\ell+1}, \check{r}_\varrho)}) \\
& \geq \breve{w}_{\check{b}}(\check{e}_\ell, \check{r}_\ell, \frac{\check{a}}{3\breve{v}(\check{e}_\ell, \check{r}_\varrho)}) \star \breve{w}_{\check{b}}(\check{e}_{\ell+1}, \check{r}_\ell, \frac{\check{a}}{3\aleph(\check{e}_\ell, \check{r}_\varrho)}) \\
& \star \breve{w}_{\check{b}}(\check{e}_{\ell+1}, \check{r}_{\ell+1}, \frac{\check{a}}{3^2\wp(\check{e}_\ell, \check{r}_\varrho)\breve{v}(\check{e}_{\ell+1}, \check{r}_\varrho)}) \\
& \star \breve{w}_{\check{b}}(\check{e}_{\ell+2}, \check{r}_{\ell+1}, \frac{\check{a}}{3^2\wp(\check{e}_\ell, \check{r}_\varrho)\aleph(\check{e}_{\ell+1}, \check{r}_\varrho)}) \\
& \star \breve{w}_{\check{b}}(\check{e}_{\ell+2}, \check{r}_{\ell+2}, \frac{\check{a}}{3^3\wp(\check{e}_\ell, \check{r}_\varrho)\wp(\check{e}_{\ell+1}, \check{r}_\varrho)\breve{v}(\check{e}_{\ell+2}, \check{r}_\varrho)}) \\
& \star \breve{w}_{\check{b}}(\check{e}_{\ell+3}, \check{r}_{\ell+2}, \frac{\check{a}}{3^3\wp(\check{e}_\ell, \check{r}_\varrho)\wp(\check{e}_{\ell+1}, \check{r}_\varrho)\aleph(\check{e}_{\ell+2}, \check{r}_\varrho)}) \\
& \star \breve{w}_{\check{b}}(\check{e}_{\ell+3}, \check{r}_\varrho, \frac{\check{a}}{3^3\wp(\check{e}_\ell, \check{r}_\varrho)\wp(\check{e}_{\ell+1}, \check{r}_\varrho)\wp(\check{e}_{\ell+2}, \check{r}_\varrho)}) \\
& \geq \breve{w}_{\check{b}}(\check{e}_\ell, \check{r}_\ell, \frac{\check{a}}{3\breve{v}(\check{e}_\ell, \check{r}_\varrho)}) \star \breve{w}_{\check{b}}(\check{e}_{\ell+1}, \check{r}_\ell, \frac{\check{a}}{3\breve{v}(\check{e}_\ell, \check{r}_\varrho)}) \star \cdots \\
& \star \breve{w}_{\check{b}}(\check{e}_{\varrho-1}, \check{r}_{\varrho-1}, \frac{\check{a}}{3^{\varrho-1}\wp(\check{e}_\ell, \check{r}_\varrho)\wp(\check{e}_{\ell+1}, \check{r}_\varrho)\cdots\breve{v}(\check{e}_{\varrho-1}, \check{r}_{\varrho-1})})
\end{aligned}$$

$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\varrho}, \breve{r}_{\varrho-1}, \frac{\breve{a}}{3^{\varrho-1}\wp(\breve{e}_{\ell}, \breve{r}_{\varrho})\wp(\breve{e}_{\ell+1}, \breve{r}_{\varrho})\cdots\aleph(\breve{e}_{\varrho}, \breve{r}_{\varrho-1})})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\varrho}, \breve{r}_{\varrho}, \frac{\breve{a}}{3^{\varrho-1}\wp(\breve{e}_{\ell}, \breve{r}_{\varrho})\wp(\breve{e}_{\ell+1}, \breve{r}_{\varrho})\cdots\wp(\breve{e}_{\varrho}, \breve{r}_{\varrho})}).$$

Apply (3.2) and (3.3), we get

$$\breve{w}_{\breve{b}}(\breve{e}_{\ell}, \breve{r}_{\varrho}, \breve{a}) \geq \breve{w}_{\breve{b}}(\breve{e}_0, \breve{r}_0, \frac{\breve{a}}{3\upsilon^{\ell}\breve{v}(\breve{e}_{\ell}, \breve{r}_{\varrho})}) \star \breve{w}_{\breve{b}}(\breve{e}_1, \breve{r}_0, \frac{\breve{a}}{3\upsilon^{\ell+1}\aleph(\breve{e}_{\ell}, \breve{r}_{\varrho})})$$
$$\star \cdots\cdots \star\, \breve{w}_{\breve{b}}(\breve{e}_0, \breve{r}_0, \frac{\breve{a}}{3^{\varrho-1}\upsilon^{\varrho}\wp(\breve{e}_{\ell}, \breve{r}_{\varrho})\wp(\breve{e}_{\ell+1}, \breve{r}_{\varrho})\cdots\wp(\breve{e}_{\varrho}, \breve{r}_{\varrho})}).$$

From (3.1), as $\ell, \varrho \to \infty$ we get

$$\breve{w}_{\breve{b}}(\breve{e}_{\ell}, \breve{r}_{\varrho}, \breve{a}) \geq 1 \text{ for all } \breve{a} > 0.$$

Therefore, bi-seq $(\breve{e}_{\ell}, \breve{r}_{\ell})$ is a Cauchy bi-seq. Since $(\Psi, \Delta, \breve{w}_{\breve{b}}, *)$ is a complete. So, bi-seq $(\breve{e}_{\ell}, \breve{r}_{\ell})$ is a convergent Cauchy bi-seq. By Proposition 2.1 the bi-seq $(\breve{e}_{\ell}, \breve{r}_{\ell})$ is biconvergent seq.

