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A Review on Mathematical Methods to Approximate $\mu-$Values

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Abstract. In this article, we review a number of iterative and analytical techniques to approximate structured singular values or μ-values which is a straightforward generalization of singular values for square and rectangular matrices. The μ-value is a well-known tool which acts as a strong link between numerical linear algebra and control theory. The computation of structured singular value provides a platform to study and discuss stability, performance, and robustness of the system. Furthermore, we review some very important literature that discusses the applications of structured singular values in different areas of engineering.

1. Introduction

The mathematical approximation to structured singular value for a given real or complex valued matrix (n-dimensional case) and a set with a block diagonal structure that represent the sets of structured uncertainties were introduced by J. C. Doyle [1] as a tool to analyze and synthesis of the linear time-invariant systems (LTI systems in control) which appears in the control theory. The methods developed by Doyle to compute structured singular values involve the minimization of the largest singular value, an admissible perturbation Δ from the uncertainty set.

The review paper gives a fairly brief introduction to some well-known methods developed by using various tools from mathematics and engineering to compute structured singular values. In this paper, we mainly focus on those numerical methods that are developed to approximate structured singular values from below. The main objective to develop such a μ-theory was to study and discuss both robustness and performance-like properties corresponding to the linear time-invariant feedback systems which does appear in control.

The computation of μ-values in an exact manner is very crucial and tedious, unfortunately it is a NP-hard problem [2]. This motivates us to develop some numerical methods for the

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approximation of structured singular values both above and below. The numerical approximation of lower bounds of structured singular values allows us to study and discuss the instability of LTI feedback systems appearing in control. Furthermore, the numerical approximation or exact computation of structured singular values from above gives a message to discuss the stability of a LTI system under control.

The organization of this article as follows: In Section 2 of this paper, we review a number of well-known iterative methods to approximate structured singular values from below while making use of some established mathematical techniques.

2. Mathematical methods to compute structured singular values

2.1. **Feedback systems analysis via structured perturbations.** A most generic approach to analyze the Linear Time Invariant (LTI) systems with structured uncertainties consists of a new generalized spectral theory of matrices is presented in [1]. The proposed theory addresses the norm-bounded type of perturbation problem with an arbitrary structure. For matrix $M \in \mathbb{C}^{n,n}$ *or* $\mathbb{R}^{n,n}$ which has a block diagonal perturbation, a mathematical function μ was introduced that provides both necessary and sufficient conditions to the structured perturbation problems. The properties of the μ function were given with the matrix algebraic terms. Techniques were developed to compute the μ in some important special cases. Furthermore, examples and discussions in computing were also provided for a much better understanding of the μ function.

The block-diagonal perturbation: The analysis based on singular values establishes a framework to develop the multi-loop generalizations to the classical single-loop techniques. But, unfortunately, the singular value techniques have their own limitations, for instance, the analysis of linear multi-variable feedback system having two multiplicative perturbations appears at the inputs and outputs.

In singular value techniques, the system can be isolated into two perturbations as a single perturbation having two-block-diagonal structure. At a later stage, the block-diagonal perturbation is written with a one full matrix perturbation.

The analysis of differential sensitivity to singular values at a single point relative to perturbations at other points, is nothing but an extension to singular values. But, unfortunately, this does not holds true for large perturbations, and this yields to directional sensitivity information.

The matrix problem of determining both necessary and sufficient conditions so that $\lambda_i(I + M\Delta) \neq 0, \ \forall i$. The partial solution to the general block-diagonal perturbation problem, and solution to three or fewer blocks is presented in [1].

In [1], a very many important properties of structured singular values are provided, these includes:

$P_1 : \mu_{\mathbb{B}_1}(A) \geq 0$, for any A.

$P_2 : \mu_{\mathbb{B}_1}(\alpha A) = |\alpha| \mu_{\mathbb{B}_1}(A)$, for any A, and for all $\alpha \in \mathbb{C}$.

$P_3 : \mu_{\mathbb{B}_1}(AB) \leq \sigma_1(A) \mu_{\mathbb{B}_1}(B)$, where $\sigma(A)$ is the largest singular value of A.

$P_4 : \mu_{\mathbb{B}_1}(\Delta) = \sigma_1(\Delta),\ \forall\ \Delta \in \mathbb{B}_1$.

P_5 : Let $\Delta_0 = \{\lambda I : \lambda \in \mathbb{C}\}$, then $\mu_{\mathbb{B}_1}(A) = \rho(A)$, where $\rho(A)$ is the spectral radius of A.

P_6 : Let $\Delta = \{\Delta_0 : \Delta \in \mathbb{C}^{n,n}\}$, then $\mu_\Delta(A) = \sigma_1(\Delta)$.

$P_7 : Sup\, \mu_{\mathbb{B}_1}(A) = \|A\|_\infty$.

P_8 : Let $\Delta = \{diag(\Delta_1, \Delta_2, \cdots, \Delta_n) : \Delta_i \in \mathbb{C}^{n,n}\}$, then $\mu_\Delta(A) = \mu_\Delta(D^{-1}MD)$, where $D = diag(d_1, \cdots, d_n),\ |d_i| \geq 0$.