As, bi-seq $(\breve{e}_{\ell}, \breve{r}_{\ell})$ is biconvergent then we can find $\chi \in \Psi \cap \Delta$ implies a limit of seq $\{\breve{e}_{\ell}\}$ and $\{\breve{r}_{\ell}\}$. Using Lemma 2.2, both seq $\{\breve{e}_{\ell}\}$ and $\{\breve{r}_{\ell}\}$ has a unique limit. From 4, assume

$$\breve{w}_{\breve{b}}(\Phi(\chi), \chi, \breve{a}) \geq \breve{w}_{\breve{b}}(\Phi(\chi), \Phi(\breve{r}_{\ell}), \frac{\breve{a}}{3\breve{v}(\Phi(\chi), \chi)}) * \breve{w}_{\breve{b}}(\Phi(\breve{e}_{\ell}), \Phi(\breve{r}_{\ell}), \frac{\breve{a}}{3\aleph(\Phi(\chi), \chi)})$$
$$* \breve{w}_{\breve{b}}(\Phi(\breve{e}_{\ell}), \chi, \frac{\breve{a}}{3\wp(\Phi(\chi), \chi)})$$

for all $\ell \in \mathbb{N}$ and $\breve{a} > 0$ and as $\ell \to \infty$ we obtain

$$\breve{w}_{\breve{b}}(\Phi(\chi), \chi, \breve{a}) \to 1*1*1 = 1.$$

From 2, we get $\Phi(\chi) = \chi$. Let $\nu \in \Psi \cap \Delta$ is one more $\mathcal{FP}$ of Φ. Then

$$\breve{w}_{\breve{b}}(\chi, \nu, \breve{a}) = \breve{w}_{\breve{b}}(\Phi(\chi), \Phi(\nu), \breve{a}) \geq \breve{w}_{\breve{b}}(\chi, \nu, \frac{\breve{a}}{\upsilon})$$

for $\upsilon \in (0, 1)$ and $\forall\ \breve{a} > 0$. Using Lemma 2.1 we obtain $\chi = \nu$. □

Example 3.1. *Let $\Psi = [0, 1]$, $\Delta = \{0\} \cup \mathbb{N} - \{1\}$ and $\breve{v}, \aleph, \wp : \Psi \times \Delta \to [1, \infty)$ be three mapping defined as $\breve{v}(\breve{e}, \breve{r}) = \breve{e} + \breve{r} + 1$, $\aleph(\breve{e}, \breve{r}) = \breve{e}^2 + \breve{r} + 1$ and $\wp(\breve{e}, \breve{r}) = \breve{e}^2 + \breve{r} - 1$. Define $\breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}) = \frac{\breve{a}}{\breve{a} + |\breve{e} - \breve{r}|}$ for all $\breve{a} > 0$ and $\breve{e} \in \Psi$ and $\breve{r} \in \Delta$. Clearly, $(\Psi, \Delta, \breve{w}_{\breve{b}}, *)$ is a complete $\aleph EF_{\mathfrak{b}}BMS$, where $*$ is a continuous $\breve{a}$-norm defined as $\sigma * \breve{b} = \sigma\breve{b}$.*

Define $\Phi : \Psi \cup \Delta \to \Psi \cup \Delta$ by

$$\Phi(\chi) = \begin{cases} \frac{\chi}{4}, & \text{if } \chi \in [0, 1], \\ 0, & \text{if } \chi \in \mathbb{N} - \{1\}, \end{cases}$$

$\forall\ \chi \in \Psi \cup \Delta$. Thus, Theorem 3.1 of all axioms are fulfilled. Hence Φ has a $\mathcal{UFP}$, i.e., $\chi = 0$.

Theorem 3.2. *Let* $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$ *be a complete* $\aleph EF_{\mathsf{b}}BMS$ *with three functions* $\breve{v}, \aleph, \wp : \Psi \times \varDelta \to [1, \infty)$ *implies that*

$$\lim_{\breve{a}\to\infty} \breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a}) = 1 \;\forall\; \breve{e} \in \Psi, \breve{r} \in \varDelta. \tag{3.4}$$

Let $\Phi : \Psi \cup \varDelta \to \Psi \cup \varDelta$ *be map as follows:*

(i) $\Phi(\Psi) \subseteq \varDelta$ *and* $\Phi(\varDelta) \subseteq \Psi$;

(ii) $\breve{w}_{\breve{b}}(\Phi(\breve{r}), \Phi(\breve{e}), \upsilon\breve{a}) \geq \breve{w}_{\breve{b}}(\breve{e}, \breve{r}, \breve{a})$, $\forall\; \breve{e} \in \Psi, \breve{r} \in \varDelta$ *and* $\breve{a} > 0$, *where* $\upsilon \in (0, 1)$.

Then Φ *has a* $\mathcal{UFP}$.

Proof. Fix $\breve{e}_0 \in \Psi$ and consider $\Phi(\breve{e}_\ell) = \breve{r}_\ell$ and $\Phi(\breve{r}_\ell) = \breve{e}_{\ell+1}$ for all $\ell \in \mathbb{N} \cup \{0\}$. Then we get $(\breve{e}_\ell, \breve{r}_\ell)$ as a bi-seq on $\aleph EF_{\mathsf{b}}$BMS $(\Psi, \varDelta, \breve{w}_{\breve{b}}, *)$. Now, we have

$$\breve{w}_{\breve{b}}(\breve{e}_1, \breve{r}_0, \breve{a}) = \breve{w}_{\breve{b}}(\Phi(\breve{r}_0), \Phi(\breve{e}_0), \breve{a}) \geq \breve{w}_{\breve{b}}(\breve{e}_0, \breve{r}_0, \frac{\breve{a}}{\upsilon})$$

for all $\breve{a} > 0$ and $\ell \in \mathbb{N}$. Using induction we obtain

$$\breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\ell, \breve{a}) = \breve{w}_{\breve{b}}(\Phi(\breve{r}_{\ell-1}), \Phi(\breve{e}_\ell), \breve{a}) \geq \breve{w}_{\breve{b}}(\breve{e}_0, \breve{r}_0, \frac{\breve{a}}{\upsilon^{2\ell}}) \tag{3.5}$$

and

$$\breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_\ell, \breve{a}) = \breve{w}_{\breve{b}}(\Phi(\breve{r}_\ell), \Phi(\breve{e}_\ell), \breve{a}) \geq \breve{w}_{\breve{b}}(\breve{e}_0, \breve{r}_0, \frac{\breve{a}}{\upsilon^{2\ell+1}}) \tag{3.6}$$

$\forall\; \breve{a} > 0$ and $\ell \in \mathbb{N}$. Consider $\ell < \varrho$, for $\ell, \varrho \in \mathbb{N}$. Then

$$\begin{aligned}
\breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\varrho, \breve{a}) \geq & \breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\ell, \frac{\breve{a}}{3\breve{v}(\breve{e}_\ell, \breve{r}_\varrho)}) \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_\ell, \frac{\breve{a}}{3\aleph(\breve{e}_\ell, \breve{r}_\varrho)}) \\
& \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_\varrho, \frac{\breve{a}}{3\wp(\breve{e}_\ell, \breve{r}_\varrho)}) \\
& \geq \breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\ell, \frac{\breve{a}}{3\breve{v}(\breve{e}_\ell, \breve{r}_\varrho)}) \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_\ell, \frac{\breve{a}}{3\aleph(\breve{e}_\ell, \breve{r}_\varrho)}) \\
& \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_{\ell+1}, \frac{\breve{a}}{3^2\wp(\breve{e}_\ell, \breve{r}_\varrho)\breve{v}(\breve{e}_{\ell+1}, \breve{r}_\varrho)}) \\
& \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+2}, \breve{r}_{\ell+1}, \frac{\breve{a}}{3^2\wp(\breve{e}_\ell, \breve{r}_\varrho)\aleph(\breve{e}_{\ell+1}, \breve{r}_\varrho)}) \\
& \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+2}, \breve{r}_\varrho, \frac{\breve{a}}{3^2\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)}) \\
& \geq \breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\ell, \frac{\breve{a}}{3\breve{v}(\breve{e}_\ell, \breve{r}_\varrho)}) \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_\ell, \frac{\breve{a}}{3\aleph(\breve{e}_\ell, \breve{r}_\varrho)}) \\
& \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_{\ell+1}, \frac{\breve{a}}{3^2\wp(\breve{e}_\ell, \breve{r}_\varrho)\breve{v}(\breve{e}_{\ell+1}, \breve{r}_\varrho)}) \\
& \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+2}, \breve{r}_{\ell+1}, \frac{\breve{a}}{3^2\wp(\breve{e}_\ell, \breve{r}_\varrho)\aleph(\breve{e}_{\ell+1}, \breve{r}_\varrho)}) \\
& \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+2}, \breve{r}_{\ell+2}, \frac{\breve{a}}{3^3\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\breve{v}(\breve{e}_{\ell+2}, \breve{r}_\varrho)})
\end{aligned}$$

$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\ell+3}, \breve{r}_{\ell+2}, \frac{\breve{a}}{3^3\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\aleph(\breve{e}_{\ell+2}, \breve{r}_\varrho)})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\ell+3}, \breve{r}_\varrho, \frac{\breve{a}}{3^3\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\wp(\breve{e}_{\ell+2}, \breve{r}_\varrho)})$$
$$\geq \breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\ell, \frac{\breve{a}}{3\breve{v}(\breve{e}_\ell, \breve{r}_\varrho)}) \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_\ell, \frac{\breve{a}}{3\breve{v}(\breve{e}_\ell, \breve{r}_\varrho)}) \star \cdots$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\varrho-1}, \breve{r}_{\varrho-1}, \frac{\breve{a}}{3^{\varrho-1}\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\cdots\breve{v}(\breve{e}_{\varrho-1}, \breve{r}_{\varrho-1})})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_\varrho, \breve{r}_{\varrho-1}, \frac{\breve{a}}{3^{\varrho-1}\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\cdots\aleph(\breve{e}_\varrho, \breve{r}_{\varrho-1})})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_\varrho, \breve{r}_\varrho, \frac{\breve{a}}{3^{\varrho-1}\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\cdots\wp(\breve{e}_\varrho, \breve{r}_\varrho)}).$$

Apply (3.2) and (3.3), we get

$$\breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\varrho, \breve{a}) \geq \breve{w}_{\breve{b}}(\breve{e}_0, \breve{r}_0, \frac{\breve{a}}{3\upsilon^{2\ell}\breve{v}(\breve{e}_\ell, \breve{r}_\varrho)}) \star \breve{w}_{\breve{b}}(\breve{e}_0, \breve{r}_0, \frac{\breve{a}}{3\upsilon^{2\ell+1}\aleph(\breve{e}_\ell, \breve{r}_\varrho)})$$
$$\star \cdots\cdots \star\, \breve{w}_{\breve{b}}(\breve{e}_0, \breve{r}_0, \frac{\breve{a}}{3^{\varrho-1}\upsilon^{2\varrho}\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\cdots\wp(\breve{e}_\varrho, \breve{r}_\varrho)}).$$

From (3.4), as $\ell, \varrho \to \infty$ we get

$$\breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\varrho, \breve{a}) \geq 1 \text{ for all } \breve{a} > 0.$$

Therefore, bi-seq $(\breve{e}_\ell, \breve{r}_\ell)$ is a Cauchy bi-seq. Since $(\Psi, \Delta, \breve{w}_{\breve{b}}, *)$ is a complete. So, bi-seq $(\breve{e}_\ell, \breve{r}_\ell)$ is a convergent Cauchy bi-seq. By Proposition 2.1 the bi-seq $(\breve{e}_\ell, \breve{r}_\ell)$ is biconvergent seq. As, bi-seq $(\breve{e}_\ell, \breve{r}_\ell)$ is biconvergent then we can find $\chi \in \Psi \cap \Delta$ implies a limit of seq $\{\breve{e}_\ell\}$ and $\{\breve{r}_\ell\}$. Using Lemma 2.2, both seq $\{\breve{e}_\ell\}$ and $\{\breve{r}_\ell\}$ have a unique limit. From 4, assume

$$\breve{w}_{\breve{b}}(\Phi(\chi), \chi, \breve{a}) \geq \breve{w}_{\breve{b}}(\Phi(\chi), \Phi(\breve{e}_\ell), \frac{\breve{a}}{3\breve{v}(\Phi(\chi), \chi)})$$
$$*\breve{w}_{\breve{b}}(\Phi(\breve{r}_\ell), \Phi(\breve{e}_\ell), \frac{\breve{a}}{3\aleph(\Phi(\chi), \chi)})$$
$$*\breve{w}_{\breve{b}}(\chi, \Phi(\breve{e}_\ell), \frac{\breve{a}}{3\wp(\Phi(\chi), \chi)}),$$

for all $\ell \in \mathbb{N}$ and $\breve{a} > 0$ and as $\ell \to \infty$ we have

$$\breve{w}_{\breve{b}}(\Phi(\chi), \chi, \breve{a}) \to 1*1*1 = 1.$$

From 2, we get $\Phi(\chi) = \chi$. Let $\nu \in \Psi \cap \Delta$ is one more $\mathcal{FP}$ of Φ. Then

$$\breve{w}_{\breve{b}}(\chi, \nu, \breve{a}) = \breve{w}_{\breve{b}}(\Phi(\nu), \Phi(\chi), \breve{a}) \geq \breve{w}_{\breve{b}}(\chi, \nu, \frac{\breve{a}}{\upsilon})$$

for $\upsilon \in (0, 1)$ and for all $\breve{a} > 0$. Using Lemma 2.1 we obtain $\chi = \nu$. □