P_9 : Let $\Delta_0 = \{diag(\Delta_1, \Delta_2, \cdots, \Delta_n) : \Delta_i \in \mathbb{C}^{n,n}\}$, then $\rho(A) < \mu_{\Delta_0}(A) < \sigma_1(A)$.

P_{10} : From P_8, and P_9 we have that $\mu_{\mathbb{B}_1}(A) = \mu_{\mathbb{B}_1}(D^{-1}AD) \leq inf\, \sigma_1(D^{-1}AD)$, where inf is taken over D.

For the differentiability properties of singular values, the necessary tools were developed in [1] to compute the **gradients** for singular values.The following new results on the computation of $\mu-$values were established.

Theorem 2.1. *The structured singular value* $\mu(M) = 1$ *if and only if* $0 \in \nabla_2(M)$, *with* $\nabla_2(M)$ *defined in [1].*

Theorem 2.2. *The largest singular value* $\sigma_{max}(M) = \mu(M)$ *if and only if* $0 \in Co\,\nabla_2$, *with* $Co\,\nabla_2$ *defined in [1], and* $n \leq 3$.

We refer interested readers to [1] to see two numerical examples and a detailed discussions regarding the computational experience.

2.2. **An iterative method to yield the lower bounds of μ.** A new iterative method in order to approximate μ-values, is developed and then investigated [3]. The developed numerical technique is based on an algorithm which is a two-level algorithm, i.e., an inner-outer algorithm. In case of the inner algorithm, we aim to formulate and then solve a system of ODEs (gradient type) corresponding to an optimization problem induced on the manifold which is yield by the defined structure. In the case of the outer algorithm, a Newton's type iteration is used for the adjustment of the desired perturbation level denoted by ϵ. The inner-outer algorithm μ-values from below instead of above.

2.2.1. *Inner-Algorithm.* Theorem 1 confess to compute an admissible $\Delta \in \mathbb{B}^*$, where $\mathbb{B}^*$ is the set consisiting upon the block diagonal matrices having only pure complex perturbations. We refer to [3] for definitions of sets of block diagonal matrices.

Theorem 2.3. *Consider that* $\Delta_{opt} = diag\,(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_2 I_{rS};\ \Delta_1, \Delta_2, \cdots, \Delta_F) : \|\Delta_{opt}\|_2 = 1$, *is an extremizer (in the sense of local) of structured epsilon spectral value set* $\Delta_\epsilon^{\mathbb{B}}(M)$. *Additionally, we assume that* $\epsilon M \Delta_{opt}$ *has a largest eigenvalue in magnitude (which is a simple eigenvalue too)* $\lambda = |\lambda| e^{\iota\theta}, 0 \leq \theta \leq 2\pi$ *having both right and left eigenvectors* x *and* y. *These eigenvectors are scaled such that* $s = e^{\iota\theta} y^* x > 0$. *Furthermore, we partition the eigenvectors as*

$$x = \left(x_1^T, \cdots, x_S^T;\ x_{S+1}^T, \cdots, x_{S+F}^T\right)^T,$$

and

$$z = M^* y = \left(z_1^T, \cdots, z_s^T;\ z_{s+1}^T, \cdots, z_{s+F}^T\right).$$

We assume that $z_k^* x_k \neq 0,\ \forall\ k = 1, 2, \cdots, S$ *and* $\|z_{S+h}\|_2 \cdot \|x_{s+h}\|_2 \neq 0,\ \forall\ h = 1, 2, \cdots, F$. *Then, we have that* $|\delta_k| = 1,\ \forall\ k = 1, 2, \cdots, S$, *and* $\|\Delta_h\|_2 = 2,\ \forall\ h = 1, 2, \cdots, F$.

An important conclusion is presented in [3] to replace full blocks with matrices which having rank one. In turn this permits to work with matrix Frobenius norm rather than that of 2-norm. The reason is that for rank-1 matrices, both matrix 2-norm and matrix Frobeneius norm are equal. This further helps to search for local extremizer.

Theorem 2.4. *[3] Let* $\Delta_{opt} = diag\,(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_2 I_{rS}; \Delta_1, \Delta_2, \cdots, \Delta_F)$ *is an extremizer (in the sense of local). Assume that* λ, x, z *such that all these quantities are considered as in above Theorem 2.3. Furthermore, suppose that the non-degeneracy conditions for full blocks holds. Then each block* Δ_h *possesses a singular value having the value exactly equal to* 1 *corresponding to the singular vectors*

$$u_h = \gamma_h \frac{z_{S+h}}{\|z_{S+h}\|_2} \qquad v_h = \gamma_h \frac{x_{S+h}}{\|x_{S+h}\|_2}, \quad \textit{for} \quad |\gamma_h| = 1\,.$$

Additionally, the matrix valued function

$$\Delta_* = diag\left(\delta_1 I_{r_1}, \delta_2 I_{r_2}, \cdots, \delta_2 I_{rs};\ u_1 v_1^*, \cdots, u_F v_F^*\right)$$

acts as an extremizer (in the sense of local), that is, $\rho(\epsilon M \Delta_{opt}) = \rho(\epsilon M \Delta_*)$.