Example 3.2. *Let* $\Psi = [0,1]$, $\Delta = \{0\} \cup \mathbb{N} - \{1\}$ *and* $\breve{v}, \aleph, \wp : \Psi \times \Delta \to [1,\infty)$ *be three mapping defined as* $\breve{v}(\check{e},\check{r}) = \check{e} + \check{r} + 1, \aleph(\check{e},\check{r}) = \check{e}^2 + \check{r} + 1$*and* $\wp(\check{e},\check{r}) = \check{e}^2 + \check{r} - 1$. *Define*

$$\breve{w}_{\breve{b}}(\check{e},\check{r},\breve{a}) = e^{-\frac{(\check{e}-\check{r})^2}{\breve{a}}},\ \forall\ \check{e} \in \Psi, \check{r} \in \Delta, \breve{a} > 0.$$

Then $(\Psi, \Delta, \breve{w}_{\breve{b}}, *)$ *is a complete* $\aleph EF_{\mathfrak{b}}BMS$ *with product* $\breve{a}$*-norm. Define* $\Phi : \Psi \cup \Delta \to \Psi \cup \Delta$ *by*

$$\Phi(\chi) = \begin{cases} \frac{1-5^{-\chi}}{7}, & \text{if } \chi \in [0,1], \\ 0, & \text{if } \chi \in \mathbb{N} - \{1\}, \end{cases}$$

$\forall\ \chi \in \Psi \cup \Delta$. *Let* $\check{e} \in [0,1]$ *and* $\check{r} \in \mathbb{N} - \{1\}$, *then*

$$\begin{aligned} \breve{w}_{\breve{b}}(\Phi(\check{e}), \Phi(\check{r}), v\breve{a}) &= \breve{w}_{\breve{b}}\left(\frac{1-5^{-\check{e}}}{7}, 0, v\breve{a}\right) \\ &= e^{-\frac{\left(\frac{1-5^{-\check{e}}}{7}\right)^2}{v\breve{a}}} \\ &\geq e^{-\frac{(\check{e}-\check{r})^2}{\breve{a}}} \\ &= \breve{w}_{\breve{b}}(\check{e},\check{r},\breve{a}). \end{aligned}$$

Thus, Theorem 3.2 of all axioms are fulfilled. Hence Φ *has a* $\mathcal{UFP}$, *i.e.,* $\chi = 0$.

Theorem 3.3. *Let* $(\Psi, \Delta, \breve{w}_{\breve{b}}, *)$ *be a complete* $\aleph EF_{\mathfrak{b}}BMS$ *with three functions* $\breve{v}, \aleph, \wp : \Psi \times \Delta \to [1,\infty)$ *and* $\Phi : \Psi \cup \Delta \to \Psi \cup \Delta$ *a map as follows:*

(1) $\Phi(\Psi) \subseteq \Psi$ *and* $\Phi(\Delta) \subseteq \Delta$;
(2) *For* $\check{e} \in \Psi, \check{r} \in \Delta$ *and* $\breve{a} > 0, \breve{w}_{\breve{b}}(\check{e},\check{r},\breve{a}) > 0 \Rightarrow \breve{w}_{\breve{b}}(\Phi(\check{e}), \Phi(\check{r}), \breve{a}) \geq \breve{w}(\breve{w}_{\breve{b}}(\check{e},\check{r},\breve{a}))$, *where* $\breve{w} : (0,1] \to (0,1]$ *is an increasing function implies that* $\lim_{\ell\to\infty} \breve{w}^{\ell}(v) = 1$ *and* $\breve{w}(v) \geq v$ *for all* $v \in (0,1]$.

Then Φ *has a* $\mathcal{FP}$.

Proof. Let $\check{e}_0 \in \Psi$ and $\check{r}_0 \in \Delta$ implies that $\Phi(\check{e}_\ell) = \check{e}_{\ell+1}$ and $\Phi(\check{r}_\ell) = \check{r}_{\ell+1}$ for all $\ell \in \mathbb{N} \cup \{0\}$, then $(\check{e}_\ell, \check{r}_\ell)$ be a bi-seq on $\aleph EF_{\mathfrak{b}}BMS$ $(\Psi, \Delta, \breve{w}_{\breve{b}}, *)$. By condition of 2 for all $\breve{a} > 0$ and condition 2, we get

$$\breve{w}_{\breve{b}}(\check{e}_\ell, \check{r}_\ell, \breve{a}) \geq \breve{w}^{\ell}(\breve{w}_{\breve{b}}(\check{e}_0, \check{r}_0, \breve{a})) \tag{3.7}$$

and

$$\breve{w}_{\breve{b}}(\check{e}_{\ell+1}, \check{r}_\ell, \breve{a}) \geq \breve{w}^{\ell}(\breve{w}_{\breve{b}}(\check{e}_1, \check{r}_0, \breve{a})). \tag{3.8}$$