The system (gradient type) of ODE's is constructed and solved in [3] whereas the solution of system of ODE's yields a local extremizer on manifold given as $\mathbb{B}_1^*$:

$$\dot{\Delta} = D_1 P_{\mathbb{B}^*}(z x^*) - D_2 \Delta\,,$$

whereas $\|x(t)\|_2 = 1$ and is corresponding to an eigenvalue (the simple one) $\lambda(t)$ of matrix $\epsilon M \Delta(t)$, $\epsilon > 0$. The matrices $D_1(t)$, $D_2(t)$ depends on $\Delta(t)$ and $P_{\mathbb{B}^*}(z x^*)$ is the orthogonal projection of matrices $z x^*$ on $\mathbb{B}^*$. The stationary points of $\dot{\Delta}(t)$ are computed with following Theorem 3.

Theorem 2.5. *[3] Let* $\Delta(t)$ *and* $z(t)$ *is maximum simple non-zero eigenvalue of the perturbed matrix* $\epsilon M \Delta$ *having both right and left eigenvectors* $x(t),\ y(t)$. *Also, we assume* $z(t) = M^* y(t)$, *then we have*

$$\frac{d}{dt}\left|\lambda(t)\right|^2 = 0 \Leftrightarrow \dot{\Delta}(t) = 0 \Leftrightarrow \Delta(t) = D\, P_{\mathbb{B}^*}\left(z(t)\, x^*(t)\right),$$

for a matrix (of diagonal nature) $D \in \mathbb{B}^*$. *Additionally, if* $z(t)$ *possesses the maximum modulus over the set* $\Delta_\epsilon^{\mathbb{B}^*}(M)$, *then the matrix* D *is positive diagonal matrix.*

In [3], the mathematical results for approximation of bounds from below for μ-values for a mixed type of both real and complex uncertainties and even for more general cases are also presented and analyzed.

2.2.2. *Outer Algorithm.* For the outer algorithm, the following Theorem 4 is given in [3] to compute the change in $\lambda(\epsilon)$ w.r.t $\epsilon > 0$.

Theorem 2.6. *[3] Let $\Delta \in \mathbb{B}^*$, and $\lambda(\epsilon)$, $\epsilon > 0$ is simple largest eigenvalue of $\epsilon M\Delta(\epsilon)$. Suppose that $x(\epsilon)$ and $y(\epsilon)$ are both right and left eigenfunctions corresponding to $\epsilon M\Delta(\epsilon)$. Let $z(\epsilon) = M^* y(\epsilon)$, then*

$$\frac{d|\lambda(\epsilon)|}{d\epsilon} = \frac{1}{|y(\epsilon)^* x(\epsilon)|}\left(\sum_{i=1}^{s} \left|z_i(\epsilon)^* x_i(\epsilon)\right| + \sum_{j=1}^{F} \left\|z_{s+j}(\epsilon)\right\| \|y_{s+j}(\epsilon)\|\right) > 0.$$

Furthermore, for the numerical statistics, the results obtained by [3] compared with Matlab function `mussv` and those obtained with new algorithm are much better than those obtained with the `mussv` function.

2.3. **Computing μ-values with low-rank ODE's based techniques.** Approximation of bounds from below corresponding to structured singular values are estimated in [4]. The presented mathematical approach is based on an algorithm which is an inner-outer algorithm. The representation of matrices for a finite symmetric groups S_n on field of numbers (only complex but not real) are considered, also the numerical testing are carried via use of the Matlab function `mussv` and that of an algorithm [3]. The comparison for μ-values bounds show an effectiveness of algorithm given in [3].

The numerical treatment of μ-values against representations for a class of matrices for the finite symmetric groups S_3 and S_4 of the field of numbers (only complex but not real) are presented in [5]. The comparison of results on bounds of μ-values are carried out via numerical experimentation on a family of matrices corresponding to symmetric groups. For the computation of μ-values, it has been considered only pure complex perturbations which are in the form block-diagonal structures.

In [6], the computation of μ-values from below are presented and analyzed. In particular the rotary electrical machines, is presented. The computation of structured singular values are presented for both real and complex uncertainties in the form of structured and unstructured matrices. For the numerical experimentation, a low-rank based numerical technical [3] is being used.

The numerical computation of μ-values from below against companion matrices with a mathematical approach based upon ODE's is developed [7]. The perturbation set consists of block diagonal matrices, which having both structured blocks (the repeated numbers of real scalar blocks), and unstructured type of blocks.

2.4. **A new mathematical technique for approximation of real μ-values.** The study a real uncorrelated parameter uncertainty, a new algorithm to compute lower bounds (to give tighter results) of structured singular values is presented in [8].

Furthermore, it is conjectured that the solutions corresponding to real μ-valued problems always look like as a 2-dimensional face of an n-dimensional hypercube in the parameter space. This conjecture provides the basis of the lower-bound algorithm, which is presented by authors in [8].

In general, the control systems are considered to be robust for the plant dynamics in the sense that they provide closed loop stability, and the robust performance. The plants transfer function is represented in term of real valued physical parameters. These physical parameters includes aerodynamic coefficients, both electrical and mechanical tools.

The real valued physical parameters are modelled as the uncertain gains in the linear feedback system. These parameters are presented with a block-diagonal structure Δ. These physical real valued parameters are written along the main diagonal of the perturbation matrix Δ. The rest of the system interconnections are described with the transfer function matrix $M(s)$ at various level of frequencies s. For the stability, robustness, and performance of the system, then one can aim to compute structured singular values of $M(s)$ with respect to Δ.

The main advantages of the new algorithm compared to methods developed by de Gaston and Chang et al are:

(a) The new algorithm consists of simple matrix operations.

(b) The iteration is on only a single variable.

(c) The new algorithm return the original values of "worst-case" parameters.

(d) The new algorithm does not require computing convex hull for real μ.

(e) For real μ, it provides an exact approximation to μ-values from below for a wide range of real matrices.

New Algorithm: Let $d = (d_1, \cdots, d_n) \in \mathbb{R}^{n,1}$, and let $\Delta = diag(d)$. Also, let $\|d\|_\infty = max|d_i|$, which denotes the size of perturbation matrix Δ. Furthermore, $\|d\|_\infty = \sigma_{max}(\Delta)$. Then, the aim is to find

$$\frac{1}{\mu} = min\{\|d\|_\infty : \ det(I - M\Delta) = 0\},$$

where "min" is over $d \in \mathbb{R}^{n,1}$, and $M \in \mathbb{C}^{n,n}$, the given matrix.

By using de Gaston's method, the exact worst-case solutions d satisfies $\|d\|_\infty = \frac{1}{\mu}$, $\Delta = diag(d)$, which have no more than two elements d_i from vector d, and does satisfy $|d_i| < \frac{1}{\mu}$.

The new algorithm aim to search over the set of solutions d to $det(I - M\Delta) = 0$, where two elements of d satisfies $|d_i| = k$. The new algorithm finds a lower bound for μ, and a solution d. These causes the matrix $(I - M\Delta)$ to have at-least one of the eigenvalue to be exactly equal to zero. We refer interested readers [8] for a complete discussion on new algorithm, and numerical experimentation's and applications.

2.5. **A nonlinear programming based methodology to approximate real structured singular values from below.**

A new formulation of real structured singular values as a mathematical problem for a non-linear programming problem have been developed in [9]. Furthermore, a new mathematical optimization based methodology called F-modified sub-gradient (F-MSG) was developed to the approximation the lower bounds of real μ-values. The F-MSG algorithm consider a much larger class of non-convex programming models as follows:

For $\epsilon \geq 0$ a small parameter η represent the compact hyper-rectangular domain

$$(\delta_1, \delta_2, \cdots, \delta_m, \lambda_1, \lambda_2) \in \eta \quad with \quad \delta_i, \lambda_i \in \mathbb{R}.$$

Then the quantity $\mu_\Delta(M)$ is obtained via following two steps.

Step-1:

$$\min_{(\delta_1,\cdots,\delta_m,\ \lambda_1,\lambda_2)} \max_i \left(|\delta_i| + (\lambda_1^r + \lambda_2^r)\right)\lambda$$

such that

$$\begin{cases} f_R(\delta_1, \cdots, \delta_m) - \lambda_1^p = 0, \\ f_I(\delta_1, \cdots, \delta_m) - \lambda_2^p = 0, \\ (\delta_1, \cdots, \delta_m, \lambda_1, \lambda_2) \in \eta. \end{cases}$$

Step-2:

$$\mu_\Delta(M) = \begin{cases} \dfrac{1}{\max|\delta_i|} & \text{if} \quad \|f_R, f_I\| < \epsilon, \\ 0 & \text{else.} \end{cases}$$

The quantity $\lambda \geq 10^5$ (called a penalty parameter) is checked and fixed. The powers in step-I are taken as $r \in \{2, 4, 6, \cdots\}$ and $p \in \{1, 3, 5, \cdots\}$. Furthermore, authors need their algorithm to study the stability analysis of an inverted pendulum while computing real structured singular value lower bounds.

An algorithm with above steps generalizes the modelling, and F-MSG algorithm is used for the solution. For $M(s)$, the definition of structured singular value is generalized to

$$\mu_\Delta(M(s)) = Sup\ \mu_\Delta(M(\mathbf{i}\omega)),$$

with "Sup" taken over $\omega \in \mathbb{R}_+$, $\mathbb{R}_+$ denotes an interval $[0, \infty]$, and $\mathbf{i} = \sqrt{-1}$. The matrix $M(s)$ is called a transfer function matrix, and ω is the frequency variable. The computation of structured singular values can be carried out various discrete levels of frequencies $[\omega_L, \omega_U] \subset \mathbb{R}_+$. Then, structured singular values is obtained by maximizing over $\omega \in [\omega_L, \omega_U]$, that is,

$$\mu_\Delta(M(s)) = max\ \mu_\Delta(M(\mathbf{i}\omega)).$$

The solution of above optimization problem is given by the following F-MSG algorithm.

F-MSG Algorithm: F-MSG algorithm was developed by Karimbeyli, and is a generalized version of the modified sub-gradient algorithm. The F-MSG algorithm is given as follows:

Initialization step: Take U_B in such a way that $U_B >> |\delta_i|\ \forall i = 1 : m$, and sufficient small $\epsilon_1,\ \epsilon_2 \geq 0$. Take $L_B = 0$, and let q be a positive number.