Let $\ell < \varrho$, for $\ell, \varrho \in \mathbb{N}$. Then

$$\begin{aligned} \breve{w}_{\breve{b}}(\check{e}_\ell, \check{r}_\varrho, \breve{a}) \geq& \breve{w}_{\breve{b}}(\check{e}_\ell, \check{r}_\ell, \frac{\breve{a}}{3\breve{v}(\check{e}_\ell, \check{r}_\varrho)}) \star \breve{w}_{\breve{b}}(\check{e}_{\ell+1}, \check{r}_\ell, \frac{\breve{a}}{3\aleph(\check{e}_\ell, \check{r}_\varrho)}) \\ &\star \breve{w}_{\breve{b}}(\check{e}_{\ell+1}, \check{r}_\varrho, \frac{\breve{a}}{3\wp(\check{e}_\ell, \check{r}_\varrho)}) \end{aligned}$$

$$\geq \breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\ell, \frac{\breve{a}}{3\breve{v}(\breve{e}_\ell, \breve{r}_\varrho)}) \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_\ell, \frac{\breve{a}}{3\aleph(\breve{e}_\ell, \breve{r}_\varrho)})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_{\ell+1}, \frac{\breve{a}}{3^2\wp(\breve{e}_\ell, \breve{r}_\varrho)\breve{v}(\breve{e}_{\ell+1}, \breve{r}_\varrho)})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\ell+2}, \breve{r}_{\ell+1}, \frac{\breve{a}}{3^2\wp(\breve{e}_\ell, \breve{r}_\varrho)\aleph(\breve{e}_{\ell+1}, \breve{r}_\varrho)})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\ell+2}, \breve{r}_\varrho, \frac{\breve{a}}{3^2\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)})$$
$$\geq \breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\ell, \frac{\breve{a}}{3\breve{v}(\breve{e}_\ell, \breve{r}_\varrho)}) \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_\ell, \frac{\breve{a}}{3\aleph(\breve{e}_\ell, \breve{r}_\varrho)})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_{\ell+1}, \frac{\breve{a}}{3^2\wp(\breve{e}_\ell, \breve{r}_\varrho)\breve{v}(\breve{e}_{\ell+1}, \breve{r}_\varrho)})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\ell+2}, \breve{r}_{\ell+1}, \frac{\breve{a}}{3^2\wp(\breve{e}_\ell, \breve{r}_\varrho)\aleph(\breve{e}_{\ell+1}, \breve{r}_\varrho)})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\ell+2}, \breve{r}_{\ell+2}, \frac{\breve{a}}{3^3\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\breve{v}(\breve{e}_{\ell+2}, \breve{r}_\varrho)})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\ell+3}, \breve{r}_{\ell+2}, \frac{\breve{a}}{3^3\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\aleph(\breve{e}_{\ell+2}, \breve{r}_\varrho)})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\ell+3}, \breve{r}_\varrho, \frac{\breve{a}}{3^3\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\wp(\breve{e}_{\ell+2}, \breve{r}_\varrho)})$$
$$\geq \breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\ell, \frac{\breve{a}}{3\breve{v}(\breve{e}_\ell, \breve{r}_\varrho)}) \star \breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \breve{r}_\ell, \frac{\breve{a}}{3\aleph(\breve{e}_\ell, \breve{r}_\varrho)}) \star \cdots$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\varrho-1}, \breve{r}_{\varrho-1}, \frac{\breve{a}}{3^{\varrho-1}\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\cdots\breve{v}(\breve{e}_{\varrho-1}, \breve{r}_{\varrho-1})})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\varrho}, \breve{r}_{\varrho-1}, \frac{\breve{a}}{3^{\varrho-1}\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\cdots\aleph(\breve{e}_{\varrho}, \breve{r}_{\varrho-1})})$$
$$\star\, \breve{w}_{\breve{b}}(\breve{e}_{\varrho}, \breve{r}_{\varrho}, \frac{\breve{a}}{3^{\varrho-1}\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\cdots\wp(\breve{e}_{\varrho}, \breve{r}_{\varrho})}).$$

Apply (3.7) and (3.8), we get

$$\breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\varrho, \breve{a}) \geq \breve{w}^\ell(\breve{w}_{\breve{b}}(\breve{e}_0, \breve{r}_0, \frac{\breve{a}}{3\breve{v}(\breve{e}_\ell, \breve{r}_\varrho)})) \star \breve{w}^\ell(\breve{w}_{\breve{b}}(\breve{e}_1, \breve{r}_0, \frac{\breve{a}}{3\aleph(\breve{e}_\ell, \breve{r}_\varrho)}))$$
$$\star \cdots\cdots \star\, \breve{w}^\ell(\breve{w}_{\breve{b}}(\breve{e}_0, \breve{r}_0, \frac{\breve{a}}{3^{\varrho-1}\wp(\breve{e}_\ell, \breve{r}_\varrho)\wp(\breve{e}_{\ell+1}, \breve{r}_\varrho)\cdots\wp(\breve{e}_{\varrho}, \breve{r}_{\varrho})})).$$

As $\ell, \varrho \to \infty$, we have $\breve{w}_{\breve{b}}(\breve{e}_\ell, \breve{r}_\varrho, \breve{a}) \to 1$ for all $\breve{a} > 0$. By Theorem 3.1. We get, if $\chi \in \Psi \cap \varDelta$ implies a unique limit of seq $\{\breve{e}_\ell\}$ and $\{\breve{r}_\ell\}$, then χ is a $\mathcal{FP}$ of $\varPhi$. We have, $\breve{w}_{\breve{b}}(\breve{e}_\ell, \chi, \breve{a}) \to \breve{a}$ for all $\breve{a} > 0$ and $\breve{w}_{\breve{b}}(\breve{e}_{\ell+1}, \varPhi(\chi), \breve{a}) = \breve{w}_{\breve{b}}(\varPhi(\breve{e}_\ell), \varPhi(\chi), \breve{a}) \geq \breve{w}(\breve{w}_{\breve{b}}(\breve{e}_\ell, \chi, \breve{a})) \geq \breve{w}_{\breve{b}}(\breve{e}_\ell, \chi, \breve{a})$, and $\breve{e}_{\ell+1} \to \varPhi(\chi)$, such that $\varPhi(\chi) = \chi$. □

Example 3.3. *Let $\Psi = \{2,4,5,6\}, \Delta = \{1,4\}, \sigma * \check{b} = \sigma\check{b}\ \forall\ \sigma, \check{b} \in [0,1]$ and $\check{v}, \aleph, \wp : \Psi \times \Delta \to [1,\infty)$ be three mapping defined as $\check{v}(\check{e},\check{r}) = \check{e}+\check{r}+1, \aleph(\check{e},\check{r}) = \check{e}^2+\check{r}+1$ and $\wp(\check{e},\check{r}) = \check{e}^2+\check{r}-1$. Define*

$$\breve{w}_{\check{b}}(\check{e},\check{r},\check{a}) = \frac{\min\{\check{e},\check{r}\}+\check{a}}{\max\{\check{e},\check{r}\}+\check{a}} \text{ for all } \check{e}\in\Psi, \check{r}\in\Delta \text{ and for all } \check{a}>0.$$

Then $(\Psi,\Delta,\breve{w}_{\check{b}},*)$ is an complete $\aleph EF_{\mathsf{b}}BMS$. Now, define $\breve{w} : (0,1] \to (0,1]$ implies that $\breve{w}(\upsilon) = \sqrt{\upsilon}$. Now, $\breve{w}(\upsilon) = \sqrt{\upsilon}$ satisfies $\breve{w}$ function.