Step 1: Take $n = 1$;

Step 2: Take (v_1^n, c_1^n) with $v_1^n \in \mathbb{R}^{2,1}$, $c_1^n > 0$, and $\phi(1)$ such that $0 < \phi(1) < q$. Set $H_n = \frac{L_B + U_B}{2},\ j = 1$, and then go to **Step 3**.

Step 3: For given (v_1^n, c_1^n) , solve the constraint satisfaction problem:

$$\textbf{Find } K \in \Omega \textbf{ such that } L(K, v_j^n, c_j^n) = f(K) + c_j^n\|h(K)\| - v_j^n h(K) \leq H_n.$$

If solution does not exist, then go to **Step 6**. If solution exists, then go to **Step 5**, otherwise go to **Step 4**.

Step 4: Update (v_1^n, c_1^n) with

$$v_{j+1}^n = v_j^n - \alpha h(K_j), \quad c_{j+1}^n = c_j^n + (1+\alpha)p_j \|h(K_j)\|,$$

where p_j is a positive scalar step size given as

$$0 \le p_j = \frac{\delta\alpha(H_n - L(K_j, v_j^n, c_j^n))}{(\alpha^2 + (1+\alpha)^2 \|h(K_j)\|^2)},$$

with $\alpha > 0,\ 0 < \delta < 2$.

Step 5: Set $U_B = f(K_j)$. If $\epsilon_2 > U_B - L_B$, $\mu_\Delta(M) = \frac{1}{U_B}$, and **STOP**; otherwise $n = n+1$, and go back to **Step 2.**

Step 6: Set $L_B = H_n$. If $\epsilon_2 > U_B - L_B,\ \ \mu_\Delta(M) = \frac{1}{U_B}$, and **STOP**; otherwise set $n = n+1$, and go back to **Step 2.**

In the algorithm, L_B, and U_B are known as lower and upper bounds of the optimization problem

$$min\ f(K)$$

while the constraints are $h(K) = 0,\ K \in \Omega$, where $K = (\delta_1, \cdots, \delta_m, \lambda_1, \lambda_2)$, $f(K) = max_i(\delta_i) + (\lambda_1^r + \lambda_2^r)\lambda$, $h(K) = [h_1(K) h_2(K)]^T$ with $h_1(K) = f_R(\delta_1, \delta_2, \cdots, \delta_m) - \lambda_1^p = 0$, $h_2(K) = f_1(\delta_1, \cdots, \delta_m) - \lambda_2^p = 0$, and Ω is a large compact hyper-rectangle having values of K.

The following theorem 2.7 shows that if the value of H_n is feasible, then the sequence K_j of the solutions to problem in **Step 3** will start converging to feasible solution of primal problem.

Theorem 2.7. *Let Ω, be a compact set and let f and h are continuous functions on Ω. Let $p_j\|h(K_j)\| + c_j^n - \|v_j^n\| > \phi(j)$ holds true. Then, $\|h(K_j)\| \to 0$ as $j \to \infty$, for every $H_n \ge \bar{H}$ if $\{K_j\},\ \ j = 1, 2, \cdots$.*

The following theorem 2.8 gives main convergence result for F-MSG algorithm.

Theorem 2.8. *Let (K_j, v_j, c_j) be an iteration calculated at steps 3 and 4 of F-MSG algorithm for $H_n = \bar{H}$. Let $\{h(K_j)\}$ denotes the bounded sequence and each (v_{j+1}, c_{j+1}) is determined for $\delta = 1$. Furthermore, if steps 3-4 generates an infinite sequence $L_j = L(K_j, v_j, c_j)$ of augmented Lagrangian, then $L_j \to \bar{H}$ as $j \to \infty$.*

2.6. **Geometrical formulation of bounds of μ-value.** Geometrical interpretation-based approach is presented in [10] to compute μ-value lower bounds against pure real perturbations. An important feature of the presented geometrical approach is resetting parametric search space. This search space is very much independent of the number of parameter repetition in the structured uncertainty matrix. An algorithm (only for lower bounds)is presented that combines the randomization and optimization methods to deal with the μ-value problem. The computational algorithm is successfully used for two extremely challenging high-order real μ-analysis problems. These problems were taken from the field of aerospace and system biology.

The geometrical analysis for the subset of the uncertain parametric space satisfy the singularity constraint in the μ-value lower bound:

$$det(I_n - M(\mathbf{i}\omega)\Delta),$$

where $det(\cdot)$ denotes the determinant of a matrix, and $\mathbf{i} = \sqrt{-1}$, the imaginary unit, and $\omega \in \mathbb{R}$, the frequency. The notation Δ denotes the diagonal matrix representing the uncertainty from the set of block-diagonal matrices $\hat{\Delta}$.

The singularity condition is written in term of both real and imaginary parts, that is,

$$f_R(\Delta) = Re\{det(I_n - M(\mathbf{i}\omega))\Delta\},$$

$$f_I(\Delta) = Img\{det(I_n - M(\mathbf{i}\omega))\Delta\},$$

where $\Delta \in \hat{\Delta} = \{diag(\delta_1 I_{r_1}, \cdots, \delta_p I_{r_p}) : \delta_i \in \mathbb{R}\}$, r_i is an element belonging to set of the natural numbers, the positive real integers, and I_{r_i} is $r_i \times r_i$ identity matrix for $i = 1 : p$. The singularity condition can be formulated as

$$F_R = \{\Delta \in \hat{\Delta} : f_R(\Delta) = 0\},$$

$$F_I = \{\Delta \in \hat{\Delta} : f_I(\Delta) = 0\},$$

and then $\mu-$value is defined as

$$\mu_\Delta(M) = \frac{1}{min\ \bar{\sigma}(\Delta)}, \quad if\ \ F_R \cap F_I \neq \phi,$$

otherwise it is equal to 0.

For $\Delta = 0$, the $det(I_n - M\Delta) = 1$, and hence $f_R(0) = 1$, and $f_I(0) = 0$. This implies that the manifold F_I does passes through the origin and the manifold F_R does not passes through the origin. We refer interested readers to see [10] for a complete and detailed discussions on the solution to the formulation of geometrical problems, and example applications.

2.7. **Detailed comparative analysis for μ-lower bound algorithm.** In [11], a comparison for significant numerical based techniques developed to approximation of μ-values from below are presented.

2.7.1. *Power Method.* The power algorithm [12] aims at solving non-convex optimization problem which is given as: For given $M \in \mathbb{C}^{n,n}$ and $\Delta = \mathrm{diag}(\Delta_1, \cdots, \Delta_N)$, where Δ is defined in [12], thus

(1) $\delta_i \in [-1, 1]$ if $\Delta_i = \delta_i I_{n_i}$.
(2) $\Delta_i^* \Delta_i = I_{n_i}$ if Δ_i is complex matrix for all i.

The local maximum is approximated via a fixed-point iteration, and the local maximum turns out to be the lower bound of structured singular values.

2.7.2. *A gain based algorithm for the approximation of μ.* In [13], μ-value problem is presented as a worse-case H_∞-problem, and then, using the algorithm developed in [14] to compute μ-values from below of pure real uncertainties, the power method [12] is used while considering the full complex blocks for numerical approximation of μ-values from below.

2.7.3. *Exponential time methods.* In [15], an experimental time method is developed for non-repeated real type of uncertainties and later generalized in [15] while making use of the theorem (the mapping theorem) of [17]: the $k\mathbb{B}_\Delta$ is given by an admissible perturbation $\Delta \rightarrow \mathrm{Det}(I - M\Delta)$ must possesses in admissible convex hull against 2^N vectors of $k\mathbb{B}_\Delta$. The position of these images with origin yields the computation of μ-values lower bounds. Algebraic methods [8] are used for the class of non-repeated uncertainties (only real but not complex) searches to destabilize the set $\Delta \in k\mathbb{B}_\Delta$ so that the all δ_i except 2 attains a maximum magnitude k, which is obtained while making use of the matrix algebraic operations; see [18] for further details.

2.7.4. *Poles migration techniques.* In the poles migration techniques, the idea is to make use of the characterization (first order) of poles variation for the system which uncertainty and are caused by a minimal change $\mathrm{d}\Delta$ of Δ. Poles start moving to imaginary axis while computing a series of such perturbations $\mathrm{d}\Delta$ of Δ. To each and every pole λ of $M(S)$ in [19], a wast number of programming problems (quadratic in nature) are being solved. The Frobenius norm of perturbations, that is, $||\mathrm{d}\Delta||_F$ bring λ on the imaginary axis. On contrary, power technique [12] is very fast to approximate μ-value lower bounds, but most oftenly it is enough from convergence problems once the pure real uncertainties. It motivates [20] to develop a three-step procedure for pure real uncertainties. The power method is used for each and every point of the considered domain of the frequency grid and gives better convergence.

2.7.5. *Optimization based techniques.* Likewise, the power algorithm [12] and the proposed techniques in [15] solve the μ-value problems while simply replacing $\rho_\mathbb{R}$ having spectral radius ρ, but they do make use of steepest ascent algorithm and conjugate gradient algorithm. These techniques solve optimization problem (non-convex) by using the standard optimization tools (the non-linear once), for instance, the `fminon` function used in Matlab toolbox

$$\min_{\Delta \in \mathbb{B}} F(\Delta), \qquad \text{such that} \quad |I - M\Delta| = 0,$$

where $\mathbb{B}$, a set of block diagonal matrices and also $F(\cdot)$ denotes maximum singular value of the matrix. A formulation singular to the above problem is considered in [9] for pure real uncertainties

$$\min_{\substack{\Delta \in \mathbb{B} \\ \lambda_1, \lambda_2}} F(\Delta) + (\lambda_1^q + \lambda_2^q) \quad \text{so that} \quad \begin{cases} \mathrm{Re}(|I - M\Delta|) = \lambda_1 p, \\ \mathrm{Re}(|I - M\Delta|) = \lambda_2 p, \end{cases}$$

where both quantities p and q are odd and even positive integers, respectively. The quantity $\lambda \geq 10^5$ is a penalty parameter. A modified sub-gradient algorithm is used to solve the optimization problem, introduced in [21].