Let $\Phi : \Psi\cup\Delta \to \Psi\cup\Delta$ be a map implies that $\Phi(2) = \Phi(4) = \Phi(1) = 4, \Phi(5) = \Phi(6) = 1$. Thus Theorem 3.3 of all axioms are fulfilled. The $\mathcal{FP}$ of Φ is $\check{e} = 4$.

Theorem 3.4. *Let $(\Psi,\Delta,\breve{w}_{\check{b}},*)$ be a complete $\aleph EF_{\mathsf{b}}BMS$ with three functions $\check{v}, \aleph, \wp : \Psi\times\Delta \to [1,\infty)$ and $\Phi : \Psi\cup\Delta \to \Psi\cup\Delta$ a map as follows:*

(i) *$\Phi(\Psi) \subseteq \Delta$ and $\Phi(\Delta) \subseteq \Psi$;*

(ii) *For $\check{e}\in\Psi, \check{r}\in\Delta$ and $\check{a}>0, \breve{w}_{\check{b}}(\check{e},\check{r},\check{a}) > 0 \Longrightarrow \breve{w}_{\check{b}}(\Phi(\check{r}),\Phi(\check{e}),\check{a}) \geq \breve{w}(\breve{w}_{\check{b}}(\check{e},\check{r},\check{a}))$.*

Then Φ has a $\mathcal{FP}$.

Proof. We can easily prove by Theorem 3.3 and Theorem 3.2. □

4. Application

In this part, we investigate the existence and unique solution of integral equations as an application of Theorem 3.1.

Theorem 4.1. *Assume that the integral equation*

$$\check{e}(\varrho) = \check{b}(\varrho) + \int_{\check{T}_1\cup\check{T}_2} \Omega(\varrho,\varsigma,\check{e}(\varsigma))d\varsigma, \ \ \varrho\in\check{T}_1\cup\check{T}_2,$$

where $\check{T}_1\cup\check{T}_2$ is a Lebesgue measurable set. Let

(1) *$\Omega : (\check{T}_1^2\cup\check{T}_2^2)\times[0,\infty) \to [0,\infty)$ and $b\in L^\infty(\check{T}_1)\cup L^\infty(\check{T}_2)$,*

(2) *there is a continuous function $\theta : \check{T}_1^2\cup\check{T}_2^2 \to [0,\infty)$ and $\upsilon\in(0,1)$ such that*

$$|\Omega(\varrho,\varsigma,\check{e}(\varsigma)) - \Omega(\varrho,\varsigma,\check{r}(\varsigma))| \leq \upsilon\theta(\varrho,\varsigma)(|\check{e}(\varrho)-\check{r}(\varrho)|),$$

for $\varrho,\varsigma\in\check{T}_1^2\cup\check{T}_2^2$,

(3) *$\sup_{\varrho\in\check{T}_1\cup\check{T}_2}\int_{\check{T}_1\cup\check{T}_2}\theta(\varrho,\varsigma)d\varsigma \leq 1$.*

Then the integral equation has a unique solution in $L^\infty(\check{T}_1)\cup L^\infty(\check{T}_2)$.

Proof. Let $\Psi = L^\infty(\check{T}_1)$ and $\Delta = L^\infty(\check{T}_2)$ be two normed linear spaces, where $\check{T}_1,\check{T}_2$ are Lebesgue measurable sets and $m(\check{T}_1\cup\check{T}_2) < \infty$.
Consider $\breve{w}_{\check{b}} : \Psi\times\Delta\times(0,\infty) \to [0,1]$ by

$$\breve{w}_{\check{b}}(\check{e},\check{r},\check{a}) = e^{-\frac{\sup_{\varrho\in\check{T}_1\cup\check{T}_2}|\check{e}(\varrho)-\check{r}(\varrho)|}{\check{a}}}.$$

for all $\check{e} \in \Psi, \check{r} \in \mathit{\Delta}$. Define $\check{v}, \aleph, \wp : \Psi \times \mathit{\Delta} \to [1, \infty)$ be three mapping defined as $\check{v}(\check{e}, \check{r}) = \check{e} + \check{r} + 1, \aleph(\check{e}, \check{r}) = \check{e}^2 + \check{r} + 1$ and $\wp(\check{e}, \check{r}) = \check{e}^2 + \check{r} - 1$. Then $(\Psi, \mathit{\Delta}, \breve{w}_{\check{b}}, \star)$ is a complete $\aleph$EF_{b}BMS. Define the mapping $\mathit{\Phi}, \Phi : L^{\infty}(\check{T}_1) \cup L^{\infty}(\check{T}_2) \to L^{\infty}(\check{T}_1) \cup L^{\infty}(\check{T}_2)$ by

$$\mathit{\Phi}(\check{e}(\varrho)) = \check{b}(\varrho) + \int_{\check{T}_1 \cup \check{T}_2} \mathit{\Omega}(\varrho, \varsigma, \check{e}(\varsigma)) d\varsigma, \ \ \varrho \in \check{T}_1 \cup \check{T}_2.$$