2.8. **Gain-based lower bound technique for both real and mixed μ-problems.** A novel lower bound technique to the approximation of both real and mixed μ-problem is presented in [13]. The new technique uses a related worse-case gain problem for the numerical approximation of real blocks and the standard power algorithm determine the complex blocks. The testing provides good bounds for μ-values. The gain-based lower bounds algorithm (LBA) compute the exact real μ-values in lower dimensions. Furthermore, it switches over to worst-case gain search for larger-dimensional problems.

Gain-Based Algorithm (GBA): Let $M \in \mathbb{C}^{n_R,n_R}$, and we consider the problem for the computing the lower bounds for pure real $\mu-$ values, that is, $\mu_{\Delta_R}(M_R)$, with $\Delta_R := \{\Delta = diag(\delta_1 I_{k_1}, \cdots, \delta_r I_{k_r}) : \delta_i \in \mathbb{R}\}$, M_R is the complex valued matrix where R denotes the real block structure used in the computation of structured singular values. It is clear from the definition of structured singular values that Δ_R does satisfies $det(I - M_R\Delta_R) = 0$, and in turn a lower bound $\frac{1}{\bar{\sigma}(\Delta_R)}$ bounded by $\mu_{\Delta_R}(M_R)$ yields. This implies that there exists $z \in \mathbb{C}^{n_R}$, $w \in \mathbb{C}^{n_R}$ which satisfies $z = M_R w$, and $w = \Delta_R z$, and such equations can be represented by Linear Fractional Transformations (LFT) $F_u(M_R, \Delta_R)$ with z, w denoting output of M_R and Δ_R.

The algebraic equations: $\begin{bmatrix} z \\ e \end{bmatrix} = \bar{M}_R \begin{bmatrix} w \\ d \end{bmatrix}$, $w = \Delta_R z$, and

$$\bar{M}_R = \begin{bmatrix} M_R & i_k \\ i_k^T M_R & 1 \end{bmatrix},$$

here $d, e \in \mathbb{C}$, the scalar disturbance and error signals. The above set of algebraic equations are well-posed and the disturbance to error relation is given by $e = F_u(\bar{M}_R, \Delta_R)d$ if $det(I - M_R\Delta_R) \neq 0$.

The GBA aims to solve the following optimization problem:

$$max\ |F_u(\bar{M}_R \Delta_R)|,$$

where **max** is taken over Δ_R, $\sigma(\bar{\Delta}_R) \leq \mu_l$, μ_l denotes the lower bound of structured singular value. The above optimization problem is non-convex and aims for the global maximizer, which is computationally expensive. Following theorem provides two interesting relations between $d - to - e$ gain and the distance of $(I - M_R\Delta_R)$ to singularity.

Theorem 2.1. *If there exists Δ_R such that $det(I - M_R\Delta_R) \neq 0$, and $|F_u(\bar{M}_R, \Delta_R)| \geq \gamma \geq 0$, then:*
1. $\exists$ a $\delta \in \mathbb{C}$, $|\delta| \leq \frac{1}{\gamma}$ such that $det(I - M_R\Delta_R - \delta i_k i_k^T) = 0$.
2. $\sigma_{min}(I - M_R\Delta_R) \leq \frac{1}{\gamma}$.

Algorithm 1: The GBA for real μ lower bound

Input: $M_R \in \mathbb{C}^{n,n}, \ \Delta_R, \ u_b, \ l_b$

Initilize $lb_{fac} = \frac{3}{4}$, and $cnt = 1$

while $cnt \leq N_{try}$ AND $l_b < u_b \ tol_{stop}$

$lb_{try} = lb + (u_b - l_b) \ lb_{fac}$,

$k := mod(cnt - 1, n_R) + 1$

$$\bar{M}_R = \begin{bmatrix} M_R & i_k \\ i_k^T M_R & 1 \end{bmatrix}$$

$\Delta_{R,try} := arg \ max|F_u(\bar{M}_R, \Delta_R)|$

if $rcound(I - M_R \Delta_{R,try}) < tol_{real}$

$lb = \frac{1}{\bar{\sigma}(\Delta_{R,try})}$

$\Delta_R = \Delta_{R,try}$

$lb_{fac} := \frac{1}{2}$

else

$lb_{fac} := max(\frac{1}{32}, \frac{lb_{fac}}{2})$

end

$cnt = cnt + 1$

end

Return: $= \Delta_{R,lb}$

Algorithm 2: The GBA for mixed μ lower bound

Input: $M, \ \Delta, \ u_b, \ l_b$

Initilize $lb_{fac} = \frac{3}{4}$, and $cnt = 1$

while $cnt \leq N_{try}$ AND $l_b < u_b \ tol_{stop}$

$lb_{try} = lb + (u_b - l_b) \ lb_{fac}$,

$k := mod(cnt - 1, n_R) + 1$

$$\bar{M}_R = \begin{bmatrix} M_R & i_k \\ i_k^T M_R & 1 \end{bmatrix}$$

$\Delta_{R,try} := arg \ max|F_u(\bar{M}_R, \Delta_R)|$

if $rcound(I - M_R \Delta_{R,try}) < tol_{real}$

$lb = \frac{1}{\bar{\sigma}(\Delta_{R,try})}$

$\Delta = diag(\Delta_{R,try}, 0)$

$lb_{fac} := \frac{1}{2}$

else

$\bar{M}_C := F_u(M, \Delta_R)$

Power iteration on $\bar{M}_C$ to find $\Delta_{C,try}$

$\Delta_{try} := diag(\Delta_{R,try}, \Delta_{C,try})$

if $rcound(I - M\Delta_{try}) < tol_{complex}$ AND $\frac{1}{\bar{\sigma}(\Delta_{try})} \geq lb$

$lb := \frac{1}{\bar{\sigma}(\Delta_{try})}$, Return := Δ, lb

2.9. **Computing tight bounds on the real μ-values.** In [23], news tools are presented for the computation of real μ-values lower bounds. These tools compute tight lower bounds for higher-order plants which are subject to pure real type of uncertainties. First approach reduces the order of uncertainty matrix (real) by using the μ-sensitivity function so that the experimental time lower bounds algorithm can be used. Search for a worse-case real destabilizing uncertainty as a nonlinear optimization problem is carried out by the second approach. The stability properties for integrated flight, and population control system for an experimental vertical take-off as well as landing aircraft configuration are studied with the help of both approaches. The results on the tight lower bounds for structured singular values are presented.

2.10. **Computation of μ via moment LMI relaxation technique.** New algorithm based upon moment LMI relation is presented by [24] in order to approximate μ-values from above to mixed real, and complex perturbations. The idea is to formulate structured singular value approximation as a non-convex polynomial optimization problem, this in turn is relaxed into a sequence of optimization problems (convex type) via moment-based relaxation methodologies. In the paper, the authors also provide the heuristic to approximate μ-values lower bounds. The numerical results on bounds for μ-values are provided which yields results for tighter bounds once compared with well-known Matlab function `mussv`.

The perturbation Δ has a block-diagonal structure and is defined in [24]. The constraint $\|\Delta\| < r$ can be rewritten as

$$\begin{bmatrix} r^2 I & \Delta \\ \Delta^H & I \end{bmatrix} > 0.$$

The following theorem provides necessary and sufficient conditions to check the robust non-singularity of the perturbed matrix $(I - M\Delta)$ for the set of structured uncertainties.

Theorem 2.9. *For $r \geq 0$, a real number, the perturbed matrix $I - M\Delta$ has at-least one of its eigenvalue to be zero for all possible uncertainties Δ if and only if the solution to following optimization problem is*

$$max\ \|x\|_2^2, \ \ s.t\ \ (I - M\Delta)x = 0, \ \begin{bmatrix} r^2 I & \Delta \\ \Delta^H & I \end{bmatrix} > 0.$$

*The **max** is taken over $x \in \mathbb{C}^{s_2}$, and $\Delta \in \hat{\Delta}$, $\hat{\Delta}$ has a block-diagonal structure.*

Result: The structured singular value of M w.r.t $\hat{\Delta}$ is given by

$$\mu_{\hat{\Delta}}(M) = \frac{1}{\sqrt{t^*}},$$

where t^* is the solution to following optimization problem:

$$t^* = min\ t\ \ s.t\ \ t \geq 0, \ \|x\|_2^2 \geq \bar{x}, \ (I - M\Delta)x = 0, \ \begin{bmatrix} r^2 I & \Delta \\ \Delta^H & I \end{bmatrix} > 0.$$

Here, $\bar{x} \geq 0$, is an arbitrary. The **min** is taken over $t \in \mathbb{R}, \ x \in \mathbb{C}^{s_2}, \ \Delta \in \hat{\Delta}$.

The following theorem show how the convergence of t^* leads to the computation of structured singular values.

Theorem 2.10. *The following results holds true:*

1. $h \geq 1$, t^*_h *is the lower bound of* t^*, *that is,* $t^*_h \leq t^*$.
2. t^*_h *converges from below to* t^*, *that is,* $t^*_h \leq t^*_{h+1} \leq t^*$, *and* $\lim_{h\to\infty} t^*_h = t^*$
3. *For any* $h \geq 1$, $det(I - M\Delta) = 0, \quad \forall \Delta \in \hat{\Delta}$.

Furthermore, $\mu_{\hat{\Delta}}(M) \leq \frac{1}{\sqrt{t^*_{h+1}}} \leq \frac{1}{\sqrt{t^*_h}}$, *and* $\lim_{h\to\infty} \frac{1}{\sqrt{t^*_h}} = \mu_{\hat{\Delta}}(M)$.

For a complete details on the computation of lower bound of structured singular values, and numericalm experimentations, we refer interested readers to see [24].

3. Conclusion

In this piece of research work, we have reviewed and presented a number of well-known mathematical methods for the computation of μ-values. Most of these mathematical methods are iterative. The main aim is to present fair introduction to numerical methods for approximation of lower bounds of μ-values. Further, we have reviewed a number of articles that discuss the practical nature of structured singular values, particularly for engineering applications.

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