Now, we have

$$\begin{aligned}
\breve{w}_{\check{b}}(\mathit{\Phi}\check{e}(\varrho), \mathit{\Phi}\check{r}(\varrho), v\check{a}) &= e^{-\sup_{\varrho \in \check{T}_1 \cup \check{T}_2} \frac{|\mathit{\Phi}\check{e}(\varrho) - \mathit{\Phi}\check{r}(\varrho)|}{v\check{a}}} \\
&= e^{-\sup_{\varrho \in \check{T}_1 \cup \check{T}_2} \frac{|\check{b}(\varrho) + \int_{\check{T}_1 \cup \check{T}_2} \mathit{\Omega}(\varrho, \varsigma, \check{e}(\varsigma)) d\varsigma - \check{b}(\varrho) - \int_{\check{T}_1 \cup \check{T}_2} \mathit{\Omega}(\varrho, \varsigma, \check{r}(\varsigma)) d\varsigma|}{v\check{a}}} \\
&= e^{-\sup_{\varrho \in \check{T}_1 \cup \check{T}_2} \frac{\left|\check{b}(\varrho) + \int_{\check{T}_1 \cup \check{T}_2} \mathit{\Omega}(\check{a}, \varsigma, \check{e}(\varsigma)) d\varsigma - \left(\check{b}(\varrho) + \int_{\check{T}_1 \cup \check{T}_2} \mathit{\Omega}(\check{a}, \varsigma, \check{r}(\varsigma)) d\varsigma\right)\right|}{v\check{a}}} \\
&\geq e^{-\sup_{\varrho \in \check{T}_1 \cup \check{T}_2} \frac{\int_{\check{T}_1 \cup \check{T}_2} |\mathit{\Omega}(\varrho, \varsigma, \check{e}(\varsigma)) - \mathit{\Omega}(\varrho, \varsigma, \check{r}(\varsigma))| d\varsigma}{v\check{a}}} \\
&\geq e^{-\sup_{\varrho \in \check{T}_1 \cup \check{T}_2} \frac{\int_{\check{T}_1 \cup \check{T}_2} v\theta(\varrho, \varsigma)(|\check{e}(\varrho) - \check{r}(\varrho)|) d\varsigma}{v\check{a}}} \\
&\geq e^{-\sup_{\varrho \in \check{T}_1 \cup \check{T}_2} \frac{\int_{\check{T}_1 \cup \check{T}_2} v\theta(\varrho, \varsigma)(|\check{e}(\varrho) - \check{r}(\varrho)|) d\varsigma}{v\check{a}}} \\
&\geq e^{-\sup_{\varrho \in \check{T}_1 \cup \check{T}_2} \frac{|\check{e}(\varrho) - \check{r}(\varrho)|}{\check{a}}} \\
&= \breve{w}_{\check{b}}(\check{e}, \check{r}, \check{a}).
\end{aligned}$$

Hence, from Theorem 3.1 of all axioms are fulfilled and thus integral equation has a unique solution. □

5. Application to Fractional Differential Equations

Fractional differential equations (FDEs) can be used to model and study physical systems with continuous distributions or interactions. They offer a framework for understanding the complex behaviors and interconnections present in various engineering systems. There are several possible applications for implicit differential equations (FDEs) in engineering research. In this part, we show that the FDE has a single solution. These kinds of differential equations are frequently utilized in engineering. These formulas offer an adaptable structure for comprehending and evaluating continuous distributions and interactions in a range of engineering domains. By merging ideas from graph mappings, Kannan mappings, and fuzzy contractions, Younis and Abdou [16] creatively developed a brand-new concept known as Kannan-graph-fuzzy contraction. We demonstrate that the following fractional differential equations have a unique solution in the sense of the Caputo derivative. Refer to this work [17] for further details.

$$\mathcal{D}_{0+}^{\eta} \check{r}(\varrho) + \mathfrak{g}(\varrho, \check{r}(\varrho)) = 0, \quad 0 < \varrho < 1, \tag{5.1}$$

with boundary conditions

$$\check{r}(0)+\check{r}'(0)=0, \quad \check{r}(1)+\check{r}'(1)=0,$$

where, $1<\eta\leq 2$, $\mathfrak{g}\colon [0,1]\times\mathbb{R}^+\to\mathbb{R}^+$ are continuous functions. Let $\Psi=(\check{C}[0,1],\mathbb{R}^+=\{\check{f}:[0,1]\to\mathbb{R}^+\}$, and $\varDelta=(\check{C}[0,1],(-\infty,0])=\{\check{f}:[0,1]\to(-\infty,0]\}$: are $\check{f}$ is a continuous function. Define $\Omega_{\mathfrak{b}}:\Psi\times\varDelta\times\mathbb{R}^+\to[0,1]$ by

$$\Omega_{\mathfrak{b}}(\check{r},\varpi,\pi)=e^{-\frac{\sup_{\varrho\in\Psi_1\cup\Psi_2}|\check{r}(\varrho)-\varpi(\varrho)|^2}{\pi}},$$

where $i*\mathfrak{b}=i\mathfrak{b}$. Consider $\check{r}\in\Psi\cup\varDelta$ solves (5.1) and for every $\check{r}\in\Psi\cup\varDelta$ is defined as

$$\begin{aligned}\check{r}(\varrho)=&\frac{1}{\Phi(\eta)}\int_0^1(1-\check{s})^{\eta-1}(1-\varrho)\mathfrak{g}(\check{s},\check{r}(\check{s}))d\check{s}\\&+\frac{1}{\Phi(\eta-1)}\int_0^1(1-\check{s})^{\eta-2}(1-\varrho)\mathfrak{g}(\check{s},\check{r}(\check{s}))d\check{s}\\&+\frac{1}{\Phi(\eta)}\int_0^{\varrho}(\varrho-\check{s})^{\eta-1}\mathfrak{g}(\check{s},\check{r}(\check{s}))d\check{s}.\end{aligned}$$

Theorem 5.1. *Consider the operators $\mathcal{T}:\Psi\cup\varDelta\to\Psi\cup\varDelta$ are given by:*

$$\begin{aligned}\mathcal{T}\check{r}(\varrho)=&\frac{1}{\Phi(\eta)}\int_0^1(1-\check{s})^{\eta-1}(1-\varrho)\mathfrak{g}(\check{s},\check{r}(\check{s}))d\check{s}\\&+\frac{1}{\Phi(\eta-1)}\int_0^1(1-\check{s})^{\eta-2}(1-\varrho)\mathfrak{g}(\check{s},\check{r}(\check{s}))d\check{s}\\&+\frac{1}{\Phi(\eta)}\int_0^{\varrho}(\varrho-\check{s})^{\eta-1}\mathfrak{g}(\check{s},\check{r}(\check{s}))d\check{s},\end{aligned}$$

Such that

(1) *for all $\check{r}\in\Psi,\varpi\in\varDelta$ and $\mathfrak{g}\colon[0,1]\times[0,\infty)\to[0,\infty)$, satisfies*

$$|\mathfrak{g}(\check{s},\check{r}(\check{s}))-\mathfrak{g}(\check{s},\varpi(\check{s}))|\leq\sigma^{\frac{1}{2}}|\check{r}(\check{s})-\varpi(\check{s})|,$$

(2)

$$\sup_{\varrho\in(0,1)}\left|\frac{1-\varrho}{\Phi(\eta+1)}+\frac{1-\varrho}{\Phi(\eta)}+\frac{\varrho^{\eta}}{\Phi(\eta+1)}\right|^2=\ell<1.$$

Then $\mathcal{T}$ has a unique solution.

Proof. Let $\check{r}\in\Psi,\varpi\in\varDelta$,

$$\begin{aligned}|\mathcal{T}\check{r}(\varrho)-\mathcal{T}\varpi(\varrho)|^2=&\Bigg|\frac{1}{\Phi(\eta)}\int_0^1(1-\check{s})^{\eta-1}(1-\varrho)(\mathfrak{g}(\check{s},\check{r}(\check{s})-\mathfrak{g}(\check{s},\varpi(\check{s}))))d\check{s}\\&+\frac{1}{\Phi(\eta-1)}\int_0^1(1-\check{s})^{\eta-2}(1-\varrho)(\mathfrak{g}(\check{s},\check{r}(\check{s})-\mathfrak{g}(\check{s},\varpi(\check{s}))))d\check{s}\\&+\frac{1}{\Phi(\eta)}\int_0^{\varrho}(\varrho-\check{s})^{\eta-1}(\mathfrak{g}(\check{s},\check{r}(\check{s})-\mathfrak{g}(\check{s},\varpi(\check{s}))))d\check{s}\Bigg|^2\end{aligned}$$

$$\begin{aligned}
&\leq \bigg(\frac{1}{\Phi(\eta)}\int_0^1 (1-\check{s})^{\eta-1}(1-\varrho)\bigg|(\mathfrak{g}(\check{s},\check{r}(\check{s})-\mathfrak{g}(\check{s},\varpi(\check{s}))))\bigg|d\check{s}\\
&+\frac{1}{\Phi(\eta-1)}\int_0^1 (1-\check{s})^{\eta-2}(1-\varrho)\bigg|(\mathfrak{g}(\check{s},\check{r}(\check{s})-\mathfrak{g}(\check{s},\varpi(\check{s}))))\bigg|d\check{s}\\
&+\frac{1}{\Phi(\eta)}\int_0^\varrho (\varrho-\check{s})^{\eta-1}\bigg|(\mathfrak{g}(\check{s},\check{r}(\check{s})-\mathfrak{g}(\check{s},\varpi(\check{s}))))\bigg|d\check{s}\bigg)^2\\
&\leq \bigg(\frac{1}{\Phi(\eta)}\int_0^1 (1-\check{s})^{\eta-1}(1-\varrho)\sigma^{\frac{1}{2}}|\check{r}(\check{s})-\varpi(\check{s})|d\check{s}\\
&+\frac{1}{\Phi(\eta-1)}\int_0^1 (1-\check{s})^{\eta-2}(1-\varrho)\sigma^{\frac{1}{2}}|\check{r}(\check{s})-\varpi(\check{s})|d\check{s}\\
&+\frac{1}{\Phi(\eta)}\int_0^\varrho (\varrho-\check{s})^{\eta-1}\sigma^{\frac{1}{2}}|\check{r}(\check{s})-\varpi(\check{s})|d\check{s}\bigg)^2\\
&=\sigma|\check{r}(\varrho)-\varpi(\varrho)|^2\bigg(\frac{1}{\Phi(\eta)}\int_0^1 (1-\check{s})^{\eta-1}(1-\varrho)d\check{s}\\
&+\frac{1}{\Phi(\eta-1)}\int_0^1 (1-\check{s})^{\eta-2}(1-\varrho)d\check{s}+\frac{1}{\Phi(\eta)}\int_0^\varrho (\varrho-\check{s})^{\eta-1}d\check{s}\bigg)^2\\
&=\sigma|\check{r}(\varrho)-\varpi(\varrho)|^2\bigg(\frac{1-\varrho}{\Phi(\eta+1)}+\frac{1-\varrho}{\Phi(\eta)}+\frac{\varrho^\eta}{\Phi(\eta+1)}\bigg)^2\\
&\leq \sigma|\check{r}(\varrho)-\varpi(\varrho)|^2\sup_{\varrho\in(0,1)}\bigg(\frac{1-\varrho}{\Phi(\eta+1)}+\frac{1-\varrho}{\Phi(\eta)}+\frac{\varrho^\eta}{\Phi(\eta+1)}\bigg)^2\\
&=\ell\sigma|\check{r}(\varrho)-\varpi(\varrho)|^2\\
&\leq \sigma|\check{r}(\varrho)-\varpi(\varrho)|^2.
\end{aligned}$$

So, we have

$$\bigg|\mathcal{T}\check{r}(\varrho)-\mathcal{T}\varpi(\varrho)\bigg|^2\leq \sigma|\check{r}(\varrho)-\varpi(\varrho)|^2,$$

i.e.,

$$-\frac{\sup_{\varrho\in[0,1]}\bigg|\mathcal{T}\check{r}(\varrho)-\mathcal{T}\varpi(\varrho)\bigg|^2}{\sigma\pi}\geq -\frac{\sup_{\varrho\in[0,1]}|\check{r}(\varrho)-\varpi(\varrho)|^2}{\pi}$$

$$\exp\bigg(-\frac{\sup_{\varrho\in[0,1]}\bigg|\mathcal{T}\check{r}(\varrho)-\mathcal{T}\varpi(\varrho)\bigg|^2}{\sigma\pi}\bigg)\geq \exp\bigg(-\frac{\sup_{\varrho\in[0,1]}|\check{r}(\varrho)-\varpi(\varrho)|^2}{\pi}\bigg),$$

thus, we have

$$\Omega_{\mathfrak{b}}(\mathcal{T}\check{r}(\varrho),\mathcal{T}\varpi(\varrho),\sigma\pi)\geq \Omega_{\mathfrak{b}}(\check{r}(\varrho),\varpi(\varrho),\pi).$$

Hence, Theorem 3.1 of all axioms are satisfied, we can find that the Caputo fracional solution (5.1).

□

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